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Schmidt's Inequality

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ABSTRACT

The main result is the computation of the best constant in the Wirtinger-Sobolev inequality

$$\|f\|_p \le C_{p,q,\theta}\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q,$$

where

$$f(\theta)=0,$$

and θ is some point in $[a,b]$, or, equivalently, the determination of the norm of the (bounded) linear map

$$A : L_q[a,b] \to L_p[a,b]$$

given by

$$Af(x) := \int_\theta^x f(t)\,dt.$$

This and other results are seen to be closely related to an inequality of Schmidt 1940.

The method of proof is elementary, and should be the main point of interest for most readers since it clearly illustrates a technique that can be applied to other situations. These include the generalisations of Hardy's inequality where $\theta=a$ and $\|\cdot\|_p$, $\|\cdot\|_q$ are replaced by weighted p, q norms, and higher order Wirtinger-Sobolev inequalities involving boundary conditions at a single point.

Key Words: Schmidt's inequality, Hardy-type inequalities, Wirtinger-Sobolev inequalities, Poincaré inequalities, Hölder's inequality, n-widths, isoperimetric calculus of variations problems

AMS (MOS) Subject Classifications: primary 41A44, 41A80, 47A30, secondary 34B10, 34L30

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1. Introduction

There has recently been considerable progress in the problem of estimating the best constant C in the inequality

$$\|D^j(f - H_\Theta f)\|_p \le C\,(b-a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \qquad \forall f \in W_q^n, \tag{1.1}$$

where $H_\Theta f$ is the Hermite interpolant to f at the some multiset Θ of n points in $[a,b]$, and $0 \le j < n$. In Shadrin [S95], the best constant was determined for $p = q = \infty$, $0 \le j < n$, and all Θ. The remaining estimates in the extensive literature on this problem were extended and put within a unified framework based on a single 'basic estimate' in Waldron [W96]. Inequalities of the form (1.1) belong to the class of Wirtinger(-Sobolev) inequalities (also called Poincaré inequalities), see, e.g., Fink, Mitrinović and Pečarić [FMP91:p66].

Towards a better understanding of what, if any, improvements to these estimates might be possible (for $p, q \ne \infty$), the best constant in (1.1) is computed in the simplest case, when $n = 1$ ($j = 0$), for $1 \le p, q \le \infty$. Here $\Theta = \{\theta\}$, a single point in $[a,b]$, and

$$H_\Theta f = f(\theta), \tag{1.2}$$

the constant polynomial which matches f at θ.

Since

$$f(x) - H_\Theta f(x) = f(x) - f(\theta) = \int_\theta^x Df(t)\,dt, \tag{1.3}$$

finding the best constant in (1.1) is equivalent to computing the norm of the linear map

$$A : L_q[a,b] \to L_p[a,b]$$

given by

$$Af(x) := \int_\theta^x f(t)\,dt,$$

and since

$$Df = D(f - f(\theta)) = D(f - H_\Theta f),$$

it is also equivalent to finding the best constant C in the inequality: for $f \in W_q^1$ with $f(\theta) = 0$,

$$\|f\|_p \le C\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q. \tag{1.4}$$

It is the last of these equivalencies which appears most commonly, and we will solve the problem in these terms. The solution is given in Theorem 4.9.

The rest of the paper is set out as follows. In Section 2, the (standard) variational approach to finding the best constant in (1.4) is outlined. In Section 3, the 'elementary argument' which allows the problem to be split into two problems with boundary conditions of the form $f(a) = 0$ (equivalently $f(b) = 0$) and thereby reduced to a 'maximisation problem' of 1 variable is given. In Section 4, the 'maximisation problem' is solved and the best constant and corresponding extremal functions (when they exist) are computed.

In Section 5, it is shown how a number of related results concerning extremal problems and n-widths can be obtained from an inequality of Schmidt 1940 [Sc40] by using simple geometric arguments (such as those in this paper).

2. The variational approach

Let $W_p^n := W_p^n[a,b]$ be the **Sobolev** space of functions f with $D^{n-1}f$ absolutely continuous on $[a,b]$ and $D^n f \in L_p := L_p[a,b]$. To solve isoperimetric extremal problems such as

$$\sup_{f\neq 0}\left\{\frac{\|f\|_p}{\|Df\|_q} : f(\theta)=0,\ f\in W_q^1\right\} = \sup\left\{\|f\|_p : f(\theta)=0,\ \|Df\|_q=1,\ f\in W_q^1\right\}, \quad (2.1)$$

the standard approach is to use the calculus of variations to find conditions for f to be a stationary point of the (Rayleigh) functional

$$J : f \mapsto \frac{\|f\|_p}{\|Df\|_q}, \qquad f\in W_q^1,\ f(\theta)=0,\ \|Df\|_q=1. \quad (2.2)$$

Since, for $1\le q\le\infty$,

$$W_q^1 \subset L_p, \qquad 0<p\le\infty,$$

it is possible to investigate (2.1) for $0<p\le 1$ also, and we will do so.

To describe the Euler-Lagrange equation for (2.1) it is convenient to define the non-linear operator

$$Q_p : f \mapsto |f|^{p-1}\,\mathrm{sign}(f), \qquad 0<p<\infty,$$

which satisfies

$$D(|f|^p) = pQ_p(f)Df, \quad (2.3)$$

and occurs when describing the cases of equality in Hölder's inequality. The notation $f_{(p)} := Q_p f$ is used by some authors.

By differentiating under the integral $\|f\|_p^p = \int_a^b |f|^p$, and using (2.3), it follows that

$$\frac{d}{d\eta}\|f+\eta g\|_p = \|f+\eta g\|_p^{1-p}\langle Q_p(f+\eta g), g\rangle, \qquad 0<p<\infty, \quad (2.4)$$

where

$$\langle f,g\rangle := \int_a^b fg.$$

Let $0<p<\infty$, $1\le q<\infty$. The condition for (2.2) to have a stationary point is that

$$\left.\frac{d}{d\eta}J(f+\eta g)\right|_{\eta=0} = 0, \quad (2.5)$$

for all $g \in W_q^1$ with $g(\theta) = 0$. Using the quotient rule for differentiation and (2.4) it follows that (2.5) is equivalent to

$$\|Df\|_q \|f\|_p^{1-p} \langle Q_p f, g\rangle - \|f\|_p \|Df\|_q^{1-q} \langle Q_q Df, Dg\rangle = 0,$$

which can be rewritten as

$$\langle Q_p f, g\rangle - \lambda^p \langle Q_q Df, Dg\rangle = 0, \tag{2.6}$$

where $\lambda = \|f\|_p$ and $\|Df\|_q = 1$. In Buslaev and Tikhomirov [BT85], such a pair (f, λ) is termed a **spectral pair** for the extremal problem (2.1) (there they use $1/\lambda$ in place of λ). Some questions of existence and uniqueness of spectral pairs (for $1 < p, q < \infty$) are investigated in Buslaev [B95].

Further information about the spectral pairs for (2.1) can be extracted from (2.6) as follows. Integrating (2.6) by parts gives

$$\langle Q_p f, g\rangle - \lambda^p \{(Q_q Df) g|_a^b - \langle D(Q_q Df), g\rangle\} = 0.$$

Since this equation holds for all $g \in C_0^\infty(a, b)$ with $g(\theta) = 0$, it follows that

$$Q_p f + \lambda^p D(Q_q Df) = 0 \tag{2.7}$$

on $(a, b) \setminus \{\theta\}$, and (in particular) f is C^1 on $[a, \theta]$ and $[\theta, b]$. For $q = 1$ ($p \neq \infty$) equation (2.7) reduces to

$$Q_p f = 0,$$

which has solution $f = 0$, indicating that there are no extremals for (2.1) in this case (see later this section for more detail). In addition to satisfying (2.7), the spectral pairs (f, λ) must satisfy the boundary condition that

$$(Q_q Df) g|_a^b = 0, \tag{2.8}$$

for all $g \in W_q^1$ with $g(\theta) = 0$.

