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The error in linear interpolation at the vertices of a simplex†

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ABSTRACT

A new formula for the error in a map which interpolates to function values at some set $\Theta \subset \mathbb{R}^n$ from a space of functions which contains the linear polynomials is given. From it *sharp* pointwise L_∞-bounds for the error in linear interpolation (interpolation by linear polynomials) to (function values at) the vertices of a simplex are obtained. The corresponding 'envelope theorem' giving the optimal recovery of functions is discussed.

This error formula reflects the geometry in a particularly appealing way. The error at any point x not lying on a line connecting points in Θ is the sum over distinct points $v, w \in \Theta$ of 1/2 the average of the second order derivative $D_{v-w}D_{w-v}f$ over the triangle with vertices x, v, w multiplied by some function which vanishes at all of the points in Θ.

Key Words: Lagrange interpolation, linear interpolation on a triangle, sharp error bounds, finite elements, Courant's finite element, multipoint Taylor formula, Kowalewski's remainder, multivariate form of Hardy's inequality, optimal recovery of functions, envelope theorems

AMS (MOS) Subject Classifications: primary 41A10, 41A44, 41A80, secondary 41A05, 65N30,

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1. Introduction

The high applicability of numerical methods based on interpolation from spaces that contain polynomials, such as the finite element method, has lead to a large literature dealing with the error in such schemes. The main contribution of this paper is an error formula for interpolation to function values at a set of points from a space that includes the linear polynomials. Since the formula covers the much studied case of linear interpolation at the vertices of a triangle (Courant's 'original' finite element [Co43]), it allows us to address some fundamental questions about the representation of errors in multivariate polynomial interpolation in this simple setting. Some questions are answered, while other perhaps suprising issues are raised.

The paper is set out as follows. In the remainder of this section we give necessary definitions. The maps of interest are cases of so-called *Lagrange maps.* We discuss the linear functional $f \mapsto \int_\Theta f$, and some of its relevant properties. This linear functional is proving to be the appropriate notation in which to express multivariate polynomial interpolants and their errors (see, e.g., de Boor [B95]). In Section 2, the error formula is given and compared with others. In Sections 3, 4 and 5, L_p-error bounds for linear interpolation, which are sharp for $p = \infty$, are obtained from the formula by using a *multivariate form of Hardy's inequality.* These bounds are compared with others in the literature, including recent work of Handscomb [H95] and Subbotin [Su90$_1$], [Su90$_2$].

Lagrange maps

We say that *Lagrange interpolation* from a space of functions P to a set of points $\Theta \subset \mathbb{R}^n$ is **correct** if for each function f (defined at least on Θ) there is a unique $p \in P$ with

$$p(v) = f(v), \quad \forall v \in \Theta.$$

The associated linear projector $f \mapsto p$ is called the **Lagrange map** (given by the space of interpolants P and points of interpolation Θ), and it is denoted by

$$L_{P,\Theta} : f \mapsto p.$$

The **Lagrange form** of a Lagrange map $L_{P,\Theta}$ is given by

$$L_{P,\Theta} f = \sum_{v\in\Theta} f(v)\, \ell_v, \tag{1.1}$$

where (1.1) uniquely defines

$$\ell_v := \ell_{v,P,\Theta} \in P,$$

the **Lagrange function** for $v \in \Theta$.

Lagrange maps with a space of interpolants that contains Π_k (the polynomials of degree $\le k$) are frequently used to interpolate to scattered data. Particular examples receiving much attention lately are maps where the interpolants include *radial basis functions.*

The linear functional $f \mapsto \int_\Theta f$

To describe the error in a Lagrange map it is often convenient to use the following linear functional called the **divided difference functional on** $\mathbb{R}^n$ by Micchelli in [M80] and analysed there and in [M79].

Definition 1.2. *For Θ the sequence $[\theta_0, \ldots \theta_k]$ of $k+1$ points in $\mathbb{R}^n$, let*

$$f \mapsto \int_\Theta f := \int_0^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1,$$

with the convention that $\int_{[\,]} f := 0$.

The value of $\int_\Theta f$ does not depend on the ordering of the points in Θ. The nature of $\int_\Theta f$ becomes more apparent by observing that

$$\int_\Theta f = \frac{1}{k!} \int_{\mathrm{conv}\,\Theta} M(\cdot|\Theta)\, f, \tag{1.3}$$

where $M(\cdot|\Theta)$ is the **simplex spline** with knots Θ (which is supported on $\mathrm{conv}\,\Theta$, the convex hull of the points in Θ). The class of functions for which $\int_\Theta f$ is defined can be determined from (1.3).

Crucial to the arguments of the paper is the following form of the *fundamental theorem of calculus*, that

$$\int_{[\Theta,v]} f - \int_{[\Theta,w]} f = \int_{[\Theta,v,w]} D_{v-w} f. \tag{1.4}$$

This is a form of the *difference identity* for simplex splines (see [M80:Th.6]) which pervades the multivariate spline literature.

We will also use the *differentiation rule*, that

$$\int_{[x_1,\ldots,x_{k+1}]} D_{z_1} D_{z_2} \cdots D_{z_k} \int_{[\cdot,\Theta]} f = \int_{[x_1,\ldots,x_{k+1},\Theta]} D_{z_1} D_{z_2} \cdots D_{z_k} f, \tag{1.5}$$

where D_z denotes the derivative in the direction z, and Θ is a finite sequence of points.

The rules (1.4) and (1.5) are the basis for a *multivariate divided difference* theory that is currently being developed (see, e.g., de Boor [B95]). In this regard, notice that the **Hermite-Genocchi formula** can be written as

$$[\theta_0, \ldots, \theta_k] f = \int_{[\theta_0,\ldots,\theta_k]} D^k f, \tag{1.6}$$

where $[\theta_0, \ldots, \theta_k] f$ is the *univariate* divided difference of f at the points $\theta_0, \ldots, \theta_k$ in $\mathbb{R}$.

Other properties, including a discussion of the continuity of the map

$$\Theta \mapsto \int_\Theta f,$$

can be found in Waldron [W96$_1$].

The special cases of $f \mapsto \int_\Theta f$ used in the paper

We will only consider $f \mapsto \int_\Theta f$ for Θ consisting of *one*, *two*, or *three* points. For a single point we simply have point evaluation

$$\int_{[u]} f = f(u). \tag{1.7}$$

If Θ consists of two distinct points, then $\int_\Theta f$ is the average of f over the line segment joining them

$$\int_{[u,v]} f = \int_0^1 f(u + t(v-u))\, dt,$$

while if Θ consists of two points that are the same, then we again have point evaluation

$$\int_{[u,u]} f = f(u).$$

If Θ consists of three points which are the vertices of a triangle $T := \mathrm{conv}\{u,v,w\}$ with area A, then $\int_\Theta f$ is half of the average of f over T

$$\int_{[u,v,w]} f = \frac{1}{2}\frac{1}{A}\int_T f. \tag{1.8}$$

If Θ consists of three points which are collinear but not all the same, then $\int_\Theta f$ is a line integral of f over the line segment $\mathrm{conv}\,\Theta$ against the B-spline with knots Θ.

