Real and complex spherical designs and their Gramian

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Abstract

If a (weighted) spherical design is defined as an integration (cubature) rule for a unitarily invariant space P of polynomials (on the sphere), then any unitary image of it is also such a spherical design. It therefore follows that such spherical designs are determined by their Gramian (Gram matrix). We outline a general method to obtain such a characterisation as the minima of a function of the Gramian, which we call a potential. This characterisation can be used for the numerical and analytic construction of spherical designs. When the space P of polynomials is not irreducible under the action of the unitary group, then the potential is not unique. In several cases of interest, e.g., spherical t-designs and half-designs, we use this flexibility to provide potentials with a very simple form. We then use our results to develop certain aspects of the theory of real and complex spherical designs for unitarily invariant polynomial spaces.

Key Words: Gramian (Gram matrix), spherical t-designs, spherical half-designs, tight spherical designs, finite tight frames, integration rules, cubature rules, cubature rules for the sphere, reproducing kernel, positive definite function, Gegenbauer polynomials, Zernike polynomials, complex spherical design, potential, frame force, codes,

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1 Introduction

Let σ be the normalised surface area measure on the unit sphere $\mathbb{S}$ in $\mathbb{R}^d$ or $\mathbb{C}^d$. A (**weighted**) **spherical design** (for P) is a sequence of points $v_1, \ldots, v_n$ in $\mathbb{S}$ and **weights** $w_1, \ldots, w_n \in \mathbb{R}$, $w_1 + \cdots + w_n = 1$, for which the integration (cubature) rule

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{j=1}^{n} w_j p(v_j), \tag{1.1}$$

holds for all p in a finite dimensional space of polynomials P. This is essentially the first of the four definitions given in [Sei01]. We say that a polynomial p (or a space of polynomials) is **integrated** by the spherical design (integration/cubature rule) if (1.1) holds. The existence of spherical designs for constant weights, i.e., $w_j = \frac{1}{n}$, and n sufficiently large, was proved in [SZ84]. In applications, it is often required that $w_j \geq 0$. The common choices for P are unitarily invariant, i.e., for U unitary, $p \circ U \in P$, $\forall p \in P$. In the real case, the unitary (inner product preserving) maps are the orthogonal transformations. For such a space P, the unitary invariance of the measure σ implies that $(Uv_j), (w_j)$ is also a spherical design when U is unitary, by the calculation

$$\sum_j w_j p(Uv_j) = \int_{\mathbb{S}} (p \circ U)(x)\, d\sigma(x) = \int_{\mathbb{S}} p(x)\, d\sigma(x), \qquad \forall p \in P.$$

The **Gramian** (or **Gram matrix**) of a sequence of vectors (v_j) is the Hermitian matrix of inner products $[\langle v_j, v_k \rangle]$. We say that sequences of vectors (v_j) and (u_j) are **unitarily equivalent** if $u_j = Uv_j$, $\forall j$, where U is unitary. Since

$$\langle u_j, u_k \rangle = \langle Uv_j, Uv_k \rangle = \langle v_j, v_k \rangle,$$

a sequence of vectors is defined up to unitary equivalence by its Gramian (see [Wal18]). Combining this with our previous observation gives:

- The real or complex spherical designs (v_j) for a unitarily invariant polynomial space P are determined by the Gramian of (v_j).

It should therefore be possible to express the condition of (v_j) being a spherical design in terms of $\langle v_j, v_k \rangle$, $1 \leq j, k \leq n$, and the weights (w_j) if these are not constant. The primary objective of this paper is to give such a characterisation for being a spherical design, which is as simple as possible. Some key features of our approach are:

- Because the reproducing kernel $K(x, y)$ for a unitarily invariant polynomial space P is a function of $\langle x, y \rangle$, we are able to find a "potential" $A_P\big([\langle v_j, v_k \rangle]\big) \geq 0$ whose zeros are the spherical designs for P.
- The potential A_P can be given by a univariate polynomial F, which is not unique when P is not irreducible under the action of the unitary group (see Table 1).
- The polynomial space P and the corresponding real or complex spherical designs can be described by a finite set of indices $L \subset \mathbb{N}$ or $\tau \subset \mathbb{N}^2$, respectively, which index the irreducible subspaces of P.

Motivated by a careful analysis of the real spherical designs [BB09], we obtain a unified theory of the most general real and complex spherical designs, which includes:

- A general variational characterisation for all types of designs (Theorem 3.1).
- New characterisations for real spherical designs, including t-designs (Theorem 4.1) and half-designs (Theorem 4.2).
- New characterisations for complex spherical designs (Theorem 5.1, Theorem 5.2).
- A natural description for real and complex projective spherical designs (§6).
- A description of the Gegenbauer polynomials that appear naturally in the analysis of complex spherical designs as orthogonal polynomials (of two variables), together with results about their products and sums (§7, §8).
- A unified approach to bounds on the numbers of points in a spherical design (§9).
- Unified results about the number of vectors in A-sets and s-angular sets (§10).

As much as possible, we treat the real and complex cases simultaneously, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

2 Basic definitions

Sometimes it is convenient to describe a weighted spherical design for P as a sequence of (possibly not unit) vectors (v_j), where the weights are inferred from the $\|v_j\|$ by

$$w_j = w_j^{(m)} := \frac{\|v_j\|^m}{\sum_\ell \|v_\ell\|^m}, \quad \text{which we call the } m\textbf{-weights}. \tag{2.2}$$

Designs with constant weights are called **unweighted**, **classical** or simply **designs**.

It is a subtle but important point, which follows from (1.1), that a spherical design depends only the restriction of P to the sphere, i.e. the space of harmonic polynomials

$$\operatorname{harm}(P) := P|_{\mathbb{S}},$$

which we will call the **harmonic part** of the polynomial space P.

Choices for the (unitarily invariant) polynomial space P of interest include:

$\Pi_t(\mathbb{R}^d)$ = polynomials of degree $\le t$ on $\mathbb{R}^d$ (**spherical t-designs**),
$\operatorname{Hom}_m(\mathbb{R}^d)$ = homogeneous polynomials of degree m on $\mathbb{R}^d$ (**spherical half-designs**),
$\operatorname{Harm}_m(\mathbb{R}^d)$ = harmonic polynomials in $\operatorname{Hom}_m(\mathbb{R}^d)$ (**spherical designs of harmonic index** m),
$\operatorname{Hom}_{t,t}(\mathbb{C}^d) = \operatorname{span}\{|\langle \cdot, v\rangle|^{2t} : v \in \mathbb{C}^d\}$ (**spherical** (t,t)**-designs**, **projective t-designs**).

The half-designs for $P = \operatorname{Hom}_{2t}(\mathbb{R}^d)$ are also called (**real**) **spherical** (t,t)**-designs**.

Example 2.1 *From the observation*

$$\int_{\mathbb{S}} \|x\|^{2k} p(x)\, d\sigma(x) = \int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_j w_j p\Big(\frac{v_j}{\|v_j\|}\Big) = \sum_j w_j \big(\|\cdot\|^{2k} p\big)\Big(\frac{v_j}{\|v_j\|}\Big),$$

it follows that a spherical design for P also integrates the spaces

$$P^- = \big\{q : \|\cdot\|^{2j} q \in P, \exists j \geq 0\big\}, \qquad P^+ = \big\{\|\cdot\|^{2k} p : p \in P, k \geq 0\big\},$$

with $\operatorname{harm}(P^-) = \operatorname{harm}(P^+) = \operatorname{harm}(P)$*. In particular, for* $P = \operatorname{Hom}_m(\mathbb{R}^d)$*, we have*

$$P^- = \operatorname{Hom}_m(\mathbb{R}^d) \oplus \operatorname{Hom}_{m-2}(\mathbb{R}^d) \oplus \operatorname{Hom}_{m-4}(\mathbb{R}^d) \oplus \cdots, \quad P^+ = \bigoplus_{k=0}^{\infty} \|\cdot\|^{2k} \operatorname{Hom}_m(\mathbb{R}^d).$$

Because of this, [KP11] refer to the spherical designs for $P = \operatorname{Hom}_m(\mathbb{R}^d)$ *as the spherical half-designs (of order m).*

Every homogeneous polynomial $p \in \operatorname{Hom}_k(\mathbb{R}^d)$ of degree k can be written uniquely

$$p(x) = \sum_{0 \leq j \leq \frac{k}{2}} \|x\|^{2j} p_{k-2j}(x) = \sum_{j=0}^{[\frac{k}{2}]} \|x\|^{2j} p_{k-2j}(x), \tag{2.3}$$

where $p_{k-2j} \in \operatorname{Harm}_{k-2j}(\mathbb{R}^d)$, and the restriction map

$$\operatorname{Harm}_k(\mathbb{R}^d) \to L_2(\mathbb{S}) : f \mapsto f|_{\mathbb{S}}$$

is injective, with image denoted $\operatorname{Harm}_k(\mathbb{S})$. We will freely identify spaces of harmonic functions on $\mathbb{R}^d$ and $\mathbb{S}$ (**solid** and **surface** spherical harmonics). It follows from the irreducibility of the summands above [ABR01], that every unitarily invariant (invariant under orthogonal transformations) polynomial space on $\mathbb{R}^d$ can be written uniquely as a direct sum

$$P = \bigoplus_{(j,k)\in\mathcal{J}} \|\cdot\|^{2j} \operatorname{Harm}_{k-2j}(\mathbb{R}^d),$$

for a subset $\mathcal{J}$ of the indices $\{(j,k) : 0 \leq j \leq \frac{k}{2}\}$. For the purpose of integration on $\mathbb{S}$, it suffices to consider the (possibly lower dimensional) space of harmonic polynomials

$$\operatorname{harm}(P) = P|_{\mathbb{S}} = \bigoplus_{\ell \in L} \operatorname{Harm}_\ell(\mathbb{S}) \approx \bigoplus_{\ell\in L} \operatorname{Harm}_\ell(\mathbb{R}^d), \quad L := \{k-2j : (j,k) \in \mathcal{J}\} \subset \mathbb{N}. \tag{2.4}$$

We note, in particular

$$\operatorname{harm}(\Pi_m(\mathbb{R}^d)) = \bigoplus_{j=0}^{m} \operatorname{Harm}_j(\mathbb{R}^d), \qquad \operatorname{harm}(\operatorname{Hom}_m(\mathbb{R}^d)) = \bigoplus_{0\leq j\leq \frac{m}{2}} \operatorname{Harm}_{m-2j}(\mathbb{R}^d).$$

A necessary condition for (v_j) to integrate $\mathrm{Hom}_m(\mathbb{R}^d)$ with the m-weights (2.2) is that

$$\begin{aligned}\int_{\mathbb{S}}\int_{\mathbb{S}}\langle x,y\rangle^m d\sigma(x)\,d\sigma(y) &= \int_{\mathbb{S}}\sum_{j=1}^n \frac{\|v_j\|^m}{\sum_\ell \|v_\ell\|^m}\langle \frac{v_j}{\|v_j\|},y\rangle^m d\sigma(y)\\ &= \sum_{j=1}^n\sum_{k=1}^n \frac{\|v_j\|^m\|v_k\|^m}{(\sum_\ell \|v_\ell\|^m)^2}\langle \frac{v_j}{\|v_j\|},\frac{v_k}{\|v_k\|}\rangle^m\\ &= \frac{1}{(\sum_\ell \|v_\ell\|^m)^2}\sum_{j=1}^n\sum_{k=1}^n \langle v_j,v_k\rangle^m. \qquad (2.5)\end{aligned}$$

To also integrate $\mathrm{Hom}_{m-1}(\mathbb{R}^d)$ with these m-weights a necessary condition is

$$\begin{aligned}\int_{\mathbb{S}}\int_{\mathbb{S}}\langle x,y\rangle^{m-1} d\sigma(x)\,d\sigma(y) &= \int_{\mathbb{S}}\sum_{j=1}^n \frac{\|v_j\|^m}{\sum_\ell \|v_\ell\|^m}\langle \frac{v_j}{\|v_j\|},y\rangle^{m-1} d\sigma(y)\\ &= \sum_{j=1}^n\sum_{k=1}^n \frac{\|v_j\|^m\|v_k\|^m}{(\sum_\ell \|v_\ell\|^m)^2}\langle \frac{v_j}{\|v_j\|},\frac{v_k}{\|v_k\|}\rangle^{m-1}\\ &= \frac{1}{(\sum_\ell \|v_\ell\|^m)^2}\sum_{j=1}^n\sum_{k=1}^n \|v_j\|\|v_k\|\langle v_j,v_k\rangle^{m-1}. \qquad (2.6)\end{aligned}$$

We will show that these conditions (for a weighted spherical m-design) are also sufficient (Theorem 4.2). It is most natural to view this result as a special case of a very general variational characterisation of spherical designs, which we now describe.

3 The variational characterisation of designs

We generalise the surface area measure σ on $\mathbb{S}$ to a measure μ on a set $\Omega \subset \mathbb{F}^d$. We say that a sequence of points $(v_1,\ldots,v_n)$ in Ω and weights $w=(w_j)$, $w_1+\cdots+w_n=1$, $w_j\ge 0$, is a **weighted (spherical) design** for a space P of functions $\Omega\to\mathbb{F}$ (or simply a P**-design**) if

$$\int_\Omega p(x)\,d\mu(x) = \sum_{j=1}^n w_j p(v_j), \qquad \forall p\in P.$$

If point evaluation on P is a continuous linear functional with respect to the inner product

$$\langle f,g\rangle_\mu := \int_\Omega f\overline{g}\,d\mu$$

given by μ (as is the case for P finite dimensional), then it can be represented by the **reproducing kernel** $K=K_P:\Omega\times\Omega\to\mathbb{F}$, which is given by

$$f(x)=\int_\Omega K(x,y)f(y)\,d\mu(y), \qquad \forall f\in P,$$

where $K(x,y)=\sum_s Y_s(x)\overline{Y_s(y)}$ for (Y_s) an orthonormal basis of P.

Theorem 3.1 *(Variational characterisation) Let μ be a measure on Ω with $\mu(\Omega) = 1$, P be a finite dimensional space of integrable functions on Ω, and H be the subspace of functions which are orthogonal to the constants, i.e.,*

$$H = P \ominus \operatorname{span}\{1\} = \{p \in P : \int_\Omega p \, d\mu = 0\}.$$

Let $\Phi = (v_1, \ldots, v_n)$, $v_j \in \Omega$, and $w = (w_j) \in \mathbb{R}^n$ be weights with $w_1 + \cdots + w_n = 1$. Write H as a direct sum $H = \oplus_\ell H^{(\ell)}$, with K_ℓ the reproducing kernel of $H^{(\ell)}$ and $c_\ell > 0$. Then

$$A_{w,c}(\Phi) = A_{P,\mu,w,c}(\Phi) := \sum_{j=1}^n \sum_{k=1}^n w_j w_k \sum_\ell c_\ell K_\ell(v_j, v_k) \geq 0, \tag{3.7}$$

with equality if and only if $(v_j), (w_j)$ is a weighted spherical design for P.

Proof: We first recall that the reproducing kernel K_ℓ for $H^{(\ell)}$ represents the point evaluations, i.e.,

$$f(x) = \int_\Omega K_\ell(x, y) f(y) \, d\mu(y), \qquad \forall f \in H^{(\ell)},$$

and that

$$K_\ell(x, y) = \sum_s Y_s^{(\ell)}(x) \overline{Y_s^{(\ell)}(y)},$$

where $(Y_s^{(\ell)})$ is an orthonormal basis for $H^{(\ell)}$. We therefore compute

$$\begin{aligned} A_{w,c}(\Phi) &:= \sum_{j=1}^n \sum_{k=1}^n w_j w_k \sum_\ell c_\ell K_\ell(v_j, v_k) \\ &= \sum_\ell c_\ell \sum_{j=1}^n \sum_{k=1}^n w_j w_k \sum_s Y_s^{(\ell)}(v_j) \overline{Y_s^{(\ell)}(v_k)} \\ &= \sum_\ell c_\ell \sum_s \left(\sum_{j=1}^n w_j Y_s^{(\ell)}(v_j) \right) \left(\sum_{k=1}^n w_k \overline{Y_s^{(\ell)}(v_k)} \right) \\ &= \sum_{\ell=1} c_\ell \sum_s \Big| \sum_{j=1}^n w_j Y_s^{(\ell)}(v_j) \Big|^2 \geq 0, \end{aligned}$$

with equality if and only if

$$\sum_{j=1}^n w_j Y_s^{(\ell)}(v_j) = 0 = \int_\Omega Y_s^{(\ell)}(x) \, d\mu(x), \qquad \forall s, \ \forall \ell.$$

which is equivalent to $(v_j), (w_j)$ being a P-design (by linearity and the fact $\sum_j w_j = 1$ ensures that the constants are integrated). $\square$

This result is essentially an abstract version of [SW09] (Theorem 3), which was for spherical designs on the real sphere, with $P = \Pi_t(\mathbb{R}^d)$. For examples of such potentials, we will use a variety of subscripts for A indicating parameters that it depends on, and which we choose to emphasize in a particular context.