In the special case $\theta = a$, the boundary condition (2.8) simply reduces to the 'dual' condition that

$$Df(b) = 0 \tag{2.9}$$

(similarly for $\theta = b$), and it is possible (for $q \neq 1$) to integrate (2.7) to obtain a spectral pair with 'eigenvalue' λ giving the solution of (2.1). Crucial to performing this integration is the fact that for the spectral pair (f, λ) giving the solution of (2.1) it can, by (1.3), be assumed that $Df \geq 0$. This integration is outlined at the end of this section.

For $\theta \neq a, b$ the boundary condition (2.8) reduces to

$$Df(a) = Df(b) = 0, \tag{2.10}$$

and the Euler-Lagrange equation splits into a pair of equations of the type $\theta = a$, which are connected by the common parameter λ. In principle, this pair of equations can be solved by using the solution for when $\theta = a$. Instead, we perform effectively this argument in terms of the inequalities (1.4). This is our 'elementary argument', and it provides the extremals in the cases when they exist – such as for $0 < p < \infty$, $1 < q < \infty$ when they satisfy (2.7). It requires the solution of (2.1) for $\theta = a$ which is stated in Lemma 2.14 below.

The solution for $\theta = a, b$

Adopting the notation of Schmidt [Sc40], let $G : [0, \infty) \to \mathbb{R}$ be the continuous function given by

$$G(u) := \begin{cases} e^u u^{-u}\Gamma(u+1), & u > 0 \\ 1, & u = 0 \end{cases} \tag{2.11}$$

where Γ is the **Gamma** function. Using the fact that

$$\Gamma(1/2) = \sqrt{\pi},$$

it is possible to compute G at the half integers

$$G(0) = 1, \quad G(1/2) = \sqrt{\frac{\pi}{2}}e^{\frac{1}{2}}, \quad G(1) = e, \quad G(3/2) = \sqrt{\frac{\pi}{6}}e^{\frac{3}{2}}, \quad G(2) = \frac{1}{2}e^2, \ldots \tag{2.12}$$

and so forth.

Then, for $0 < p \le \infty$, $1 \le q \le \infty$, let

$$C(p,q) := \frac{G(1/p + 1/q')}{G(1/p)G(1/q')} = \frac{(\frac{1}{p} + \frac{1}{q'})^{-\frac{1}{p}-\frac{1}{q'}}}{(\frac{1}{p})^{-\frac{1}{p}}(\frac{1}{q'})^{-\frac{1}{q'}}} \frac{\Gamma(1 + \frac{1}{p} + \frac{1}{q'})}{\Gamma(1+\frac{1}{p})\Gamma(1+\frac{1}{q'})}, \tag{2.13}$$

where q' denotes the *conjugate exponent* of q, and $1/\infty$ is to be interpreted (in the usual way) as 0. Observe that

$$0 < C(p,q) \le 1.$$

Using (2.12), we compute that

$$C(1,1) = C(\infty,\infty) = 1, \quad C(1,2) = C(2,\infty) = 1/\sqrt{3}, \quad C(1,\infty) = 1/2,$$
$$C(2,1) = C(\infty,2) = 1, \quad C(2,2) = 2/\pi.$$

More generally, for $0 < p \le \infty$, $1 \le q \le \infty$,

$$C(p,1) = C(\infty,q) = 1, \quad C(p,\infty) = \left(\frac{1}{p+1}\right)^{1/p}, \quad C(1,q) = \left(\frac{1-1/q}{2-1/q}\right)^{1-\frac{1}{q}}.$$

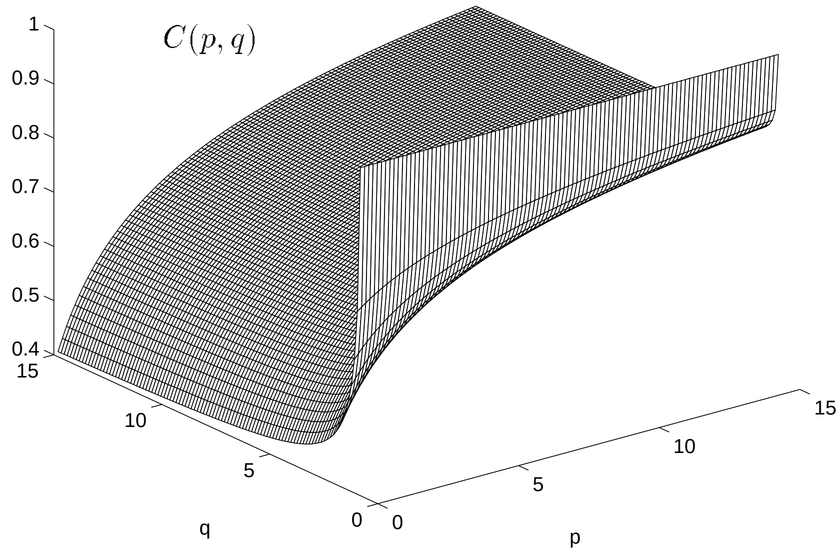

Fig. 2.1. The graph of $(p,q) \mapsto C(p,q)$ over $(0,15] \times [1,15]$. Notice that $C(p,1) = 1$.

The following result is essentially due to Schmidt [Sc40:(20),p306] (see Section 5).

Lemma 2.14. *Let $0 < p \le \infty$, $1 \le q \le \infty$. Then, for all $f \in W_q^1[a,b]$ satisfying*

$$f(a) = 0, \qquad \textit{(equivalently } f(b) = 0\textit{)} \tag{2.15}$$

there is the sharp inequality

$$\|f\|_p \le C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q, \tag{2.16}$$

where $C(p,q)$ is defined by (2.13).

In the cases $0 < p < \infty$, $1 < q < \infty$, for $[a,b] = [0,1]$, equality holds in (2.16) if and only if f is a scalar multiple of the 1-1 and onto function

$$E_{p,q} : [0,1] \to [0,1]$$

defined as the (unique) solution of the initial value problem

$$Df = I_{p,q}(1-f^p)^{\frac{1}{q}}, \qquad f(0) = 0, \tag{2.17}$$

where

$$I_{p,q} := \int_0^1 \frac{d\xi}{(1-\xi^p)^{1/q}} = \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{q'})}{p\Gamma(\frac{1}{p}+\frac{1}{q'})} = \frac{\Gamma(\frac{1}{p}+1)\Gamma(\frac{1}{q'})}{\Gamma(\frac{1}{p}+\frac{1}{q'})}. \tag{2.18}$$

These extremal functions come from the solution of (2.7) (see the end of this section), and their uniqueness (for $1 < p, q < \infty$) is a special case of Buslaev [B95:Th.4]. From (2.17), it is easily seen that $E_{p,q}$ is strictly increasing, concave, and satisfies

$$\begin{aligned} E_{p,q}(0) &= 0, \qquad E_{p,q}(1) = 1, \\ DE_{p,q}(0) &= I_{p,q}, \qquad DE_{p,q}(1) = 0. \end{aligned}$$

Further, for the case $p = q = 2$,

$$E_{2,2}(x) = \sin(\frac{\pi}{2}x).$$

This example provides motivation for the nonlinear spectral theory developed by Buslaev and Tikhomirov [BT85] (and others) to describe the stationary points of functionals such as (2.2). The functions $E_{p,q}$ can be expressed as the p-th power of the inverse of an *incomplete Gamma function*.

The numerical solution of (2.17) provides no obstacles. In our case, the graphs of $E_{p,q}$ appearing in this paper were done using MATLAB to compute the values of $E_{p,q}$ (at equally spaced points) by the *Runge-Kutta method of order* 4.

Extremals exist unless $q = 1$, $p \neq \infty$

In the cases $p = \infty$ and $q = 1, \infty$ it is possible to identify extremal functions for (2.16), when they exist, by taking the appropriate limits of the initial value problem (2.17), as follows. As before, let $[a, b] = [0, 1]$.

