4. Univariate examples

There are many examples of *univariate* interpolation operators which reproduce the linear polynomials. We shall apply Theorem 2.1 only to the best known of these, the *Lagrange interpolation* and *Bernstein operators.* We leave the other examples to the reader. These include the variation diminishing splines, with Θ infinite, the Meyer–König and Zeller, Baskakov, Szasz–Mirakjan operators, and the continuous version (2.9) for the Beta, Picard operators, etc.

Example 4.1. Let $L := L_\Theta$ be the operator of *Lagrange interpolation* at Θ a (finite) set of two or more points in $[a, b]$, with 'Lagrange form'

$$Lf(x) = \sum_{v\in\Theta} f(v)\ell_v(x), \qquad \ell_v := \prod_{\substack{\theta\in\Theta \\ \theta\neq v}} \frac{(\cdot - \theta)}{(v-\theta)}.$$

Since L reproduces Π_1, using the Hermite-Genocchi formula (1.7) to change to divided difference notation, the error formula (2.2) gives

$$f(x) - Lf(x) = \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \ell_v(x)\ell_w(x)(v-w)(w-v)\,\mathrm{d}x, v, wf, \qquad \forall f \in C[a,b].$$

This holds for $f \in C[a, b]$ because, in view of the Hermite-Genocchi formula, the argument of Theorem 2.1 can be done entirely in terms of divided differences.

Example 4.2. Let L be

$$B_k : C[0,1] \to C[0,1], \qquad k = 1, 2, \ldots$$

the *Bernstein operator* of order k, which is given by

$$B_k f(x) := \sum_{i=0}^{k} f\left(\frac{i}{k}\right) \binom{k}{i} x^i (1-x)^{k-i}.$$

This operator reproduces Π_1, and so Theorem 2.1 gives the error formula

$$f(x) - B_k f(x) = \frac{1}{2} \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \left(\frac{i}{k} - \frac{j}{k}\right)\left(\frac{j}{k} - \frac{i}{k}\right) x^{i+j} (1-x)^{2k-i-j} \mathrm{d}x, \frac{i}{k}, \frac{j}{k} f,$$

which, as in the previous example, is valid for all $f \in C[0, 1]$. Notice that this expresses the error as a linear combination of second order divided differences each with a negative coefficient.