We call the $A_{w,c}(\Phi)$ of (3.1) a **potential** for the P-designs (with weights w), given by

$$K := \sum_\ell c_\ell K_\ell.$$

The potential for P when there is a single summand and $c_1 = 1$ is called the **canonical potential**, and is denoted by A_P. This can be obtained by taking $\oplus_\ell H^{(\ell)}$ to be an orthogonal direct sum and $c_\ell = 1$, $\forall \ell$. It is also convenient at times to add a (positive) constant b_0 to a potential, to obtain a **potential with constant** $B(\Phi) = A_{w,c}(\Phi) + b_0$, with (3.7) then replaced by $B(\Phi) \geq b_0$.

There are *many* possible potentials for P-designs, and so P-designs are "universally optimal distributions of points" (cf. [CK07]) for the above class of such potentials. Given a tractable formula for a potential, P-designs can be constructed numerically (for sufficiently many points) by minimising it (to zero) [BGM$^+$21], [EW25]. We now investigate the case of real and complex spherical designs in detail, where $K(x, y)$ is a univariate function F of $\langle x, y\rangle$, which we also refer to as (giving) a **potential**, and hence $A_{w,c}(\Phi)$ is indeed function of the Gramian of Φ. Potentials which depend on triples of points, and hence not the Gramian in general, have recently been considered by [BFG$^+$22].

4 Real spherical designs

We have observed in (2.4) that every unitarily invariant polynomial space P restricted to the real sphere has the form

$$P|_{\mathbb{S}} = P_L := \bigoplus_{\ell \in L} \mathrm{Harm}_\ell(\mathbb{S}) \qquad \text{(orthogonal direct sum)}, \tag{4.8}$$

with $L \subset \mathbb{N} = \{0, 1, 2, \ldots\}$ an index set (which is finite for $P|_{\mathbb{S}}$ finite-dimensional). For the irreducible subspace $\mathrm{Harm}_\ell(\mathbb{R}^d)$, the reproducing kernel is

$$K_\ell^{(d)}(x, y) := \|x\|^\ell \|y\|^\ell Q_\ell^{(d)}\Big(\frac{\langle x, y\rangle}{\|x\|\|y\|}\Big), \tag{4.9}$$

where $Q_k = Q_k^{(d)}$ are the orthogonal polynomials for the Gegenbauer weight for $\lambda = \frac{d-2}{2}$, i.e., $(1 - x^2)^{\frac{d-3}{2}}$ on $[-1, 1]$, with the normalisation $Q_k^{(d)}(1) = \dim(\mathrm{Harm}_k(\mathbb{R}^d))$ [DGS77]. These satisfy

$$\begin{aligned} Q_\ell^{(d)}(x) &= (2\ell + d - 2) \sum_{j=0}^{[\ell/2]} (-1)^j \frac{d(d + 2)\cdots(d + 2\ell - 2j - 4)}{2^j j!(\ell - 2j)!} x^{\ell-2j} \\ &= C_\ell^{(\frac{d}{2})}(x) - C_{\ell-2}^{(\frac{d}{2})}(x) = \frac{2\ell + d - 2}{d - 2} C_\ell^{(\frac{d-2}{2})}(x), \end{aligned} \tag{4.10}$$

where $C_\ell^{(\lambda)}$ are the Gegenbauer (ultraspherical) polynomials, with $C_\ell^{(\lambda)} := 0,\ \ell < 0$. They are given by $Q_0(x) = 1$, $Q_1(x) = dx$, and the three-term recurrence

$$\lambda_{k+1} Q_{k+1}(x) = x Q_k(x) - (1 - \lambda_{k-1}) Q_{k-1}(x), \qquad \lambda_k := \frac{k}{2k + d - 2}.$$

Since surface area measure on the sphere is unitarily invariant, for U unitary, we have

$$(f \circ U)(x) = \int_{\mathbb{S}} K(Ux, y) f(y)\, d\sigma(y) = \int_{\mathbb{S}} K(Ux, Uy)(f \circ U)(y)\, d\sigma(y).$$

Hence the reproducing kernel $K(x, y)$ of a unitarily invariant space P is a function of $\langle x, y\rangle$, and hence the potential is a function of the Gramian $[\langle v_j, v_k\rangle]$. The direct calculation of the formula (4.9) is called the *addition formula* (see [BHS19]). We also observe, that for the real sphere

$$\langle x, y\rangle = \frac{1}{2}\big(\|x\|^2 + \|y\|^2 - \|x - y\|^2\big) = 1 - \frac{1}{2}\|x - y\|^2, \tag{4.11}$$

so that the reproducing kernel on $\mathbb{S}$ can also be thought of as a function of the "squared distance" $\|x - y\|^2$, which is the direction generalised by Delsarte spaces (see [Hog92]).

Since $\mathrm{Harm}_0(\mathbb{S}) = \mathrm{span}\{1\}$, a potential for P of the form (4.8), with weights w_j, is given by

$$A_{w,c}(\Phi) = A_{F,w}(\Phi) := \sum_{j=1}^n \sum_{k=1}^n w_j w_k F(\langle v_j, v_k\rangle), \tag{4.12}$$

where F is the *univariate* polynomial given by

$$F(x) := \sum_{\ell \in L\setminus\{0\}} c_\ell Q_\ell(x) = \sum_{\ell \in L\setminus\{0\}} c_\ell \big\{C_\ell^{(\frac{d}{2})}(x) - C_{\ell-2}^{(\frac{d}{2})}(x)\big\}. \tag{4.13}$$

Indeed, every univariate polynomial $F = \sum_k c_k Q_k$ with $c_k \geq 0$ gives a potential for the unitarily invariant polynomial space $P = P_L$ of (4.8), where $L := \{k > 0 : c_k > 0\}$. We will say that such a polynomial F is a **potential** for P_L. The function F of (4.13) is an example of what Schoenberg [Sch42] calls a **positive definite function on the sphere**, i.e., a continuous function $F : [-1, 1] \to \mathbb{R}$ for which the right-hand side of (4.12)

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k F(\langle v_j, v_k\rangle) = w^T [F(\langle v_j, v_k\rangle)]_{j,k=1}^n w$$

is nonnegative for all choices of points $(v_j) \subset \mathbb{S}(\mathbb{R}^d)$ and $w = (w_j) \in \mathbb{R}^n$, $n = 1, 2, \ldots$. These are characterised by the fact that their Fourier series in (Q_ℓ) has only nonnegative coefficients [DX13]. In this terminology, we can paraphrase our observation:

- A positive definite function F on the real sphere which is a polynomial is a potential for a spherical P-design, where $P = P_L$ is given by the correspondence (4.13).

We observe that the value of the potential $A_{F,w}(\Phi)$ depends only on the set of angles and their weighted multiplicities (see Section 9), i.e.,

$$\mathrm{Ang}(\Phi) = \{\langle v_j, v_k\rangle\}_{1\le j,k\le n}, \qquad m_{w,\alpha} = \sum_{\substack{j,k \\ \langle v_j, v_k\rangle = \alpha}} w_j w_k, \quad \alpha \in A.$$

Example 4.1 *For spherical designs of harmonic index m [BOT15], $P = \mathrm{Harm}_m(\mathbb{R}^d)$, i.e., $L = \{m\}$, and there is a unique (up to a scalar) potential $Q_m(x)$ given by (4.10).*

Example 4.2 *(Tight frames) For $P = \mathrm{Hom}_2(\mathbb{R}^d)$, $P|_{\mathbb{S}} = \mathrm{Harm}_2(\mathbb{S}) \oplus \mathrm{span}\{1\}$, i.e., $L = \{0, 2\}$, and*

$$Q_2^{(d)}(x) = \frac{1}{2} d(d+2)\Big(x^2 - \frac{1}{d}\Big).$$

gives the canonical potential. The zeros of this potential are the unit norm tight frames, and if the 2-weights $w_j = w_j^{(2)}$ given by (2.2) are taken, then one obtains the variational characterisation of tight frames [Wal03] (take $m = 2$ in (4.21) of Theorem 4.2), [BF03].

For spherical half-designs of order m, i.e.,

$$P|_{\mathbb{S}} = \mathrm{Hom}_m(\mathbb{R}^d)|_{\mathbb{S}} = \mathrm{Harm}_m(\mathbb{S}) \oplus \mathrm{Harm}_{m-2}(\mathbb{S}) \oplus \mathrm{Harm}_{m-4}(\mathbb{S}) \oplus \cdots, \quad m \ge 1,$$

the sum over ℓ for $c_\ell = 1$ in (4.13) is telescoping, with $C_0^{(\frac{d}{2})}(x) = 1$, and we obtain:

Example 4.3 *The canonical potential for spherical half-designs of order m is given by*

$$F(x) = \begin{cases} C_m^{(\frac{d}{2})}(x), & m \text{ odd;} \\ C_m^{(\frac{d}{2})}(x) - 1, & m \text{ even,} \end{cases}$$

i.e.,

$$A_{w,1}(\Phi) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k C_m^{(\frac{d}{2})}(\langle v_j, v_k\rangle) - \varepsilon, \qquad \varepsilon := \begin{cases} 0, & m \text{ odd;} \\ 1, & m \text{ even.} \end{cases} \tag{4.14}$$

Since the univariate polynomials $C_\ell^{(\frac{d}{2})}$ are even or odd, with monomial coefficients of alternating sign, it turns out that we can choose the c_ℓ to obtain a potential for the spherical half-designs with a very simple form. Let

$$b_m(\mathbb{R}^d) := \int_{\mathbb{S}}\int_{\mathbb{S}} \langle x, y\rangle^m d\sigma(x)\, d\sigma(y) = \begin{cases} 0, & m \text{ odd;} \\ \frac{1\cdot 3\cdot 5\cdots(m-1)}{d(d+2)\cdots(d+m-2)}, & m \text{ even.} \end{cases} \tag{4.15}$$

Lemma 4.1 *(Half-designs) A potential for the spherical designs for $P = \mathrm{Hom}_m(\mathbb{R}^d)$, equivalently $L = \{m, m-2, \ldots\}$, is given by the polynomial $F(x) = x^m - b_m(\mathbb{R}^d)$, i.e.,*

$$A_{F,w}(\Phi) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \langle v_j, v_k\rangle^m - b_m(\mathbb{R}^d),$$

where $b_m(\mathbb{R}^d)$ is given by (4.15).

Proof: The index set is $L=\{m-2a:1\le m-2a\le m\}=\{m-2a:0\le a\le \frac{m-1}{2}\}$. Let

$$c_{m,a}:=\frac{1}{2^a a!}\prod_{r=1}^{a}(d+2(m-r)),\qquad 0\le a\le \frac{m-1}{2}.$$

Then a simple calculation, using the explicit formula of (4.10), gives

$$F(x):=\frac{m!}{d(d+2)\cdots(d+2m-2)}\sum_{0\le a\le\frac{m-1}{2}} c_{m,a}Q_{d,m-2a}(x)=x^m-b_m(\mathbb{R}^d),$$

so that we have a potential

$$A_{F,w}(\Phi)=\sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\Big\{\langle v_j,v_k\rangle^m-b_m(\mathbb{R}^d)\Big\}=\sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\langle v_j,v_k\rangle^m-b_m(\mathbb{R}^d),$$

as claimed. □

This result was proved for m even by Venkov [Ven01] (constant weights) and [KP11] (nonnegative weights), by using a different method.

We can now describe various variational conditions for being a real spherical t-design.

Theorem 4.1 *Let $w_1+\cdots+w_n=1$, $w_j\in\mathbb{R}$, $v_j\in\mathbb{S}$, and $t\ge 1$. Then $(v_j),(w_j)$ is a weighted spherical t-design if and only if there is equality in the inequalities*

$$\sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\langle v_j,v_k\rangle^t\ge b_t(\mathbb{R}^d),\qquad \sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\langle v_j,v_k\rangle^{t-1}\ge b_{t-1}(\mathbb{R}^d). \tag{4.16}$$

In particular, (v_j) is a spherical t-design if and only if

$$\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle v_j,v_k\rangle^m=\int_{\mathbb{S}}\int_{\mathbb{S}}\langle x,y\rangle^m\,d\sigma(x)\,d\sigma(y)=b_m(\mathbb{R}^d),\qquad m\in\{t-1,t\}. \tag{4.17}$$

Proof: Since $\Pi_t(\mathbb{R}^d)|_{\mathbb{S}}=\mathrm{Hom}_t(\mathbb{R}^d)|_{\mathbb{S}}\oplus\mathrm{Hom}_{t-1}(\mathbb{R}^d)|_{\mathbb{S}}$, the spherical t-designs are precisely the spherical half-designs of order t which are also half-designs of order $t-1$, and the result follows directly from Lemma 4.1. □

The condition (4.17) for $m\in\{1,2,\ldots,t\}$ is well known (see [GS79], Theorem 4.4). The following condition given by the canonical potential (4.14) appears to be new.

Corollary 4.1 *The unit vectors $(v_j)\subset\mathbb{R}^d$ give a weighted spherical t-design if and only if*

$$\sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\,C_m^{(\frac{d}{2})}(\langle v_j,v_k\rangle)=\begin{cases}0, & m\ \text{odd};\\ 1, & m\ \text{even},\end{cases}\qquad m\in\{t-1,t\}, \tag{4.18}$$

which can also be written as

$$\sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\left\{C_t^{(\frac{d}{2})}(\langle v_j,v_k\rangle)+C_{t-1}^{(\frac{d}{2})}(\langle v_j,v_k\rangle)\right\}=1. \tag{4.19}$$

The condition that spherical t-designs are characterised by their canonical potentials for $\mathrm{Harm}_\ell(\mathbb{R}^d)$ vanishing, i.e.,

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k \, C_\ell^{(\frac{d-2}{2})}(\langle v_j, v_k\rangle) = 0, \qquad \ell \in \{1,2,\ldots,t\},$$

is well known, as is the generalisation to spherical designs of harmonic index t [ZBB+17] (Lemma 2.1).

We now show that the necessary conditions (2.5) and (2.6), which are given in Sidel'nikov [Sid74] (Corollary 1) for unit vectors, are sufficient.

Theorem 4.2 *Let (v_j) be vectors in $\mathbb{R}^d$, not all zero, with corresponding m-weights, i.e.,*

$$w_j := \frac{\|v_j\|^m}{\sum_\ell \|v_\ell\|^m}. \tag{4.20}$$

Then (v_j) gives a weighted spherical half-design of order m if and only if there is equality in the inequality

$$\sum_{j=1}^n \sum_{k=1}^n \langle v_j, v_k\rangle^m \ge b_m(\mathbb{R}^d)\Bigl(\sum_{\ell=1}^n \|v_\ell\|^m\Bigr)^2. \tag{4.21}$$

Moreover, this is also a weighted spherical m-design if and only if in addition there is equality in

$$\sum_{j=1}^n \sum_{k=1}^n \|v_j\|\|v_k\|\langle v_j, v_k\rangle^{m-1} \ge b_{m-1}(\mathbb{R}^d)\Bigl(\sum_{\ell=1}^n \|v_\ell\|^m\Bigr)^2. \tag{4.22}$$

Proof: Take w_j given by (4.2) in Lemma 4.1 and Theorem 4.1. $\square$

To find spherical t-designs with nonnegative weights, one can minimise the single potential

$$A(\Phi) := \sum_{j=1}^n \sum_{k=1}^n \bigl(\langle v_j, v_k\rangle^t + \|v_j\|\|v_k\|\langle v_j, v_k\rangle^{t-1}\bigr) \ge c_t(\mathbb{R}^d) + c_{t-1}(\mathbb{R}^d),$$

over the compact set of $\Phi = (v_j)$ with $\sum_\ell \|v_\ell\|^m = 1$, where equality gives a weighted spherical t-design for the t-weights.

