

A CHARACTERIZATION OF PROJECTIVE UNITARY EQUIVALENCE OF FINITE FRAMES AND APPLICATIONS*

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Abstract. Many applications of finite tight frames (e.g., the use of SICs and mutually unbiased bases (MUBs) in quantum information theory and harmonic frames for the analysis of signals subject to erasures) depend only on the vectors up to projective unitary equivalence. It is well known that two finite sequences of vectors in inner product spaces are unitarily equivalent if and only if their respective inner products (Gramian matrices) are equal. Here we present a corresponding result for the projective unitary equivalence of two sequences of vectors (lines) in inner product spaces, i.e., that a finite number of (Bargmann) projective (unitary) invariants are equal. This result is equivalent to finding a rank-one matrix completion of a certain matrix. We give an algorithm to recover the sequence of vectors (up to projective unitary equivalence) from a small subset of these projective invariants and apply it to SICs, MUBs, and harmonic frames. We also extend our results to the projective similarity of vectors.

Key words. projective unitary equivalence, Gramian matrix, Gramian, low-rank matrix completion, harmonic frame, equiangular tight frame, symmetric informationally complete positive operator valued measure (SIC-POVM), mutually unbiased bases (MUBs), triple products, Bargmann invariants, frame graph, cycle space, chordal graph, projective symmetry group

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1. Introduction.

1.1. Motivation. Finite frames provide redundant and stable expansions, which have numerous applications [1]. These include quantum measurements and codes using SICs [2] [3], [4] and mutually unbiased bases (MUBs) [5], [6], [7], and signal analysis using harmonic frames which are robust to erasures [8], [9], [10]. They are also prominent in the theory of real and complex spherical t -designs [11], [12] and spherical 2-distance sets [13]. Many applications of finite frames, such as those above, depend only on the vectors up to projective unitary equivalence. This point of view naturally leads to the notion of a *fusion frame* (where the vectors are replaced by one-dimensional orthogonal projections and more generally d -dimensional orthogonal projections). Such applications motivate our study of when sequences of vectors are projectively unitarily equivalent. In addition, the study of lines in $\mathbb{R}^d$ and $\mathbb{C}^d$, in particular, equiangular lines [14], is effectively the study of configurations of unit vectors up to projective unitary equivalence.

1.2. Key definitions. Finite sequences of vectors $\Phi = (v_j)$ and $\Psi = (w_j)$ in (real or complex) inner product spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are *unitarily equivalent* if there is a unitary map $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$w_j = Uv_j \quad \forall j$$

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and *projectively unitarily equivalent* if there is a unitary map U and unit scalars c_j , such that

$$w_j = c_j U v_j \quad \forall j,$$

or equivalently

$$w_j w_j^* = U(v_j v_j^*) U^* \quad \forall j$$

A finite spanning sequence of vectors for an inner product space is also called a *finite frame*.

Orthogonal bases can be generalized as follows (cf. [1], [15]). We say a sequence of n vectors (v_j) is a *tight frame* for a d -dimensional inner product space $\mathcal{H}$ if for some $A > 0$

$$\|f\|^2 = \frac{1}{A} \sum_j |\langle f, v_j \rangle|^2 \quad \forall f \in \mathcal{H}.$$

By the polarization identity this is equivalent to the expansion

$$f = \frac{1}{A} \sum_j \langle f, v_j \rangle v_j \quad \forall f \in \mathcal{H},$$

Examples of tight frames of more than d vectors include SICs, MUBs, and harmonic frames.

The *Gramian (matrix)* of $\Phi = (v_j)_{j=1}^n$ is

$$\text{Gram}(\Phi) = [\langle v_k, v_j \rangle]_{j,k=1}^n.$$

We take our inner products to be linear in the first variable. It is well known (cf. [16]) that

- Φ and Ψ are unitarily equivalent if and only if

$$(1.1) \quad \text{Gram}(\Phi) = \text{Gram}(\Psi),$$

- Φ and Ψ are projectively unitarily equivalent if and only if

$$(1.2) \quad \text{Gram}(\Psi) = C^* \text{Gram}(\Phi) C,$$

where C is the diagonal matrix with diagonal entries c_j .

Clearly, (1.1) can be used to verify unitary equivalence, as can (1.2) be used to verify projective unitary equivalence for real inner product spaces (where $c_j \in \{-1, 1\}$; cf. Example 2.3). For complex inner product spaces, we have no knowledge of c_j , other than $|c_j| = 1$, and so (1.2), i.e.,

$$(1.3) \quad \langle w_k, w_j \rangle = c_k \overline{c_j} \langle v_k, v_j \rangle \quad \forall j, k,$$

does not provide a practical method for verifying projective unitary equivalence in general (see section 8). To our knowledge, projective unitary equivalence has been calculated in only a few cases, which have a very specific structure. These are $\mathcal{H} = \mathbb{R}^d$ by considering all possible $c_j = \pm 1$ in (1.2) or by using two-graphs in the case of equiangular lines [17] for a specific construction of MUBs via symplectic spreads and Hadamard matrices [18], [19], [20], for SICs [21], and for the equal-norm tight frames of n vectors in $\mathbb{C}^2$ robust to $n - 2$ erasures [10]. Here we present a general method for determining whether any two finite sequence of vectors are projectively unitarily equivalent.

The question of determining projective unitary equivalence between frames via (1.2) has a structural form similar to that of the question of when a frame $\Phi = (v_j)$ is *scalable*, i.e., there are scalars c_j for which $\Psi = (c_j v_j)$ is a tight frame [22], [23]. Since a frame is tight if and only if its Gramian is a scalar multiple of an orthogonal projection matrix and $\text{Gram}(\Psi) = C^* \text{Gram}(\Phi) C$, it follows that if Φ is not tight, then such a scaling is possible if and only if the scalars c_j don't all have the same modulus.

Following [24], we define the *m-vertex Bargmann invariants* or *m-products* of a sequence of n vectors $\Phi = (v_j)$ to be

$$(1.4) \quad \Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m}) := \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_m}, v_{j_1} \rangle, \quad 1 \leq j_1, \dots, j_m \leq n.$$

In particular (cf. [21]), we define the *triple products* to be

$$(1.5) \quad T_{jkl} := \Delta(v_j, v_k, v_l) = \langle v_j, v_k \rangle \langle v_k, v_l \rangle \langle v_l, v_j \rangle.$$

We observe that the *m-products* are *projective unitary invariants*, e.g., for $m = 3$

$$\begin{aligned} \Delta(c_j U v_j, c_k U v_k, c_l U v_l) &= \langle c_j U v_j, c_k U v_k \rangle \langle c_k U v_k, c_l U v_l \rangle \langle c_l U v_l, c_j U v_j \rangle \\ &= c_j \overline{c_k} \langle U v_j, U v_k \rangle c_k \overline{c_l} \langle U v_k, U v_l \rangle c_l \overline{c_j} \langle U v_l, U v_j \rangle \\ &= \langle v_j, v_k \rangle \langle v_k, v_l \rangle \langle v_l, v_j \rangle \\ (1.6) \quad &= \Delta(v_j, v_k, v_l). \end{aligned}$$

We define the *frame graph* (cf. [25]; also known as the *correlation network* [26]) of a sequence of vectors (v_j) to be the graph with vertices $\{v_j\}$ (or the indices j themselves) and

$$\text{an edge between } v_j \text{ and } v_k, j \neq k \iff \langle v_j, v_k \rangle \neq 0.$$

Clearly, projectively unitarily equivalent frames have the same frame graph.

1.3. Outline. We will show (Theorem 3.1) that a sequence of vectors (v_j) is determined up to projective unitary equivalence by all its *m-products*. Our proof relies on the fact that certain small subsets of the *m-products* are sufficient. These depend on which of the *m-products* are nonzero, which is conveniently encapsulated by the *frame graph*.

