Open Problems

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Which is the best error formula for linear interpolation on a triangle?

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Interpolation by polynomials has played a central role in the development of Approximation Theory from the very beginning (of the differential calculus). The ideas of polynomial interpolation have inspired and been reflected in many major developments, such as the theory of splines, and continues to be a rich source of problems. In light of this, and the current interest in approximating multivariate functions (with radial basis functions, subdivision schemes, wavelets, shift–invariant spaces, etc), it is suprising that only a tiny fraction of the literature on polynomial interpolation is concerned with *multivariate polynomial interpolation*.

Below are a number of questions, stated in the simple setting of $\mathbb{R}^2$, which were raised by the discovery of new error formula for linear interpolation (see Waldron [W96$_1$]). They provide an introduction to some important questions about the error in multivariate interpolation and (as in the univariate case) are accessible to the techniques of advanced calculus. It is intended that this article be read along side [W96$_1$], which defines much of the notation used here, and from which the general forms of the problems stated here are apparent.

For any points $u, v, w \in \mathbb{R}^2$, let

$$\int_{[u,v,w]} f := \int_0^1 \int_0^s f(u + s(v-u) + t(w-v))\, dt\, ds,$$

which is 1/2 the average of f over the triangle with vertices u, v, w when the points do not lie on a line. This is a special case of the *divided difference functional* (see, e.g., [W96$_1$]).

From now on, let $u, v, w \in \mathbb{R}^2$ be the vertices of a triangle T (the convex hull of u, v, w). There is a unique linear polynomial matching the values $f(u), f(v), f(w)$, called the *Lagrange interpolant* or *linear interpolant* to f, which is given by

$$Lf := f(u)\ell_u + f(v)\ell_v + f(w)\ell_w \in \Pi_1, \qquad (1)$$

where (1) uniquely determines the *Lagrange polynomials* $\ell_u, \ell_v, \ell_w \in \Pi_1$.

The problems of this paper concern the structure of two (of the many) formulæ for the error in linear interpolation. The first is the *Kowalewski–Ciarlet–Wagschal multipoint Taylor formula*

$$\begin{aligned} f(x) - Lf(x) = &- \ell_u(x) \int_{[x,x,u]} D^2_{u-x} f \\ &- \ell_v(x) \int_{[x,x,v]} D^2_{v-x} f \\ &- \ell_w(x) \int_{[x,x,w]} D^2_{w-x} f, \qquad \forall f \in C^2(T), \end{aligned} \tag{2}$$

of Ciarlet and Wagschal [CW71:Th.1,2]. The second is

$$\begin{aligned} f(x) - Lf(x) = &\ \ell_u(x)\ell_v(x) \int_{[x,u,v]} D_{u-v} D_{v-u} f \\ &+ \ell_v(x)\ell_w(x) \int_{[x,v,w]} D_{v-w} D_{w-v} f \\ &+ \ell_w(x)\ell_u(x) \int_{[x,w,u]} D_{w-u} D_{u-w} f, \qquad \forall f \in C^2(T), \end{aligned} \tag{3}$$

which was given in [W96$_1$], and will be referred to as the *'new formula'*. Here, D_z denotes differentiation in the direction $z = (z_1, z_2)$, i.e.,

$$D_z f = z_1 D_1 f + z_2 D_2 f.$$

It was shown in [W96$_2$] that these are instances of a family of error formulæ (for linear interpolation) which are indexed by real measures on $\mathbb{R}^2$ of unit mass.

Both of these formulæ yield the *sharp* pointwise error bound, that for $x \in T$

$$|f(x) - Lf(x)| \le \frac{1}{2}(R^2 - \|x - c\|^2)\, |f|_{2,\infty,T}, \qquad \forall f \in W^2_\infty(T), \tag{4}$$

where R, c are the radius and centre of the unique circle containing the points u, v, w, and $|f|_{k,p,\Omega}$ is the seminorm measuring the $L_p(\Omega)$–norm of the k–th derivative of $f \in W^k_p(\Omega)$, $\Omega \subset \mathbb{R}^n$, which is defined in [W96$_1$], and given by

$$|f|_{k,p,\Omega} := \Big\| \, x \mapsto \sup_{\substack{\xi_1,\ldots,\xi_k \in \mathbb{R}^n \\ \|\xi_i\| \le 1}} |D_{\xi_1} \cdots D_{\xi_k} f(x)| \, \Big\|_{L_p(\Omega)}, \qquad 1 \le p \le \infty, \tag{5}$$

for $f \in C^k(\Omega)$.