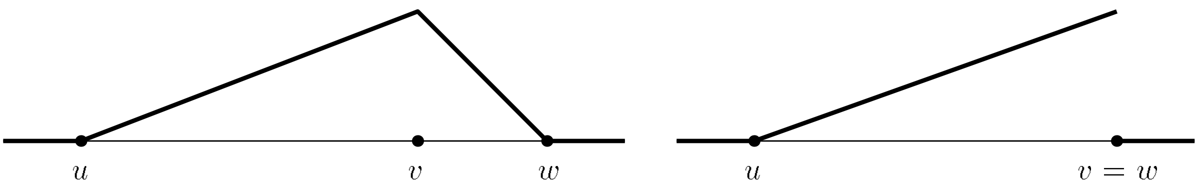

Fig 1.1 The B-splines corresponding to three collinear points

If Θ consists of three points all the same, then we have point evaluation divided by 2

$$\int_{[u,u,u]} f = \frac{f(u)}{2}.$$

2. The error formula

In this section we give an error formula for a Lagrange map whose space of interpolants contains Π_1 (the linear polynomials). This formula involves only second order derivatives of the function interpolated (the form desired by numerical analysts). The idea of the proof below is to use the 'difference identity' (1.4) in just the right way so as to introduce these second order derivatives. Let $D_y f$ denote the derivative of f in the direction y.

Theorem 2.1. *Suppose that $\bar{\Omega}$ (the closure of Ω) is starshaped with respect to Θ. If $L_{P,\Theta}$ is a Lagrange map with*

$$\Pi_1 \subset P \subset C(\bar{\Omega}),$$

then for all $f \in C^2(\bar{\Omega})$

$$f(x) - L_{P,\Theta} f(x) = \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \ell_v(x)\ell_w(x) \int_{[x,v,w]} D_{v-w}D_{w-v} f, \qquad \forall x \in \bar{\Omega}. \tag{2.2}$$

where the sum is taken over all 2-element subsets of Θ.

Proof: Since P contains Π_0 (the constants), it follows from (1.1) that

$$\sum_{v\in\Theta} \ell_v = 1. \tag{2.3}$$

This, together with (1.7) and the 'difference identity' (1.4), gives

$$f(x) - L_{P,\Theta} f(x) = \sum_{v\in\Theta} \left(\int_{[x]} f - \int_{[v]} f \right) \ell_v(x) = \sum_{v\in\Theta} \left(\int_{[x,v]} D_{x-v} f \right) \ell_v(x). \tag{2.4}$$

Since P contains Π_1 (the linear polynomials), and each coordinate of $(\cdot - v)$ is a linear polynomial, it follows from (1.1) that

$$x - v = \sum_{w\in\Theta} (w-v)\ell_w(x). \tag{2.5}$$

Substituting (2.5) into (2.4), and using the linearity of $y \mapsto D_y$ gives

$$f(x) - L_{P,\Theta} f(x) = \sum_{v\in\Theta} \sum_{w\in\Theta} \ell_v(x)\ell_w(x) \int_{[x,v]} D_{w-v} f. \tag{2.6}$$

The double summation in (2.6) is over all ordered pairs (v, w) where $v \neq w$ (the terms for $v = w$ are zero). By summing the pairs (v, w) and (w, v) first, we obtain the following sum over the unordered pairs $\{v, w\}$

$$f(x) - L_{P,\Theta} f(x) = \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \ell_v(x)\ell_w(x) \left(\int_{[x,v]} D_{w-v} f - \int_{[x,w]} D_{w-v} f \right). \tag{2.7}$$

Finally, by the 'difference identity' (1.4) again,

$$\int_{[x,v]} D_{w-v} f - \int_{[x,w]} D_{w-v} f = \int_{[x,v,w]} D_{v-w} D_{w-v} f,$$

which gives the result. $\square$

This error formula reflects the geometry in a particularly appealing way. The error at any point x not lying on a line connecting points in Θ is the sum over distinct points $v, w \in \Theta$ of $1/2$ the average of the second order derivative $D_{v-w}D_{w-v}f$ over the triangle $\mathrm{conv}[x, v, w]$ multiplied by the function $\ell_v \ell_w$ which vanishes at all of the points in Θ.

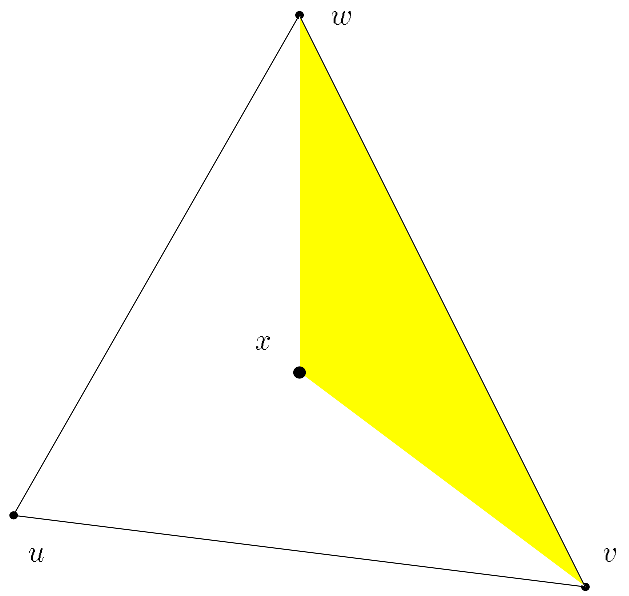

Fig 2.1 The region of integration for the derivative $D_{v-w}D_{w-v}f$ occuring in the error formula (2.2)

The special case of this formula when $P = \Pi_1$ is the first in a family of error formulæ for **Chung-Yao interpolation** from Π_k recently obtained by de Boor [B95]. In Chung-Yao interpolation, see [CY77] for more details, the points of interpolation are the intersections of certain sets of hyperplanes. Interpolation from Π_1 (the linear polynomials) is often referred to as *linear interpolation* (for obvious reasons), a term which is also used (by some) for any interpolation scheme where the interpolation operator is linear (for equally obvious reasons). In this paper we will always mean the former.

For the special case of linear interpolation at u, v, w the vertices of a triangle in $\mathbb{R}^2$, there are many formulæ which express the error in terms of second order derivatives. Most of these formulæ require the points u, v, w to lie in some special position, and do not transform in a simple way under an affine change of variables. For example, Gregory [G75] gives a formula for when the points are $(0,0)$, $(1,0)$, $(0,1)$ that involves four line integrals and five area integrals of the partial derivatives $D_1^2 f$, $D_1 D_2 f$, and $D_2^2 f$. More recently, Sauer and Xu [SX95] have given the formula

$$f(x) - L_{\Theta,\Pi_1} f(x) = \ell_v(x) \int_{[x,u,v]} D_{x-v} D_{v-u} f + \ell_w(x) \int_{[x,u,w]} D_{x-w} D_{w-u} f, \qquad (2.8)$$

and similar ones for Lagrange interpolation from Π_k. The formula (2.8), though of similar form to (2.2), has the disadvantage that it is not symmetric in the points u, v, w, that is, it depends on some ordering of them.