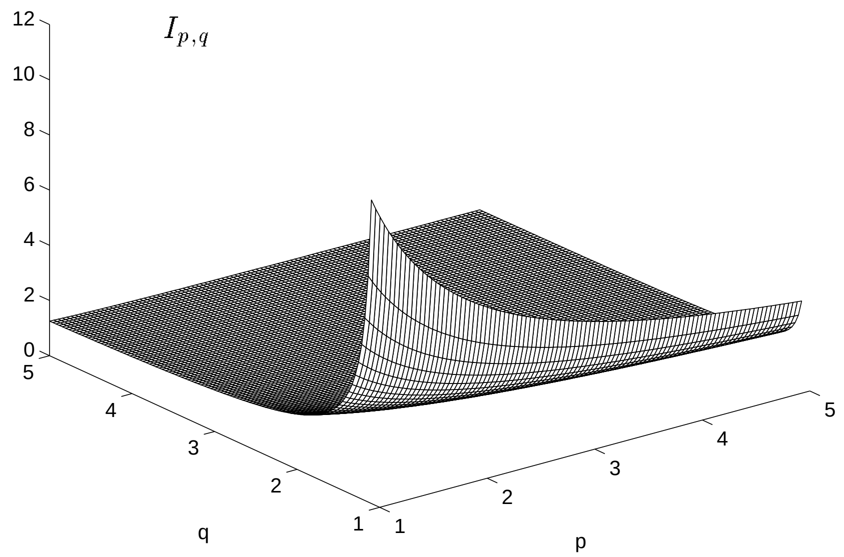

Fig. 2.2. The graph of $(p, q) \mapsto I_{p,q} = DE_{p,q}(0)$ over $[1, 5] \times [1.1, 5]$ indicating

the assymptote at $q = 1$.

For the cases $p = \infty$, or $q = \infty$, observe that

$$\lim_{p\to\infty} I_{p,q} = 1, \qquad \lim_{q\to\infty} I_{p,q} = 1,$$

and the linear polynomial

$$E_{\infty,q}(x) := E_{p,\infty}(x) := x, \qquad 0 < p \le \infty,\ 1 \le q \le \infty, \tag{2.19}$$

which is the solution of the initial value problem

$$Df = 1, \qquad f(0) = 1,$$

gives equality in (2.16). This is the only extremal (upto a scalar multiple) in these cases.

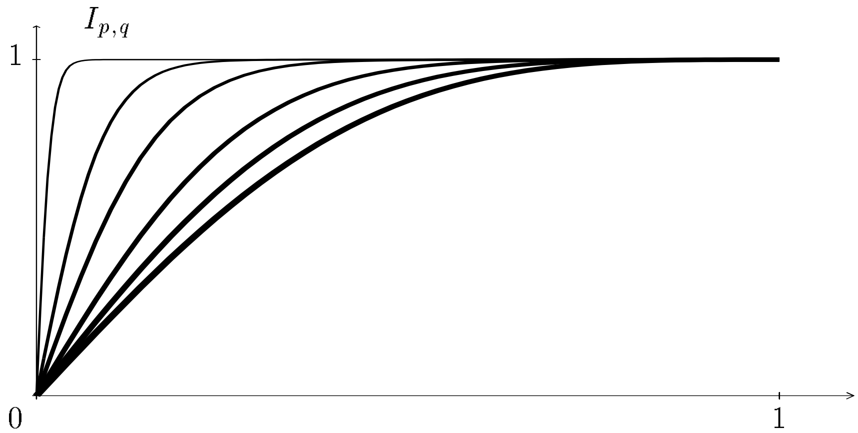

Fig. 2.3. Graphs of the extremals $E_{p,q}$ showing the behaviour as $q \to 1^+$ (thinner lines). The example depicted is $p = 2$, $q = 1.4,\ 1.3,\ 1.2,\ 1.1,\ 1.05, 1.01$.

For $q = 1$ and $p \ne \infty$,

$$\lim_{q\to 1+} I_{p,q} = \infty,$$

and no extremal functions for (2.16) exist. Suppose to the contrary, that there was a function $f \in W_q^1[0,1] \subset C[0,1]$ with

$$f(0) = 0, \qquad \|Df\|_q = 1, \qquad \|f\|_p = C(p,1) = 1.$$

Then,

$$|f(x)| = \left|\int_0^x Df\right| \le \int_0^1 |Df| = 1.$$

But, the only continuous functions with $|f| \le 1$ and $\|f\|_p = 1$, $0 < p < \infty$ are $f = \pm 1$, neither of which satisfies $f(0) = 0$.

Thus, extremals $E_{p,q}$ exist for (2.16) unless $q = 1$, $p \neq \infty$. In Fink [F74:Lem.2], it is claimed incorrectly that (for $1 \leq p, q \leq \infty$) extremals exist unless $(p, q) = (1, 1)$ or (∞, ∞). The argument given there mistakenly concludes that since

$$C(p, q) = C(q', p')$$

(which the use G in (2.13) emphasizes), if extremals exist for the choice of norms (p, q), then they must exist for the choice (q', p').

Integrating the Euler-Lagrange equation for $\theta = a$.

In this subsection we outline how the Euler-Lagrange equation (2.7) can be integrated in the case that $\theta = a$ (where (2.7) holds on [a,b]), and $q \neq 1$. As before, let $[a, b] = [0, 1]$.

The standard way of doing this, see, e.g. Fink [F74] and Tikhomirov [T76], is to use the 'dual' boundary condition (2.9) to write (2.7) as a pair of differential equations in f and y, where y is defined by

$$Dy = \frac{1}{\lambda^p} Q_p f, \qquad y(1) = 0$$

(or some variation thereof). In this case one obtains

$$y + Q_q Df = 0, \qquad Dy = \frac{1}{\lambda^p} Q_p f, \qquad f(0) = 0, \qquad y(1) = 0, \tag{2.20}$$

which Tikhomirov and Buslaev [BT85] term a 'canonical system' of equations (for the extremal problem). The solution is then obtained by performing (effectively) the following integration.

By (2.3),

$$D(Q_q Df) = (q-1)|Df|^{q-1} D^2 f,$$

so that (2.7) can be written as

$$Q_p f + \lambda^p (q-1)|Df|^{q-1} D^2 f = 0. \tag{2.21}$$

Multiplying (2.21) by Df gives

$$(Q_p f) Df + \lambda^p (q-1) Q_q(Df) D^2 f = 0, \tag{2.22}$$

where f satisfies the boundary conditions

$$f(0) = 0, \qquad Df(1) = 0. \tag{2.23}$$

Using (2.3) equation (2.22) can be integrated to obtain

$$\frac{|f|^p}{p} + \lambda^p (q-1) \frac{|Df|^q}{q} = H \lambda^p. \tag{2.24}$$

To determine the constant H, integrate (2.24) over $[0,1]$ to obtain

$$\frac{1}{p} + \frac{q-1}{q} = \frac{1}{p} + \frac{1}{q'} = H. \tag{2.25}$$

By (1.3), it can be assumed that $Df \geq 0$, and so the extremal function f is nonnegative. To simplify calculations we normalise f to obtain an extremal E with

$$E(1) = 1.$$

Evaluating (2.24) at 1 gives

$$f(1) = (Hp)^{1/p}\lambda,$$

and so

$$E := E_{p,q} := \frac{f}{(Hp)^{1/p}\lambda}.$$

This (normalised) extremal then satisfies

$$\frac{DE}{(1-E^p)^{1/q}} = \frac{1}{\lambda} H^{1-\frac{1}{q'}-\frac{1}{p}} \frac{(q')^{1/q}}{p^{1/p}}.$$

Integrating the above from 0 to 1 gives

$$\frac{1}{\lambda} H^{1-\frac{1}{q'}-\frac{1}{p}} \frac{(q')^{1/q}}{p^{1/p}} = \int_0^1 \frac{d\xi}{(1-\xi^p)^{1/q}} = \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{q'})}{p\Gamma(\frac{1}{p}+\frac{1}{q'})} = I_{p,q},$$

where $I_{p,q}$ is defined by (2.18). In particular, one obtains that

$$\lambda = \frac{(\frac{1}{p}+\frac{1}{q'})^{-\frac{1}{p}-\frac{1}{q'}}}{(\frac{1}{p})^{-\frac{1}{p}}(\frac{1}{q'})^{\frac{-1}{q'}}} \frac{\Gamma(1+\frac{1}{p}+\frac{1}{q'})}{\Gamma(1+\frac{1}{p})\Gamma(1+\frac{1}{q'})} = C(p,q),$$

and E is the solution of (2.17).