If $X = \{v_j\}$ is antipodal (centrally symmetric), i.e., $X = -X$, then it gives a spherical half-design of odd order m, for every m. We will say a spherical half-design of odd order is **nontrivial** if its vectors span $\mathbb{R}^d$ and it is not antipodal.

Example 4.4 *The $d+1$ vertices of a regular simplex in $\mathbb{R}^d$ are a nontrivial example of a spherical half-design of order $m=1$, via direct calculation of (4.21)*

$$\sum_j \sum_k \langle v_j, v_k\rangle^1 = (d+1) + \{(d+1)^2 - (d+1)\}\Bigl(\frac{-1}{d}\Bigr) = 0.$$

Hardin and Sloane [HS96] give various half-designs for $\mathbb{R}^3$, e.g., they give 11, 13 *and* 15*-point spherical* 3*-designs which are nontrivial half-designs of order* 3.

It follows from the definition of designs (1.1), that a weighted spherical half-design of order m is a weighted spherical-half design of orders $m-2, m-4, \ldots$, with the same weights. Expressing this observation in terms of (4.21) gives:

Example 4.5 *Let (v_j) be a sequence in $\mathbb{R}^d$ and m be an odd positive integer. If*

$$\sum_{j=1}^n \sum_{k=1}^n \langle v_j, v_k \rangle^m = 0,$$

then
$$\sum_{j=1}^n \sum_{k=1}^n \|v_j\|^{m-\ell} \|v_k\|^{m-\ell} \langle v_j, v_k \rangle^\ell = 0, \qquad \ell = 1, 3, 5, \ldots, m.$$

Example 4.6 *(Sharp configurations) In [CK07], an f-potential energy for a finite set points $\mathcal{C}$ on the real sphere by*

$$\sum_{\substack{x,y \in \mathcal{C} \\ x \neq y}} f(\|x-y\|^2),$$

where $f := (0,4] \to [0,\infty)$ is any decreasing continuous function. In view of (4.11), this can be written as

$$\sum_{\substack{x,y \in \mathcal{C} \\ x \neq y}} F(\langle x, y \rangle), \qquad F(t) := f(2(1-t)), \quad -1 \leq t < 1.$$

A subset $\mathcal{C}$ of $\mathbb{S}$ is a **sharp configuration** *(or* **code***) if there are m inner products between distinct points and it is a spherical $(2m-1)$-design. It is shown by Cohn and Kumar [CK07] (also see [BHS19] §5.7) that if $\mathcal{C}$ is a sharp configuration or the vertices of the 600-cell, and the above f is completely monotonic, i.e., $(-1)^k f^{(k)}(t) \geq 0$, $k \geq 0$, equivalently, F is absolutely monotonic, i.e., $F^{(k)}(t) \geq 0$, $k \geq 0$, then*

$$\sum_{\substack{x,y \in \mathcal{C}' \\ x \neq y}} f(\|x-y\|^2) \geq \sum_{\substack{x,y \in \mathcal{C} \\ x \neq y}} f(\|x-y\|^2),$$

for any other set of points with $|\mathcal{C}'| = |\mathcal{C}|$, i.e., $\mathcal{C}$ is a uniformly optimal distribution of points on the sphere (for all such f).

5 Complex spherical designs

The complex unitary matrices for $\mathbb{C}^d$ are a subgroup of the unitary matrices for $\mathbb{R}^{2d}$ (the orthogonal group), and so irreducible subspaces under the action of the orthogonal group may not be irreducible under the action of the complex unitary group.

For the complex sphere $\mathbb{S}$, the harmonic functions $\mathrm{Harm}_k(\mathbb{C}^d) \approx \mathrm{Harm}_k(\mathbb{R}^{2d})$ can be further decomposed into orthogonal (complex) unitarily invariant irreducible subspaces

$$\mathrm{Harm}_k(\mathbb{C}^d) = H(k,0) \oplus H(k-1,1) \oplus \cdots \oplus H(0,k),$$

where $H(p,q)$ consists of all harmonic homogeneous polynomials on $\mathbb{C}^d$ that have degree p in the variables $z_1,\ldots,z_d$ and degree q in the variables $\overline{z_1},\ldots,\overline{z_d}$ (see [Rud80]). Thus the unitarily invariant subspaces of polynomials restricted to the complex sphere have the form

$$P|_{\mathbb{S}} = P_\tau := \bigoplus_{(p,q)\in\tau} H(p,q), \qquad \text{(orthogonal direct sum)}$$

for τ a finite subset of indices from $\{(j,k): j,k\ge 0\}$. Thus, the most general complex spherical design is one which integrates P_τ, which we call a (**spherical**) **τ-design**. Aspects of these τ-designs have been studied by [MOP11], [RS14] and [MW24].

There is a subtlety in defining classes of complex spherical designs, as the τ defining a class is not unique (as is the L in the real case), as we now see.

The reproducing kernel $K_{pq}=K_{pq}^{(d)}$ for $H(p,q)$ (and hence for any unitarily invariant space of polynomials) has been calculated explicitly by [Fol75] for $d>1$ as a function of $\langle z,w\rangle$

$$K_d^{(p,q)}(z,w) = Q_{pq}^{(d)}(\langle z,w\rangle), \qquad z,w\in\mathbb{C}^d,$$

where $Q_{pq}=Q_{pq}^{(d)}$ is the univariate polynomial

$$\begin{aligned} Q_{pq}^{(d)}(z) &:= c_{pq}^{(d)}\, z^{p-\min\{p,q\}}\overline{z}^{q-\min\{p,q\}}\frac{P_{\min\{p,q\}}^{(d-2,|p-q|)}(2|z|^2-1)}{P_{\min\{p,q\}}^{(d-2,|p-q|)}(1)} \\ &= \frac{p+q+d-1}{(d-1)!}\sum_{j=0}^{\min\{p,q\}}(-1)^j\frac{(d+p+q-j-2)!}{j!(p-j)!(q-j)!}z^{p-j}\overline{z}^{q-j}, \quad z\in\mathbb{C}, \end{aligned} \tag{5.23}$$

and

$$c_{pq}^{(d)} := \dim(H(p,q)) = \frac{(p+q+d-1)(p+d-2)!(q+d-2)!}{p!q!(d-1)!(d-2)!} = Q_{pq}^{(d)}(1). \tag{5.24}$$

The second of these formulas also holds for $d=1$, where $H(p,q)=0$, unless $p=0$ or $q=0$, in which case $H(p,0)=\operatorname{span}\{z^p\}$, $H(0,q)=\operatorname{span}\{\overline{z}^q\}$. We also have

$$\operatorname{Hom}(p,q)=\operatorname{Hom}_{p,q}(\mathbb{C}^d)|_{\mathbb{S}} = H(p,q)\oplus H(p-1,q-1)\oplus\cdots.$$

We observe that the expansion for $Q_{pq}(z)$ in terms of the monomials $z^j\overline{z}^k$ has real coefficients, so that

$$\overline{Q_{pq}(z)} = Q_{pq}(\overline{z}) = Q_{qp}(z),$$

and so the canonical potentials for $H(p,q)$ and $H(q,p)$ are equal, by the calculation

$$\begin{aligned} A_{w,H(p,q)}(\Phi) = \overline{A_{w,H(p,q)}(\Phi)} &= \sum_j\sum_k w_jw_k\overline{Q_{pq}(\langle v_j,v_k\rangle)} \\ &= \sum_j\sum_k w_jw_kQ_{qp}^{(d)}(\langle v_j,v_k\rangle) = A_{w,H(q,p)}(\Phi), \end{aligned} \tag{5.25}$$

as are those for $\mathrm{Hom}(p,q)$ and $\mathrm{Hom}(q,p)$. Since the canonical potentials for $H(p,q)$ and $H(q,p)$ are equal, the class of spherical designs for some unitarily invariant polynomial spaces are equal. For a set of indices τ, we define

$$\tau^{\mathrm{rev}} := \{(q,p) : (p,q) \in \tau\}.$$

By the **class** of a spherical τ-design we mean the maximal unitarily invariant subspace that it integrates, or the indices τ^* of that subspace.

Proposition 5.1 *The class of complex spherical designs for P_L and P_K are equal if and only if*

$$L \cup L^{\mathrm{rev}} \cup \{0\} = K \cup K^{\mathrm{rev}} \cup \{0\}.$$

In particular, the class (of indices) for any τ-design is

$$\tau^* = \tau \cup \tau^{\mathrm{rev}} \cup \{0\}.$$

Proof: Since every complex spherical design integrates the constants, we can add $0 = (0,0)$ to the set of indices L without changing the class of the spherical designs it gives. Similarly, since $H(p,q)$ and $H(q,p)$ have the same canonical potential, by (5.25), we may add (q,p) for $(p,q) \in L$. Thus if $L \cup L^{\mathrm{rev}} \cup \{0\} \subset K \cup K^{\mathrm{rev}} \cup \{0\}$, then a spherical design for P_K is spherical design for P_L. This gives the forward implication.

The converse follows from the fact that the canonical potentials for different classes differ by at least one term, and then a linear algebra argument. □

We will not labour the point, but the classes of complex spherical designs are given by the possible choices for τ^*, and for a τ-design we will refer to τ^* as the **canonical indices** for the design, and use $\tau^* \setminus \{(0,0)\}$ to calculate the **canonical potential**.

We will carry over terminology from the real case, e.g., we say that a univariate polynomial

$$F = \sum_{(p,q)} f_{pq} Q_{pq}, \qquad f_{pq} \geq 0, \tag{5.26}$$

gives a potential $A_{F,w}(\Phi)$ for the unitarily invariant polynomial space

$$P = P_\tau = \bigoplus_{(p,q)\in\tau} H(p,q), \qquad \tau := \{(p,q) : f_{pq} > 0, (p,q) \neq (0,0)\}.$$

Example 5.1 *(Balanced sets) Let $P_\tau = H(1,0)$, i.e., $\tau = \{(1,0)\}$. Then*

$$\tau^* = \{(0,0),(1,0),(0,1)\}, \qquad Q_{10}^{(d)}(z) = dz,$$

so that the τ-designs (v_j) are characterised by

$$\sum_j \sum_k \langle v_j, v_k \rangle = \langle \sum_j v_j, \sum_k v_k \rangle = \| \sum_j v_j \|^2 = 0 \quad \Longrightarrow \quad \sum_j v_j = 0,$$

i.e., the sum of their vectors is zero, and they are said to be **balanced**.

We have the complex version of Example 4.2.

Example 5.2 *(Complex tight frames) For* $P = \mathrm{Hom}_{1,1}(\mathbb{C}^d)$, $P|_{\mathbb{S}} = H(1,1) \oplus H(0,0)$, *i.e.,* $\tau = \tau^* = \{(1,1),(0,0)\}$, *and*

$$Q_{11}^{(d)}(z) = d(d+1)\Bigl(|z|^2 - \frac{1}{d}\Bigr),$$

gives the canonical potential (which is zero for $d = 1$*)*

$$A_{P,w}(\Phi) = d(d+1)\sum_j \sum_k w_j w_k \Bigl(|\langle v_j, v_k\rangle|^2 - \frac{1}{d}\Bigr).$$

The zeros of this potential are the unit norm tight frames for $\mathbb{C}^d$*. For vectors* (v_j) *in* $\mathbb{C}^d$*, by taking the* 2*-weights given by (2.2), one obtains the variational characterisation of tight frames [Wal03].*

The functions $F : \mathbb{D} \to \mathbb{C}$, on the complex unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$, of (5.26) that give a potential, are in general complex-valued. They do satisfy

$$\overline{F(z)} = F(\overline{z}),$$

and so the potential $A_{F,w}$ that F gives is real-valued, since we may group terms in (4.12)

$$w_j w_k\bigl(F(\langle v_j, v_k\rangle) + F(\langle v_k, v_j\rangle)\bigr) = w_j w_k\bigl(F(\langle v_j, v_k\rangle) + \overline{F(\langle v_j, v_k\rangle)}\bigr) = 2 w_j w_k \Re\bigl(F(\langle v_j, v_k\rangle)\bigr).$$

It is possible to develop a theory of **positive definite functions on the complex sphere** which include such polynomials F (see [MP01], [MPP17]).

The canonical potential for

$$P = \mathrm{Hom}_2(\mathbb{C}^d), \qquad P|_{\mathbb{S}} = H(2,0) \oplus H(1,1) \oplus H(0,2) \oplus H(0,0),$$

is given by

$$F(z) = Q_{20}^{(d)}(z) + Q_{11}^{(d)}(z) + Q_{20}^{(d)}(z) = \frac{1}{2}d(d+1)\Bigl(z^2 + 2z\overline{z} + \overline{z}^2 - \frac{2}{d}\Bigr) = 2d(d+1)\Bigl(\Re(z)^2 - \frac{1}{2d}\Bigr).$$

Since this potential is zero if and only if (v_j) is tight frame for $\mathbb{R}^{2d}$, we have the following.

Proposition 5.2 *Let* (v_j) *be a sequence of* n *vectors in* $\mathbb{C}^d$*, not all zero. Then* (v_j) *is tight frame for* $\mathbb{C}^d$ *if and only if there is equality in the inequality*

$$\sum_j \sum_k |\langle v_j, v_k\rangle|^2 \ge \frac{1}{d}\Bigl(\sum_\ell \|v_\ell\|^2\Bigr)^2.$$

Moreover, it is also a tight frame for $\mathbb{R}^{2d}$ *if and only if, in addition, there is equality in the inequality*

$$\sum_j \sum_k \langle v_j, v_k\rangle^2 \ge 0.$$

Proof: The first statement is just the variational characterisation of tight frames for $\mathbb{C}^d$ given in Example 5.2. For the vectors (v_j) in $\mathbb{C}^d$ to be a tight frame for $\mathbb{R}^{2d}$ the potentials $A_{H(1,1)}(\Phi)$ and $A_{H(2,0)}(\Phi) = A_{H(0,2)}(\Phi)$ must be minimised for the 2-weights. The first is minimised by being a tight frame for $\mathbb{C}^d$, and the second if and only if there is equality in

$$A_{H(2,0)}(\Phi) = \frac{1}{2}d(d+1)\sum_j\sum_k \frac{\|v_j\|^2}{C}\frac{\|v_k\|^2}{C}\langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|}\rangle^2 = \frac{d(d+1)}{2C^2}\sum_j\sum_k \langle v_j, v_k\rangle^2 \geq 0,$$

where $C = \sum_\ell \|v_\ell\|^2 > 0$. □

This result appears in [Wal25] (Theorem 3.1) where it is obtained in a similar way.

Example 5.3 *For $v_j = z_j = x_j + iy_j \in \mathbb{C}$, not all zero, (v_j) is a tight frame for $\mathbb{C}$ (since the first inequality holds). The condition for the vectors (x_j, y_j) to be a tight frame for $\mathbb{R}^2$ is that*

$$\sum_j\sum_k (z_j\overline{z_k})^2 = \Bigl(\sum_j z_j^2\Bigr)\Bigl(\sum_k \overline{z_k}^2\Bigr) = \Bigl|\sum_j z_j^2\Bigr|^2 = 0,$$

i.e., $\sum_j z_j^2 = 0$. This characterisation of tight frames for $\mathbb{R}^2$ (in polar form) is given in [Fic01] and [HKLW07] (where the vectors z_j^2 are called diagram vectors).

The potentials for $P = \mathrm{Hom}(2,2) = H(2,2) \oplus H(1,1) \oplus H(0,0)$ are given by

$$\begin{aligned} F(z) &= c_1 Q_{11}^{(d)}(z) + c_2 Q_{22}^{(d)}(z) \\ &= c_1(d+1)(d|z|^2 - 1) + c_2\frac{1}{4}d(d+3)\bigl((d^2+3d+2)|z|^4 - 4(d+1)|z|^2 + 2\bigr). \end{aligned}$$

The $F(z)$ for the canonical potential ($c_1 = c_2 = 1$) has nonzero terms in 1, $|z|^2$, $|z|^4$. The term in $|z|^2$ can be cancelled by choosing $c_1 = (d+3)c_2$, which gives

$$F(z) = |z|^4 - \frac{2}{d(d+1)}.$$

We now seek a similar 'telescoping' sum for a general $\mathrm{Hom}(p,q)$, to obtain an analogue of Lemma 4.1.