Besides (2.2), there is one other formula for the error in a Lagrange map whose space of interpolants contains the linear polynomials. Like (2.2), this formula, the **multipoint Taylor formula** given below, is symmetric in the points of Θ.

Multipoint Taylor formula ([CW71]) 2.9. *Suppose that $\bar{\Omega}$ is starshaped with respect to Θ. If $L_{P,\Theta}$ is a Lagrange map with*

$$\Pi_1 \subset P \subset C(\bar{\Omega}),$$

then for all $f \in C^2(\bar{\Omega})$

$$f(x) - L_{P,\Theta}f(x) = -\sum_{v\in\Theta} \ell_v(x) \int_{[x,x,v]} D^2_{v-x}f, \qquad \forall x \in \bar{\Omega}. \tag{2.10}$$

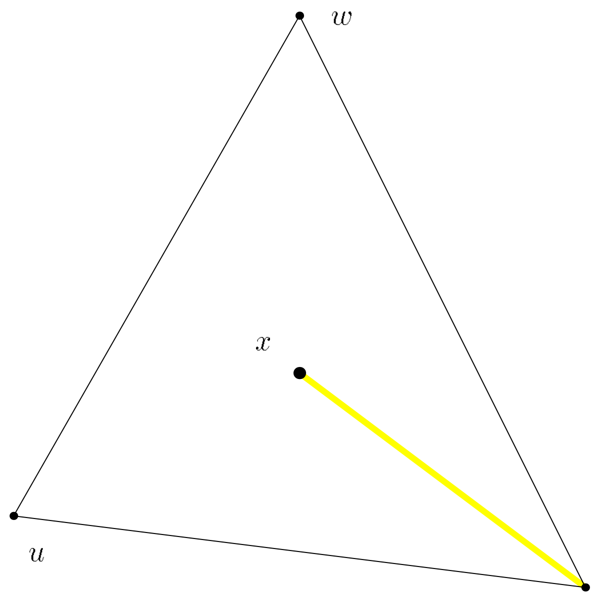

Fig 2.2 The region of integration for the derivative $D^2_{v-x}f$ occuring in the multipoint Taylor formula (2.10)

The *multipoint Taylor formula* (2.10) is the multivariate form of **Kowalewski's remainder** (see [K32:p21-24], or Davis's book [D75:p71]). Recently, it has been shown in [W96*] that (2.2) and (2.10) are special cases of a family of error formulæ indexed by certain real measures of unit mass.

There exists a version of the multipoint Taylor formula for the derivative of the error, but it is not clear whether there is such a formula for (2.2). For example, simply differentiating (2.2) using the 'differentiation rule' (1.5) gives

$$D_y\big(f - L_{P,\Theta}f\big)(x) = \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \ell_v(x)\ell_w(x) \int_{[x,x,v,w]} D_y D_{v-w} D_{w-v} f$$
$$+ \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \big(D_y\ell_v(x)\ell_w(x) + \ell_v(x)D_y\ell_w(x)\big) \int_{[x,v,w]} D_{v-w}D_{w-v}f,$$

which has third order derivatives of f (in addition to the desired second order derivatives).

Comparison of the relative merits of (2.2) and the multipoint Taylor formula (2.10) turns out to involve a complex set of issues. Ultimately, it is seen that in some situations one formula is to be favoured over the other, while in other situations it is the reverse.

First consider the case when L is the map of linear interpolation to the points $v, w \in \mathbb{R}$. In this case, using the Hermite-Genocchi formula (1.6) to change to divided difference notation, our error formula (2.2) is the standard formula

$$f(x) - Lf(x) = (x-v)(x-w)\,[x,v,w]f, \tag{2.11}$$

while the multipoint Taylor formula (2.10) is *Kowalewski's remainder*

$$f(x) - Lf(x) = \frac{(x-w)}{(v-w)}(v-x)^2\,[x,x,v]f + \frac{(x-v)}{(w-v)}(w-x)^2\,[x,x,w]f. \tag{2.12}$$

Here, for standard applications, such as obtaining L_p-bounds on the error (see, e.g., [W96$_2$]), the error formula (2.2) is to be preferred, as is indicated by the relative obscurity of Kowalewski's formula.

For linear interpolation in $\mathbb{R}^n$ when $n > 1$, the question of which is the best formula becomes more complicated. If $n = 2$, then both formulæ have 3 terms, while for $n > 2$ our error formula has $n(n+1)/2$ terms which is more than the multipoint Taylor formula which has only $n+1$ terms. However, from a structural point of view our formula is perhaps a little more pleasing, since (like the standard univariate formula) it involves quadratic polynomials multiplying averages of derivatives (over triangles), as opposed to cubic polynomials multiplying averages of derivatives This form makes obvious certain features of linear interpolation, such as the fact that the error in approximation to a quadratic polynomial is a quadratic polynomial.

More substantive differences between the formulæ become apparent when they are used to obtain L_p-error bounds. These questions are considered in the remaining sections, and ultimately lead to a number of open problems (see Research Problems Section of this issue) whose solution will provide the answers (or raise a whole new set of questions).

3. Preliminary facts and some notation

In this section we outline those techniques and notations needed to obtain L_p-error bounds in Sections 4 (sharp L_∞-bounds) and 5.

The multivariate form of Hardy's inequality

The following inequality is extremely useful for obtaining L_p-bounds from many of the error formulæ for multivariate polynomial interpolation schemes, such as the formula (2.2) and the multipoint Taylor formula. Let

$$(a)_n := (a)(a+1)(a+2)\cdots(a+n-1),$$

the **shifted factorial function**, and denote the cardinality of Θ by $\#\Theta$.

Multivariate form of Hardy's inequality ([W96$_1$]) 3.1. *Let Θ be a finite sequence in $\mathbb{R}^n$, and let Ω be an open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ. If $m - n/p > 0$, then the rule*

$$H_{m,\Theta} f(x) := \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f$$

induces a positive bounded linear map $H_{m,\Theta} : L_p(\Omega) \to L_p(\Omega)$ with norm

$$\|H_{m,\Theta}\| \leq \frac{1}{(m-1)!(m-n/p)_{\#\Theta}} \to \infty \qquad as \qquad m - n/p \to 0^+. \tag{3.2}$$

This upper bound for $\|H_{m,\Theta}\|$ is sharp when Θ involves only one point, i.e., when

$$\Theta = [v, \ldots, v],$$

and when $p = \infty$ (with the norm taken on only for the constant functions).

The inequality given by (3.2) was used to obtain L_p-error bounds from a wide variety of formulæ in [W96$_1$]. There it is referred to as the *multivariate form of Hardy's inequality*, since in the univariate case when $m = 1$, $\Theta = \{0\}$, and $\Omega = (0, \infty)$, it reduces to **Hardy's inequality**.