3. The elementary argument

The key to solution of (2.1) for $a < \theta < b$ presented below is the observation that functions $f \in W_q^1[a,b]$ with $f(\theta) = 0$ are of the form

$$f(x) = \begin{cases} g(x), & a \leq x \leq \theta \\ h(x), & \theta \leq x \leq b \end{cases} \tag{3.1}$$

where

$$g \in W_q^1[a,\theta] \qquad \text{with} \qquad g(\theta) = 0,$$

$$h \in W_q^1[\theta, b] \qquad \text{with} \qquad h(\theta) = 0,$$

together with the fact that

$$\int_a^b |f|^p = \int_a^\theta |g|^p + \int_\theta^b |h|^p, \qquad 0 < p < \infty,$$

$$\int_a^b |Df|^q = \int_a^\theta |Dg|^q + \int_\theta^b |Dh|^q, \qquad 1 \le q < \infty.$$

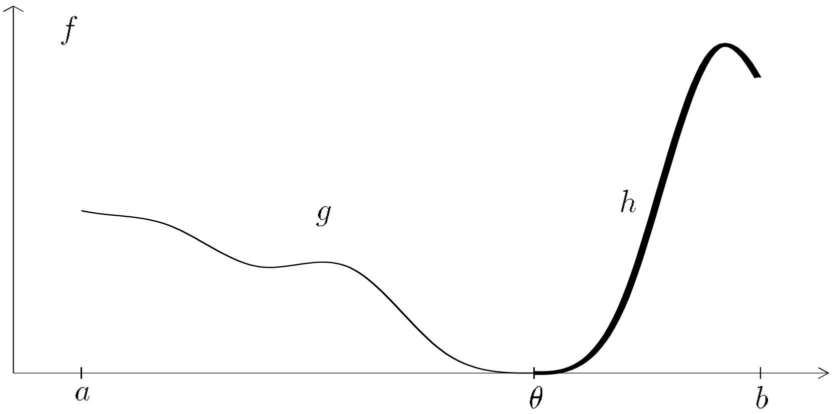

Fig. 3.1. The splitting of f into g and h (thicker).

It will be convenient to have (2.16) in the form

$$\sup_{\substack{f(a)=0 \\ \int_a^b |Df|^q = \alpha}} \int_a^b |f|^p = \alpha^{\frac{p}{q}} C(p,q)^p (b-a)^{p(1+\frac{1}{p}-\frac{1}{q})}, \qquad 0 < p < \infty,\ 1 \le q < \infty, \tag{3.2}$$

where $\alpha > 0$, (and the supremum is over functions in $W_q^1[a,b]$). It is to be understood that (3.2) also holds when the condition $f(a) = 0$ is replaced by $f(b) = 0$.

For simplicity, assume without loss of generality that $[a,b] = [0,1]$. Let $0 \le \theta \le 1$ and $0 < p < \infty$, $1 \le q < \infty$. Then, by using the splitting (3.1) and (3.2), we compute that the solution of (2.1) satisfies

$$\begin{aligned}
&\sup \left\{ \|f\|_p : f(\theta) = 0,\ \|Df\|_q = 1,\ f \in W_q^1 \right\} \\
&= \sup_{0 \le A \le 1} \left\{ \left(\int_0^\theta |g|^p + \int_\theta^1 |h|^p \right)^{\frac{1}{p}} : g(\theta) = h(\theta) = 0,\ \int\limits_0^\theta |Dg|^q = A,\ \int\limits_\theta^1 |Dh|^q = 1 - A \right\} \\
&= \sup_{0 \le A \le 1} \left(\sup_{\substack{g(\theta)=0 \\ \int_0^\theta |Dg|^q = A}} \int_0^\theta |g|^p + \sup_{\substack{h(\theta)=0 \\ \int_\theta^1 |Dh|^q = 1-A}} \int_\theta^1 |h|^p \right)^{1/p} \\
&= C(p,q) \max_{0 \le A \le 1} \left(A^{\frac{p}{q}} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}} (1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})} \right)^{1/p}.
\end{aligned} \tag{3.3}$$

Thus, the problem (2.1) has been reduced to the maximisation problem of 1 variable of finding

$$M(p,q,\theta)^p := \max_{0\le A\le 1} f(A), \tag{3.4}$$

where

$$f := f_{p,q,\theta} : A \mapsto A^{\frac{p}{q}}\theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}}(1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}. \tag{3.5}$$

This maximum is found in the next section.

Application to Hardy-type inequalities

The careful reader will notice that the argument just outlined also applies to a variety of similar situations – some of interest.

One such example is when $\|\cdot\|_p$, $\|\cdot\|_q$ are replaced by weighted p,q norms $\|\cdot\|_p^*$, $\|\cdot\|_q^*$. The corresponding inequalities

$$\|f\|_p^* \le C\,\|Df\|_q^*, \qquad \forall f \in W_q^1, \tag{3.6}$$

where $f(a)=0$, analogous to (2.16), are called **Hardy-type** inequalities. The original Hardy's inequality is the case $p=q>1$, where

$$\|f\|_p^* := \Big(\int_0^\infty \Big|\frac{f(x)}{x}\Big|^p\,dx\Big)^{1/p}, \qquad \|f\|_q^* := \|f\|_{L_p[0,\infty)}, \qquad 1<p<\infty,$$

with the condition $f(0)=0$. Here the best constant is $C=p/(p-1)$. Often Hardy's inequality is stated with f in the form

$$f(x) = \int_0^x g(t)\,dt.$$

There is considerable interest in Hardy-type inequalities, see, e.g., the monograph of Opic and Kufner [OK90]. The author has made no attempt to translate the various conditions for the existence of an inequality of the form (3.6) and estimates for the best constant to when the condition $f(a)=0$ is replaced by $f(\theta)=0$ with θ some point inside the interval of interest (which has left endpoint a).

Another situation of interest where the argument applies is higher order Wirtinger inequalities

$$\|f\|_p \le C\,(b-a)^{n+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \qquad \forall f \in W_q^n,$$

where f satisfies boundary conditions at a single point θ, $a<\theta<b$, to which could be added boundardy conditions at the endpoints (same conditions at both points). A particular case of note is when f vanishes to order n at θ. For $\theta=a$ this extremal problem has recently been investigated by Buslaev [B95].

4. The best constant

In this section, the solution of (2.1) is completed by finding the points A^* where the maximum (3.4) is attained, and by treating the cases $p = \infty$, $q = \infty$ using 'continuity' arguments.

We require the maximum over $0 \le A \le 1$ of

$$f := f_{p,q,\theta} : A \mapsto A^{\frac{p}{q}} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}} (1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}, \tag{3.5}$$

where $0 < \theta < 1$. Since the second derivative of f is

$$D^2 f(A) = \frac{p}{q}\left(\frac{p}{q} - 1\right) \left\{ A^{\frac{p}{q}-2} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}-2} (1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})} \right\},$$

where the term inside the $\{\ \}$ is positive, f is either convex, linear, or concave, depending on the values of p, q.

To describe the extremal functions we need the following. Suppose $0 < p \le \infty$, $1 < q \le \infty$ or $p = \infty$, $q = 1$. For $0 \le \theta < 1$, let

$$E_{p,q}^{\theta+}(x) := \begin{cases} 0, & 0 \le x \le \theta \\ E_{p,q}\left(\frac{x-\theta}{1-\theta}\right), & \theta \le x \le 1 \end{cases} \tag{4.1}$$

which is the continuous function supported on $[\theta, 1]$ obtained from $E_{p,q}$ by an affine change of variables. Here $E_{p,q}$ is defined by (2.17) and (2.19). Similarly, for $0 < \theta \le 1$, let

$$E_{p,q}^{\theta-}(x) := \begin{cases} E_{p,q}\left(\frac{\theta-x}{\theta}\right), & 0 \le x \le \theta \\ 0, & \theta \le x \le 1 \end{cases} \tag{4.2}$$

which is supported on $[0, \theta]$.