Lemma 5.1 *A direct calculation gives*

$$\sum_{j=0}^{\min\{p,q\}} \frac{(d-1)!p!q!}{j!(p+q+d-1-j)!} Q_{p-j,q-j}^{(d)}(z) = z^p\overline{z}^q. \tag{5.27}$$

The complex analogue of (4.15) is

$$b_{p,q}(\mathbb{C}^d) := \int_{\mathbb{S}}\int_{\mathbb{S}} \langle z,w\rangle^p \overline{\langle z,w\rangle}^q \, d\sigma(z)\, d\sigma(w) = \begin{cases} 0, & p \neq q; \\ \frac{(d-1)!p!}{(d-1+p)!}, & p = q. \end{cases} \tag{5.28}$$

Theorem 5.1 *A potential for* $\mathrm{Hom}(p,q)=\mathrm{Hom}_{p,q}(\mathbb{C}^d)$ *is given by*

$$F(z)=z^p\overline{z}^q-b_{p,q}(\mathbb{C}^d), \tag{5.29}$$

where $b_{p,q}(\mathbb{C}^d)$ *is given by (5.28), and a potential for* $\mathrm{Hom}_m(\mathbb{C}^d)$ *is given by*

$$F(z)=(z+\overline{z})^m-2^m b_m(\mathbb{R}^{2d})=2^m\bigl\{\Re(z)^m-b_m(\mathbb{R}^{2d})\bigr\}. \tag{5.30}$$

where $b_m(\mathbb{R}^{2d})$ *is given by (4.15).*

Proof: We note that all the coefficients in the expansion (5.27) are positive. Hence, for $p\neq q$, this gives a potential for $\mathrm{Hom}(p,q)=\oplus_j H(p-j,q-j)$, and for $p=q$, this is a potential for $\mathrm{Hom}(p,p)$ plus the $Q_{00}^{(d)}(z)$ constant term

$$\frac{(d-1)!p!p!}{p!(p+d-1)!}Q_{00}^{(d)}(z)=\frac{(d-1)!p!}{(p+d-1)!}=\frac{1}{\binom{p+d-1}{p-1}}=b_{p,p}(\mathbb{C}^d).$$

This gives (5.29) in both cases.

A potential for $\mathrm{Hom}_m(\mathbb{C}^d)=\bigoplus_j \mathrm{Hom}(j,m-j)$ is given by

$$F(z)=\sum_{j=0}^m\binom{m}{j}\bigl(z^j\overline{z}^{m-j}-b_{j,m-j}(\mathbb{C}^d)\bigr)=(z+\overline{z})^m-\sum_{j=0}^m\binom{m}{j}b_{j,m-j}(\mathbb{C}^d).$$

The constant term subtracted above is zero, unless m is even, in which case it is

$$\binom{m}{\frac{m}{2}}b_{\frac{m}{2},\frac{m}{2}}(\mathbb{C}^d)=\frac{m!}{(\frac{m}{2})!(\frac{m}{2})!}\frac{(d-1)!(\frac{m}{2})!}{(d-1+\frac{m}{2})!}=2^m\frac{m!}{2^{\frac{m}{2}}(\frac{m}{2})!}\frac{(d-1)!}{2^{\frac{m}{2}}(d-1+\frac{m}{2})!}=2^m b_m(\mathbb{R}^{2d}).$$

Hence we obtain (5.30). $\square$

The nonnegativity of the potential $A_{F,w}(\Phi)$ given by (5.29) is given as Lemma 3.3 in [RS14], with equality asserted when it holds for all (p,q) in a lower set τ for a unit-norm τ-design Φ. The complex spherical designs for the special case $\mathrm{Hom}(p,p)$ are "projective designs", which we will discuss in the next section.

Example 5.4 *A potential for the holomorphic polynomials* $\mathrm{Hom}(k,0)=H(k,0)$ *is given by* $F(z)=z^k$*, and for the holomorphic polynomials of degree* $\leq k$ *a potential is given by any linear combination of* $z,z^2,\ldots,z^k$ *with positive coefficients.*

We now give the weighted version of Theorem 5.1.

Theorem 5.2 *Let* $m=p+q$*. For any vectors* $v_1,\ldots,v_n$ *in* $\mathbb{C}^d$*, not all zero, we have*

$$\sum_{j=1}^n\sum_{k=1}^n\langle v_j,v_k\rangle^p\overline{\langle v_j,v_k\rangle}^q\geq b_{p,q}(\mathbb{C}^d)\Bigl(\sum_{\ell=1}^n\|v_\ell\|^{p+q}\Bigr)^2, \tag{5.31}$$

with equality if and only (v_j) *is an* m*-weighted spherical design for* $\mathrm{Hom}(p,q)$*, and*

$$\sum_{j=1}^n\sum_{k=1}^n(\Re\langle v_j,v_k\rangle)^m\geq b_m(\mathbb{R}^{2d})\Bigl(\sum_{\ell=1}^n\|v_\ell\|^m\Bigr)^2, \tag{5.32}$$

with equality if and only if (v_j) *is an* m*-weighted spherical half-design of order* m*.*

Proof: Let $m = p+q$, and $C = \sum_\ell \|v_\ell\|^m > 0$. Then the potential given by (5.29) for the m-weights is

$$A_{F,w}(\Phi) = \sum_j \sum_k \frac{\|v_j\|^m \|v_k\|^m}{C^2} \Big\{ \langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|} \rangle^p \overline{\langle \frac{v_j}{\|v_j\|}, \frac{v_k}{\|v_k\|} \rangle}^q - b_{p,q}(\mathbb{C}^d) \Big\}$$
$$= \frac{1}{C^2} \sum_j \sum_k \langle v_j, v_k \rangle^p \overline{\langle v_j, v_k \rangle}^q - b_{p,q}(\mathbb{C}^d).$$

Multiplying this by C^2 and rearranging gives (5.31).

For $P = \mathrm{Hom}(m) = \mathrm{Hom}_m(\mathbb{C}^d)$ a similar argument using (5.30) gives (5.32). □

We will refer to spherical designs for $\mathrm{Hom}(p,q)$ as **spherical** (p,q)**-designs**, which generalises the definition of complex spherical (t,t)-designs. We now show that the canonical potential for these has a simple form.

Lemma 5.2 *The complex Gegenbauer polynomials satisfy*

$$\sum_{j=0}^{\min\{p,q\}} Q^{(d)}_{p-j,q-j}(z) = \frac{d}{p+q+d} Q^{(d+1)}_{pq}(z), \qquad (5.33)$$

equivalently,

$$Q^{(d)}_{p,q}(z) = \frac{d}{p+q+d} Q^{(d+1)}_{pq}(z) - \frac{d}{p+q+d-2} Q^{(d+1)}_{p-1,q-1}(z). \qquad (5.34)$$

Proof: The equivalence of (5.33) and (5.34) is obvious. Thus it suffices to prove (5.34) by direct calculation from (5.23), i.e., by equating coefficients. □

From Lemma 5.2, we have

Example 5.5 *(Spherical (p,q)-designs) The canonical potential for spherical (p,q)-designs, i.e., $P = \mathrm{Hom}(p,q) = \mathrm{Hom}_{p,q}(\mathbb{C}^d)$ is given by*

$$\sum_{j=0}^{\min\{p,q\}} Q^{(d)}_{p-j,q-j}(z) = \frac{d}{p+q+d} Q^{(d+1)}_{pq}(z).$$

6 Projective spherical designs

We will say that a (weighted) spherical design (v_j) for the polynomial space P is a **projective spherical design** if $(c_j v_j)$ is a spherical design for all choices of unit scalars c_j. Clearly such a design can be thought of as a sequence of lines. Since a projective design depends on the (v_j) up to unitary equivalence and multiplication by unit scalars, it follows from [CW16] that it can be characterised in terms of its m-products

$$\Delta(v_{j_1}, \ldots, v_{j_m}) := \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_m}, v_{j_1} \rangle, \qquad 1 \le j_1, \ldots, j_m \le n.$$

Hence the reproducing kernels $K(x,y)$ which are invariant under this equivalence, i.e., replacing (x,y) with $(c_x Ux, c_y Uy)$, where U is unitary and c_x, c_y are unit scalars, are those which are functions of $\langle x,y \rangle^2$ and $|\langle x,y \rangle|^2$, respectively. Thus the polynomial spaces giving projective spherical designs in the real and complex cases are

- The spaces P_L consisting of even polynomials on $\mathbb{R}^d$, i.e., $L \subset \{0, 2, 4, \ldots\}$.
- The spaces P_τ of polynomials on $\mathbb{C}^d$, where $\tau = \tau^* \subset \{(0,0), (1,1), \ldots\}$.

This notion of projective designs generalises to designs on the Grassmannian [EG19].

It follows from the multiplication rules for Gegenbauer polynomials (Theorem 8.1), that the univariate functions F giving potentials for projective designs are closed under multiplication.

The projective spherical designs for the (projectively unitarily invariant) spaces

$$P_L = \mathrm{Hom}(2t) = \mathrm{Harm}(0) \oplus \mathrm{Harm}(2) \oplus \cdots \oplus \mathrm{Harm}(2t), \quad L = 0, 2, \ldots, 2t,$$

$$P_\tau = \mathrm{Hom}(t,t) = H(0,0) \oplus H(1,1) \oplus \cdots \oplus H(t,t), \quad \tau = \{(0,0), (1,1), \ldots, (t,t)\},$$

are of particular interest. In the real case, these are the **spherical half-designs** of order $2t$ [KP11] and the real **spherical (t,t)-designs** [Wal17]. In the complex case, they are known as **projective t-designs** on Delsarte spaces [Hog90],[Wal20], complex **spherical semi-designs** of order $2t$ [KP17], and complex **spherical (t,t)-designs** [Wal17]. The potentials of Lemma 4.1 and Theorem 5.1 for these are

$$F(x) = x^{2t} - b_{2t}(\mathbb{R}^d), \qquad F(z) = |z|^{2t} - b_{t,t}(\mathbb{C}^d).$$

We may combine the characterisations of Theorem 4.2 and Theorem 5.2 to obtain

Example 6.1 *For any vectors $v_1, \ldots, v_n$ in $\mathbb{F}^d$, not all zero, we have*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k \rangle|^{2t} \geq c_t(\mathbb{F}^d) \Big(\sum_{\ell=1}^n \|v_\ell\|^{2t} \Big)^2,$$

where

$$c_t(\mathbb{R}^d) := b_{2t}(\mathbb{R}^d) = \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{d(d+2)\cdots(d+2t-2)}, \qquad c_t(\mathbb{C}^d) := b_{t,t}(\mathbb{C}^d) = \frac{1}{\binom{t+d-1}{t}},$$

with equality if and only if $(v_j) \subset \mathbb{F}^d$ is a spherical (t,t)-design for $\mathbb{F}^d$.

This characterisation is given in [KP11],[KP17] and [Wal17], and the inequalities were first given by Sidel'nikov [Sid74] and Welch [Wel74].

In Table 1, we summarise our calculations of potentials from Sections 4, 5 and 6.

Table 1: Selected real and complex spherical P-designs and their potentials. The inclusion of zero or not in the index set is for aesthetics. Those marked with a * are canonical potentials.

P-design	P and the index set L or τ	Potential F	Comments
harmonic index m	$\mathrm{Harm}_m(\mathbb{R}^d)$	$Q_m^{(d)}(x)$	Example 4.1 *
	$\{m\}$		
real tight frame	$\mathrm{Hom}_2(\mathbb{R}^d)$	$x^2 - \frac{1}{d}$	Example 4.2 *
	$\{2\}$		
real half-design	$\mathrm{Hom}_m(\mathbb{R}^d)$	$x^m - b_m(\mathbb{R}^d)$	Lemma 4.1
	$\{m, m-2, \ldots\}$	$C_m^{(\frac{d}{2})}(x) - 1,\ m$ even	Example 4.3 *
		$C_m^{(\frac{d}{2})}(x),\ m$ odd	
spherical t-design	$\Pi_t(\mathbb{R}^d)$	$C_t^{(\frac{d}{2})}(x) + C_{t-1}^{(\frac{d}{2})}(x) - 1$	Corollary 4.1 *
	$\{1, 2, \ldots, t\}$	$\sum_{m=t,t-1} c_m\big(x^m - b_m(\mathbb{R}^d)\big)$	Theorem 4.1
harmonic index (p,q)	$H(p,q)$	$Q_{pq}^{(d)}(z)$	Equation (5.23) *
	$\{(p,q)\}$		
complex tight frame	$\mathrm{Hom}_{1,1}(\mathbb{C}^d)$	$\|z\|^2 - \frac{1}{d}$	Example 5.2 *
	$\{(1,1)\}$		
spherical (p,q)-design	$\mathrm{Hom}_{p,q}(\mathbb{C}^d)$	$z^p\overline{z}^q - b_{p,q}(\mathbb{C}^d)$	Theorem 5.1
	$\{(p,q), (p-1, q-1), \ldots\}$	$\frac{d}{p+q+d} Q_{pq}^{(d+1)}(z), \quad p \neq q$	Example 5.5 *
spherical (t,t)-design	$\mathrm{Hom}_{t,t}(\mathbb{C}^d)$	$\|z\|^{2t} - c_t(\mathbb{C}^d)$	Example 6.1
	$\{(1,1), \ldots (t,t)\}$		
complex half-design	$\mathrm{Hom}_m(\mathbb{C}^d)$	$\Re(z)^m - b_m(\mathbb{R}^{2d})$	Theorem 5.1
	$\{(p,q) : p+q = m, m-2, \ldots\}$		
spherical t-design	$\Pi_t(\mathbb{C}^d)$	$\sum_{m=t-1,t} c_m\big(\Re(z)^m - b_m(\mathbb{R}^{2d})\big)$	Theorem 5.1
	$\{(p,q) : p+q = 1, 2, \ldots, t\}$		

7 Orthogonality of the Gegenbauer polynomials

The polynomials $Q_k^{(d)}$ of (4.10) that give the reproducing kernel for $\mathrm{Harm}_k(\mathbb{R}^d)$ are orthogonal with respect to the Gegenbauer weight $(1-x^2)^{\frac{d-2}{2}-\frac{1}{2}}$ on $[-1,1]$. Indeed

$$\langle Q_j, Q_k\rangle_{\mathrm{geg}} = \begin{cases} Q_j(1) = \dim(\mathrm{Harm}_k(\mathbb{R}^d)), & j = k; \\ 0, & j \neq k, \end{cases} \tag{7.35}$$

where

$$\langle f, g\rangle_{\mathrm{geg}} := \frac{\Gamma(\frac{1}{2}d)}{\sqrt{\pi}\Gamma(\frac{1}{2}d-\frac{1}{2})} \int_{-1}^{1} f(x)g(x)\,(1-x^2)^{\frac{d-3}{2}}\,dx.$$

We will refer to these, and the polynomials $Q_{pq}^{(d)}$ of (5.23) giving the reproducing kernel for $H(p,q)$ as (real or complex) **Gegenbauer polynomials**. The terms "Jacobi" and "disk" polynomial are sometimes used for the latter [RS14], [MOP11].

We now show the polynomials $Q_{pq}^{(d)}$ are orthogonal with respect to the Gegenbauer weight $(1-|z|^2)^{d-2}$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ in $\mathbb{C}$ (cf. [MOP11]).