We will need only the special cases that: for $1 - n/p > 0$

$$\Big\| x \mapsto \int_{[x,v,w]} f \Big\|_{L_p(T)} \leq \frac{1}{(1-n/p)(2-n/p)} \|f\|_{L_p(T)}, \qquad \forall f \in L_p(T), \tag{3.3}$$

and for $2 - n/p > 0$

$$\Big\| x \mapsto \int_{[x,x,v]} f \Big\|_{L_p(T)} \leq \frac{1}{(2-n/p)} \|f\|_{L_p(T)}, \qquad \forall f \in L_p(T), \tag{3.4}$$

where $T = \bar{\Omega}$ is starshaped with respect to v, w (respectively v). In particular, to investigate the extremal functions for the sharp inequalities given in Section 4, we need the consequence of (3.3) that, for all $x \in T$:

$$\Big| \int_{[x,v,w]} f \Big| \leq \frac{1}{2} \|f\|_{L_\infty(T)}, \qquad \forall f \in C(T), \tag{3.5}$$

for which there is equality *only* when f is constant on $\operatorname{conv}\{x, v, w\}$ with the size of that constant value equal to $\|f\|_{L_\infty(T)}$. This inequality also follows (immediately) from the mean value theorem (for positive measures).

Norming W_p^k

Many L_p-error bounds can be conveniently described by using the co-ordinate independent seminorms

$$f \mapsto |f|_{k,p,T}$$

defined as follows. Let $W_p^k(T)$ be the *Sobolev space* of functions with derivatives of order up to k in $L_p(T)$. To each $f \in W_p^k(T)$ associate the function $|D^k f| \in L_p(T)$, which measures the size of its k-th derivative, and is given by the rule

$$|D^k f|(x) := \sup_{\substack{u_1,\ldots,u_k \in \mathbb{R}^n \\ \|u_i\| \le 1}} |D_{u_1} \cdots D_{u_k} f(x)| = \sup_{\substack{u \in \mathbb{R}^n \\ \|u\|=1}} |D_u^k f(x)|, \tag{3.6}$$

where the derivatives $D_{u_1} \cdots D_{u_k} f$ are computed from any (fixed) choice of representatives for the k-th order partial derivatives of f. Here $\|\cdot\|$ denotes the *Euclidean norm.* The equality of the two suprema follows from a classical result of Banach on the norm of a symmetric multilinear mapping (see, e.g., Harris [Har96]). This definition of $|D^k f|$ is consistent with its standard univariate interpretation. From (3.6), it is easy to see that $|D^k f|$ is well-defined and satisfies

$$|D_{u_1} \cdots D_{u_k} f| \le |D^k f| \, \|u_1\| \cdots \|u_k\| \qquad a.e. \tag{3.7}$$

for all $u_1, \ldots, u_k \in \mathbb{R}^n$. For $k = 1$ and 2, $|D^k f|$ relates to the usual notions of first and second derivative as follows: $|Df|$ is the norm of the *gradient* of f

$$|Df| = \|\mathrm{grad} f\| = \sqrt{(D_1 f)^2 + \cdots + (D_n f)^2}, \tag{3.8}$$

and $|D^2 f|$ is the spectral radius of the *Hessian* of f, which in the bivariate case can be computed by

$$|D^2 f| = |D_1^2 f + D_2^2 f| + \sqrt{(D_1^2 f - D_2^2 f) + 4 D_1 D_2 f}. \tag{3.9}$$

The $L_p(T)$-norm of $|D^k f|$ gives a seminorm on $W_p^k(T)$

$$f \mapsto |f|_{k,p,T} := \| \, |D^k f| \, \|_{L_p(T)}, \tag{3.10}$$

with

$$|f|_{0,p,T} = \|f\|_{L_p(T)}.$$

4. Sharp pointwise L_∞-error bounds for linear interpolation

The main result of this section is a *sharp* pointwise L_∞-error bound for linear interpolation. By *linear interpolation* we mean interpolation by linear polynomials to function values at $n+1$ points in $\mathbb{R}^n$. These $n+1$ points are necessarily *affinely independent*, i.e., the vertices of a simplex in $\mathbb{R}^n$. This simplex will be denoted

$$T := \mathrm{conv}\,\Theta,$$

its *diameter* by

$$h := \mathrm{diam}\,\Theta = \max_{v,w \in \Theta} \|v - w\|,$$

and the map of linear interpolation by L_Θ.

Theorem 4.1. *Suppose that L_Θ is the map of linear interpolation at Θ. Let c be the centre and R the radius of the (unique) sphere containing Θ. Then, for each $x \in T$, there is the sharp inequality*

$$|f(x) - L_\Theta f(x)| \le \frac{1}{2}(R^2 - \|x-c\|^2) \, |f|_{2,\infty,T}, \quad \forall f \in W^2_\infty(T). \tag{4.2}$$

Equality in (4.2) occurs when

$$f \in Q := \operatorname{span}\{q\} \oplus \Pi_1, \tag{4.3}$$

where q is the quadratic polynomial obtained by taking the square of the Euclidean norm, i.e.,

$$q := \|\cdot\|^2 : (x_1, \ldots, x_n) \mapsto x_1^2 + x_2^2 + \cdots + x_n^2, \tag{4.4}$$

and these are the only $C^2(T)$ functions giving equality in (4.2) for $x \in T \setminus \Theta$.
In particular, there is the sharp inequality

$$\|f - L_\Theta f\|_{L_\infty(T)} \le \frac{1}{2}(R^2 - d^2) \, |f|_{2,\infty,T}, \qquad \forall f, \tag{4.5}$$

where d is the distance of c from T, i.e.,

$$d := \operatorname{dist}(c, T) = \min_{x \in T} \|x - c\|.$$

Special cases of (4.5) of interest include the following:
(a) *If $c \in T$, then there is the sharp inequality*

$$\|f - L_\Theta f\|_{L_\infty(T)} \le \frac{1}{2} R^2 \, |f|_{2,\infty,T}, \qquad \forall f. \tag{4.6}$$

(b) *For the bivariate case ($n = 2$), if $c \notin T$, then there is the sharp inequality*

$$\|f - L_\Theta f\|_{L_\infty(T)} \le \frac{1}{8} h^2 \, |f|_{2,\infty,T}, \qquad \forall f. \tag{4.7}$$

The inequalities (4.5), (4.6) and (4.7) are sharp when $f \in Q$.

Proof: In the interests of simplicity (2.2) was stated for $f \in C^2(\bar\Omega)$. However, as its proof indicates, it also holds (more generally) for $f \in W^2_\infty(\bar\Omega)$. The details of this fact are a little technical, involving such things as the use of the multivariate form of Hardy's inequality 3.1 to show that the divided difference functionals that appear in (2.2) and its proof remain well defined for $f \in W^2_\infty(T)$.

For a general $f \in W^2_\infty(T)$ the function

$$x \mapsto \ell_v(x)\ell_w(x) \int_{[x,v,w]} D_{v-w} D_{w-v} f \tag{4.8}$$

occuring in (2.2) is only defined a.e. To (4.8) apply (3.3) (with $p=\infty$), followed by (3.7) (with $u_1 = v - w$, $u_2 = w - v$) to obtain

$$\begin{aligned}\Big|\ell_v(x)\ell_w(x)\int_{[x,v,w]} D_{v-w}D_{w-v}f\Big| &\le \ell_v(x)\ell_w(x)\Big\|\int_{[\cdot,v,w]} D_{v-w}D_{w-v}f\Big\|_{L_\infty(T)} \\ &\le \frac{1}{2}\ell_v(x)\ell_w(x)\|D_{v-w}D_{w-v}f\|_{L_\infty(T)} \\ &\le \frac{1}{2}\ell_v(x)\ell_w(x)\|v-w\|^2 |f|_{2,\infty,T},\end{aligned} \tag{4.9}$$

for a.e. x, which gives:

$$|f(x) - L_\Theta f(x)| \le \frac{1}{2}\sum_{v\neq w}\ell_v(x)\ell_w(x)\|v-w\|^2 \, |f|_{2,\infty,T}, \qquad \forall f\in W_2^\infty(T). \tag{4.10}$$

Here the fact that the Lagrange polynomials ℓ_v are non-negative on T was used.