The case $1 \le q \le p < \infty$

Since f is convex when $p > q$, and linear when $p = q$, it attains its maximum at an endpoint given by

$$A^* = \begin{cases} 0, & 0 \le \theta < 1/2 \\ 0, 1, & \theta = 1/2,\ p > q \\ [0,1], & \theta = 1/2,\ p = q \\ 1, & 1/2 < \theta \le 1. \end{cases}$$

Thus, since

$$\max\{\theta, 1-\theta\} = 1/2 + |1/2 - \theta|,$$

we obtain

$$M(p, q, \theta) = (1/2 + |1/2 - \theta|)^{1+\frac{1}{p}-\frac{1}{q}}, \qquad 1 \le q \le p < \infty. \tag{4.3}$$

For $0 \le \theta < 1/2$, $q \ne 1$, the corresponding extremal function is $E_{p,q}^{\theta+}$, and this is the unique extremal upto a multiplication by a constant. Similarly, for $1/2 < \theta \le 1$, the extremal function is $E^{\theta-}_{p,q}$. For $\theta = 1/2$, $1 < q < p$, there are two extremal functions $E^{1/2+}_{p,q}$ and $E^{1/2-}_{p,q}$ (corresponding to $A^* = 0, 1$ respectively). For $\theta = 1/2$, $p = q > 1$, any (nontrivial) linear combination of $E^{1/2+}_{p,q}$ and $E^{1/2-}_{p,q}$ is an extremal.

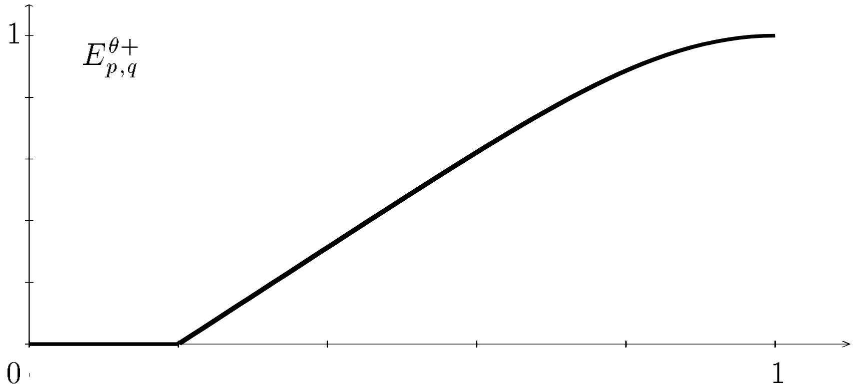

Fig. 4.1. Behaviour of the extremal function $E^{\theta+}_{p,q}$ when $0 < \theta < 1/2$ and $p > q > 1$. The example depicted is $\theta = 1/5$, $p = 5$, $q = 2$.

The case $0 < p < q < \infty$, $1 \le q < \infty$

If $p < q$, then f is concave, and so we need to compute any local maxima of f. Since the first derivative of f is

$$Df(A) = \frac{p}{q} A^{\frac{p}{q}-1} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} - \frac{p}{q}(1-A)^{\frac{p}{q}-1}(1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})},$$

f has a stationary point when

$$A^{-\alpha}\theta^{\beta} = (1-A)^{-\alpha}(1-\theta)^{\beta},$$

where

$$\alpha := 1 - p/q > 0, \qquad \beta := p(1 + 1/p - 1/q) > 0.$$

This has one solution

$$A^* = \frac{(1-\theta)^{-\beta/\alpha}}{\theta^{-\beta/\alpha} + (1-\theta)^{-\beta/\alpha}},$$

which is inside $(0,1)$ (since $0 < \theta < 1$). Thus, f has a maximum at A^* given by

$$f(A^*) = \frac{(1-\theta)^{-\beta(1-\alpha)/\alpha}\theta^{\beta} + \theta^{-\beta(1-\alpha)/\alpha}(1-\theta)^{\beta}}{(\theta^{-\beta/\alpha} + (1-\theta)^{-\beta/\alpha})^{1-\alpha}} = \left(\theta^{\beta/\alpha} + (1-\theta)^{\beta/\alpha}\right)^{\alpha},$$

and we obtain

$$M(p,q,\theta) = \left(\theta^{1+1/(\frac{1}{p}-\frac{1}{q})} + (1-\theta)^{1+1/(\frac{1}{p}-\frac{1}{q})}\right)^{\frac{1}{p}-\frac{1}{q}}, \qquad p < q. \tag{4.4}$$

Let $f = g + h$ given by

$$g := G\,E_{p,q}^{\theta-},\ G > 0, \qquad h := H\,E_{p,q}^{\theta+},\ H > 0,$$

be the extremal function which attains the supremum in (3.3). Then

$$\int_0^\theta |Dg|^q = G^q\theta^{1-q}\|E_{p,q}\|_q^q = A^*, \qquad \int_\theta^1 |Dh|^q = H^q(1-\theta)^{1-q}\|E_{p,q}\|_q^q = 1 - A^*,$$

so that

$$\frac{G}{H} = \left(\frac{A^*}{1-A^*}\frac{(1-\theta)^{1-q}}{\theta^{1-q}}\right)^{1/q} = \left(\frac{\theta}{1-\theta}\right)^{1+\frac{p}{q-p}}.$$

Thus, the extremal functions are (scalar multiples of)

$$\theta^{1+\frac{p}{q-p}}E_{p,q}^{\theta-} \pm (1-\theta)^{1+\frac{p}{q-p}}E_{p,q}^{\theta+}. \tag{4.5}$$

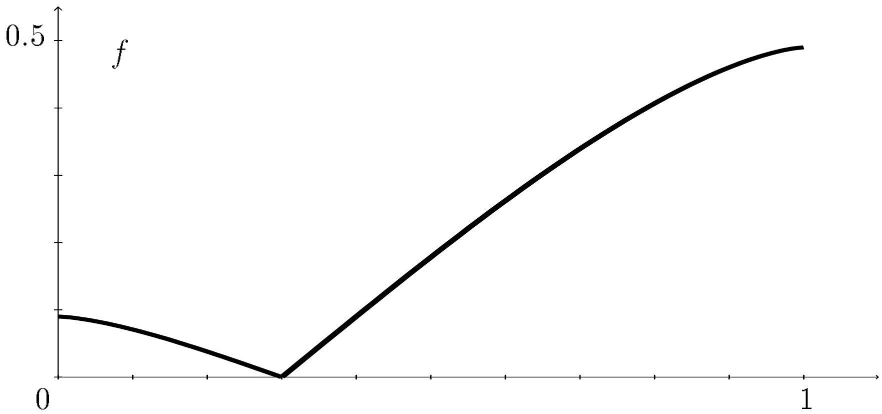

Fig. 4.2. Behaviour of the extremal function $f := \theta^{1+\frac{p}{q-p}}E_{p,q}^{\theta-} + (1-\theta)^{1+\frac{p}{q-p}}E_{p,q}^{\theta+}$ when $0 < \theta < 1/2$ and $p < q$. The example depicted is $\theta = 3/10$, $p = 2$, $q = 3$.

The cases $p = \infty$, $q = \infty$

From (4.3), (4.4) we see that for $0 < p < \infty$, $1 \le q < \infty$ the maximum $M(p, q, \theta)$ depends only on $1/p - 1/q$ and θ, i.e.,

$$M(p, q, \theta) = W(\frac{1}{p} - \frac{1}{q}, \theta),$$

where

$$W(t, \theta) := \begin{cases} (1/2 + |1/2 - \theta|)^{1+t}, & -1 \le t \le 0 \\ (\theta^{1+\frac{1}{t}} + (1-\theta)^{1+\frac{1}{t}})^t, & 0 < t < \infty \end{cases} \tag{4.6}$$

It is easily seen from (2.17) that the extremal functions do not depend only on $1/p - 1/q$ and θ. Similary, $C(p, q)$ is not a function of $1/p - 1/q$.