Proposition 7.1 *For $d > 1$, the (complex) Gegenbauer polynomials $Q_{pq}^{(d)}$ of* (5.23) *are orthogonal with respect to the inner product*

$$\begin{aligned}\langle f, g\rangle_{\mathbb{D},d} &:= \frac{d-1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^{d-2}\,dA(z) \\ &= \frac{d-1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\,(1-r^2)^{d-2} r\,dr\,d\theta,\end{aligned}$$

where A is the area measure on $\mathbb{C} = \mathbb{R}^2$, and

$$\langle Q_{pq}, Q_{k\ell}\rangle_{\mathbb{D},d} = \begin{cases} Q_{pq}(1) = \dim(H(p,q)), & (p,q) = (k,\ell); \\ 0, & (p,q) \neq (k,\ell). \end{cases} \tag{7.36}$$

Proof: We write $z = re^{i\theta}$, so that

$$Q_{pq}^{(d)}(z) = c_{pq}^{(d)}\,(re^{i\theta})^{p-m}(r\overline{e^{i\theta}})^{q-m}\frac{P_m^{(d-2,|p-q|)}(2r^2-1)}{P_m^{(d-2,|p-q|)}(1)}, \qquad m := \min\{p,q\}.$$

By integrating in θ first, we see that $\langle Q_{pq}^{(d)}, Q_{p'q'}^{(d)}\rangle_{\mathbb{D},d} = 0$, except when $p-q = p'-q'$. In this case, we may suppose, without loss of generality, that $p-q = p'-q' = k \geq 0$, i.e., $m = q$, $m' = q'$, so that $(re^{i\theta})^{p-m}(r\overline{e^{i\theta}})^{q-m}(r\overline{e^{i\theta}})^{p'-m'}(re^{i\theta})^{q'-m'} = r^{2k}$, and

$$\langle Q_{pq}^{(d)}, Q_{p'q'}^{(d)}\rangle_{\mathbb{D},d} = 2c_{pq}^{(d)}c_{p'q'}^{(d)}\int_0^1 r^{2k}P_m^{(d-2,k)}(2r^2-1)P_{m'}^{(d-2,k)}(2r^2-1)\,(1-r^2)^{d-2}\,r\,dr.$$

By making the change of variables $x = 2r^2-1$, the integral in r above becomes

$$\int_{-1}^{1}\left(\frac{1+x}{2}\right)^k P_m^{(d-2,k)}(x)P_{m'}^{(d-2,k)}(x)\,\left(\frac{1-x}{2}\right)^{d-2}\frac{1}{4}dx,$$

which is zero, unless $m = m'$, in which case $(p,q) = (p',q')$. The calculation of the constant for the nonzero inner product is a straight forward calculation. □

This orthogonality of the polynomials $Q^{(d)}_{pq}$ can also be proved from the reproducing kernel property, by using the orthogonality of the $H(p,q)$, and the result (see §1.4.5 of [Rud80]) that for $\mathbb{S}$ the unit ball in $\mathbb{C}^d$ and f a univariate function

$$\int_{\mathbb{S}} f(\langle z,w\rangle)\,d\sigma(w) = \frac{d-1}{\pi}\int_{\mathbb{D}} f(\zeta)(1-|\zeta|^2)^{d-2}\,dA(\zeta), \qquad z\in\mathbb{C}^d,\ \|z\|=1.$$

In particular, we can calculate the orthogonality constant

$$\begin{aligned} Q^{(d)}_{pq}(1) = \overline{Q^{(d)}_{pq}(\langle z,z\rangle)} &= \int_{\mathbb{S}} Q^{(d)}_{pq}(\langle z,w\rangle)\overline{Q^{(d)}_{pq}(\langle z,w\rangle)}\,d\sigma(w) \\ &= \frac{d-1}{\pi}\int_{\mathbb{D}} |Q^{(d)}_{pq}(\zeta)|^2(1-|\zeta|^2)^{d-2}\,dA(\zeta) = \langle Q^{(d)}_{pq}, Q^{(d)}_{pq}\rangle_{\mathbb{D},d}. \end{aligned}$$

Since $\mathrm{Harm}_k(\mathbb{C}^d) = \bigoplus_{p+q=k} H(p,q)$, we have

$$Q^{(2d)}_k\big((x,y)\big) = \sum_{p+q=k} Q^{(d)}_{pq}(z), \qquad z = x+iy\in\mathbb{C}^d.$$

For $d=1$, the polynomials $Q^{(1)}_{pq}(z)$ can be viewed as being orthogonal with respect to the "singular Gegenbauer weight"

$$\langle f,g\rangle_{\mathbb{D},d} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta.$$

8 Products of the Gegenbauer polynomials

To construct positive functions F giving a potential for a spherical design, it is useful to know the Gegenbauer expansion for a product of Gegenbauer polynomials. In the real case, there is the following celebrated formula dating back to Rogers (1895)

$$Q^{(d)}_k Q^{(d)}_l = \sum_{j=0}^{\min\{k,l\}} \frac{(k+\nu)(l+\nu)(k+l-2j)!(\nu)_j(\nu)_{k-j}(\nu)_{l-j}(2\nu)_{k+l-j}}{\nu(k+l+\nu-j)j!(k-j)!(l-j)!(\nu)_{k+l-j}(2\nu)_{k+l-2j}} Q^{(d)}_{k+l-2j}, \tag{8.37}$$

where $\nu := \frac{d-2}{2}$, $(\nu)_j$ is the Pochhammer symbol, and the coefficients are clearly positive. We will denote by $k\cdot l$ the degrees of the Gegenbauer polynomials occuring in (8.37), i.e.,

$$k\cdot l := \{k+l-2j : 0\le j\le m\}, \qquad m := \min\{k,l\}, \tag{8.38}$$

which we extend to subsets in the natural way

$$K\cdot L := \bigcup_{\substack{k\in K\\ l\in L}} k\cdot l. \tag{8.39}$$

As an example, for $L=\{1,4\}$, we have $L\cdot L = \{0,2,3,4,5,6,8\}$.

Lemma 8.1 *If F and G give potentials for P_L and P_K (on the real sphere), then FG gives a potential for $P_{L\cdot K}$.*

Proof: Multiply out the Gegenbauer expansions for F and G and use (8.37). □

We now present the analogue of Lemma 8.1 for the product of complex Gegenbauer polynomials. It can be shown (see [Rud80]) that

$$H(p,q)H(r,s) \subset \sum_{j=0}^{\mu} H(p+r-j, q+s-j), \quad \mu := \min\{p,s\} + \min\{q,r\}, \tag{8.40}$$

where there is equality for $d \geq 3$. Motivated by this, for indices (p,q) and (r,s), we define the operation

$$(p,q)\cdot(r,s) := \bigcup_{j=0}^{\mu} \{(p+r-j, q+s-j)\}, \quad \mu := \min\{p,s\} + \min\{q,r\}, \tag{8.41}$$

which we extend to subsets of indices as in (8.39). As examples, we have

$$(p,q)\cdot(q,p) = \{(0,0),(1,1),\ldots,(k,k)\}, \quad k = p+q,$$

so that $(0,0) \in \mathcal{U}\cdot\mathcal{U}^{\mathrm{rev}}$, when $\mathcal{U}$ is nonempty, and

$$(p,q)\cdot(p,q) = \{(2p,2q),(2p-1,2q-1),\ldots,(2p-2m,2q-2m)\}, \quad m = \min\{p,q\}.$$

In view of (8.40) and (8.41), it follows that

$$P_L P_K \subset \sum_{(p,q)\in L\cdot K} H(p,q).$$

The corresponding analogue of the Roger's formula (8.37) is

$$Q_{pq}^{(d)} Q_{rs}^{(d)} = \sum_{j=0}^{\mu} c_{p+r-j,q+s-j}^{(d)} Q_{p+r-j,q+s-j}^{(d)}, \quad \mu := \min\{p,s\} + \min\{q,r\},$$

with nonnegative coefficients (see [CW18]). The nonnegativity of the coefficients follows from the Schur product theorem. This gives the complex analogue of Lemma 8.1.

Theorem 8.1 *The product of functions giving a potential for the real or complex sphere gives a potential. Indeed, if F and G are give potentials for $P_{\mathcal{U}}$ and $P_{\mathcal{V}}$, then FG gives a potential for a subspace of $P_{\mathcal{U}\cdot\mathcal{V}}$, which is all of $P_{\mathcal{U}\cdot\mathcal{V}}$ for the real sphere and the complex sphere when $d \geq 3$.*

The following particular case will be useful.

Corollary 8.1 *If $F = \sum_{(p,q)} f_{pq} Q_{pq}$, $f_{pq} \geq 0$ is a potential for the complex sphere, then $G = \overline{Q_{ab}} F / Q_{ab}(1)$ is a potential with constant term $g_0 = \langle G, 1\rangle_{\mathbb{D},d} = f_{ab}$.*

Proof: Since $\overline{Q_{ab}} = Q_{ba}$, G is a potential, and its constant term is

$$g_0 = \langle \overline{Q_{ab}} F / Q_{ab}(1), 1\rangle_{\mathbb{D},d} = \frac{1}{Q_{ab}(1)} \langle \sum_{(p,q)} f_{pq} Q_{pq}, Q_{ab}\rangle_{\mathbb{D},d} = \frac{f_{ab}}{Q_{ab}(1)} \langle Q_{ab}, Q_{ab}\rangle_{\mathbb{D},d} = f_{ab},$$

as desired. □

Example 8.1 *Since* $\overline{z} = \overline{Q_{10}(z)}/Q_{10}(1)$*, we have that if* F *is a potential for the complex sphere, then* $G(z) = \overline{z}F(z)$ *is a potential with constant term* $g_0 = f_{10}$*.*

Remark 8.1 *It is shown in [Rud80] (Theorem 12.5.10), that for* $d = 2$ *the only time there is not equality in (8.40) is when* $(p, q) = (r, s)$*, in which case*

$$H(p,q)H(r,s) = \sum_{\substack{j=0 \\ j \text{ even}}}^{\mu} H(p+r-j, q+s-j), \quad \mu = 2\min\{p,q\}.$$

If the product $\cdot$ *of (8.41) is modified in this case to* $\cdot_2$*, with*

$$(p,q) \cdot_2 (p,q) := \bigcup_{\substack{j=0 \\ j \text{ even}}}^{\mu} \{(2p-j, 2q-j)\} \subset (p,q)\cdot(p,q), \qquad \mu := 2\min\{p,q\},$$

then the product FG *of Theorem 8.1, is a potential for (all of)* $P_{\mathcal{U}\cdot_2\mathcal{V}}$*.*

9 Bounds for real and complex spherical designs

Here we consider the relationship between our results on potentials and the seminal paper [DGS77] on codes and spherical designs. In particular, we seek to understand the given bounds on the number of vectors in real spherical designs, and then extend them in a natural way to complex spherical designs.

It is now convenient to allow the univariate polynomial F giving a potential for real or complex spherical designs $\Phi = (v_j)_{j=1}^n$ to have a (possibly nonzero) constant term f_0 in its Gegenbauer expansion, so that the potential (with constant) satisfies

$$n^2 A_F(\Phi) = \sum_j \sum_k F(\langle v_j, v_k \rangle) \geq n^2 f_0,$$

with equality if and only if (v_j) is a design. A very specific way equality can be achieved is by having each nondiagonal term in the sum be constant, i.e.,

$$F(\langle v_j, v_k \rangle) = c, \quad \forall j \neq k. \tag{9.42}$$

The value of the constant c depends strongly on the choice of F, indeed

$$n^2 A_F(\Phi) = nF(1) + (n^2 - n)c = n^2 f_0 \quad \Longrightarrow \quad c = \frac{nf_0 - F(1)}{n-1}.$$

In actuality, the number of equations in (9.42) depends on the number of **angles** of Φ, i.e., size of the set of inner products

$$\mathrm{Ang}(\Phi) := \{\langle v_j, v_k \rangle : v_j \neq v_k\} \subset [-1, 1),$$

and the value of the potential depends on these angles and their multiplicities. i.e.,

$$n^2 A_F(\Phi) = nF(1) + \sum_{\alpha \in \mathrm{Ang}(\Phi)} m_\alpha F(\alpha) \geq n^2 f_0, \tag{9.43}$$

where $m_\alpha > 0$ is the multiplicity of the angle α as an entry of the Gramian $[\langle v_j, v_k\rangle]$.

We are now in a position to give a transparent statement and proof of Theorem 4.3 of [DGS77]. This is the key result which gives upper bounds for the number of points in codes and designs, given a suitable choice of potential F.

Let $A \subset [-1,1)$. A finite set $X=(v_j)$ of n unit vectors in $\mathbb{R}^d$ is an **A-code** if its angles $\mathrm{Ang}(X)$ are contained in A.

Theorem 9.1 *(Upper bound) Let $F=\sum_k f_k Q_k$ be a polynomial with $f_k \geq 0$, $f_0 > 0$, i.e., a potential with a positive constant f_0, for which*

$$F(\alpha) \leq 0, \qquad \alpha \in A \subset [-1,1). \tag{9.44}$$

Then the size n of any A-code X satisfies

$$n \leq \frac{F(1)}{f_0}, \tag{9.45}$$

with equality if and only if the angles of X are roots of F, and X is a spherical P-design for

$$P = P_L = \bigoplus_{k\in L} \mathrm{Harm}_k(\mathbb{S}), \quad L=\{k : f_k > 0\}.$$

Proof: Suppose that X is an A-code. Then applying (9.44) to (9.43) gives

$$n^2 f_0 \leq n^2 A_F(\Phi) = nF(1) + \sum_{\alpha\in\mathrm{Ang}(\Phi)} m_\alpha F(\alpha) \leq nF(1),$$

which gives the inequality (9.45). There is equality above when $F(\alpha)=0$, $\alpha \in \mathrm{Ang}(X)$, and $A_F(\Phi)=f_0$, i.e., X is a spherical design for the potential F, i.e., for $P=P_L$. □

The weighted version of (9.43) is

$$A_F(\Phi) = F(1)\Big(\sum_j w_j^2\Big) + \sum_{\alpha\in A} F(\alpha)\Big(\sum_{\substack{j,k\\ \langle v_j,v_k\rangle=\alpha}} w_j w_k\Big) \geq f_0,$$

which allows Theorem 9.1 to be generalised, with (9.45) becoming

$$n \leq \frac{F(0)}{f_0}\big(n\sum_j w_j^2\big).$$

Before giving examples of Theorem 9.1, we consider the corresponding lower bound on n given by Theorem 5.10 of [DGS77], which can be established using a similar method, where F is a difference of potentials. For $F=\sum_k f_k Q_k$ a univariate polynomial, with Gegenbauer coefficients $f_k \in \mathbb{R}$, we call

$$F = f_0 + F^+ - F^-, \qquad F^+ := \sum_{\substack{k\neq 0\\ f_k>0}} f_k Q_k, \quad F^- := -\sum_{\substack{k\neq 0\\ f_k<0}} f_k Q_k,$$

its decomposition into potentials. We have the following version of (9.43)

$$\sum_j\sum_k F(\langle v_j,v_k\rangle) = n^2 f_0 + n^2 A_{F^+}(\Phi) - n^2 A_{F^-}(\Phi) = nF(1) + \sum_{\alpha\in\mathrm{Ang}(\Phi)} m_\alpha F(\alpha). \tag{9.46}$$

Theorem 9.2 *(Lower bound) Let $P = P_L$ be a unitarily invariant space of polynomials, and $F = \sum_k f_k Q_k$ a be a univariate polynomial with*

$$\{k : f_k > 0\} = L \cup \{0\}. \qquad F(\alpha) \geq 0, \quad \forall \alpha \in [-1, 1].$$

Then the size n of any P-design Φ satisfies

$$n \geq \frac{F(1)}{f_0}, \tag{9.47}$$

with equality if and only if the angles of Φ are roots of F, and Φ is a spherical P_K-design for $K = \{k : f_k < 0\}$.

Proof: By assumption, F^+ is a potential for $P = P_L$, and so $A_{F^+}(\Phi) = 0$. Thus (9.46) reduces to

$$n^2 f_0 - n^2 A_{F^-}(\Phi) - nF(1) = \sum_{\alpha \in \mathrm{Ang}(\Phi)} m_\alpha F(\alpha).$$

Since $F(\alpha) \geq 0$, we obtain the inequality

$$n^2 f_0 - nF(1) \geq n^2 A_{F^-}(\Phi),$$

with equality when $F(\alpha) = 0$, $\alpha \in \mathrm{Ang}(\Phi)$. Since F^- is a potential for P_K and $f_0 > 0$, we have

$$n^2 f_0 - nF(1) \geq n^2 A_{F^-}(\Phi) \geq 0 \quad \Longrightarrow \quad n \geq \frac{F(1)}{f_0},$$

which gives (9.47). Moreover, there is equality above when $A_{F^-}(\Phi) = 0$, i.e., when Φ is a spherical P_K-design. □

The original statement of Theorem 9.2 in [DGS77] (Theorem 5.10) was for spherical t-designs, i.e., $L = \{0, 1, 2, \ldots, t\}$ (see Example 9.1). The weighted version of this result can be obtained, in the obvious way, giving the lower bound

$$n \geq \frac{F(0)}{f_0} \big(n \sum_j w_j^2\big).$$

The easiest way to find an F satisfying $F(x) \geq 0$ on $[-1, 1]$, which we will refer to as a "nonnegative potential", is to take the square of an appropriate univariate polynomial.