For a given $x \in T \setminus \{u, v, w\}$, the only $C^2(T)$–functions giving equality in (4) are those from the 4–dimensional space of quadratics

$$Q := \Pi_1 \oplus \operatorname{span}\{\|\cdot\|^2\},$$

which leads to the first question:

Problem I. For fixed $x \in T \setminus \{u, v, w\}$, are the quadratic polynomials $f \in Q$ the only functions from $W^2_\infty(T)$ giving equality in (4)?

I would imagine that the conditions for equality given by the proof of (4) from formulæ (2), (3) (and others) can be shown to be sufficiently restrictive that there is equality *only* for $f \in Q$ (as is the case under the additional assumption that $f \in C^2(T)$). If not, then (my calculations indicate) it is likely that there is equality only for certain piecewise polynomial functions. Whatever the case, it should be possible to settle this final piece of the puzzle for the L_∞–bounds by using an appropriate representation for f and a bit of determination (while developing some useful techniques).

A more substantial question is that of obtaining good L_p–bounds. Let h be the diameter of $\{u, v, w\}$, i.e., $h = \max\{\|v-u\|, \|w-u\|, \|w-v\|\}$. Much of the work on multivariate polynomial interpolation has been motivated by a desire of the finite element community for bounds of the type

$$\|f - Lf\|_{L_p(T)} \le C\, h^2\, |f|_{2,p,T}, \qquad \forall f \in W^2_p(T), \tag{6}$$

which are crucial to the error analysis of the finite element method (see, e.g., Ciarlet [C78:Ch.3]). It is well-known (see, e.g., [C78:Th15.3]) that an inequality of the form (6) holds with $C := C_{p,T}$ denoting the best constant. In [W96$_1$], using what was termed the *multivariate form of Hardy's inequality*, the following bounds were obtained: from (2) that

$$C_{p,T} \le \frac{3}{8(1-2/p)(1-1/p)}, \qquad p > 1,$$

and from (3) the bound

$$C_{p,T} \le \frac{3}{2(1-1/p)}, \qquad p > 2.$$

Whether these are the best bounds that one can hope to obtain from each formula (which they aren't for $p = \infty$) is the next question. In particular, can an L_2–error bound be obtained from the new formula (3)?

Problem II. Which formula gives the best L_p-error bounds for $p < \infty$? (or when is the multivariate form of Hardy's inequality sharp?)

Abstractly the difficulty is as follows. Let X be the Banach space consisting of the second derivatives $D^2 f$ (a multilinear map) of functions $f \in W^2_p(T)$, with norm given by

$$\|D^2 f\|_X := |f|_{2,p,T}.$$

This norm is equivalent to the one obtained by thinking of X as the (closed) subspace of $L_p(T) \times L_p(T) \times L_p(T)$ given by the ordered triples

$$(D_1^2 f, D_1 D_2 f, D_2^2 f), \qquad f \in W^2_p(T). \tag{7}$$

The existence of a smallest constant $C := C_{p,T}$ in (6) is equivalent to being able to write the error as

$$f - Lf = \mathcal{R}(D^2 f),$$

where $\mathcal{R} : X \to L_p(T)$ is a bounded linear map with

$$\|\mathcal{R}\| = C_{p,T}\, h^2.$$

Each of the formulæ (2) and (3) expresses the error as

$$f - Lf = \mathcal{R}(D^2 f) = \sum_{i=1}^{3} \mathcal{R}_i(D^2 f),$$

for certain linear maps $\mathcal{R}_i$ (defined initially for $f \in C^2(T)$). To bound $C_{p,T}$ using these formulæ one would hope to show that each $\mathcal{R}_i$ induces a *bounded* linear map $\mathcal{R}_i : X \to L_p(T)$, thereby obtaining the bound