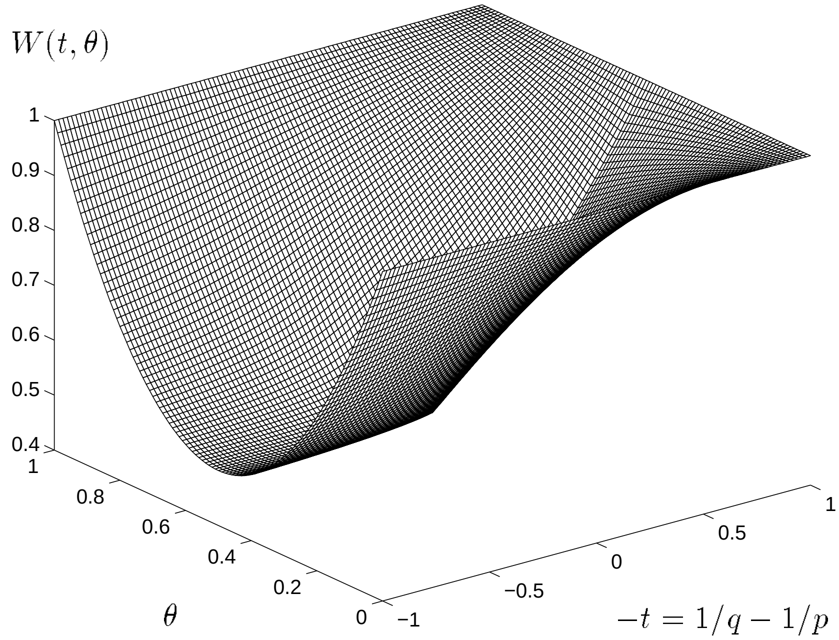

Fig. 4.3. Graph of $(t,\theta) \mapsto W(t,\theta)$ for $-1 \le t \le 1$. This range of t corresponds to the values $1 \le p, q \le \infty$.

Notice that the values $t = -1, 0, 1$ correspond to the following values of p, q

$$
\begin{aligned}
&p = \infty,\ q = 1 \qquad (t = -1)\\
&p = q \qquad (t = 0)\\
&p = 1,\ q = \infty \qquad (t = 1)
\end{aligned}
$$

Below are the corresponding functions $\theta \mapsto W(t,\theta)$.

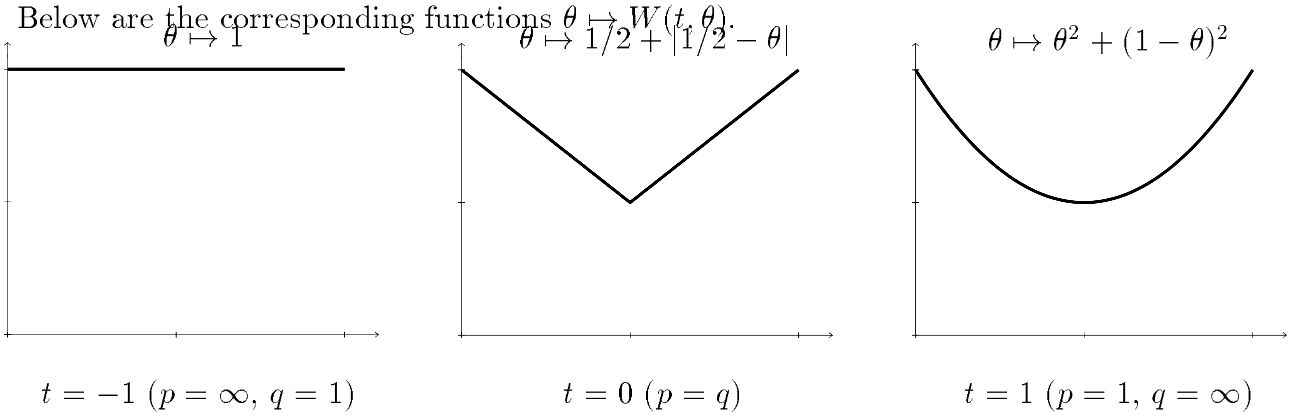

Fig. 4.4. The graph of $\theta \mapsto W(t,\theta)$ when $t = -1, 0, 1$.

Since the solution of (2.1) is a bounded and continuous function of $0 < p < \infty$, $1 \le q < \infty$, it would be expected the remaining values ($p = \infty$ and $q = \infty$) could be obtained by taking the appropriate limits. The argument is as follows.

The case $p = \infty$, $1 \leq q < \infty$. For $f \in W_q^1$ with $f(\theta) = 0$, there is the sharp inequality

$$\|f\|_{p^*} \leq W(\frac{1}{p^*} - \frac{1}{q}, \theta) C(p^*, q) \, \|Df\|_q, \tag{4.7}$$

where $0 < p^* < \infty$ is fixed. Since $f \in W_q^1 \subset L_{p^*}$, for all $0 < p^* \leq \infty$, the limit of (4.7) as $p^* \to \infty$ can be taken to obtain

$$\|f\|_\infty \leq W(\frac{1}{\infty} - \frac{1}{q}, \theta) C(\infty, q) \, \|Df\|_q,$$

which is sharp, since for each $p^* < \infty$ an f with $\|Df\|_q = 1$ which gives as close to equality in (4.7) as desired can be chosen.

The case $1 \leq p \leq \infty$, $q = \infty$. For $f \in W_\infty^1$ with $f(\theta) = 0$, there is the sharp inequality

$$\|f\|_p \leq W(\frac{1}{p} - \frac{1}{q^*}, \theta) C(p, q^*) \, \|Df\|_{q^*}, \tag{4.8}$$

where $1 \leq q^* < \infty$. As before, the limit as $q^* \to \infty$ can be taken to obtain the sharp inequality

$$\|f\|_p \leq W(\frac{1}{p} - \frac{1}{\infty}, \theta) C(p, \infty) \, \|Df\|_\infty.$$

Summary of the results

These cases just considered combine to give the main result.

Theorem 4.9. *Let $0 < p \leq \infty$, $1 \leq q \leq \infty$ and $a \leq \theta \leq b$. Then, for all $f \in W_q^1[a, b]$ satisfying*

$$f(\theta) = 0,$$

there is the sharp inequality

$$\|f\|_p \leq W(\frac{1}{p} - \frac{1}{q}, \frac{\theta - a}{b - a}) C(p, q) \, (b - a)^{1 + \frac{1}{p} - \frac{1}{q}} \|Df\|_q, \tag{4.10}$$

where $1/2 \leq W \leq 1$ is defined by (4.6), and $0 < C \leq 1$ is defined by (2.13).

The inequality (4.10) will be referred to as **Schmidt's inequality** (see below). Inequalities of this type are used in the spectral analysis of certain ordinary differential equations (see, e.g., Brown, Hinton and Schwabik [BHS95]).

The case when θ is the midpoint of the interval $[a, b]$

The only case where the best constant in (4.10) has been investigated for $\theta \neq a, b$ is in Tikhomirov [T76:§2.5.2], where

$$\theta = \frac{a + b}{2},$$

the midpoint of the interval, and $1 < p < \infty$, $1 \le q < \infty$. Tikhomirov used this best constant in his calculation of the n-width $s_n(B_q^1, L_p)$ given in Theorem 5.21.

For simplicity, suppose that $[a, b] = [0, 1]$ and $\theta = 1/2$. For this choice of θ, we have from (4.6), that

$$W(\frac{1}{p} - \frac{1}{q}, \frac{1}{2}) = \begin{cases} \frac{1}{2}, & p < q \\ (\frac{1}{2})^{1+\frac{1}{p}-\frac{1}{q}}, & p \ge q, \end{cases} \tag{4.11}$$

for $0 < p \le \infty$, $1 \le q \le \infty$. For $1 < p \le q < \infty$, this agrees with the result of Tikhomirov [T76:p127]. But, for $1 \le q < p < \infty$, Tikhomirov claims the best constant (4.10) is

$$\frac{1}{2} C(p, q), \tag{4.12}$$

rather than the larger constant

$$\left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} C(p, q),$$

given by (4.11). Tikhomirov's constant (4.12) would be correct, if it could be assumed that the extremal function was symmetric about $\theta = (a + b)/2$ (as it is in the case $p \le q$). However, as we have seen, for $1 < q < p \le \infty$ the extremal function is only supported on half of the interval $[0, 1]$. Indeed, we compute that the best constant in (4.10) must be at least as large as

$$\frac{\|E_{p,q}^{1/2+}\|_p}{\|DE_{p,q}^{1/2+}\|_q} = \frac{(1/2)^{1/p}\|E_{p,q}\|_p}{(1/2)^{1/q-1}\|DE_{p,q}\|_q} = \left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} C(p, q), \qquad 1 < q < p \le \infty,$$

where

$$\frac{1}{2} < \left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} < 1,$$

and this is the best constant.