The next part of the proof relies on the fact that

$$x = \sum_{v\in\Theta} v\ell_v(x), \quad 1 = \sum_{v\in\Theta}\ell_v(x), \tag{4.11}$$

which follows from (1.1), and is effectively the observation that $(\ell_v(x) : v\in\Theta)$ are the *barycentric co-ordinates* of x with respect to Θ. With $\langle\cdot,\cdot\rangle$ denoting the *Euclidean inner product*, the quadratic polynomial (of x) occurring in (4.10) can be expanded and simplified using (4.11) in the following way.

$$\begin{aligned}&\frac{1}{2}\sum_{v\neq w}\ell_v(x)\ell_w(x)\|v-w\|^2 \\ &\qquad= \frac{1}{4}\sum_v\sum_w \ell_v(x)\ell_w(x)\|v-w\|^2 \\ &\qquad= \frac{1}{4}\sum_v\sum_w \ell_v(x)\ell_w(x)(\|v\|^2 - 2\langle v,w\rangle + \|w\|^2) \\ &\qquad= \frac{1}{4}\sum_v \ell_v(x)\|v\|^2 - \frac{1}{2}\sum_v\ell_v(x)\langle v,\sum_w w\ell_w(x)\rangle + \frac{1}{4}\sum_w\ell_w(x)\|w\|^2 \\ &\qquad= \frac{1}{2}\left(\sum_v\|v\|^2\ell_v(x) - \langle\sum_v v\ell_v(x), x\rangle\right) \\ &\qquad= \frac{1}{2}\left(\sum_v\|v\|^2\ell_v(x) - \|x\|^2\right).\end{aligned} \tag{4.12}$$

Since

$$x \mapsto \sum_v \|v\|^2\ell_v(x) - \|x\|^2 \tag{4.13}$$

is the unique quadratic polynomial which is zero at the points in Θ and has the quadratic part of its Taylor series at the origin equal to $-q = -\|\cdot\|^2$, it must be equal to

$$R^2 - \|\cdot - c\|^2.$$

This gives (4.2) with equality for $f \in Q$. It is shown at the end of this section that when $f \in C^2(T)$ (and $x \notin \Theta$) these are the only cases of equality.

The sharp inequality

$$\|f - L_\Theta f\|_{L_\infty(T)} \leq \frac{1}{2} \max_{x \in T}(R^2 - \|x - c\|^2) \, |f|_{2,\infty,T}, \qquad \forall f,$$

follows immediately from (4.2), and the constant

$$\max_{x \in T}(R^2 - \|x - c\|^2) = R^2 - \min_{x \in T} \|x - c\|^2 = R^2 - d^2,$$

giving (4.5). Finally the special cases.

Case (a). If $c \in T$, then $R^2 - d^2 = R^2$.

Case (b). If $c \notin T$, then x^* the (unique) choice of $x \in T$ which minimises $\|x - c\|$ must lie in some facet F of T, since when x is in the interior of T it may be moved closer to c (thereby reducing $\|x - c\|$). In the bivariate case ($n = 2$), T is an obtuse angled triangle with F its largest side and x^* is the midpoint of F (see Fig. 4.1.). Since the line segment from c to x^* is orthogonal to the facet F which has length h, Pythagoras's theorem gives

$$d^2 + (h/2)^2 = R^2,$$

and so

$$\frac{1}{2}(R^2 - d^2) = \frac{1}{8}h^2.$$

□

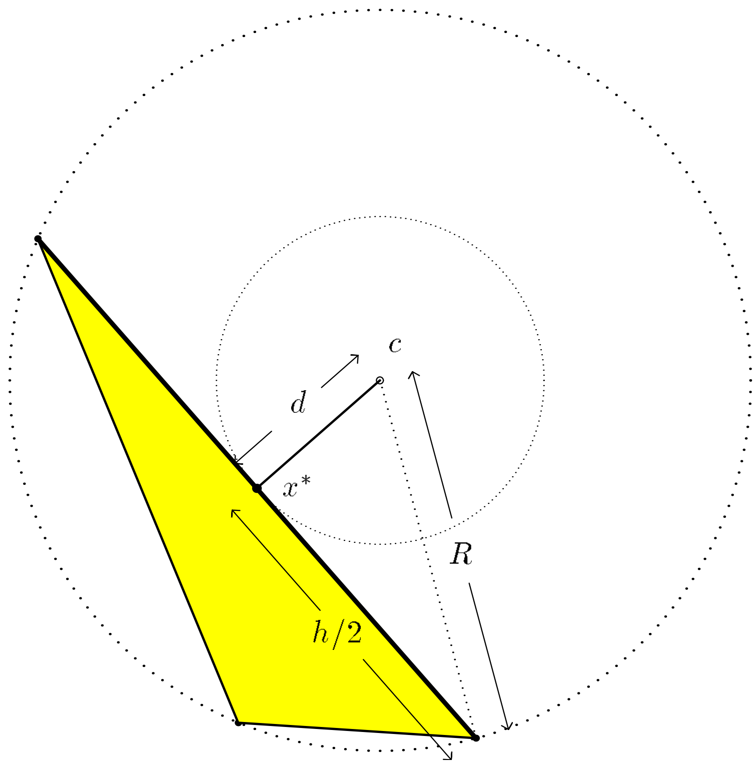

Fig. 4.1. The situation for an obtuse angled triangle: showing the triangle T (shaded), the facet F (thick side), and x^* the closest point to the center c

Remark 4.14. It is interesting to observe that Theorem 4.1 can also be obtained from the multipoint Taylor formula as follows. Using (3.4) in place of (3.3), the argument used for (4.10) can be applied to the multipoint Taylor formula (2.10) to obtain

$$|f(x) - L_\Theta f(x)| \leq \frac{1}{2} \sum_v \ell_v(x) \|v - x\|^2 \, |f|_{2,\infty,T} \qquad \forall f. \tag{4.15}$$

The polynomial $\sum_v \ell_v \|v - \cdot\|^2$ occurring in (4.15) appears to be cubic (which may explain why Theorem 4.1 was not obtained earlier). But, it is infact a quadratic giving the sharp bound (4.2) as is shown by the expansion:

$$\begin{aligned}
\frac{1}{2} \sum_v \ell_v(x) \|v - x\|^2 &= \frac{1}{2} \sum_v \ell_v(x) (\|v\|^2 - 2\langle x, v\rangle + \|x\|^2) \\
&= \frac{1}{2} \sum_v \|v\|^2 \ell_v(x) - \langle x, \sum_v v \ell_v(x) \rangle + \frac{1}{2}\|x\|^2 \\
&= \frac{1}{2} \sum_v \|v\|^2 \ell_v(x) - \langle x, x\rangle + \frac{1}{2}\|x\|^2 \\
&= \frac{1}{2} \left(\sum_v \|v\|^2 \ell_v(x) - \|x\|^2 \right),
\end{aligned}$$

which is similar to (4.12). Theorem 4.1 can also be obtained from Sauer and Xu's formula (2.8), but the proof is more involved. $\square$

Comparison with the sharp L_∞-bounds of Handscomb and Subbotin

The inequality (4.2) is well-known in the *univariate* case (see, e.g., Davis [D75:p57]), but is not known for $n > 1$.