In section 2, generalizing the results of [21] for SICs, we show that if the common frame graph of $\Phi = (v_j)$ and $\Psi = (w_j)$ is complete, then they are projectively unitarily equivalent if and only if their 3-products (triple products) are equal (Theorem 2.2). Later we will show this condition extends (Corollary 5.1), e.g., to the case when the frame graph is chordal. We apply this result to sequences of equiangular lines (including SICs), then give an example to show that the 3-products do not determine projective unitary equivalence in general (Example 2.5).

In sections 3, 4, and 5, we show that projective unitary equivalence is characterized by the *m-products* (Theorem 3.1). This leads to methods for calculating the projective symmetry group (see [27] for details). We show that it is sufficient to consider only a small subset of these projective invariants, which can be determined from the frame graph (Corollary 3.2). We give an algorithm for constructing all sequences with given *m-products* (Theorem 4.1) and consider the classification of MUBs (Corollary 5.4).

In sections 6 and 7, we apply our results to the classification of sequences of vectors up to (projective) similarity (Theorem 6.2) and the classification of harmonic frames up to projective unitary equivalence (Theorem 7.2).

In section 8, we explore a fascinating connection between our results and the rank-one completions of partial matrices, which was pointed out by one of the referees. Namely, determining projective unitary equivalence is equivalent to finding a rank-one matrix completion (which is already understood). Low-rank matrix completions play an important role in the fields of compressed sensing and phase retrieval, where they have been extensively studied. Since we know in advance exactly where the nonzero entries are and exactly what the rank is, sampling results such as those in [28] are not applicable.

2. Complete frame graphs. A sequence of $n \geq d$ unit vectors (v_j) in $\mathbb{C}^d$ is *equiangular* if for some $C \geq 0$

$$|\langle v_j, v_k \rangle| = C, \quad j \neq k.$$

For $C > 0$, such a sequence has a complete frame graph (no zero inner products), as does a generic sequence of vectors. In [21], it was shown that d^2 equiangular vectors in $\mathbb{C}^d$ are characterized up to projective unitary equivalence by their triple products (3-products). Here we modify the argument to when the frame graph is complete. We then show, by an example, that this result does not extend to a general sequence of vectors.

The *angles* of a sequence of vectors $\Phi = (v_j)$ are the $\theta_{jk} \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ defined by

$$\langle v_j, v_k \rangle = |\langle v_j, v_k \rangle| e^{i\theta_{jk}}, \quad \langle v_j, v_k \rangle \neq 0.$$

Since $\langle v_j, v_k \rangle = \overline{\langle v_k, v_j \rangle}$, these satisfy

$$\theta_{jk} = -\theta_{kj}.$$

A sequence of vectors may have few angles, e.g., an orthogonal basis has no angles.

LEMMA 2.1. *Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors in Hilbert spaces, with angles θ_{jk} and θ'_{jk} . Then Φ and Ψ are projectively unitarily equivalent if and only if*

1. *their Gramians have entries with equal moduli, i.e.,*

$$|\langle w_j, w_k \rangle| = |\langle v_j, v_k \rangle| \quad \forall j, k;$$

2. *their angles are “gauge equivalent,” i.e., there exist $\phi_j \in \mathbb{T}$ with*

$$\theta'_{jk} = \theta_{jk} + \phi_j - \phi_k \quad \forall j, k.$$

Proof. First suppose that Φ and Ψ are projectively unitarily equivalent, i.e., $w_j = c_j U v_j$, where U is unitary and $c_j = e^{i\phi_j}$. Then

$$\begin{aligned} e^{i\theta'_{jk}} |\langle w_j, w_k \rangle| &= \langle w_j, w_k \rangle = \langle c_j U v_j, c_k U v_k \rangle = c_j \overline{c_k} \langle U v_j, U v_k \rangle = e^{i(\phi_j - \phi_k)} \langle v_j, v_k \rangle \\ &= e^{i(\phi_j - \phi_k)} e^{i\theta_{jk}} |\langle v_j, v_k \rangle|. \end{aligned}$$

By equating the moduli and then the arguments we obtain 1 and 2.

Conversely, suppose that 1 and 2 hold. Let $\tilde{v}_j := e^{i\phi_j} v_j$. Then

$$\begin{aligned} \langle \tilde{v}_j, \tilde{v}_k \rangle &= \langle e^{i\phi_j} v_j, e^{i\phi_k} v_k \rangle = e^{i(\phi_j - \phi_k)} \langle v_j, v_k \rangle = e^{i(\phi_j - \phi_k)} e^{i\theta_{jk}} |\langle v_j, v_k \rangle| \\ &= e^{i\theta'_{jk}} |\langle w_j, w_k \rangle| = \langle w_j, w_k \rangle. \end{aligned}$$

Thus (w_j) is unitarily equivalent to $(\tilde{v}_j)$, which is projectively unitarily equivalent to (v_j) , and so Ψ and Φ are projectively unitarily equivalent. $\square$

We observe that $|\langle v_j, v_k \rangle|$ can be calculated from the triple products of (1.5), since

$$(2.1) \quad T_{jjj} = \langle v_j, v_j \rangle^3 = \|v_j\|^6, \quad T_{jjk} = \langle v_j, v_j \rangle |\langle v_j, v_k \rangle|^2 = T_{jjj}^{\frac{1}{3}} |\langle v_j, v_k \rangle|^2.$$

THEOREM 2.2 (characterization). *Let $\Phi = (v_j)_{j \in J}$ and $\Psi = (w_j)_{j \in J}$ be finite sequences of vectors in Hilbert spaces. Then*

1. *Φ and Ψ are unitarily equivalent if and only if their Gramians are equal, i.e.,*

$$\langle v_j, v_k \rangle = \langle w_j, w_k \rangle \quad \forall j, k;$$

2. *if the frame graphs of Φ and Ψ are complete, then they are projectively unitarily equivalent if and only if their triple products are equal, i.e.,*

$$\langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle = \langle w_j, w_k \rangle \langle w_k, w_\ell \rangle \langle w_\ell, w_j \rangle \quad \forall j, k, \ell.$$

Proof. The condition for unitary equivalence is well known. It is included in the theorem only for the purpose of comparison. We now prove 2.

First suppose that Φ and Ψ are projectively unitarily equivalent, i.e., $w_j = c_j U v_j$. Then by (1.6) their triple products are equal.

Conversely, suppose that Φ and Ψ have the same triple products, and their common frame graph is complete, i.e., all the triple products are nonzero.

It follows from (2.1) that their Gramians have entries with equal moduli, i.e.,

$$|\langle v_j, v_k \rangle| = |\langle w_j, w_k \rangle| \quad \forall j, k.$$

Let θ_{jk} and θ'_{jk} be the angles of Φ and Ψ . Since the triple products have the polar form

$$T_{jkl} = \langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle = e^{i(\theta_{jk} + \theta_{k\ell} + \theta_{\ell j})} |\langle v_j, v_k \rangle \langle v_k, v_\ell \rangle \langle v_\ell, v_j \rangle|,$$

we obtain

$$\theta_{jk} + \theta_{k\ell} + \theta_{\ell j} = \theta'_{jk} + \theta'_{k\ell} + \theta'_{\ell j}.$$

Fix ℓ , and rearrange this, using $\theta_{k\ell} = -\theta_{\ell k}$ and $\theta'_{k\ell} = -\theta'_{\ell k}$, to get

$$\theta'_{jk} = \theta_{jk} + (\theta_{\ell j} - \theta'_{\ell j}) + (\theta_{k\ell} - \theta'_{k\ell}) = \theta_{jk} + (\theta_{\ell j} - \theta'_{\ell j}) - (\theta_{\ell k} - \theta'_{\ell k}) = \theta_{jk} + \phi_j - \phi_k,$$

where $\phi_j := \theta_{\ell j} - \theta'_{\ell j}$, i.e., the angles of Φ and Ψ are gauge equivalent. Since the conditions of Lemma 2.1 hold, it follows that Φ and Ψ are projectively unitarily equivalent. $\square$

The real case is closely connected with the theory of *two-graphs* (cf. [17]) as follows.