The cases when the extremal functions are splines

With a mind to identifying phenomenon that might also hold for other Wirtinger inequalities of the form (1.1), we now consider those cases where the extremals for (2.1) are (polynomial) splines. This occurs only when

$$\begin{aligned} &p = 1, \ q = 2 \qquad \text{(quadratic splines)} \\ &p = \infty, \quad \text{or} \quad q = \infty \qquad \text{(linear splines)}. \end{aligned}$$

For $0 < p < \infty$, $1 < q < \infty$, the extremal is a spline if and only if $E_{p,q}$ the solution of (2.17) is a polynomial. Originally, I had hoped to show this was the case only when $p = 1$, $q = 2$, by showing that for other values of p, q a polynomial could not satisfy the differential equation assymptotically. But, since there is no reason (for general p, q) why a polynomial solution should satisfy (2.17) outside the interval $[0, 1]$, this argument fails. However, Evsei Dyn'kin pointed out that, by considering *analytic continuations* of a (polynomial) solution that satisfies (2.17), it is possible to show the solution is a polynomial only in the case $p = 1$, $q = 2$. Here is that argument.

Let $0 < p < \infty$, $1 < q < \infty$. $E_{p,q}$ is the (unique) solution of the differential equation (2.17), which can be rewritten as

$$Df = F(f), \qquad f(0) = 0,$$

where

$$F : [0, 1] \to \mathbf{C} : y \mapsto I_{p,q}(1 - y^p)^{1/q}.$$

Suppose that

$$E := E_{p,q} = f|_{[0,1]},$$

where f is a polynomial of degree n.

Since $Df(0) \neq 0$, it is possible to choose a closed path $\gamma_1 : [0, 1] \to \mathbf{C}$ winding once around 0 which is sufficiently small that $\gamma_1^* := f \circ \gamma_1$ (its image under f) winds once around 0 and doesn't pass through 1. Since γ_1^* doesn't pass through 0 or 1 (the branch points of the function F), there is an analytic continuation $\tilde{F}$ of F to $\gamma_1^*([0, 1])$. As a point w_1 moves once around the curve γ_1^*, the value $\tilde{F}(w_1)$ remains unchanged, i.e.,

$$(1 - w_1^p)^{1/q} = (1 - e^{2\pi i p} w_1^p)^{1/q},$$

where the powers of p and $1/q$ denote the unique branches giving the continuation $\tilde{F}$. In particular, this implies $e^{2\pi i p} = 1$, i.e., p is an integer.

Next, a similar argument is performed at 1 (the other branch point of F). Let g be the polynomial of degree n defined by

$$g := f - 1.$$

Since p is an integer, by the binomial expansion

$$1 - f^p = 1 - (1 + g)^p = -pg - \sum_{i=2}^{p} \binom{p}{i} g^i,$$

and so g satisfies

$$Dg = G(g) \qquad \text{on} \qquad [0, 1],$$

where

$$G : [0, 1] \to \mathbf{C} : y \mapsto I_{p,q}\, y^{1/q}(-p - \sum_{i=2}^{p} y^{i-1})^{1/q}.$$

Let z_0 be a zero of g of multiplicity $1 \leq m \leq n$ (one such zero is 1). If $\gamma_2 : [0, 1] \to \mathbf{C}$ is a closed path making a sufficiently small loop around 0 and not passing through any of (the finite number of) points where $1 + (1 + g)^p = 0$, then $\gamma_2^* := g \circ \gamma_2$ (its image under g)

contains a point in $[0,1]$ and passes through no branch points of G, and hence G has an analytic continuation $\tilde{G}$ to γ_2^*. Thus, $\tilde{G} \circ g$ provides an analytic continuation of $Dg|_{[0,1]}$ to γ_2, and so, by the uniqueness of analytic continuations,

$$Dg = \tilde{G} \circ g \qquad \text{on} \qquad \gamma_2([0,1]). \tag{4.13}$$

In particular, $Dg(z_2) = 0$, so that $m > 1$. From the assymptotic expansion of each side of (4.13) as $z \to z_0$, it follows that

$$m - 1 = \frac{m}{q}, \tag{4.14}$$

so m depends only on q, and hence

$$m = \frac{n}{k}, \qquad \text{for some integer } k.$$

Similarly, taking the assyptotic expansion (about z_0) as $|z| \to \infty$ gives

$$n - 1 = \frac{np}{q}. \tag{4.15}$$

Combining (4.14) and (4.15) gives

$$p = \frac{n-1}{n} q = \frac{mk-1}{mk} \frac{m}{m-1} = 1 + \frac{k-1}{k(m-1)},$$

which is an integer only if

$$\frac{k-1}{k(m-1)} = 0,$$

i.e., if $p = 1$, $q = 2$. In this case

$$E_{1,2}(x) = -x(x-2),$$

a quadratic polynomial with zeros at $x = 0, 2$.

Thus, for $0 < p < \infty$, $1 < q < \infty$, the extremal function (4.5) is a spline only when $p = 1$, $q = 2$, giving the extremal

$$f(x) := \begin{cases} \theta^2 - x^2, & 0 \le x \le \theta \\ \pm((1-\theta)^2 - (1-x)^2), & \theta \le x \le 1. \end{cases} \tag{4.16}$$

These are *perfect quadratic splines* with a double knot at θ satisfying

$$Df(0) = Df(1) = 0,$$

and giving the best constant

$$\frac{1}{\sqrt{3}} \sqrt{\theta^3 + (1-\theta)^3}.$$

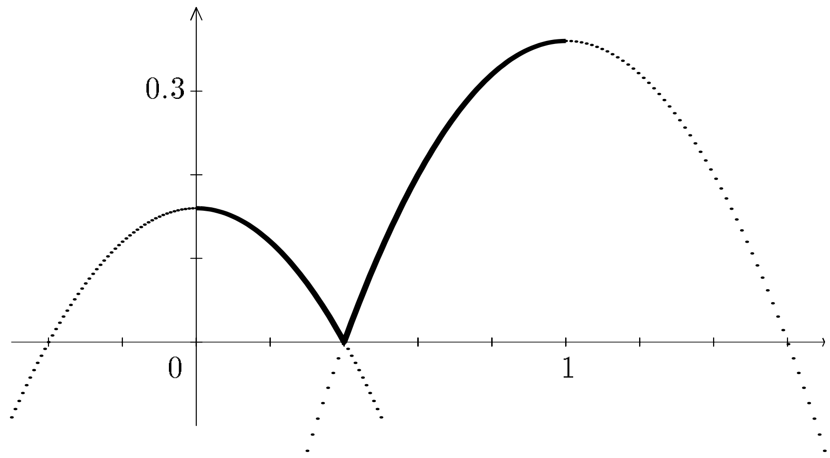

Fig. 4.5. Graph of an extremal for $p=1$, $q=2$ showing the quadratic pieces.

For $p=\infty$, motivated by the limiting case of the extremal for when $1\le q\le p<\infty$, we observe that for $0\le\theta\le 1/2$ the linear spline

$$E^{\theta+}_{\infty,q}(x)=\begin{cases}0, & 0\le x\le\theta\\ \frac{x-\theta}{1-\theta}, & \theta\le x\le 1\end{cases}$$

is an extremal giving the best constant

$$(1-\theta)^{1-1/q}$$

in (4.10). Similarly, for $1/2\le\theta\le 1$ the linear spline

$$E^{\theta-}_{\infty,q}(x)=\begin{cases}\frac{\theta-x}{\theta}, & 0\le x\le\theta\\ 0, & \theta\le x\le 1\end{cases}$$

is an extremal. It does not appear that the fact this extremal is a spline is a special case of some more general result. For example, in Waldron [W96*] it is shown that for $1<q<\infty$ the extremal giving the best constant in the Wirtinger inequality

$$\|f-H_\Theta f\|_\infty\le C\,\|D^n f\|_q,\qquad \forall f\in W^n_q,$$

where $H_\Theta f$ is the Hermite interpolant to f at a multiset Θ of n points in $[a,b]$, is a spline if and only if $\#\Theta=1$ (the case just considered) or

$$q=2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \ldots$$

(equivalently when q' is an integer).