The inequalities (4.6) and (4.7) were recently proved by Handscomb [H95] for the *bivariate* case, i.e., when T is a *triangle.* His proof uses bounds for the error in univariate interpolation in a clever way, and in the case of (4.6) could be extended to the multivariate case $n > 2$. There the condition $c \in T$ (respectively $c \notin T$) is stated in the equivalent way that the triangle T be *acute angled* (respectively *obtuse angled*).

The inequality (4.7) does not extend to $n > 2$, since in this case for given h, R there is an interval of possible values for d (depending on the geometry of the points Θ). For example, when $n = 3$ the constant $\frac{1}{2}(R^2 - d^2)$ occurring in (4.5) can as small as $h^2/8$ (exactly two of the points are a distance h from each other), or as large as $h^2/6$ (exactly three of the points are a distance h from each other).

In the bivariate case (when T is a triangle)

$$\sup\{R^2/h^2 : T \text{ acute angled}\} = 1/3,$$

with the supremum attained (only) when T is an equilateral triangle. Thus from (4.6) and (4.7) it follows that for *all* triangles

$$\|f - L_\Theta f\|_{L_\infty(T)} \leq \frac{1}{6} h^2 \, |f|_{2,\infty,T}, \qquad \forall f, \tag{4.16}$$

which is *sharp* if and only if T is an *equilateral triangle.* The inequality (4.16) was proved in Subbotin [Su90$_1$] using an argument similar to that of Handscomb. Suprisingly the argument involves a claim that there is sharpness in (4.16) not for some $f \in Q$, but for a certain *cubic* polynomial (Example 1 of [Su90$_1$]). At the end of this section, it is shown that when $f \in C^2(T)$ there is (nontrivial) equality in (4.2) for $x \notin \Theta$ *only* for $f \in Q$, and it is pointed out why Subbotin's calculation is in error.

More generally, for $n \geq 1$ it can be shown that

$$\sup\{R^2/h^2 : c \in T\} = \frac{n}{2(n+1)},$$

with the supremum attained (only) when the points in Θ are *equidistant* from each other. In this way one obtains the $n > 2$ analogue of (4.16) that

$$\|f - L_\Theta f\|_{L_\infty(T)} \leq \frac{1}{4} \frac{n}{n+1} h^2 \, |f|_{2,\infty,T}, \qquad \forall f, \tag{4.17}$$

which is sharp when the points in Θ are equal distances from each other. This inequality (4.17) was proved by Subbotin [Su90$_2$:Th.1] where, this time correctly, the sharpness was demonstrated by considering an appropriate *quadratic* polynomial $f \in Q$, namely

$$f = \frac{1}{2}h^2 \sum_{\substack{\{v,w\}\subset\Theta \\ v\neq w}} \ell_v \ell_w,$$

which we recognise as the polynomial given by (4.12), with each occurence of $\|v-w\|$ replaced by h.

Geometric interpretation of the result, the optimal recovery of functions and envelope theorems

Suppose that the following information about $f \in W^2_\infty(T)$ is known:

$$f(v), \ v \in \Theta \qquad \text{(its values at the points } \Theta) \tag{4.18}$$

and

$$|D^2 f| \le K \quad \text{on} \quad T \qquad (\text{i.e., } |f|_{2,\infty,T} \le K). \tag{4.19}$$

If μ is any continuous linear functional on $W^2_\infty(T)$, then it follows from an observation of Golomb and Weinberger [GW59] that the possible values of $\mu(f)$ form a bounded interval

$$\mathtt{L} \le \mu(f) \le \mathtt{U},$$

where the values of the endpoints may or may not be attained. The reason for this, is that the set of f satisfying (4.18) and (4.19) is bounded with respect to the norm

$$f \mapsto |f|_{2,\infty,T} + \sum_{v\in\Theta} |f(v)|$$

(which is equivalent to any of the usual norms for $W^2_\infty(T)$), and so its image under the bounded linear map μ (the set of possible values for $\mu(f)$) is a bounded convex set, i.e., an interval.

In particular, considering the (continuous) linear functional of point evaluation at x, there exist functions $\mathtt{L}, \mathtt{U}$ for which

$$\mathtt{L}(x) \le f(x) \le \mathtt{U}(x), \qquad \forall x \in T,$$

and these bounds cannot be improved in the sense that there exists an f taking any value strictly between them. For obvious reasons, some authors refer to these functions L and U that enclose f as (*lower* and *upper*) *envelopes* for f.

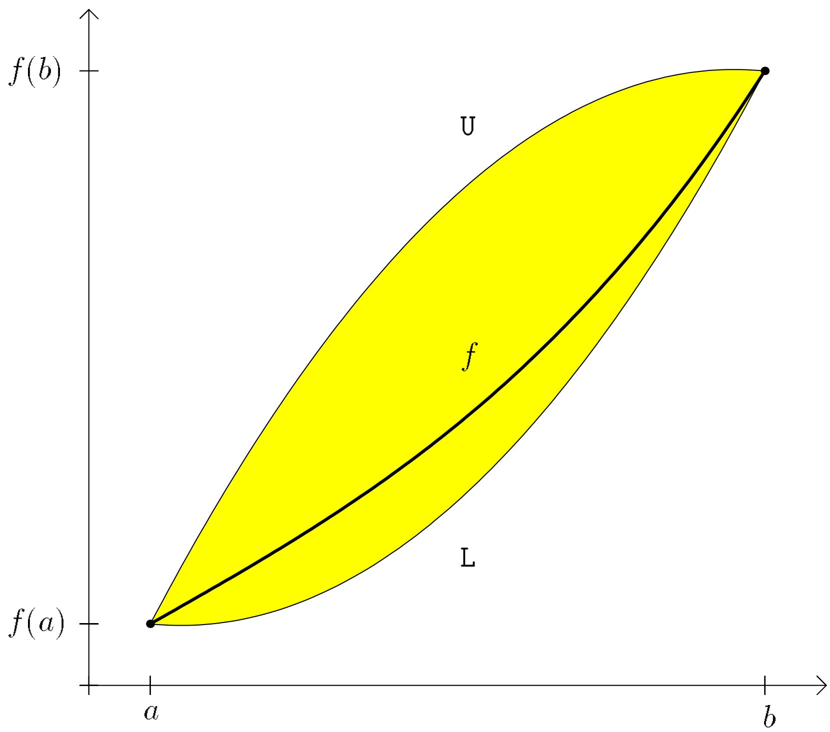

Fig. 4.2. The lower and upper envelopes L, U bounding the (shaded) region where the univariate function f must lie, given that its values at a, b are known and its second derivative is bounded by some (known) constant on $[a \,..\, b]$.