Corollary 9.1 *Let $E \subset \mathbb{N}$ be a nonempty finite set of indices, and $L = E \cdot E$. Then*

$$F := \Big(\sum_{k \in E} Q_k\Big)^2, \tag{9.48}$$

gives a nonnegative potential for P_L, and the number of points n in a P_L-design satisfies

$$n \geq \frac{F(1)}{f_0} = \sum_{k \in E} Q_k(1) = \sum_{k \in E} \dim(\mathrm{Harm}_k(\mathbb{R}^d)). \tag{9.49}$$

Proof: Clearly, F is nonnegative, and by Lemma 8.1, it is a potential for $L = E \cdot E$. Thus, we may apply Theorem 9.2. By the orthogonality relations (7.35), we have

$$f_0 = \langle F, 1\rangle_{\text{geg}} = \langle\Big(\sum_{k\in E} Q_k\Big)^2, 1\rangle_{\text{geg}} = \langle\sum_{k\in E} Q_k, \sum_{\ell\in E} Q_\ell\rangle_{\text{geg}} = \sum_{k\in E}\langle Q_k, Q_k\rangle_{\text{geg}} = \sum_{k\in E} Q_k(1),$$

so that

$$\frac{F(1)}{f_0} = \sum_{k\in E} Q_k(1) = \sum_{k\in E} \dim(\text{Harm}_k(\mathbb{R}^d)),$$

and we obtain the desired estimate. $\square$

Corollary 9.2 *Let $E \subset \mathbb{N}$ be a finite set of even indices or of odd indices, and*

$$F := \Big(\frac{Q_1}{d} + 1\Big)\Big(\sum_{k\in E} Q_k\Big)^2, \qquad L = \{0,1\} \cdot (E \cdot E). \tag{9.50}$$

Then F gives a nonnegative potential for P_L, and the number of points n in a P_L-design satisfies

$$n \geq \frac{F(1)}{f_0} = 2\sum_{k\in E} Q_k(1) = 2\sum_{k\in E} \dim(\text{Harm}_k(\mathbb{R}^d)). \tag{9.51}$$

Proof: The proof is similar to that of Corollary 9.1. The first factor of F satisfies

$$\frac{Q_1(x)}{d} + 1 = x + 1 \geq 0, \qquad x \in [-1, 1],$$

and so F gives a nonnegative potential for L. The polynomial $Q_1(\sum_k Q_k)^2$ is odd, so has zero integral with respect to the Gegenbauer weight, and we have

$$f_0 = \langle F, 1\rangle_{\text{geg}} = \langle\Big(\frac{Q_1}{d} + 1\Big)\Big(\sum_{k\in E} Q_k\Big)^2, 1\rangle_{\text{geg}} = \langle\Big(\sum_{k\in E} Q_k\Big)^2, 1\rangle_{\text{geg}} = \sum_{k\in E} Q_k(1) > 0.$$

Further,

$$F(1) = 2\Big(\sum_{k\in E} Q_k(1)\Big)^2,$$

and so we obtain the desired estimate from Theorem 9.2. $\square$

We observe that the choice of the polynomial $\sum_{k\in E} Q_k$ in Corollaries 9.1 and 9.2 is optimal. Indeed, if a different convex combination $\sum_k c_k Q_k$ is taken in the potential F, then term $\sum_k Q_k(1)$ in the lower estimate becomes

$$M = \frac{\Big(\sum_k c_k Q_k(1)\Big)^2}{\sum_k c_k^2 Q_k(1)}.$$

By Cauchy-Schwarz, we have

$$\Big(\sum_k c_k Q_k(1)\Big)^2 = \Big(\sum_k c_k \sqrt{Q_k(1)}\sqrt{Q_k(1)}\Big)^2 \leq \Big(\sum_k c_k^2 Q_k(1)\Big)\Big(\sum_k Q_k(1)\Big),$$

so that

$$M \le \sum_k Q_k(1),$$

with equality if and only if $c_k = 1$, $\forall k$.

Spherical designs which give equality in one of the bounds of Theorems 9.1 and 9.2, are said to be **tight** (not to be confused with tight frames). Such designs are very special, and have played a prominent role in the theory of spherical designs: since the angles of tight designs are roots of F, it is possible to investigate their existence.

This following example is Theorems 5.11 and 5.12 of [DGS77].

Example 9.1 *(Spherical t-designs) These are given by $L = \{0, 1, 2, \dots, t\}$, which can be obtained by the following choices (for t even and odd).*

$$\begin{aligned} t = 2e: &\qquad L = E \cdot E, \quad E = \{0, 1, 2, \dots, e\}, \\ t = 2e+1: &\qquad L = \{0,1\} \cdot (E \cdot E), \quad E = \{e, e-2, e-4, \dots\}. \end{aligned}$$

The corresponding estimates (9.49) and (9.51) are

$$n \ge \sum_{k=0}^{e} \dim\bigl(\mathrm{Harm}_k(\mathbb{R}^d)\bigr) = \dim(\Pi_e(\mathbb{R}^d)|_{\mathbb{S}}) = \binom{e+d-1}{d-1} + \binom{e+d-2}{d-1}, \quad t = 2e,$$

$$n \ge 2 \sum_{0 \le j \le e/2} \dim\bigl(\mathrm{Harm}_{e-2j}(\mathbb{R}^d)\bigr) = 2\dim(\mathrm{Hom}_e(\mathbb{R}^d)|_{\mathbb{S}}) = 2\binom{e+d-1}{d-1}, \quad t = 2e+1.$$

In [BRV13], it is shown that there exist spherical t-designs whose number of points has this order of growth in t, i.e., with $n \ge c_d t^{d-1}$.

Example 9.2 *(Spherical half-designs) The spherical half-designs of even order $m = 2t$ are given by*

$$L = \{2t, 2t-2, 2t-4, \dots\} = E \cdot E, \quad E = \{t, t-2, t-4, \dots\}.$$

By applying Corollary 9.1, we obtain the estimate

$$n \ge \sum_{0 \le j \le t/2} \dim\bigl(\mathrm{Harm}_{t-2j}(\mathbb{R}^d)\bigr) = \dim(\mathrm{Hom}_t(\mathbb{R}^d)|_{\mathbb{S}}) = \binom{t+d-1}{d-1}.$$

In [DGS77], it is shown that for $t = 2m+1$ odd, a tight spherical t-design consists the vectors of a tight spherical-half design of order $2m$, and its negatives.

A second natural way to try and find suitable potentials, is to optimise over all possible such potentials. This is the "linear programming method" of [DGS77], e.g., from Theorem 9.2, one has the lower bound for P_L-designs

$$n \ge \max\Bigl\{ \frac{F(1)}{f_0} : F = \sum_k f_k Q_k, \{k : f_k > 0\} = L \cup \{0\}, F \ge 0 \text{ on } [-1, 1] \Bigr\},$$

and the following example.

Example 9.3 *(Spherical designs of harmonic index t) A polynomial F giving potential for the spherical designs for $P_L = \mathrm{Harm}_t(\mathbb{R}^d)$ (and no larger space), to which we can apply Theorem 9.2, has the form $F = Q_t + c$, for $c \geq b := -\min_{x\in[-1,1]} Q_k(x) > 0$. The corresponding estimate $n \geq F(1)/f_0 = Q_t(1)/c + 1$ is optimised by taking $c = b$. This estimate, and variants of it, can be found in [ZBB+17].*

The linear programming method has recently been applied to real spherical (t,t)-designs [Boy20], [BBD+25a], [BBD+25b] (weighted designs).

We now consider bounds for complex designs. As before, we will take polynomials of the form

$$F = \sum_{(p,q)} f_{pq} Q_{pq}, \qquad f_{pq} \in \mathbb{R}.$$

These are polynomials of a complex variable with real coefficients, and so take complex values in general (unless $f_{pq} = f_{qp}$, $\forall (p,q)$). Nevertheless, they do have $F(1), f_0 \in \mathbb{R}$, and, most importantly,

$$\sum_{\alpha\in A} F(\alpha) m_\alpha = \sum_{\alpha\in A} \Re(F(\alpha)) m_\alpha. \tag{9.52}$$

The last equation follows since $\overline{F(z)} = F(\overline{z})$, and an angle α and $\overline{\alpha}$ appear with the same multiplicity $m_\alpha = m_{\overline{\alpha}}$ in (9.43), so the sum of the corresponding pair of terms is

$$\begin{aligned} m_\alpha F(\alpha) + m_{\overline{\alpha}} F(\overline{\alpha}) &= m_\alpha F(\alpha) + m_\alpha \overline{F(\alpha)} = 2 m_\alpha \Re(F(\alpha)) \\ &= m_\alpha \Re(F(\alpha)) + m_{\overline{\alpha}} \Re(\overline{F(\alpha)}) = m_\alpha \Re(F(\alpha)) + m_{\overline{\alpha}} \Re(F(\overline{\alpha})). \end{aligned}$$

In view of (9.52), the extension of Theorems 9.1 and 9.2 to the complex case become obvious, and we combine them.

Theorem 9.3 *(Upper and Lower bounds) Let $F = \sum_{(p,q)} f_{pq} Q_{pq}$ be a polynomial with*

$$f_{pq} \in \mathbb{R}, \qquad f_0 = f_{00} > 0, \qquad \tau = \tau^+ := \{(p,q) : f_{pq} > 0\}, \quad \tau^- := \{(p,q) : f_{pq} < 0\},$$

and $A \subset \{z \in \mathbb{C} : |z| \leq 1, z \neq 1\}$. Then

(i) If F is a potential, i.e., $\tau^- = \{\}$, and $\Re(F(\alpha)) \leq 0$, $\forall \alpha \in A$, then the size n of any A-code X satisfies

$$n \leq \frac{F(1)}{f_0},$$

with equality if and only if the angles of X are roots of F, and X is a τ-design

(ii) If $\Re(F(\alpha)) \geq 0$, $\alpha \in \{z \in \mathbb{C} : |z| \leq 1\}$, then the size n of any spherical τ-design X satisfies

$$n \geq \frac{F(1)}{f_0},$$

with equality if and only if the angles of X are roots of F, and X is a τ^--design.

This result can be found in [RS14], but with $F(\alpha)$ in place of our $\Re(F(\alpha))$, as the subtlety that $F(\alpha)$ can be complex was not considered. For a finite set of indices $\mathcal{E}$,

$$Q_{\mathcal{E}} := \sum_{(p,q)\in\mathcal{E}} Q_{pq}. \tag{9.53}$$

As for real designs, we will say that a complex spherical design X is **tight** if it meets one of the bounds of Theorem 9.3, i.e.,

$$n = |X| = \frac{F(1)}{f_0}, \qquad \text{the angles of } X \text{ are roots of } F.$$

Corollary 9.3 *Let $\mathcal{E} \subset \mathbb{N}^2$ be a nonempty finite set of indices. Then*

$$F := |Q_{\mathcal{E}}|^2 = \Big(\sum_{(p,q)\in\mathcal{E}} Q_{pq}\Big)\Big(\sum_{(r,s)\in\mathcal{E}^{\text{rev}}} Q_{rs}\Big) \geq 0, \tag{9.54}$$

is a potential for $P_{\mathcal{E}\cdot\mathcal{E}^{\text{rev}}}$, and the number of points n in a $(\mathcal{E}\cdot\mathcal{E}^{\text{rev}})$-design satisfies

$$n \geq \dim(P_{\mathcal{E}}) = \sum_{(p,q)\in\mathcal{E}} \dim\big(H(p,q)\big), \tag{9.55}$$

with equality if and only if the angles of the design are roots of the polynomial $Q_{\mathcal{E}}$.

Proof: Since $\overline{Q_{pq}} = Q_{qp}$, we have the equality in (9.54), and the multiplication rule of Theorem 8.1 implies that $F = Q_{\mathcal{E}}\overline{Q_{\mathcal{E}}}$ is a potential for $\mathcal{E}\cdot\mathcal{E}^{\text{rev}}$, with

$$f_{00} = \langle F, 1\rangle_{\mathbb{D},d} = \langle \sum_{(p,q)\in\mathcal{E}} Q_{pq}, \sum_{(r,s)\in\mathcal{E}} Q_{rs}\rangle_{\mathbb{D},d} = \sum_{(p,q)\in\mathcal{E}} \langle Q_{pq}, Q_{pq}\rangle_{\mathbb{D},d} = \sum_{(p,q)\in\mathcal{E}} Q_{pq}(1).$$

Since $F = |Q_{\mathcal{E}}|^2 \geq 0$, by construction, Theorem 9.3 gives the estimate

$$n \geq \frac{F(1)}{f_{00}} = \frac{|Q_{\mathcal{E}}(1)|^2}{Q_{\mathcal{E}}(1)} = Q_{\mathcal{E}}(1) = \sum_{(p,q)\in\mathcal{E}} Q_{pq}(1) = \sum_{(p,q)\in\mathcal{E}} \dim\big(H(p,q)\big) = \dim(P_{\mathcal{E}}),$$

with equality if and only if the angles are roots of $Q_{\mathcal{E}}$. $\square$

The inequality (9.55) is given in [RS14] (Theorem 4.2) for $\mathcal{U}$ a lower set, via a different argument. We now explain, and give the argument, which is classical and neat. A "convolution" product $*$ on indices (and sets of indices) is given in [RS14] by

$$\mathcal{E} * \mathcal{E} := \mathcal{E} + \mathcal{E}^{\text{rev}} \subset \mathcal{E}\cdot\mathcal{E}^{\text{rev}}.$$

For $\mathcal{E}$ a lower set, one has $\mathcal{E} * \mathcal{E} = \mathcal{E}\cdot\mathcal{E}^{\text{rev}}$, and so Theorem 9.3 can be applied. Let X be a spherical $(\mathcal{E}\cdot\mathcal{E}^{\text{rev}})$-design, and (f_j) be an orthonormal basis for $P_{\mathcal{E}} = \oplus_{(p,q)\in\mathcal{E}} H(p,q)$. It follows from (8.40) that

$$f_j\overline{f_k} \in \Big(\bigoplus_{(p,q)\in\mathcal{E}} H(p,q)\Big)\Big(\bigoplus_{(r,s)\in\mathcal{E}^{\text{rev}}} H(r,s)\Big) \subset \bigoplus_{(p,q)\in\mathcal{E}\cdot\mathcal{E}^{\text{rev}}} H(p,q),$$

and so we have, by the cubature rule, that

$$\frac{1}{|X|}\sum_{x\in X} f_j(x)\overline{f_k(x)} = \int_{\mathbb{S}} f_j\overline{f_k}\,d\sigma = \delta_{jk},$$

i.e., $(f_j|_X)$ is orthonormal in $\mathbb{C}^X$, and hence $n=|X|\ge \dim(P_{\mathcal{E}})$.

We observe that the sets $\mathcal{U}=\mathcal{E}\cdot\mathcal{E}^{\mathrm{rev}}=\mathcal{E}^{\mathrm{rev}}\cdot\mathcal{E}=\mathcal{E}^{\mathrm{rev}}\cdot(\mathcal{E}^{\mathrm{rev}})^{\mathrm{rev}}$ in Corollary 9.3 are "symmetric" in the sense that $\mathcal{U}=\mathcal{U}^{\mathrm{rev}}$, i.e.,

$$(p,q)\in\mathcal{E}\cdot\mathcal{E}^{\mathrm{rev}} \quad\Longleftrightarrow\quad (q,p)\in\mathcal{E}\cdot\mathcal{E}^{\mathrm{rev}}.$$

The indices (p,p), which give the projective designs, are called **projective indices**. Here is an example where $\mathcal{E}$ is not a lower set.