Example 2.3 (equiangular lines in $\mathbb{R}^d$). Suppose that $\Phi = (v_j)$ is a sequence of $n > d$ equiangular unit vectors (lines) in $\mathbb{R}^d$, i.e., there is an $\alpha > 0$ with

$$\langle v_j, v_k \rangle = \pm \alpha, \quad j \neq k.$$

Then the Gramian matrix has the form

$$G_\Phi = \text{Gram}(\Phi) = I + \alpha S_\Phi,$$

where $S = S_\Phi$ is a *Seidel matrix*, i.e., S is symmetric, with zero diagonal, and off diagonal entries ± 1 . Moreover, each Seidel matrix is associated with a sequence of

equiangular lines. Each Seidel matrix S is in turn associated with the graph $\text{gr}(S)$ which has an edge between $j \neq k$ if and only if $S_{jk} = -1$. Let $\mathcal{C}$ be the set of all diagonal matrices with diagonal entries ± 1 . Then the projective unitary equivalence class of Φ is uniquely determined by all the possible Gramian matrices of its members, i.e.,

$$\mathcal{G} := \{CG_{\Phi}C^* : C \in \mathcal{C}\},$$

and hence all the possible Seidel matrices,

$$\mathcal{S} := \{CG_{\Phi}C^* : C \in \mathcal{C}\},$$

and in turn the corresponding graphs $\text{gr}(\mathcal{S})$. The set of graphs $\text{gr}(\mathcal{S})$ is called the *switching class* of $\text{gr}(S_{\Phi})$, or a *two-graph*. Since the frame graph of Φ is complete, Theorem 2.2 gives that projective unitary equivalence class of Φ (equivalently $\mathcal{G}$, $\mathcal{S}$, or $\text{gr}(\mathcal{S})$) is in 1–1 correspondence with the triple products of Φ . It suffices to consider only those triple products with distinct indices, since if an index is repeated twice or thrice, then by (2.1) the triple products depend only on α . In this way, the two-graph is in 1–1 correspondence with the triple products

$$\{T_{jkl} = \pm\alpha^3 : j, k, \ell \text{ are distinct}\}.$$

Since these triple products take only two values, which are independent of the ordering of the indices, they can be described by giving the collection of the subsets $\{j, k, \ell\}$ where they take one of these values. This association leads to the equivalent definition of a two-graph as a set of (unordered) triples chosen from a finite vertex set X such that every unordered quadruple from X contains an even number of triples of the two-graph.

Example 2.4 (equiangular lines in $\mathbb{C}^d$). If Φ is a sequence of n equiangular unit vectors (lines) in $\mathbb{C}^d$, with $C > 0$, then up to projective unitary equivalence Φ is determined by its triple products. This result was given in [21] for the special case $n = d^2$. Such a configuration has $C = \frac{1}{\sqrt{d+1}}$ and is known as a SIC or a SIC-POVM (symmetric informationally complete positive operator valued measure).

We now give an example to show that projective unitary equivalence is not always characterized by the triple products if the frame graph is not complete. We observe that the m -products are closed under complex conjugation, i.e.,

$$(2.2) \quad \overline{\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m})} = \Delta(v_{j_m}, \dots, v_{j_2}, v_{j_1}).$$

Example 2.5 (n -cycle). Let (e_j) be the standard basis vectors in $\mathbb{C}^n$. Fix $|z| = 1$, and let

$$v_j := \begin{cases} e_j + e_{j+1}, & 1 \leq j < n, \\ e_n + ze_1, & j = n. \end{cases}$$

Then the frame graph of (v_j) is the n -cycle $(v_1, \dots, v_n)$, and so the only nonzero m -products for distinct vectors are

$$(2.3) \quad \Delta(v_j) = \|v_j\|^2 = 2, \quad 1 \leq j \leq n,$$

$$(2.4) \quad \Delta(v_j, v_{j+1}) = |\langle v_j, v_{j+1} \rangle|^2 = 1, \quad 1 \leq j < n,$$

$$(2.5) \quad \Delta(v_1, v_2, \dots, v_n) = z,$$

and their complex conjugates. Therefore different choices of z give projectively inequivalent frames. Thus, for $n > 3$, the vectors (v_j) are not defined up to projective unitary equivalence by their triple products.

3. Characterization of projective unitary equivalence. We now show that a sequence of n vectors is determined up to projective unitary equivalence by its m -products for $1 \leq m \leq n$. This is done by showing that if Φ and Ψ have the same m -products, then we can find unit modulus scalars c_j satisfying (1.3).

The diagonal entries of the Gramian of Φ are given by the 1-products and the moduli of its off diagonal entries by the 2-products with distinct arguments, i.e.,

$$(3.1) \quad \Delta(v_j) = \|v_j\|^2, \quad \Delta(v_j, v_k) = |\langle v_j, v_k \rangle|^2, \quad j \neq k.$$

We recall the following facts:

- Every finite graph Γ has a spanning tree (forest) $\mathcal{T}$.
- For each edge $e \in \Gamma \setminus \mathcal{T}$, there is a unique cycle in $e \cup \mathcal{T}$ called the *fundamental cycle* (corresponding to e).

THEOREM 3.1 (characterization). *Two sequences (v_j) and (w_j) of n vectors are projectively unitarily equivalent if and only if their m -products are equal, i.e.,*

$$\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m}) = \Delta(w_{j_1}, w_{j_2}, \dots, w_{j_m}), \quad 1 \leq j_1, \dots, j_m \leq n, \quad 1 \leq m \leq n.$$

Proof. We have already observed that projectively unitarily equivalent sequences have the same m -products. We therefore suppose that Φ and Ψ have the same m -products and will show that we can choose $c_1, \dots, c_n$ so that (1.3) holds. By (3.1), the Gramians of Φ and Ψ are block diagonal (with entries having the same moduli), with blocks given by the vertices of the connected components of the common frame graph. We therefore assume without loss of generality that there is a single block, i.e., the frame graph Γ is connected.

Spanning tree argument. Find a spanning tree $\mathcal{T}$ of Γ with root vertex r . By working outward from the root r , we can multiply the vertices $v \in \Gamma \setminus \{r\}$ by unit scalars c_v so that for an edge $\{v_j, v_k\} \in \mathcal{T}$, (1.3) holds, i.e.,

$$\langle w_k, w_j \rangle = c_k \overline{c_j} \langle v_k, v_j \rangle.$$

In this way, we can choose $c_1, \dots, c_n$ so that (1.3) holds for all edges $\{v_j, v_k\} \in \mathcal{T}$.