Theorem 4.1 provides the solution of the *optimal recovery problem* of determining L and U as follows. Since f satisfies (4.19), inequality (4.2) gives

$$|f(x) - L_\Theta f(x)| \le \frac{1}{2} K\,(R^2 - \|x-c\|^2),$$

which can be rewritten as

$$L_\Theta f(x) - \frac{1}{2} K (R^2 - \|x-c\|^2) \le f(x) \le L_\Theta f(x) + \frac{1}{2} K\,(R^2 - \|x-c\|^2). \tag{4.20}$$

Since (4.20) is sharp for those $f \in Q$ with $|f|_{2,\infty,T} = K$ (which is nonempty for any given data $f(v)$, $v \in \Theta$), it provides the envelopes for f, which we now state as a corollary.

Corollary (Envelope Theorem) 4.21. *Suppose that the definitions of Theorem 4.1 hold. If the value of $f \in W^2_\infty(T)$ is known at the points Θ, and $|D^2 f| \le K$ on T, then*

$$\mathtt{L}(x) \le f(x) \le \mathtt{U}(x), \qquad \forall x \in T, \tag{4.22}$$

where

$$\begin{aligned} \mathtt{L}(x) &:= L_\Theta f(x) - \frac{1}{2} K\,(R^2 - \|x-c\|^2), \\ \mathtt{U}(x) &:= L_\Theta f(x) + \frac{1}{2} K\,(R^2 - \|x-c\|^2), \end{aligned} \tag{4.23}$$

and there exists a function f taking any of the values allowed by (4.22). In particular, the quadratic functions $\mathtt{L}, \mathtt{U}$ *match f at Θ and satisfy* $|D^2\mathtt{L}|, |D^2\mathtt{U}| = K$ *on T.*

Notice that the envelope functions $\mathtt{L}, \mathtt{U}$ given by (4.23) can be computed from Θ and the values given for $f(v)$, $v \in \Theta$.

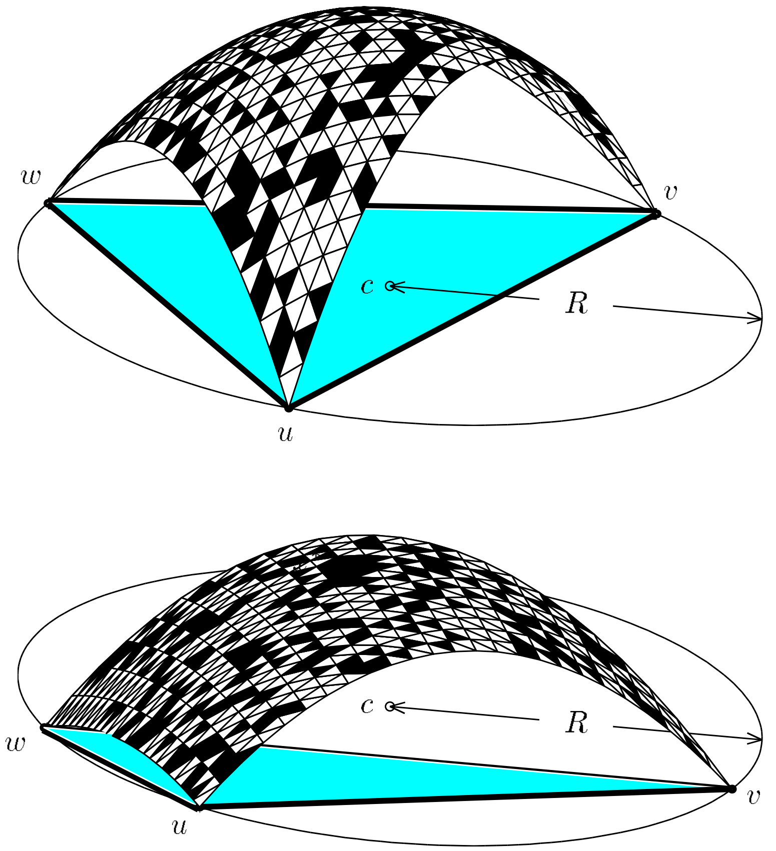

Fig. 4.3. Examples of the upper envelopes for a bivariate function f which is zero at $\Theta = \{u, v, w\}$ and has $|D^2 f| \leq K$ on the triangle T (shaded), showing the circle containing the points Θ. The first example is an acute angled triangle and the second an obtuse angled triangle.

Perhaps the best known 'envelope theorem' is the result of Gaffney–Powell [GP76] and Micchelli–Rivlin–Winograd [MRW76] which shows that if the values of a *univariate* function f is known at $m + k$ points in $[a \mathinner{..} b]$ and $|D^k f| \leq K$ on $[a \mathinner{..} b]$, then f must lie between two *perfect splines* of degree k. Corollary 4.21 is a *multivariate* generalisation of the case $k = 2$ (with $m = n - 1$). Though a trivial generalisation in the sense that the envelope functions are such simple multivariate splines (quadratic polynomials), it is important in view of the lack of results on the optimal recovery of multivariate functions from such information, and the possibility that it may lead towards more significant results.

In this regard, it is worth mentioning a related result of Shvartsman (see [B94]) which has a geometric form similar to (4.2). Let η be the centre of a ball of radius ρ containing

Θ. Then, using *Jensen's inequality*, Shvartsman proves that for all $x \in T$

$$|f(x) - L_\Theta f(x)| \le \sqrt{\rho^2 - \|x-\eta\|^2}\, |f|_{1,\infty,T}, \qquad \forall f \in W^1_\infty(T),$$

and so, in particular,

$$|f(x) - L_\Theta f(x)| \le \sqrt{R^2 - \|x-c\|^2}\, |f|_{1,\infty,T} \qquad \forall f.$$

These inequalities are not sharp for general $x \in T$.

Extremal functions

For a general $f \in W^2_\infty(T)$ the function (4.8) used in the proof of (4.2) is only defined a.e., and so (for fixed $x \in T$) it does not make sense to ask what conditions on f give equality in (4.9). However, if $f \in C^2(T)$, then (4.8) defines a continuous function, and the conditions giving equality in (4.9) are known. In this way, it is now shown that the set E of $C^2(T)$ functions giving equality in (4.2) for a fixed $x \in T \setminus \Theta$ is exactly Q.