$$C_{p,T}\, h^2 \le \sum_{i=1}^{3} \|\mathcal{R}_i\|.$$

In both cases the maps $\mathcal{R}_i$ can be obtained from each other by an affine change of variables which interchanges the vertices of the triangle T, and so they are either all bounded or all unbounded. Offhand, it would seem suprising if there were values of p for which they were all unbounded, yet (despite being so closely related) added together to give a bounded map. However, the following indicates that this could well be the situation for $p = 1$ in the multipoint Taylor formula (2). There the maps $\mathcal{R}_i$ are of the form

$$\mathcal{R}_i(D^2 f)(x) = -\ell_v(x) \int_{[x,x,v]} D^2_{v-x} f.$$

And it follows, from the multivariate (form of) Hardy's inequality (and details given in [W96$_3$]), that, for $p > 1$, this induces a bounded linear map $\mathcal{R}_i : X \to L_p(T)$ with

$$\|\mathcal{R}_i\| \le \frac{h^2}{2(1 - 1/p)} \to \infty, \qquad \text{as } p \to 1^+,$$

while the rule

$$Af(x) := \int_{[x,x,v]} f \tag{8}$$

does not induce a bounded linear map $L_1(T) \to L_1(T)$. For the new formula (3) the maps are of the form

$$\mathcal{R}_i(D^2 f)(x) = \ell_v(x)\ell_w(x) \int_{[x,v,w]} D_{v-w} D_{w-v} f,$$

and the multivariate Hardy's inequality fails to guarantee a bounded extension to a map $X \to L_p(T)$ for $p > 2$. Since the ordered triple in (7) could equally well be replaced by

$$(D_{u-v}D_{v-u}f, D_{v-w}D_{w-v}f, D_{w-u}D_{u-w}f),$$

extending this $\mathcal{R}_i$ is equivalent to showing that the rule

$$Af(x) := \ell_v(x)\ell_w(x)\int_{[x,v,w]} f$$

induces a bounded linear map $L_p(T) \to L_p(T) : f \mapsto Af$. Answering the following question about sharpness in the multivariate Hardy's inequality would go a long way towards settling this (and understanding the cases of equality in the multivariate Hardy's inequality).

Problem III. Does the rule

$$Hf(x) := \int_{[x,v,w]} f = \frac{1}{2} \text{ the average of } f \text{ over the triangle with vertices } x, v, w$$

induce a bounded linear map $L_2(T) \to L_2(T) : f \mapsto Hf$?

This is a deceptively difficult problem. Start by determining the conditions for (near) equality in each step of the proof (most notably the use of the integral form of Minkowski's inequality) of the multivariate Hardy's inequality: for $p > 2$

$$\|Hf\|_{L_p(T)} \le \frac{1}{(1-2/p)(2-2/p)}\|f\|_{L_p(T)}, \qquad \forall f \in C(T),$$

which shows that H induces a bounded linear map $L_p(T) \to L_p(T)$ for all $p > 2$. Also, be aware of the fact that the map A given by (8) has eigenfunctions $f = \|\cdot - v\|^\alpha$, $\alpha + 2 > 0$, with corresponding eigenvalues $\lambda = 1/(2+\alpha)$. From this, it can be shown that A induces a bounded linear map $L_p(T) \to L_p(T)$ only when $p > 1$, and the norm of this map can be computed exactly.

Much of the literature on the error in linear interpolation involves the following L_2–bounds:

Problem IV. What are the best constants C in the inequalities

$$\|f - Lf\|_{L_2(T)} \le Ch^2 \,|f|_{2,2,T}, \qquad \forall f \in W_2^2(T), \tag{9}$$

and

$$|f - Lf|_{1,2,T} \le Ch\,|f|_{2,2,T}, \qquad \forall f \in W_2^2(T). \tag{10}$$

How do they depend on the geometry of T, and which functions give equality?