Completing cycles. It remains only to show that (1.3) also holds for all edges $e = \{v_j, v_k\} \in \Gamma \setminus \mathcal{T}$. Let $(v_j, v_k, v_{\ell_1}, \dots, v_{\ell_r})$ be the fundamental cycle given by the edge $e = \{v_j, v_k\}$. Since the m -products are equal, and the other edges in this cycle belong to $\mathcal{T}$, we obtain

$$\begin{aligned} \Delta(w_j, w_k, w_{\ell_1}, \dots, w_{\ell_r}) &= \langle w_j, w_k \rangle \langle w_k, w_{\ell_1} \rangle \langle w_{\ell_1}, w_{\ell_2} \rangle \cdots \langle w_{\ell_r}, w_j \rangle \\ &= \langle w_j, w_k \rangle c_k \overline{c_{\ell_1}} \langle v_k, v_{\ell_1} \rangle c_{\ell_1} \overline{c_{\ell_2}} \langle v_{\ell_1}, v_{\ell_2} \rangle \cdots c_{\ell_r} \overline{c_j} \langle v_{\ell_r}, v_j \rangle \\ &= (c_k \overline{c_j} \langle w_j, w_k \rangle) \langle v_k, v_{\ell_1} \rangle \langle v_{\ell_1}, v_{\ell_2} \rangle \cdots \langle v_{\ell_r}, v_j \rangle \\ &= \langle v_j, v_k \rangle \langle v_k, v_{\ell_1} \rangle \langle v_{\ell_1}, v_{\ell_2} \rangle \cdots \langle v_{\ell_r}, v_j \rangle \\ &= \Delta(v_j, v_k, v_{\ell_1}, \dots, v_{\ell_r}), \end{aligned}$$

and cancellation gives (1.3) for the edge $\{v_j, v_k\} \in \Gamma \setminus \mathcal{T}$. $\square$

Above we associated the (directed) m -cycle $(v_{j_1}, \dots, v_{j_m})$ in the frame graph with the nonzero m -product $\Delta(v_{j_1}, \dots, v_{j_m})$. For $m \geq 3$, all m -products can be calculated from those corresponding to simple cycles, since if a cycle crosses at a , we have

$$(3.2) \quad \Delta(v_1, \dots, v_s, a, w_1, \dots, w_t, a) = \Delta(v_1, \dots, v_s, a) \Delta(w_1, \dots, w_t, a).$$

We call a subset of the m -products (or the corresponding indices) a *determining set* for the m -products if all m -products can be determined from them.

The *cycle space* of a finite graph Γ is the set of its *Eulerian subgraphs* (those with vertices of even degree). This can be viewed as a $\mathbb{Z}_2$ -vector space, where the addition is the symmetric difference of sets. From this, it follows that the cycle space is spanned by the simple cycles (and its elements are disjoint unions of cycles). If the sum of two simple cycles is a simple cycle (in the frame graph), then the corresponding m -product can be determined from those of the summands

$$\begin{aligned} & \Delta(v_1, \dots, v_s, e_1, \dots, e_r) \Delta(w_1, \dots, w_t, e_r, \dots, e_1) \\ &= \Delta(e_1, e_2) \cdots \Delta(e_{r-1}, e_r) \Delta(v_1, \dots, v_s, e_1, w_1, \dots, w_t, e_r). \end{aligned}$$

Combining these observations, we have that a determining set for the m -products of Φ is given by the 2-products and the m -products corresponding to a basis for the cycle space of the frame graph. The fundamental cycles corresponding to a spanning tree (forest) of a finite graph form a basis for the cycle space, called a *fundamental cycle basis*. We therefore have the following strengthening of Theorem 3.1.

COROLLARY 3.2. *A finite frame Φ , with frame graph Γ , is determined up to projective unitary equivalence by a determining set for the m -products, e.g.,*

1. *the 2-products,*
2. *the m -products, $3 \leq m \leq n$, corresponding to a fundamental cycle basis (for the cycle space of Γ) formed from a spanning tree (forest) $\mathcal{T}$ for Γ .*

In particular, if M is the number of edges of $\Gamma \setminus \mathcal{T}$, then it is sufficient to know all of the 2-products and M of the m -products, $3 \leq m \leq n$.

Proof. It suffices to verify the condition of Theorem 3.1 for a determining set. $\square$

We now illustrate Theorem 3.1 by constructing all the possible Gramian matrices for a sequence of vectors $\Psi = (w_j)$ which is projectively unitarily equivalent to a given sequence $\Phi = (v_j)$ by using only the m -products of Φ .

Example 3.3. Let $\Phi = (e_j)$ be an orthonormal basis for $\mathbb{C}^3$, which has a Gramian matrix

$$\text{Gram}(\Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the frame graph is totally disconnected (see Figure 1), and so each projectively unitarily equivalent frame (w_j) is determined by the 1-products and 2-products using (3.1), i.e.,

$$\langle w_j, w_j \rangle = \langle v_j, v_j \rangle = 1, \quad |\langle w_k, w_j \rangle|^2 = |\langle v_k, v_j \rangle|^2 = 0, \quad j \neq k \implies \langle w_k, w_j \rangle = \delta_{jk}.$$

Thus (w_j) and Φ have the same Gramian and hence are unitarily equivalent. Alternatively, by (1.2), one has that the Gramians of all projectively unitarily equivalent frames are given by $C^* \text{Gram}(\Phi) C = \text{Gram}(\Phi)$.

Now we give an example where the spanning tree and cycle completing arguments are not trivial.

Example 3.4. Let $\Phi = (v_j)$ be three equally spaced unit vectors in $\mathbb{R}^2$, viewed as vectors in $\mathbb{C}^2$. These have Gramian matrix

$$\text{Gram}(\Phi) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix},$$

and the frame graph is complete (see Figure 1). We now construct the Gramian matrices of every $\Psi = (w_j)$ which is projectively unitarily equivalent to Φ . A spanning tree with root v_1 is given by the path v_1, v_2, v_3 , and so the spanning tree argument gives

$$\langle w_1, w_2 \rangle = c_1 \bar{c}_2 \langle v_1, v_2 \rangle = -\frac{1}{2} c_1 \bar{c}_2, \quad \langle w_2, w_3 \rangle = c_2 \bar{c}_3 \langle v_2, v_3 \rangle = -\frac{1}{2} c_2 \bar{c}_3,$$

where c_1 is fixed (any choice will do) and c_2, c_3 are free unit modulus complex numbers. We only require the possible values of $\langle w_1, w_3 \rangle$. These are determined by completing the cycle (v_1, v_2, v_3) by adding the corresponding edge $\{v_1, v_3\}$, which gives the condition

$$\Delta(w_1, w_2, w_3) = \Delta(v_1, v_2, v_3),$$

i.e.,

$$\left(-\frac{1}{2} c_1 \bar{c}_2\right) \left(-\frac{1}{2} c_2 \bar{c}_3\right) \langle w_3, w_1 \rangle = \left(-\frac{1}{2}\right)^3 \implies \langle w_1, w_3 \rangle = -\frac{1}{2} c_1 \bar{c}_3.$$

Thus all possible Gramians for Ψ (projectively unitarily equivalent to Φ) are given by

$$\text{Gram}(\Psi) = \begin{pmatrix} 1 & -\frac{1}{2}a & -\frac{1}{2}ab \\ -\frac{1}{2}\bar{b} & 1 & -\frac{1}{2}b \\ -\frac{1}{2}\bar{a}\bar{b} & -\frac{1}{2}\bar{b} & 1 \end{pmatrix}, \quad |a| = |b| = 1.$$

Here we write $a = c_1 \bar{c}_2$, $b = c_2 \bar{c}_3$ to emphasize the fact there are two free parameters.

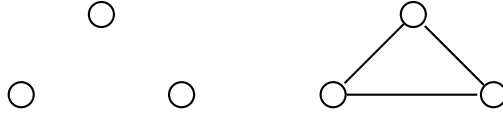


FIG. 1. The frame graph of an orthonormal basis for $\mathbb{C}^3$ (Example 3.3), and the frame graph for three equiangular vectors in $\mathbb{C}^2$ (Example 3.4).

4. Reconstruction from the m -products. We now state the characterization in a way which summarizes how all projectively unitary frames can be constructed from a small determining set of m -products.