By (3.5), there is equality in (4.9) only if

$$D_{v-w} D_{w-v} f \text{ is constant on } \operatorname{conv}\{x, v, w\}, \tag{4.24}$$

and

$$|D^2_{v-w} f(x)| = \|v-w\|^2 |D^2 f|(x) = \|v-w\|^2 |f|_{2,\infty,T}. \tag{4.25}$$

The condition (4.25) is a statement about the quadratic form given by

$$y \mapsto D^2_y f(x) = y^T H y$$

where H is the Hessian matrix of f evaluated at x. It says precisely that each vector $v - w$ is an eigenvector of H with eigenvalue of maximum modulus $\rho = |D^2 f|(x) = |f|_{2,\infty,T}$. All of these eigenvectors correspond to one of the two possible eigenvalues $\pm\rho$, since for $u, v, w \in \Theta$ two of the vectors $v-u, w-v, u-w$ correspond to the same eigenvalue, and so the third, being a difference of the other two, must also. This implies H is a scalar multiple of the identity matrix, i.e.,

$$D^2_y f(x) = \pm |f|_{2,\infty,T} \|y\|^2. \tag{4.26}$$

Since the $(n+2)$-dimensional space Q is contained in E, to prove that $E = Q$ it is sufficient to show that the only function from E that is annihilated by some $n+2$ linear functionals which are linearly independent over Q is the zero function. We now show this for the $n+2$ linear functionals consisting of point evaluation at each $v \in \Theta$ together with

$$f \mapsto D^2_y f(x),$$

for some y. If $D^2_y f(x) = 0$ for $f \in E$, then (4.26) implies $|f|_{2,\infty,T} = 0$, and so f is a linear polynomial. But the only linear polynomial which is zero at each point in Θ is the zero polynomial, and we conclude $E = Q$.

An immediate consequence of this is that the only functions $f \in C^2(T)$ giving equality in (4.16), for T an equilateral triangle, are $f \in Q$. In Subbotin [Su90$_1$:Ex.1] it is claimed that for the equilateral triangle with vertices

$$\Theta := \{(0,0),(h,0),(h/2,\sqrt{3}h/2)\}$$

there is equality in (4.16) for the cubic polynomial given by

$$f(x,y) := \frac{1}{2}M\left[y(h_0-y)+\frac{h_0-y}{h_0}(h-x)x\right], \qquad h_0 := \frac{\sqrt{3}}{2}h,$$

(which vanishes at Θ). There it is supposed that $|f|_{2,\infty,T} = M$, and it is observed that

$$f(c) = \frac{1}{6}Mh^2, \qquad c = (h/2, \sqrt{3}h/6),$$

giving the sharpness. However, it is not true that $|f|_{2,\infty,T} = M$. Indeed, with $\xi = (\xi_1,\xi_2)$ one has that

$$D_\xi^2 f(0,0) = -M\left(\xi_1^2+\xi_2^2+h\xi_1\xi_2\right), \tag{4.27}$$

and so $|f|_{2,\infty,T} \ge M(1+h/2)$.

5. The corresponding L_p-bounds

In this section the multivariate form of Hardy's inequality is applied to (2.2) to obtain L_p-error bounds for linear interpolation for $1 \le p \le \infty$. For small p the results obtained in this way are not a significant improvement over those already in the literature. As before, let $T := \operatorname{conv}\Theta$.

Proposition 5.1. *Suppose L_Θ is the map of linear interpolation at Θ, and $1 \le p \le \infty$. Then, the best constant in the inequality*

$$\|f - L_\Theta f\|_{L_p(T)} \le C \sum_{\substack{\{v,w\}\subset\Theta \\ v\ne w}} \|D_{v-w}D_{w-v}f\|_{L_p(T)}, \qquad \forall f \in W_p^2(T), \tag{5.2}$$

depends only on n and p. This best constant, which will be denoted by $C_{n,p}$, satisfies

$$C_{n,p} \le \frac{1}{4(1-n/p)(2-n/p)}, \qquad p > n, \tag{5.3}$$

which is sharp when $p=\infty$, i.e.,

$$C_{n,\infty} = 1/8. \tag{5.4}$$

Proof: Inequalities of the form (5.2) with the seminorm

$$f \mapsto \sum_{v\ne w} \|D_{v-w}D_{w-v}f\|_{L_p(T)} \tag{5.5}$$

replaced by other equivalent seminorms such as $|\cdot|_{2,p,T}$, and C a constant depending on T also, are well-known (see, e.g., Ciarlet [C78:Th.3.1.4]). Thus, (5.2) holds with a constant which depends on T in addition to n and p. An affine change of variables shows this constant to work for all T, and so in particular we may choose the best possible constant $C_{n,p}$, which depends only on n and p.

It follows from the multivariate form of Hardy's inequality 3.1 that (2.2) holds for $\forall f \in W_p^2(T)$ when $p > n$. It can easily be shown that:

$$\|\ell_v \ell_w\|_{L_\infty(T)} = 1/4,$$

with the maximum taken at the midpoint of the line segment with endpoints v and w. Using this fact, together with (3.3), one obtains from (2.2) that

$$\begin{aligned}\|f - L_\Theta f\|_{L_p(T)} &\le \frac{1}{4} \sum_{\substack{\{v,w\}\subset\Theta\\ v\neq w}} \Big\| x \mapsto \int_{[x,v,w]} D_{v-w} D_{w-v} f \Big\|_{L_p(T)} \\ &\le \frac{1}{4(1-n/p)(2-n/p)} \sum_{\substack{\{v,w\}\subset\Theta\\ v\neq w}} \|D_{v-w} D_{w-v} f\|_{L_p(T)}.\end{aligned}$$

Finally, we prove the sharpness asserted in (5.4). Let f be a quadratic polynomial with all the derivatives that occur in (2.2) zero except for $D_{v-w}D_{w-v}f$. Then, by (1.8)

$$\int_{[x,v,w]} D_{v-w} D_{w-v} f = \frac{1}{2} D_{v-w} D_{w-v} f,$$

so (2.2) reduces to

$$f - L_\Theta f = \frac{1}{2} \ell_v \ell_w / 2\, D_{v-w} D_{w-v} f.$$

Taking the $L_p(T)$-norm of this gives the lower bound

$$C_{n,p} \ge \frac{1}{2} \frac{\|\ell_v \ell_w\|_{L_p(T)}}{\|1\|_{L_p(T)}},$$

which is sharp when $p = \infty$. □

The seminorm (5.5), though convenient for our purposes, is not usually used. Instead, for the purposes of comparison, we give a result using the more usual $|f|_{2,p,T}$.

From (3.7) it follows that

$$\|D_{v-w} D_{w-v} f\|_{L_p(T)} \le \|v - w\|^2 \, |f|_{2,p,T},$$

and so by Proposition 5.1 we have, that for $p > n$

$$\|f - L_\Theta f\|_{L_p(T)} \le \frac{n(n+1)}{8(1-n/p)(2-n/p)} h^2 \, |f|_{2,p,T}, \qquad \forall f \in W_p^2(T), \tag{5.6}$$

where, as before, $h := \operatorname{diam} T$. For $p < \infty$, the best bound in the literature was obtained by Arcangeli and Gout [AG76:Th.1-1] from the multipoint formula (2.10) by using (the equivalent of) (3.4). They proved that for $p > n/2$

$$\|f - L_\Theta f\|_{L_p(T)} \le \frac{n+1}{2-n/p} h^2 \, |f|_{2,p,T}, \qquad \forall f \in W_p^2(T). \tag{5.7}$$

Since the inequality (5.6) improves upon (5.7) only when $n < 8$ and $p > 8n/(8-n)$, it provides only minor improvements. Particularly distressing is that in the case of greatest practical interest, when $p = n = 2$, (5.6) does not provide a bound. Whether this is a limitation of the error formula (2.2), or of the argument used to obtain (5.6) is an important question. This, and other questions arising from the this paper are elaborated upon in a set of research problems (see this issue).

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