Example 9.4 *(Projective indices) For any $\mathcal{E}$ of the form*

$$\mathcal{E}=\{(p,q),(p-1,q-1),\ldots,(p-m,q-m)\},\qquad 0\le m\le\min\{p,q\}, \tag{9.56}$$

we have

$$\mathcal{E}\cdot\mathcal{E}^{\mathrm{rev}}=\{(0,0),(1,1),\ldots,(t,t)\},\qquad t=p+q,$$

a sequence of consecutive projective indices. Hence for any $\mathcal{U}$, we have

$$\{(0,0),(1,1),\ldots,(t,t)\}\subset\mathcal{U}\cdot\mathcal{U}^{\mathrm{rev}},\qquad t:=\max_{(p,q)\in\mathcal{U}}(p+q).$$

Since we can take $\mathcal{E}=\{(p,q)\}$, a single point, we observe that $\mathcal{U}\cdot\mathcal{U}^{\mathrm{rev}}$ cannot be a "small" set.

Example 9.5 *(Spherical (t,t)-designs) Applying Corollary 9.3 for the set $\mathcal{E}$ of (9.56) for $p+q=t$ and $m=\min\{p,q\}$, gives the estimate*

$$n\ge\sum_{j=0}^{\min\{p,q\}}\dim\bigl(H(p-j,q-j)\bigr)=\dim\bigl(\mathrm{Hom}(p,q)\bigr)=\binom{p+d-1}{d-1}\binom{q+d-1}{d-1},$$

for n the number of points in a spherical (t,t)-design for $\mathbb{C}^d$. The best estimate from those above is obtained for the choice $p=\lfloor\frac{t}{2}\rfloor$, which gives

$$n\ge\begin{cases}\binom{k+d-1}{d-1}^2, & t=2k;\\ \binom{k+d-1}{d-1}\binom{k+d}{d-1}, & t=2k+1.\end{cases} \tag{9.57}$$

This estimate improves that given in [Wal18] (Exercise 6.22), [Boy20], i.e.,

$$n\ge\binom{t+d-1}{d-1}.$$

The bound, and function $Q_{\mathcal{E}}$ with $F = |Q_{\mathcal{E}}|^2$, for the first few values of t are (respectively)

$$\begin{aligned}
t=1: &\qquad n \ge d, &\qquad & Q_{10}(z) = dz, \\
t=2: &\qquad n \ge d^2, &\qquad & Q_{11}(z) + Q_{00}(z) = d\big((d+1)|z|^2 - 1\big), \\
t=3: &\qquad n \ge \frac{1}{2}d^2(d+1), &\qquad & Q_{21}(z) + Q_{10}(z) = \frac{1}{2}d(d+1)z\big((d+2)|z|^2 - 2\big).
\end{aligned}$$

The reverse inequality appears in [Hog82] (Theorem 3.2) as an absolute bound. The roots of the above polynomials give the absolute value $|z|$ of the angles for "tight" designs. The possible values for $|z|^2$ in a tight design for $t = 1, 2, \ldots, 5$ are

$$\{0\}, \quad \{\frac{1}{d+1}\}, \quad \{0, \frac{2}{d+2}\}, \quad \{\frac{\sqrt{2(d+1)/(d+2)} \pm 2}{d+3}\}, \quad \{0, \frac{\sqrt{3(d+1)/(d+3)} \pm 3}{d+4}\}.$$

Tight designs exist for $t = 1$ (orthonormal bases) and $t = 2$ (complex equiangular lines). There are also known examples for $t = 3$ of 6 lines at "angles" $|z|^2 = 0, \frac{1}{2}$ in $\mathbb{C}^2$, 40 lines at angles $0, \frac{1}{3}$ in $\mathbb{C}^4$ and 126 lines at angles $0, \frac{1}{4}$ in $\mathbb{C}^6$ (see [Hog82], [HW21]). For $t = 5$, an example of 12 lines in $\mathbb{C}^2$ was given in [HW21], with angles $\frac{1}{2}(\frac{1}{\sqrt{5}} \pm 1)$.

Here is an example where $\mathcal{E}$ is a lower set.

Example 9.6 *(The simplex) Let $\mathcal{E} = \{(0,0), (1,0)\}$, which is a lower set. Then*

$$\tau = \mathcal{E} \cdot \mathcal{E}^{\text{rev}} = \{(0,0), (0,1), (1,0), (1,1)\},$$

and the bound and polynomial $Q_{\mathcal{E}}$ for the class of τ-designs given by Corollary 9.3 is

$$n \ge d+1, \qquad Q_{\mathcal{E}}(z) = Q_{00}(z) + Q_{10}(z) = dz + 1.$$

Here the τ-designs are the balanced unit norm tight frames (see Examples 5.1 and 5.2). For such a design to be tight, it must have $d+1$ vectors with angles $\frac{-1}{d}$. There is a unique such configuration, given by the $d+1$ vertices of the regular simplex in $\mathbb{R}^d \subset \mathbb{C}^d$.

The polynomials of Example 9.5, i.e., $Q_{\mathcal{E}_t}(z)$ where

$$\mathcal{E}_t = \{(k+\varepsilon-j, k-j)\}_{0 \le j \le k}, \qquad t = 2k+\varepsilon, \quad \varepsilon = 0, 1,$$

appear, implicitly, in [Hog82]. We now explain this connection, and show that they are Jacobi polynomials, which has implications for the location and spacing of their roots. Hoggar defines polynomials of degree k by

$$Q_k^{\varepsilon}(x) := \frac{(md)_{2k+\varepsilon}}{(m)_{k+\varepsilon} k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{{}_i(k+m+\varepsilon-1)}{{}_i(2k+md+\varepsilon-2)} x^{k-i}, \qquad \varepsilon = 0, 1, \tag{9.58}$$

$${}_i(x) = x(x-1)\cdots(x-i+1), \qquad (x)_i = x(x+1)\cdots(x+i-1),$$

which depend on $\varepsilon = 0, 1$, and a parameter m, with $m = \frac{1}{2}$ being the real case, and $m = 1$ the complex case. It is easily verified that these are related to our Gegenbauer polynomials as follows

$$Q_{2k+\varepsilon}(x) = x^\varepsilon Q_k^\varepsilon(x^2), \qquad m = \frac{1}{2},$$

$$Q_{k+\varepsilon,k}(z) = z^\varepsilon Q_k^\varepsilon(|z|^2), \qquad m = 1.$$

It follows from the orthogonality relations for the Gegenbauer polynomials that the Q_k^ε are orthogonal polynomials of degree k on $[0, 1]$, for a Jacobi weight (depending on ε). The polynomial $Q_{\mathcal{E}_t}(z) = Q_{\mathcal{E}_{2k+\varepsilon}}(z)$ appears in [Hog82] as $z^\varepsilon R_k^\varepsilon(|z|^2)$, and, by Lemma 5.2, and (5.23), it can be expressed as

$$\begin{aligned} Q^{(d)}_{\mathcal{E}_{2k+\varepsilon}}(z) &= \sum_{j=0}^{k} Q^{(d)}_{j+\varepsilon,j}(z) \\ &= \frac{d}{2k+\varepsilon+d} Q^{(d+1)}_{k+\varepsilon,k}(z) \\ &= \dim\bigl(\mathrm{Hom}(k+\varepsilon, k)\bigr) z^\varepsilon \frac{P_k^{(d-1,\varepsilon)}(2|z|^2-1)}{P_k^{(d-1,\varepsilon)}(1)} \\ &= \frac{1}{(d-1)!} \sum_{j=0}^{k} (-1)^j \frac{(d+2k+\varepsilon-j-1)!}{j!(k-j)!(k+\varepsilon-j)!} z^\varepsilon (|z|^2)^{k-j}. \end{aligned} \tag{9.59}$$

We now give a complex analogue of Corollary 9.2. Let

$$\mathcal{S}_1 := \{(0, 0), (1, 0), (0, 1)\}.$$

Corollary 9.4 *Let $\mathcal{E}$ be a finite set of indices, with the property that*

$$(p \pm 1, q), (p, q \pm 1) \notin \mathcal{E}, \quad \forall (p, q) \in \mathcal{E},$$

and $\tau = \mathcal{S}_1 \cdot (\mathcal{E} \cdot \mathcal{E}^{\mathrm{rev}})$. Then the number of points n in a spherical τ-design satisfies

$$n \geq 2 \dim(P_\mathcal{E}) = 2 \sum_{(p,q)\in\mathcal{E}} \dim\bigl(H(p, q)\bigr),$$

with equality if and only if the angles are -1 or the roots of $Q_\mathcal{E} = \sum_{(p,q)\in\mathcal{E}} Q_{pq}$.

Proof: Since $z = \frac{1}{d} Q_{10}(z)$ and $\overline{z} = \frac{1}{d} Q_{01}(z)$, it follows from Theorem 8.1 that a potential for the τ-designs is given by

$$F(z) := \Bigl(\frac{z + \overline{z}}{2} + 1\Bigr) |Q_\mathcal{E}|^2 \geq 0.$$

Let $(p, q) \in \mathcal{E}$. Since $(1, 0) \cdot (p, q) = \{(p + 1, q), (p, q - 1)\}$, (8.40) gives

$$zQ_{pq}(z) = \frac{1}{d} Q_{10}(z) Q_{pq}(z) \in H(p + 1, q) \oplus H(p, q - 1) \perp P_\mathcal{E},$$

i.e., $zQ_{pq}(z)$ is orthogonal to $P_{\mathcal{E}}$, and similarly for $\overline{z}Q_{pq}(z)$, so that

$$f_{00} = \langle F, 1\rangle_{\mathbb{D},d} = \langle\Big(\frac{z+\overline{z}}{2}+1\Big)Q_{\mathcal{E}}, Q_{\mathcal{E}}\rangle_{\mathbb{D},d} = \sum_{(p,q)\in\mathcal{E}} \langle Q_{pq}, Q_{pq}\rangle_{\mathbb{D},d} = \sum_{(p,q)\in\mathcal{E}} Q_{pq}(1).$$

Thus, by Theorem 9.3, we have the estimate

$$n \geq \frac{F(1)}{f_{00}} = \frac{2|Q_{\mathcal{E}}(1)|^2}{Q_{\mathcal{E}}(1)} = 2Q_{\mathcal{E}}(1) = 2\dim(P_{\mathcal{U}}),$$

with the equality as stated. □

10 Absolute and special bounds for complex designs

We now give an important variant of the upper bound in Theorem 9.3, where

- The condition that F be a potential, or even have real Gegenbauer coefficients, is weakened.
- The condition $F(\alpha) \leq 0$, $\forall\alpha \in A$ is strengthened to $F(\alpha) = 0$, $\forall\alpha \in A$.
- The set A and F depend on each other.
- The upper bound only depends on which Gegenbauer coefficients of F are nonzero.

Theorem 10.1 *Let $X = (v_j)$ be a sequence of n unit vectors in $\mathbb{C}^d$, and*

$$F = \sum_{(p,q)} f_{pq}Q_{pq}, \qquad f_{pq} \in \mathbb{C},$$

be a polynomial with

$$F(1) = 1, \qquad A := \{z \in \mathbb{C} : |z| \leq 1, F(z) = 0\}, \qquad \tau := \{(p,q) : f_{pq} \neq 0\}.$$

Then the $n\times n$ matrix $M = [F(\langle v_j, v_k\rangle)]_{j,k=1}^n$ satisfies

$$\operatorname{rank}(M) \leq \min\Big\{n, \sum_{(p,q)\in\tau} \dim(H(p,q))\Big\}. \tag{10.60}$$

In particular, if X is an A-code, i.e., $F(\langle v_j, v_k\rangle) = 0$, $j \neq k$, then

$$n \leq \sum_{(p,q)\in\tau} \dim(H(p,q)), \tag{10.61}$$

with equality if and only if

$$F = \frac{1}{n}\sum_{(p,q)\in\tau} Q_{pq}, \tag{10.62}$$

in which case the angles of X are roots of F, and X is a spherical τ-design if and only if $(0,0) \in \tau$.

Proof: Let $H_{pq} = (W_j^{(p,q)})$ be a row vector whose entries are an orthonormal basis for $H(p,q)$, then the reproducing kernel for $H(p,q)$ is

$$K_{pq}(\xi,\eta) = \sum_j W_j^{(p,q)}(\xi)\overline{W_j^{(p,q)}(\eta)} = H_{pq}(\xi)H_{pq}(\eta)^* = Q_{pq}(\langle \xi,\eta\rangle).$$

Using this, we may write M as a product of matrices

$$M = H_X \operatorname{diag}(f_{pq}I_{\dim(H(p,q))})_{(p,q)\in\tau}H_X^*, \qquad H_X := [H_{pq}(v_j)]_{1\le j\le n,(p,q)\in\tau}.$$

The $\sum_{(p,q)\in\tau}\dim(H(p,q))$ columns of H_X are an (appropriately ordered) orthonormal basis for $P_\tau = \oplus_{(p,q)\in\tau}H(p,q)$ sampled at the n points $v_j \in X$. Given the size of H_X, we obtain (10.60) from $\operatorname{rank}(M) \le \operatorname{rank}(H_X)$.

Now suppose that X is an A-code. Then $M = I$, which is rank n, and (10.60) implies (10.61). Further, suppose that there is equality in (10.61), i.e., H_X is square, and hence invertible. Since $\operatorname{diag}(f_{pq}I_{\dim(H(p,q))})_{(p,q)\in\tau}$ is congruent to I, by Sylvester's law of inertia, its eigenvalues are all positive, i.e., $f_{pq} > 0$, $\forall(p,q)\in\tau$. Thus F is a potential for τ. By Corollary 8.1, $G = F\overline{Q_{pq}} = FQ_{qp}$ is a potential, with constant $g_{00} = f_{pq}Q_{pq}(1)$. Thus $F\overline{Q_{pq}} - f_{pq}Q_{pq}(1)$ is a potential, with zero constant, and we have

$$\sum_j\sum_k \big(F\overline{Q_{pq}} - f_{pq}Q_{pq}(1)\big)(\langle v_j,v_k\rangle) = nF(1)\overline{Q_{pq}}(1) - n^2 f_{pq}Q_{pq}(1) = nQ_{pq}(1)(\frac{1}{n} - f_{pq}) \ge 0,$$

which implies that

$$f_{pq} \le \frac{1}{n}, \qquad (p,q)\in\tau. \tag{10.63}$$

This gives

$$1 = F(1) = \sum_{(p,q)\in\tau} f_{pq}Q_{pq}(1) \le \frac{1}{n}\sum_{(p,q)\in\tau} Q_{pq}(1) = \frac{1}{n}\sum_{(p,q)\in\tau}\dim(H(p,q)) = 1,$$

so we must have equality in (10.63) throughout, and we obtain (10.62). Finally, we observe that $G = F - f_{00}$ is a potential for τ-designs, with zero constant, and calculate

$$\sum_j\sum_k G(\langle v_j,v_k\rangle) = nF(1) - n^2 f_{00} = n^2(\frac{1}{n} - f_{00}) = \begin{cases} 0, & (0,0)\in\tau; \\ n, & (0,0)\notin\tau, \end{cases}$$

so that X is a τ-design if and only if $(0,0)\in\tau$. □

We will refer to any sequence (v_j) of unit vectors in $\mathbb{C}^d$ as "a design", which is warranted since it is indeed a $\{(0,0)\}$-design. The obvious way to apply Theorem 10.1 is for an F constructed to vanish at certain prescribed angles. In this regard, we say that a polynomial F is an **annihilator** (polynomial) for a design, or a collection of designs, if $F(1) = 1$ and all angles of the designs are roots of F. Sometimes the condition $F(1) = 1$ is replaced by $F(1) \ne 0$. Heuristically, we desire that

- F has a large zero set, i.e., the collection of designs that it annihilates is large.

- $F \in P_\tau$, for some P_τ of small dimension (thereby giving a good bound).

Corollary 10.1 *If $F = \sum_{(p,q)} f_{pq} Q_{pq}$ is an annihilator polynomial of a design X, then*

$$n = |X| \le \dim(P_\tau) = \sum_{(p,q)\in\tau} \dim(H(p,q)), \qquad \tau := \{(p,q) : f_{pq} \ne 0\},$$

with equality if and only if the angles of X are roots of $\sum_{(p,q)\in\tau} Q_{pq}$.

This result appears in [RS14] Theorem 4.2, for $\tau = \mathcal{S}$ a lower set. There, a design with an annihilator polynomial in $\operatorname{span}\{z^p\overline{z} : (p,q) \in \mathcal{S}\}$ (which equals $P_\mathcal{S}$, for $\mathcal{S}$ a lower set) is called an $\mathcal{S}$-code. Bounds which follow from Corollary 10.1 will be called **absolute bounds**, as they are in the real and projective cases.