THEOREM 4.1 (reconstruction). Suppose $\Phi = (v_j)$ is a frame of n vectors. Let Γ be the frame graph of Φ , $\mathcal{T}$ be a spanning tree (forest) for Γ , and

$N = \text{the number of edges in } \mathcal{T},$

$M = \text{the number of edges in } \Gamma \setminus \mathcal{T}.$

Then the collection of all Gramians $G = [\langle w_k, w_j \rangle]$ of frames $\Psi = (w_j)$ which are unitarily projectively equivalent to Φ can be parameterized by N free variables. More precisely, for each of the N edges $\{v_j, v_k\} \in \mathcal{T}$ (choose an order) we have a free variable

$$\langle w_j, w_k \rangle = |\langle v_j, v_k \rangle| a_{(j,k)}, \quad |a_{(j,k)}| = 1,$$

and for the remaining M edges $e = \{v_j, v_k\} \in \Gamma \setminus \mathcal{T}$, $\langle w_j, w_k \rangle$ is uniquely determined by equality of the m -products of Φ and Ψ for the fundamental cycle given by e .

Example 4.2. Let $\Phi = (v_j)$ be the two MUBs (see section 5) for $\mathbb{C}^2$ given by

$$\Phi = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right), \quad \text{Gram}(\Phi) = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

The frame graph Γ of Φ is the 4-cycle (v_1, v_3, v_2, v_4) . A spanning tree $\mathcal{T}$ is given by the path v_1, v_3, v_2, v_4 . Corresponding to the three edges of $\mathcal{T}$, we have three free variables

$$\langle w_1, w_3 \rangle = \frac{a}{\sqrt{2}}, \quad \langle w_3, w_2 \rangle = \frac{\bar{b}}{\sqrt{2}}, \quad \langle w_2, w_4 \rangle = \frac{c}{\sqrt{2}}.$$

The remaining inner product $\langle w_4, w_1 \rangle = \frac{1}{\sqrt{2}}\bar{z}$ is determined by the fundamental cycle given by $\{v_1, v_4\}$, i.e., by completing the 4-cycle

$$\langle w_1, w_3 \rangle \langle w_3, w_2 \rangle \langle w_2, w_4 \rangle \langle w_4, w_1 \rangle = \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle \langle v_2, v_4 \rangle \langle v_4, v_1 \rangle,$$

which gives

$$\bar{a}b\bar{c}\bar{z} = -1.$$

Thus all the Gramians of Ψ which are projectively unitarily equivalent to Φ are given by

$$G = \begin{pmatrix} 1 & 0 & \frac{\bar{a}}{\sqrt{2}} & -\frac{\bar{a}b\bar{c}}{\sqrt{2}} \\ 0 & 1 & \frac{\bar{b}}{\sqrt{2}} & \frac{\bar{c}}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 1 & 0 \\ -\frac{abc}{\sqrt{2}} & \frac{c}{\sqrt{2}} & 0 & 1 \end{pmatrix}, \quad |a| = |b| = |c| = 1.$$

This particular Φ is in fact determined up to projective unitary equivalence by just its 2-products. This is because Sylvester's criterion for G (as function of a, b, c, z) to be positive semidefinite gives

$$\det(G) = -\frac{1}{4} \frac{(bz + ac)^2}{abcz} = -\frac{1}{4} \left| \frac{bz}{ac} + 1 \right|^2 \geq 0 \implies \frac{bz}{ac} + 1 = 0 \implies z = -\frac{ac}{b}.$$

In contrast, the (v_j) of Example 2.5 for $n = 4$ also has frame graph a 4-cycle, but it is not determined up to projective unitary equivalence by its 2-products (and triple products).

Example 4.3. Let $\Phi = (v_j)$ be four equiangular vectors with $C > 0$. These exist in $\mathbb{C}^2$ (a SIC) and in $\mathbb{R}^3$ (the vertices of the tetrahedron). The frame graph of Φ is complete, and $M = 6 - 3 = 3$. Spanning trees (see Figure 2) include

$$\mathcal{T}_p := \text{the path } v_1, v_2, v_3, v_4,$$

$$\mathcal{T}_s := \text{the star graph with internal vertex } v_1 \text{ and leaves } v_2, v_3, v_4.$$

For $\mathcal{T}_p$, the fundamental cycles given by the edges $\{v_1, v_4\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$ are

$$(v_1, v_2, v_3, v_4), \quad (v_1, v_2, v_3), \quad (v_2, v_3, v_4).$$

For $\mathcal{T}_s$, the fundamental cycles given by the edges $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$ are

$$(v_1, v_2, v_3), \quad (v_1, v_2, v_4), \quad (v_1, v_3, v_4).$$

Thus Φ is determined up to projective unitary equivalence by its 2-products and either of the following sets of m -products:

$$\begin{aligned} &\Delta(v_1, v_2, v_3, v_4), \quad \Delta(v_1, v_2, v_3), \quad \Delta(v_2, v_3, v_4), \\ &\Delta(v_1, v_2, v_3), \quad \Delta(v_1, v_2, v_4), \quad \Delta(v_1, v_3, v_4). \end{aligned}$$

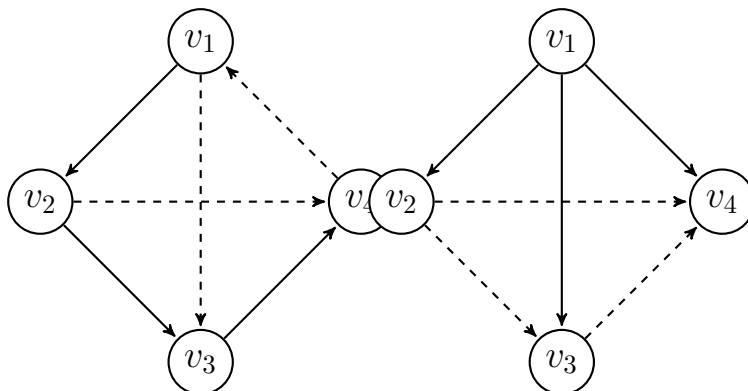


FIG. 2. The spanning trees $\mathcal{T}_p$ and $\mathcal{T}_s$ (and cycle completions) of Example 4.3.

5. Triple products, equiangular lines, and MUBs. The following special case of Corollary 3.2 is often useful.

COROLLARY 5.1 (triple products). *A finite frame Φ is determined up to projective unitary equivalence by its triple products (3-products) if the cycle space of its frame graph is spanned by 3-cycles (and so the cycle space has a basis of 3-cycles).*

Proof. The 2-products can be deduced from the triple products by using

$$\Delta(v_j, v_j, v_j) = \langle v_j, v_j \rangle^3, \quad \Delta(v_j, v_j, v_k) = \langle v_j, v_j \rangle |\langle v_j, v_k \rangle|^2. \quad \square$$

This generalizes Theorem 2.2.

Example 5.2 (chordal graphs). A graph is said to be *chordal* (or *triangulated*) if each of its cycles of four or more vertices has a chord, and so the cycle space is spanned by the 3-cycles. Hence a frame is determined by its triple products if its frame graph is chordal. The extreme cases are the empty graph (orthogonal bases) where there are no cycles and the complete graph where all subsets of three vectors lie on a 3-cycle (equiangular lines).

We now give an example (Corollary 5.4) where the cycle space of the frame graph has a basis of 3-cycles, but the frame graph is not chordal.

DEFINITION 5.3. A family of orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ for $\mathbb{C}^d$ is said to be *mutually unbiased* if for $r \neq j$

$$|\langle v, w \rangle|^2 = \frac{1}{d}, \quad v \in \mathcal{B}_r, \quad w \in \mathcal{B}_j.$$

We call $\mathcal{B}_1, \dots, \mathcal{B}_k$ a sequence of k MUBs.