The best constants C (and functions giving equality) in these and other similar inequalities (involving equivalent seminorms) can be expressed as the solution of a variational problem. No analytical solutions have been found, but there have been several papers giving numerical solutions, most recently the work of Siganevich [S88] (penalty method and projection-grid method), and Handscomb [H95] (finite element method). For (9), we have seen that a constant independent of T can be chosen, while for (10), it was understood by Zlámal [Z68] that the best constant C depends on the smallest angle in T. Further refinements of Zlámal's *minimum angle condition* were given by Babuška and Aziz [BA76] and Jamet [J76] (also see the discussion in Rippa [R92]), and it should be possible to gain an even better understanding using numerical solutions. The univariate analogues of (9) and (10), where L is linear interpolation at points $a, b \in \mathbb{R}$ and T is replaced by the interval $[a, b]$, have analytical solutions (the functions giving equality are sinusoidal). Using this fact together with a clear understanding of the dependence of the best constant on the geometry of T (which could involve R, c), it may eventually be possible to give an analytic solution for the best constant and functions giving equality in one or both of (9), (10). In this regard, be aware of the monograph [MPV89] which contains a wealth of information about inequalities relating the various geometric attributes of triangles (and simplicies).

To deal with (10), it is necessary to understand the behaviour of the derivative of the error, which is the next question.

Problem V. Analyse the error formulæ for the derivative of the error in linear interpolation.

There are a number of formulæ for the derivative of the error in linear interpolation, including a version of the multivariate Taylor formula, and a formulæ of Gregory [G75] involving two line integrals and three area integrals. The discussion of Problem II applies here, and, in light of (4), (sharp) L_∞–bounds on the derivative of the error might be a realistic first step.

In the same vein, building on techniques used for the previous problems, one might consider error formulæ for quadratic interpolation at the simplex points, i.e., interpolation by quadratics at the points u, v, w and their midpoints $(u+v)/2, (v+w)/2, (w+u)/2$.

Problem VI. Analyse the error formulæ for quadratic interpolation at the simplex points (or more general configurations).

The most recent formula for the error in quadratic interpolation (of the many that exist) is a generalisation of the new formula (3) given in de Boor [B95]. That paper also proposes a multivariate divided difference and outlines some connections with algebraic geometry that are relevant to all the problems posed here. Also, be aware of the bound of Subbotin [Su90:Th.2].

The final problem concerns the possibility of extending the 'envelope theorem' of Gaffney–Powell [GP76] and Micchelli–Rivlin–Winograd [MRW76] to the multivariate setting.

Problem VII. The optimal recovery of multivariate functions from function values. Find the function s giving the sharp inequality, for given $x \in T$, that

$$|f(x) - Lf(x)| \leq s(x)\, |f|_{1,\infty,T}, \qquad \forall f \in W^1_\infty(T). \tag{11}$$

Suppose that Θ is a finite set of points in $\mathbb{R}^n$ with convex hull S, and k is a positive integer for which

$$f \mapsto |f|_{k,\infty,S} + \sum_{v\in\Theta} |f(v)|$$

is a norm on $W_\infty^k(S)$ (see (5) for the definition of $|f|_{k,\infty,S}$). In [W96$_1$:p.14], it is outlined why the following information about $f \in W_\infty^k(S)$:

$$f(v), \ v \in \Theta \qquad \text{(its values at the points } \Theta) \tag{12}$$

and

$$|D^k f| \le K \quad \text{on} \quad S \qquad (\text{i.e., } |f|_{k,\infty,S} \le K),$$

implies that f must lie between two functions

$$\mathtt{L}(x) \le f(x) \le \mathtt{U}(x), \qquad \forall x \in S,$$

giving the best possible bounds at each point. In the univariate case, $n = 1$, it was shown by [GP76] and [MRW76] that if the number of points in Θ is at least k, then $\mathtt{L}, \mathtt{U}$ are splines that can be computed (see also de Boor [B77]). It follows from (4), that in the bivariate case, $n = 2$, when $\Theta = \{u, v, w\}$, $S = T$ and $k = 2$ that $\mathtt{L}, \mathtt{U}$ are also splines (quadratic polynomials). This suggests that in other multivariate situations $\mathtt{L}, \mathtt{U}$ may also be (computable) splines. The easiest case to investigate is $n = 2$, $\Theta = \{u, v, w\}$, $S = T$ and $k = 1$, which is equivalent to Problem VII. Presumably the function s is some spline, and it will be possible to tell whether there is any hope of extending the 'envelope theorem' to other multivariate examples. Related work of Pavel Shvartsman on Problem VII is mentioned in [W96$_1$:p.17].

The author would be delighted to provide further information, problems, etc, to those interested in the problems of *multivariate polynomial interpolation* touched upon here.

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