We start with an obvious example (see [RS14] Corollary 4.3).

Example 10.1 *Let X be a design with $m = |A|$ angles $A \subset \{z \in \mathbb{C} : |z| \le 1, z \ne 1\}$. Then*

$$F(z) := \prod_{\alpha\in A} \frac{z-\alpha}{1-\alpha} = \sum_{k=0}^{m} f_{k0} Q_{k0}(z),$$

gives the estimate

$$n = |X| \le \sum_{k=0}^{m} \dim(H(k,0)) = \sum_{k=0}^{m} \binom{k+d-1}{d-1} = \binom{m+d}{d}.$$

Complex codes X with two and three angles (inner products) have been studied by [NS16], [NS18]. For two angles [RS14], [NS16] gives the bound

$$n = |X| \le \begin{cases} 2d+1, & d \text{ is odd;} \\ 2d, & d \text{ is even,} \end{cases} \tag{10.64}$$

which is clearly better than that of Example 10.1 ($m = 2$), i.e.,

$$n = |X| \le \frac{1}{2}(d+1)(d+2).$$

Interestingly, the only case where these two bounds coincide is for $d = 1$, in which case

$$F(z) = \frac{1}{3}\big(Q_{20}^{(1)}(z) + Q_{10}^{(1)}(z) + Q_{00}^{(1)}(z)\big) = \frac{1}{3}(z^2+z+1).$$

The zeros of this polynomial are the primitive third roots of unity, and the third roots of unity $X = \{1, \omega, \omega^2\}$ gives the unique three-vector two angle code for $\mathbb{C}^1$ attaining the bound (see Table 1, [NS16]).

If α is an angle of design, then so is its conjugate $\overline{\alpha}$, i.e., two-angle designs have angles $\{\alpha, \overline{\alpha}\}$. These have the same real part, and so are both roots of the polynomial

$$F(z) = \Re(z) - \Re(\alpha) = \frac{z+\overline{z}}{2} - \frac{\alpha+\overline{\alpha}}{2} = \frac{Q_{10}(z)}{2d} + \frac{Q_{01}(z)}{2d} - \Re(\alpha) Q_{00}(z).$$

Thus, by applying Theorem 10.1, we obtain essentially the bound of (10.64).

Example 10.2 *The collection of designs X whose angles have a fixed real part $a \in \mathbb{R}$, e.g., two-angle designs with angles $\{\alpha, \overline{\alpha}\}$, where $\Re(\alpha) = a$, satisfy the bound*

$$n = |X| \le \begin{cases} 2d+1, & a \neq 0; \\ 2d, & a = 0. \end{cases}$$

Further, if there is equality above for $a \neq 0$, then

$$Q_{10}(\alpha) + Q_{01}(\alpha) + Q_{00}(\alpha) = d\alpha + d\overline{\alpha} + 1 = 2d\Re(\alpha) + 1 = 0 \quad \Longrightarrow \quad a = \Re(\alpha) = -\frac{1}{2d}.$$

The two-angle complex designs are an example of complex equiangular lines. The Gramian matrix which determines a two-angle complex design has a particularly simple form, which can be associated naturally with a graph (on its elements) via a conference matrix. In this way, the two-angle complex designs have been classified in [NS16] (also see the corresponding complex equiangular lines in [Ren07], [Wal18] Exercises 12.11 and 12.12).

If a design X has three angles, then they must be a complex conjugate pair $\alpha, \overline{\alpha}$, with $a = \Re(\alpha) = \Re(\overline{\alpha})$, and a real angle $b \in \mathbb{R}$. Since b is a root of the polynomial $z - b$, we have the following annihilator polynomial

$$\begin{aligned} F(z) &= (\Re(z) - \Re(\alpha))(z - b) \\ &= \frac{Q_{20}(z)}{d(d+1)} + \frac{Q_{11}(z)}{2d(d+1)} - \Big(\frac{b}{2} + a\Big)\frac{Q_{10}(z)}{d} - b\frac{Q_{01}(z)}{2d} + \Big(ab + \frac{1}{2d}\Big)Q_{00}(z). \end{aligned} \tag{10.65}$$

For a generic a, b (all coefficients above nonzero) this gives

$$n \le Q_{20}(1) + Q_{11}(1) + Q_{10}(1) + Q_{01}(1) + Q_{00}(1) = \frac{d(3d+5)}{2}. \tag{10.66}$$

The bound given in [NS16] for three angle complex designs is

$$n = |X| \le \begin{cases} 4, & d = 1; \\ d(d+2), & d \ge 2. \end{cases} \tag{10.67}$$

These agree for $d = 1$ ($n \le 4$), where the F for four vectors meeting the bound (10.66) becomes

$$F(z) = \frac{1}{4}(z^2 + 2|z|^2 - 1 + z + \overline{z}),$$

which has roots $-1, i, -i$, which are the three angles of $X = \{1, -1, i, -i\} \subset \mathbb{C}^1$.

For $d \ge 2$, the bound (10.67) improves the generic bound (10.66). In particular, for $d = 2$, the bounds are $n \le 8$ and $n \le 11$. By specifying a particular form for the three angle design, the number of terms in (10.65) can be reduced, giving a sharper bound. To get $n \le 8$, one must choose

$$\frac{b}{2} + a = 0, \quad ab + \frac{1}{2d} = 0 \quad \Longrightarrow \quad a = \pm\frac{1}{2\sqrt{d}}, \; b = \mp\frac{1}{\sqrt{d}} \; (d = 2).$$

Given that [NS16] show that there is a unique three-angle design of eight vectors in $\mathbb{C}^2$ given by the two-angle design of four vectors with $a=0$ (equiangular lines) multiplied by ± 1 ($b=-1$), there can be no three-angle design with the above parameters.

For $d=3$ the bounds are $n \le 15$ and $n \le 21$, whilst the maximal number of vectors in a three-angle design is calculated to be $n=9$ (one can take an orthonormal basis multiplied by the three roots of unity, $a=-1/2$, $b=0$). Moreover, [NS16] Theorem 14 shows that equality in (10.67) can only be obtained for $d=1,2$ (where $b=-1$).

Theorem 10.2 *Let X be a design for which the real part of its angles can take s possible values. This includes the complex designs with $m=2s$ angles, none real. Then*

$$n = |X| \le \sum_{p+q \le s} \dim(H(p,q)) = \binom{s+2d-1}{2d-1} + \binom{s+2d-2}{2d-1}. \tag{10.68}$$

Proof: Let $a_1, \ldots, a_s \in \mathbb{R}$ be the possible real parts of the angles. Then

$$F(z) = \prod_{j=1}^{s} \frac{\Re(z) - a_j}{1 - a_j} = \prod_{j=1}^{s} \frac{\frac{z+\overline{z}}{2} - a_j}{1 - a_j} = \sum_{p+q \le s} f_{pq} Q_{pq}(z),$$

is an annihilator polynomial for X. The Gegenbauer expansion above follows because the $\binom{s+2}{2}$ polynomials $(Q_{pq}^{(d)}(z))_{p+q \le s}$ are linearly independent, and hence are a basis for the space of polynomials of degree s in the variables z and $\overline{z}$. Applying Corollary 10.1, then gives the result, where the upper bound is calculated via

$$\begin{aligned}\sum_{p+q \le s} \dim(H(p,q)) &= \dim(\mathrm{Hom}_s(\mathbb{R}^{2d})) + \dim(\mathrm{Hom}_{s-1}(\mathbb{R}^{2d})) \\ &= \binom{s+2d-1}{2d-1} + \binom{s-1+2d-1}{2d-1}.\end{aligned}$$

□

For $s=1$, we recover Example 10.2.

Example 10.3 *($s=2$) For four-angle designs (10.68) gives the estimate*

$$n = |X| \le d(2d+3),$$

which has the same growth in d as the estimates for three-angle designs. As for the case $s=1$, better estimates can be obtained if the values of the angles are constrained. Indeed

$$(\Re(z)-a)(\Re(z)-b) = \frac{Q_{20}(z)+Q_{11}(z)+Q_{02}(z)}{2d(d+1)} - \frac{a+b}{2d}\{Q_{10}(z)+Q_{01}(z)\} + \Big(ab + \frac{1}{2d}\Big),$$

so that if $a \ne b$ are the real parts of the angles, then

$$n \le \begin{cases} d(2d+1), & b=-a; \\ d(2d+1)-1, & b=-a=\pm\frac{1}{\sqrt{2d}}. \end{cases}$$

We conclude this section, by showing that the estimates for the number of lines in $\mathbb{C}^d$ (projective designs) naturally follow from Corollary 10.1. For lines, or projective designs, there is no meaningful notion of angle $\langle v_j, v_k\rangle \in \mathbb{C}$, but rather the projective invariants

$$|\langle v_j, v_k\rangle|^2 = a,$$

are of interest. These are sometimes referred to as angles (particularly for projective designs), e.g., as in "complex equiangular lines". Fortunately, there simple and natural annihilator polynomials for this "angle", i.e.,

$$F(z) = |z|^2 - a, \quad a \neq 0, \qquad F(z) = z,\ F(z) = \overline{z}, \quad a = 0.$$

Let $X = (v_j)$ be unit vectors giving a set of lines in $\mathbb{C}^d$. If the angles $|\langle v_j, v_k\rangle|^2$, $j \neq k$, take s possible values $A \subset [0, 1)$, then an annihilator polynomial for X is given by

$$F(z) := z^\varepsilon \prod_{a\in A\setminus\{0\}} \frac{|z|^2 - a}{1-a} = \sum_{j=0}^{s-\varepsilon} f_{j+\varepsilon,j} Q_{j+\varepsilon,j}(z), \qquad \varepsilon = \varepsilon(A) := \begin{cases} 0, & 0 \notin A; \\ 1, & 0 \in A. \end{cases}$$

Such configurations are said to be an A**-set** or s**-angular**. We may apply Corollary 10.1, to obtain the estimate for A-sets given in [DGS77] (Theorem 6.1), i.e.,

$$n = |X| \le \sum_{j=0}^{s-\varepsilon} \dim(H(j+\varepsilon, j)). \tag{10.69}$$

Theorem 10.3 *Let $X = (v_j)$ be unit vectors giving a set of s-angular lines in $\mathbb{C}^d$, i.e., whose the angles $x = |z|^2 = |\langle v_j, v_k\rangle|^2$, $j \neq k$, take s possible values $A \subset [0, 1)$. Then*

$$n = |X| \le \binom{s+d-1}{d-1}\binom{s-\varepsilon+d-1}{d-1}, \qquad \varepsilon = \varepsilon(A) := \begin{cases} 0, & 0 \notin A; \\ 1, & 0 \in A. \end{cases} \tag{10.70}$$

Further, the angles of the sets of s-angular lines giving equality in (10.70) are roots of

$$f(x) = x^\varepsilon P_{s-\varepsilon}^{(d-1,\varepsilon)}(2x-1). \tag{10.71}$$

Proof: The upper bound of (10.69) simplifies, using Lemma 5.2 and (5.24), to

$$\begin{aligned} \sum_{j=0}^{s-\varepsilon} Q_{j+\varepsilon,j}^{(d)}(1) &= \frac{d}{2s-\varepsilon+d} Q_{s,s-\varepsilon}^{(d+1)}(1) \\ &= \frac{d}{2s-\varepsilon+d} \frac{(2s-\varepsilon+d)(s+d-1)!(s-\varepsilon+d-1)!}{s!(s-\varepsilon)!d!(d-1)!} \\ &= \binom{s+d-1}{d-1}\binom{s-\varepsilon+d-1}{d-1}, \end{aligned}$$

which gives (10.70). The (tight) s-angular designs giving equality in (10.70) have inner products $z = \langle v_j, v_k \rangle$, $j \neq k$, which are roots of the polynomial

$$\sum_{j=0}^{s-\varepsilon} Q^{(d)}_{j+\varepsilon,j}(z) = \frac{d}{2s-\varepsilon+d} Q^{(d+1)}_{s,s-\varepsilon}(z). \tag{10.72}$$

Therefore, in view of (9.59), the s distinct possible angles $x = |z|^2 = |\langle v_j, v_k \rangle|$ are roots of the polynomial (10.71). $\square$

By way of comparison with (10.70), the bound (9.57) for spherical (t,t)-designs can be written as

$$n = |X| \geq \binom{s+d-1}{d-1}\binom{s-\varepsilon+d-1}{d-1}, \qquad t = 2s-\varepsilon, \quad \varepsilon = 0, 1,$$

where the angles $|z| = |\langle v_j, v_k \rangle|$ of the tight spherical (t,t)-designs are roots of the same polynomial (10.72). In this way,

- There is a 1–1 correspondence between the tight complex spherical (t,t)-designs with $t = 2s-\varepsilon$, $\varepsilon = 0, 1$, and the tight s-angular designs which have 0 as an angle if and only if $\varepsilon = 1$.

Upper bounds obtained from annihilator polynomials which are potentials are known as **special bounds**. We now show, by example, how the special and absolute bounds for s-angular lines (projective designs) can be obtained directly from our general results.

The annihilator polynomial for one angle

$$F(z) = (|z|^2 - a) = \frac{Q_{11}(z)}{d(d+1)} + \frac{1-ad}{d} Q_{00}(z),$$

gives the absolute bounds

$$n \leq Q_{11}(1) + Q_{00}(1) = d^2, \quad a \neq \frac{1}{d}, \qquad n \leq Q_{21}(1) = d^2 - 1, \quad a = \frac{1}{d}.$$

We observe that for $d = 2$ there are tight such designs, four complex equiangular lines and the vertices of the equilateral triangle, respectively. We may apply Theorem 9.3, subject to the condition that the above F is a potential, to obtain the special bounds

$$n \leq \frac{F(1)}{f_0} = \frac{1-a}{\frac{1-ad}{d}} = \frac{d(1-a)}{1-da}, \qquad a > \frac{1}{d}.$$

The special bound holds for $a = 0$, as $n \leq d$, which is considerably sharper than the absolute bound, but this cannot be proved by taking the limit $a \to 0$. It can be proved using Example 8.1 applied to annihilator polynomial $F(z) = z$. We illustrate this general process for 2-angular lines. Consider the following annihilator polynomials lines with angles $\{0,a\}$ and $\{a,b\}$

$$\begin{aligned}
F(z) &= z(|z|^2-a) \\
&= \frac{Q_{21}(z)}{\frac{1}{2}d(d+1)(d+2)} + \frac{Q_{10}(z)}{d}\Big(\frac{2}{d+1}-a\Big), \\
F(z) &= (|z|^2-a)(|z|^2-b) \\
&= \frac{Q_{22}(z)}{\frac{1}{4}d(d+1)(d+2)(d+3)} + \Big(-a-b+\frac{4}{d+2}\Big)\frac{Q_{11}(z)}{d(d+1)} \\
&\qquad + \frac{d(d+1)ab-(d+1)(a+b)+2}{d(d+1)}Q_{00}(z).
\end{aligned}$$

By using Example 8.1, and applying Theorem 9.3 to the potential $\overline{z}F(z)$, the first gives the special bound

$$n \le \frac{F(1)}{f_{10}} = \frac{1-a}{\frac{1}{d}(\frac{2}{d+1}-a)} = \frac{d(d+1)(1-a)}{2-(d+1)a}, \qquad a < \frac{2}{d+1}, \tag{10.73}$$

for sets with angles $\{0,a\}$. For sets with angles $\{a,b\}$, direct application of Theorem 9.3 to the second potential F gives the special bounds

$$n \le \frac{F(1)}{f_{00}} = \frac{(1-\alpha)(1-\beta)}{\frac{d(d+1)ab-(d+1)(a+b)+2}{d(d+1)}} = \frac{d(d+1)(1-\alpha)(1-\beta)}{d(d+1)ab-(d+1)(a+b)+2}, \tag{10.74}$$

which holds for

$$a+b \le \frac{4}{d+2}, \qquad d(d+1)ab-(d+1)(a+b)+2>0.$$

We observe that the limit $\beta \to 0$ of this bound, gives the bound for angles $\{0,\alpha\}$, but does not prove it. The bounds (10.73) and (10.74) were originally obtained using the polynomials Q_k^ε of (9.58) (see [DGS91], [Hog92]).

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