We now show that for three or more MUBs the cycle space of the frame graph is spanned by the 3-cycles. This is not case for two MUBs (see Example 4.2).

COROLLARY 5.4 (MUBs). *A frame Φ consisting of three or more MUBs in $\mathbb{C}^d$, $d \geq 2$, is determined up to projective unitary equivalence by its triple products.*

Proof. It suffices to show that the cycle space of the frame graph Γ of Φ has a basis of 3-cycles. To this end, let $\mathcal{B}_j$, $j = 1, \dots, k$, be the MUBs for $\mathbb{C}^d$, so that Γ is a complete k -partite graph (with partite sets $\mathcal{B}_j$). Fix $v_1 \in \mathcal{B}_1$ and $v_2 \in \mathcal{B}_2$. A spanning tree $\mathcal{T}$ for Γ is given by taking an edge from v_1 to each vertex of $\mathcal{B}_j$, $j \neq 1$, and an edge from v_2 to each vertex of $\mathcal{B}_1 \setminus v_1$. Each of the remaining edges of $\Gamma \setminus \mathcal{T}$ gives a fundamental cycle. These have two types (see Figure 3):

1. $\frac{1}{2}d^2(k-1)(k-2)$ edges between vertices in $\mathcal{B}_r$ and $\mathcal{B}_s$, $r, s \neq 1$, which give fundamental 3-cycles (involving v_1),
2. $(d-1)((k-1)d-1)$ edges between vertices $u \in \mathcal{B}_1 \setminus v_1$ and $w \in \cup_{j \neq 1} \mathcal{B}_j \setminus v_2$, which give fundamental 4-cycles (u, w, v_1, v_2) . These can be written as a sum (symmetric difference) of the 3-cycles (u, w, v_2) and (v_1, v_2, w) .

Thus the cycle space is spanned by 3-cycles. $\square$

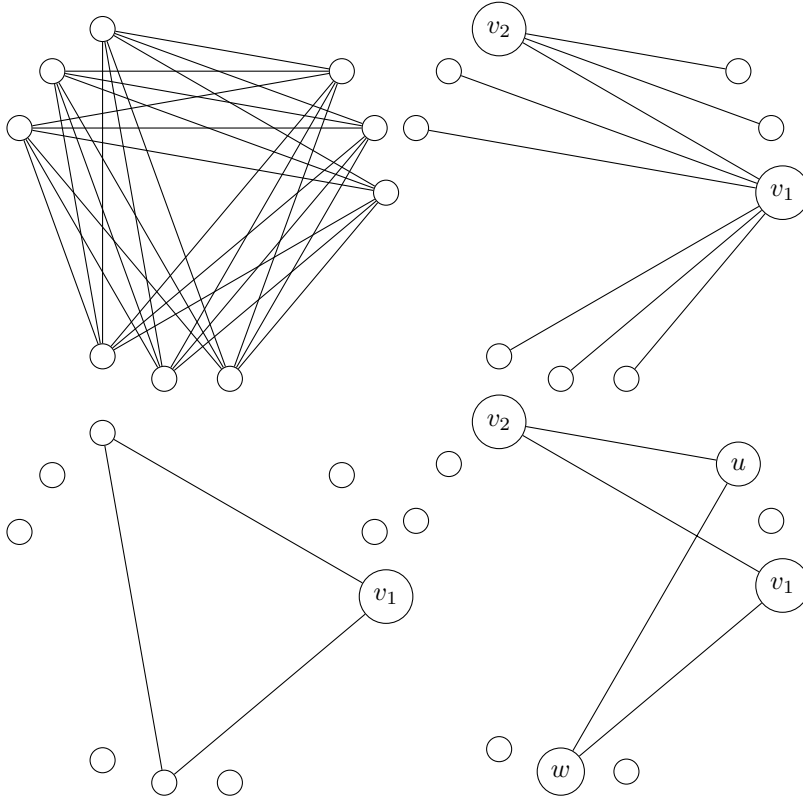


FIG. 3. *The proof of Corollary 5.4 for MUBs $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ in $\mathbb{C}^3$. The frame graph Γ , the spanning tree $\mathcal{T}$, and fundamental cycles of type 1 and 2.*

The maximal number of MUBs is of interest in quantum information theory. For d a prime, or a power of a prime, the maximal number of MUBs in $\mathbb{C}^d$ is $d+1$; see [20], [5], [6] for constructions. These have a special (Heisenberg) structure, which has been used to classify them up to projective unitary equivalence; see [7], [19], [20]. Our classification using 3-products does not presuppose any structure on the MUBs.

There exist graphs which are not chordal, with every edge on a 3-cycle (as is the

case for the frame graph of three or more MUBs), but for which the cycle space is not spanned by 3-cycles (see Figure 4).

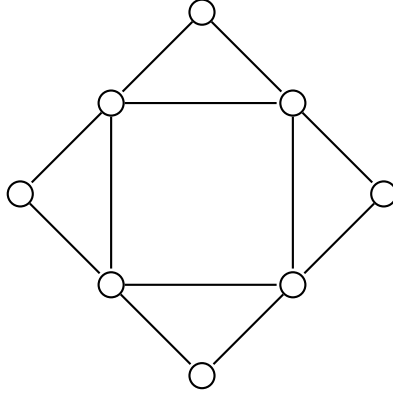


FIG. 4. A nonchordal graph for which each edge is on a 3-cycle.

6. Similarity and m -products for vector spaces. Using the theory of frames for vector spaces [29], one can give analogous results for vector spaces, where the role of unitary equivalence is played by “similarity” and the role of m -products by “canonical m -products.” This allows the “projective symmetry group” to be defined in a very general setting (see [27]).

Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors which span vector spaces X and Y over a subfield $\mathbb{F}$ of $\mathbb{C}$. We say that Φ and Ψ are *similar* if there is an invertible linear map $Q : X \rightarrow Y$ with

$$w_j = Qv_j \quad \forall j$$

and *projectively similar* if there is an invertible linear map $Q : X \rightarrow Y$ and unit scalars c_j with

$$w_j = c_j Qv_j \quad \forall j.$$

For a finite sequence $\Phi = (v_j)_{j \in J}$ in X the *synthesis map* is

$$V = [v_j]_{j \in J} : \mathbb{F}^J \rightarrow X : a \mapsto \sum_j a_j v_j.$$

The subspace of all linear dependencies between the vectors of Φ is

$$\text{dep}(\Phi) := \ker(V) = \{a \in \mathbb{F}^J : \sum_j a_j v_j = 0\},$$

and we denote the orthogonal projection onto $\text{dep}(\Phi)^\perp$ (orthogonal complement) by P_Φ .

We have following characterization of similarity in terms of linear dependencies.

LEMMA 6.1 (see[29]). *Let $\Phi = (v_j)_{j \in J}$ and $\Psi = (w_j)_{j \in J}$ be spanning sequences for the $\mathbb{F}$ -vector spaces X and Y . Then the following are equivalent:*

- (a) Φ and Ψ are similar, i.e., there is an invertible linear map $Q : v_j \mapsto w_j$.
- (b) $\text{dep}(\Phi) = \text{dep}(\Psi)$ (the dependencies are equal).
- (c) $P_\Phi = P_\Psi$ (the associated projections are equal).

The proof of Lemma 6.1 shows that $\Phi = (v_j)$ is similar to columns of $P = P_\Phi$. These columns (Pe_j) span a subspace of $\mathbb{F}^J$, which inherits the Euclidean inner product. Indeed

$$\langle Pe_j, Pe_k \rangle = P_{jk},$$

i.e., the Gramian of (Pe_j) is $P = P_\Phi$. We will call the m -products of (Pe_j) the *canonical m -products* of (v_j) and denote them

$$(6.1) \quad \Delta_C(v_{j_1}, \dots, v_{j_m}) := \Delta(Pe_{j_1}, \dots, Pe_{j_m}) = P_{j_1 j_2} P_{j_2 j_3} \cdots P_{j_m j_1}.$$

In this way, we may apply Theorem 2.2.

THEOREM 6.2 (characterization). *Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors in vector spaces over a subfield $\mathbb{F}$ of $\mathbb{C}$ which is closed under complex conjugation. Then*

1. Φ and Ψ are similar if and only if $P_\Phi = P_\Psi$,
2. Φ and Ψ are projectively similar if and only if their canonical m -products (for a determining set) are equal.

Proof. The first follows from Lemma 6.1 and implies that Φ and Ψ are projectively similar, i.e.,

$$w_j = c_j Q v_j = Q(c_j v_j) \quad \forall j$$

if and only if $\Psi = (w_j)$ and $\Phi' = (c_j v_j)$ are similar, for some choice of unit scalars (c_j) , i.e.,

$$(6.2) \quad P_\Psi = P_{(c_j v_j)} = C^* P_\Phi C.$$

Here the last equality follows by a simple calculation. Since Φ and Ψ are similar to $(P_\Phi e_j)$ and $(P_\Psi e_j)$, which have Gramian matrices P_Φ and P_Ψ , it follows from (1.2) that (6.2) is equivalent to $(P_\Phi e_j)$ and $(P_\Psi e_j)$ being projectively unitarily equivalent, and by Theorem 2.2, this is equivalent to their m -products, i.e., the canonical m -products of (v_j) and (w_j) being equal. $\square$

For the case of projective similarity, one can calculate the c_j and Q in $w_j = c_j Q v_j$ explicitly, as we now illustrate.

Example 6.3. Suppose that $\Phi = (v_j)$ spans a two-dimensional space, i.e.,

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then $\text{dep}(\Phi) = \text{span}\{u\}$, $u = (\alpha, \beta, \gamma)$, so that

$$P_\Phi = I - uu^* = \begin{pmatrix} 1 - |\alpha|^2 & -\alpha\bar{\beta} & -\alpha\bar{\gamma} \\ -\bar{\alpha}\beta & 1 - |\beta|^2 & -\beta\bar{\gamma} \\ -\bar{\alpha}\gamma & -\bar{\beta}\gamma & 1 - |\gamma|^2 \end{pmatrix}.$$

The canonical 2-products are uniquely determined by $|a|, |b|, |c|$, e.g.,

$$\Delta_C(v_1, v_1) = (1 - |\alpha|^2)^2, \quad \Delta_C(v_1, v_2) = |-\bar{\alpha}\beta|^2 = |\alpha|^2 |\beta|^2,$$

as is the canonical 3-product corresponding to the unique 3-cycle

$$\Delta_C(v_1, v_2, v_3) = (-\bar{\alpha}\beta)(-\bar{\beta}\gamma)(-\alpha\bar{\gamma}) = -|\alpha|^2 |\beta|^2 |\gamma|^2.$$

Thus if $\Psi = (w_j)$ is given by

$$\tilde{\alpha} w_1 + \tilde{\beta} w_2 + \tilde{\gamma} w_3 = 0, \quad |\tilde{\alpha}|^2 + |\tilde{\beta}|^2 + |\tilde{\gamma}|^2 = 1,$$

then

1. Φ is similar to Ψ if and only if $P_\Psi = P_\Phi$, i.e.,

$$\tilde{\alpha}\tilde{\beta} = \alpha\bar{\beta}, \quad \tilde{\alpha}\tilde{\gamma} = \alpha\bar{\gamma}, \quad \tilde{\beta}\tilde{\gamma} = \beta\bar{\gamma}.$$

2. Φ is projectively similar to Ψ if and only if their canonical m -products are equal, i.e.,

$$|\tilde{\alpha}| = |\alpha|, \quad |\tilde{\beta}| = |\beta|, \quad |\tilde{\gamma}| = |\gamma|.$$

When Ψ and Φ are projectively similar, i.e., $w_j = c_j Q v_j$ (the canonical m -products are equal), one has $P_\Psi = C^* P_\Phi C$. Here, suppose $\alpha, \beta, \gamma \neq 0$; then we have

$$\overline{c_1} c_2 \alpha \bar{\beta} = \tilde{\alpha} \tilde{\beta}, \quad \overline{c_1} c_3 \alpha \bar{\gamma} = \tilde{\alpha} \tilde{\gamma}, \quad \overline{c_2} c_3 \beta \bar{\gamma} = \tilde{\beta} \tilde{\gamma} \implies c_2 = \frac{\tilde{\alpha}}{\alpha} \frac{\beta}{\tilde{\beta}} c_1, \quad c_3 = \frac{\tilde{\alpha}}{\alpha} \frac{\gamma}{\tilde{\gamma}} c_1,$$

where Q is defined by

$$Qv_1 := \overline{c_1} w_1, \quad Qv_2 := \overline{c_2} w_2 = \frac{\alpha}{\tilde{\alpha}} \frac{\tilde{\beta}}{\beta} \overline{c_1} w_2.$$

7. Projectively equivalent harmonic frames. Let G be a finite abelian group of order n (written additively) with irreducible characters $\xi \in \hat{G}$. Here $\hat{G}$ is known as the character group (which is isomorphic to G). Let $J \subset G$ with $|J| = d$; then any tight frame which is unitarily equivalent to the equal-norm tight frame for $\mathbb{C}^J \approx \mathbb{C}^d$ given by

$$\Phi_J = (\xi|_J)_{\xi \in \hat{G}}$$

is called a *harmonic frame* and is said to be *cyclic* if G is a cyclic group. This is the class of tight frames which are the orbit of a group of unitary transformations on $\mathbb{C}^d$, which is isomorphic to G (see [30], [31]). The harmonic frames were studied up to unitary equivalence in [32], [31]. We now recount some of the basic details.

Let G be a fixed finite abelian group. Subsets J and K of G are *multiplicatively equivalent* if there is an automorphism $\sigma : G \rightarrow G$ for which $K = \sigma J$. In this case,

$$\hat{\sigma} : \hat{G} \rightarrow \hat{G} : \chi \mapsto \chi \circ \sigma^{-1}$$

is an automorphism of $\hat{G}$, and

$$\langle \xi|_J, \eta|_J \rangle = \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle,$$

i.e., Φ_J and Φ_K are unitarily equivalent after reindexing by the automorphism $\hat{\sigma}$.

The *translations* of G are the bijections

$$\tau_b : G \rightarrow G : j \mapsto j - b, \quad b \in G,$$

and we say K is a *translate* of J if $K = J - b$, i.e., $K = \tau_b J$. We define the affine group of G to be the group of bijections $\pi : G \rightarrow G$ generated by the translations and automorphisms, i.e., the $|G| |\text{Aut}(G)|$ maps of the form

$$\pi(g) = \sigma(g) - b, \quad \sigma \in \text{Aut}(G), \quad b \in G.$$

If $K = \pi J$ for some π in the affine group, we say J and K are *affinely equivalent*.

LEMMA 7.1. *If J and K are subsets of a finite abelian group G , which are translates of each other, then the harmonic frames Φ_J and Φ_K are projectively unitarily equivalent.*

Proof. Suppose $K = J - b$. Since $\Phi_J = (\xi|_J)_{\xi \in \hat{G}}$, we need to show

$$\xi|_K = c_\xi U(\xi|_J), \quad \xi \in \hat{G},$$

where $U : \mathbb{C}^J \rightarrow \mathbb{C}^K$ is unitary. Let $U_b : \mathbb{C}^J \rightarrow \mathbb{C}^K$ be the unitary map

$$(U_b v)(k) := v(k + b), \quad k \in K.$$

Since ξ is a character, we have

$$(U_b \xi|_J)(k) = \xi|_J(k + b) = \xi(k + b) = \xi(k)\xi(b) = \xi|_K(k)\xi(b),$$

and so we can take $U = U_b$ and $c_\xi = 1/\xi(b)$. $\square$

Conversely, that projective unitary equivalence implies J and K are translates of each other appears to be true.

THEOREM 7.2. *Suppose J and K are subsets of a finite abelian group G . Then*

1. *if J and K are translates, then Φ_J and Φ_K are projectively unitarily equivalent;*
2. *if J and K are multiplicatively equivalent, then Φ_J and Φ_K are unitarily equivalent after reindexing by an automorphism;*
3. *if J and K are affinely equivalent, then Φ_J and Φ_K are projectively unitarily equivalent after reindexing by an automorphism.*

Proof. The first part is Lemma 7.1, the second is given in [31] (Theorem 3.5), and the third follows by combining the first two. $\square$

Example 7.3. Let $p > 2$ be a prime. Then all harmonic frames of p vectors in $\mathbb{C}^2$ are projectively unitarily equivalent up to reindexing (to p equally spaced vectors in $\mathbb{R}^d$). This follows since there is a unique affine map, taking a sequence of two distinct elements of $\mathbb{Z}_p$ to any other. In particular, allowing for reindexing, the two harmonic frames of three vectors in $\mathbb{C}^2$ which are unitarily inequivalent (one is real, one is complex) are projectively unitarily equivalent.

The conditions of 1, 2, and 3 of Theorem 7.2 say that J and K are in the same orbit under action of the *group of translations*, the *automorphism group*, and the *affine group* on the subsets of G , respectively. Using this, we were able to calculate the various equivalences using the computer algebra package MAGMA [33]. The results of these calculations for the cyclic harmonic frames are summarized in Figure 5. These indicate that the number of projective unitary equivalence classes is much smaller than the number of unitary equivalence classes (up to any reindexing). There are just a few cases where the number of equivalence classes is smaller than that predicted by the group theoretic calculations, because there is a reindexing which is *not* an automorphism which makes harmonic frames equivalent. This was previously observed in the case of unitary equivalence [31]. In these cases the larger group theoretic estimate is given in the row below in Figure 5.

$d = 2$			$d = 3$			$d = 4$		
n	uni	proj	n	uni	proj	n	uni	proj
2	1	1	3	1	1	5	2	1
3	2	1	4	3	1	6	9	3
4	3	2	5	3	1	7	7	2
5	3	1	6	11	3	8	21	6
6	6	3	7	7	2		23	5
7	4	1	8	16	4	9	23	4
8	7	3		17			24	
9	6	2	9	15	3	10	53	9
10	9	3	10	29	4		54	
11	6	1	11	17	2	11	34	4
12	13	5	12	56	9	12	138	21
13	7	1		57			141	
14	12	3	13	25	3			
15	13	3						

$d = 5$			$d = 6$			$d = 7$		
n	uni	proj	n	uni	proj	n	uni	proj
5	1	1	6	1	1	7	1	1
6	4	1	7	2	1	8	4	1
7	4	1	8	11	3	9	8	2
8	19	4	9	16	3	10	32	4
	20		10	55	9	11	34	4
9	23	4		56		12	228	25
	24		11	48	6		234	
10	67	9						
11	48	6						

FIG. 5. The number of unitary and projective unitary equivalence classes (up to reindexing) of cyclic harmonic frames of n vectors in $\mathbb{C}^d$, $d = 2, \dots, 7$. When the group theoretic estimate of Theorem 7.2 is larger (because there are reindexings which are not automorphisms) it is given in the row below.

8. Rank-one matrix completions. We now show that determining projective unitary equivalence is equivalent to a rank-one matrix completion. This leads to an alternative derivation of our results, as kindly pointed out by one of the referees.

The condition (1.2) for projective unitary equivalence can be written as

$$(8.1) \quad \text{Gram}(\Phi) = C \text{Gram}(\Psi) C^* = \text{Gram}(\Psi) \circ (cc^*),$$

where $c = (c_j)$ and $\circ$ is the Hadamard matrix product. Here $A = cc^*$ is a self-adjoint rank-one unimodular matrix, whose existence is equivalent to the projective unitary equivalence of Φ and Ψ . Some of the entries of $A = cc^*$ can be determined by pointwise division, i.e.,

$$(8.2) \quad a_{jk} = c_j \overline{c_k} = (cc^*)_{jk} = \frac{\text{Gram}(\Phi)_{jk}}{\text{Gram}(\Psi)_{jk}} = \frac{\langle v_k, v_j \rangle}{\langle w_k, w_j \rangle}, \quad \langle v_k, v_j \rangle \neq 0.$$

The matrix A restricted to the *pattern*

$$\mathcal{P} := \{(j, k) : \langle v_k, v_j \rangle \neq 0\}$$

is called a $\mathcal{P}$ -*matrix* or *partial matrix supported on $\mathcal{P}$* . The possible ranks of completions of partial matrices were considered in [34]. We require the following result (see [35] for a more direct proof).

THEOREM 8.1 (see[34]). *A nonzero partial matrix has a rank-one completion if and only if it has the zero row or column property and the cycle property.*

The *zero row or column property* is that if $a_{jk} = 0$, $(j, k) \in \mathcal{P}$, then either $a_{j\tilde{k}} = 0 \forall (\tilde{k}) \in \mathcal{P}$ (zero row) or $a_{\tilde{j}k} = 0 \forall (\tilde{j}) \in \mathcal{P}$ (zero column). Since $A = cc^*$ has unimodular entries, this is trivially satisfied.

For A an $m \times n$ partial matrix supported on $\mathcal{P}$ consider the bipartite graph with vertices $\{1, \dots, m\}$ (row vertices) and $\{1, \dots, n\}$ (column vertices) and an edge from j to k with weight a_{jk} , $(j, k) \in \mathcal{P}$. For $\Gamma = (v_0, v_1, \dots, v_{2k})$, $v_0 = v_{2k}$, a cycle in this edge-weighted directed graph, the *cycle property* is that

$$\prod_{r=0}^{k-1} a_{v(2r), v(2r+1)} = \prod_{r=0}^{k-1} a_{v(2r), v(2r-1)},$$

i.e., the product of the weights for the edges from a row vertex to column vertex is equal to the product of the weights for edges from a column vertex to a row vertex. The *cycle condition* is that the cycle property hold for all cycles. For our partial matrix $A = cc^*$, consider the cycle $\Gamma = (j_1, j_1, j_2, j_2, \dots, j_m, j_m)$, where the first occurrence of a j_r is as a row vertex and the second as a column vertex (or vice versa). Then the cycle property is

$$a_{j_1 j_1} a_{j_2 j_2} \cdots a_{j_m j_m} = a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{m-1} j_m}.$$

By (8.2) and cross multiplication, this can be written as

$$\langle w_{j_1}, w_{j_2} \rangle \langle w_{j_2}, w_{j_3} \rangle \cdots \langle w_{j_{m-1}}, w_{j_m} \rangle = \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_{m-1}}, v_{j_m} \rangle,$$

i.e., the nonzero m -products of Ψ and Φ are equal. It can be shown that the cycle property corresponding to the other cycles in $\mathcal{P}$ reduces to an equality of nonzero m -cycles. In this way one can obtain a proof of Theorem 3.1 as a corollary of Theorem 8.1.

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