For $q=\infty$, we observe that the linear polynomial

$$f(x):=x-\theta$$

is an extremal (as is $|f|$) giving the best constant

$$\|f\|_p = \left(\frac{1}{p+1}\right)^{1/p} (\theta^{1+p} + (1-\theta)^{1+p})^{1/p}.$$

This is a special case of a more general result in Waldron [W96:Th.4.1].

For $q = 1$, $p \neq \infty$ there is no extremal function for (4.10). If there were, then by the computation of the best constant, the part supported on the largest of the intervals $[0, \theta]$ and $[\theta, 1]$ would be (after an affine change of variables) an extremal for the case $\theta = 0$, for which we earlier showed no extremal exists.

5. Schmidt's inequality

There are several inequalities of the form

$$\|f\|_p \leq C\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q$$

where f belongs to some class of functions, with best constant and extremals related to $C(p, q)$ and $E_{p,q}$ respectively, which are closely related to the following result of Schmidt.

Schmidt's inequality([Sc40:(4),p302]) 5.1. *Let $0 < p \leq \infty$, $1 \leq q \leq \infty$. Then, for all $f \in W_q^1[a, b]$ satisfying*

$$f(a) = f(b), \qquad \max_{t \in [a,b]} f(t) + \min_{t \in [a,b]} f(t) = 0, \tag{5.2}$$

there is the sharp inequality

$$\|f\|_p \leq \frac{1}{4} C(p, q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.3}$$

where $C(p, q)$ is defined by (2.13).

Under the same hypotheses as in 5.1, Schmidt also proves the sharp inequality

$$\frac{1}{b-a} \int_a^b \log|f| \leq \log\left(\frac{1}{4G(1/q')}(b-a)^{1-\frac{1}{q}} \|Df\|_q\right),$$

where G is defined by (2.11).

Schmidt's proof of 5.1 does not use the calculus of variations, but instead uses Hölder's inequality in a very clever way. We now outline this nice argument.

Schmidt's Hölder inequality argument

We may assume without loss of generality that $[a, b] = [0, 1]$, and the (periodic) function f has been normalised to obtain ξ with

$$\max \xi = 1, \qquad \min \xi = -1 \qquad \text{(multiplying by a constant)}$$

and

$$\xi(0) = 0, \quad \xi(\sigma) = 1, \quad \xi(\sigma_0) = 0, \quad \xi(\sigma') = -1, \quad \xi(1) = 0, \qquad \text{(by shifting)}$$

where $0 < \sigma < \sigma_0 < \sigma' < 1$.

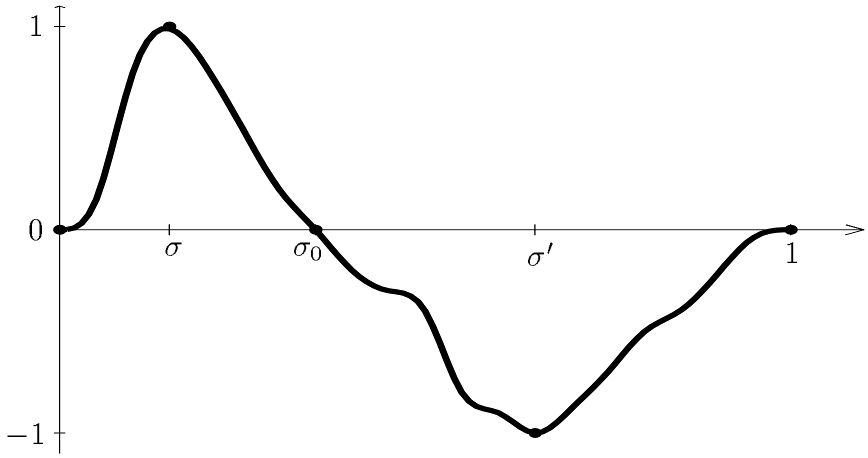

Fig 5.1. Graph showing the behaviour of ξ at the points $0 < \sigma < \sigma_0 < \sigma' < 1$.

If

$$\eta : [-1, 1] \to [0, \infty)$$

is a nonnegative even function, then the area under it over the intervals $[-1, 0]$ and $[0, 1]$ are equal. This area F (for Fläche) can be expressed by using lengths of the curve $t \mapsto \xi(t)$ as parameterisations for $[-1, 0]$ and $[0, 1]$ giving

$$F := \int_0^1 \eta = \int_0^{\sigma} (\eta \circ \xi) D\xi = -\int_{\sigma}^{\sigma_0} (\eta \circ \xi) D\xi = -\int_{\sigma_0}^{\sigma'} (\eta \circ \xi) D\xi = \int_{\sigma'}^{1} (\eta \circ \xi) D\xi,$$

which leads to

$$4F \leq \int_0^1 (\eta \circ \xi) |D\xi|. \tag{5.4}$$

Since $q \geq 1$, Hölder's inequality can be applied to (5.4), giving

$$1 \leq \frac{1}{4F} \|\eta \circ \xi\|_{q'} \|D\xi\|_q. \tag{5.5}$$

Let

$$A := \int_0^1 |\xi|^p = \|\xi\|_p^p < 1.$$

Then, for $q > 1$ and $p < \infty$, choosing $\eta : x \mapsto \eta(x)$ to be the curve given by

$$|x|^p + \eta(x)^{q'} = 1, \tag{5.6}$$

gives

$$\|\eta \circ \xi\|_{q'} = (1-A)^{1/q'}.$$

With this choice of η, (5.5) can be rewritten as

$$\|\xi\|_p = A^{1/p} \le \frac{1}{4F}(1-A)^{1/q'} A^{1/p} \, \|D\xi\|_q,$$

giving

$$\|\xi\|_p \le \frac{M_{p,q}}{4F_{p,q}} \, \|D\xi\|_q, \tag{5.7}$$

where $F_{p,q}$ is the area under one arc of the curve given by (5.6), and

$$M_{p,q} := \max_{0\le u\le 1} (1-u)^{1/q'} u^{1/p}. \tag{5.8}$$

This inequality (5.7) is precisely (5.3).

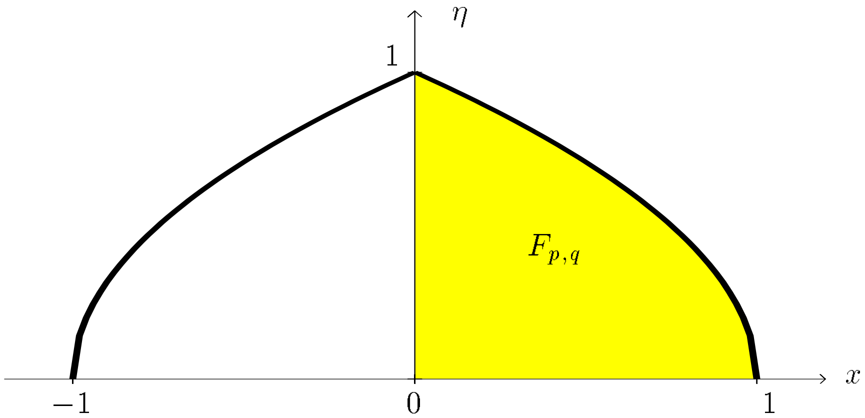

Fig 5.2. Graph of the curve $|x|^p + \eta^{q'} = 1$ showing the area $F_{p,q}$. The example depicted is $p = 1$, $q = 2$.

For the record

$$F_{p,q} = \int_0^1 \eta = \int_0^1 (1-\xi^p)^{1/q'} \, d\xi = \frac{\Gamma(1+\frac{1}{p})\Gamma(1+\frac{1}{q'})}{\Gamma(1+\frac{1}{p}+\frac{1}{q'})},$$

and the maximum in (5.8) occurs for

$$u = \frac{\frac{1}{p}}{\frac{1}{p}+\frac{1}{q'}}$$

giving (after simplification)

$$M_{p,q} = \frac{(\frac{1}{p}+\frac{1}{q'})^{-\frac{1}{p}-\frac{1}{q'}}}{(\frac{1}{p})^{-\frac{1}{p}}(\frac{1}{q'})^{-\frac{1}{q'}}}.$$

The cases $q = 1, \infty$ and $p = \infty$ are obtained by simple 'continuity' arguments.

Extremals exist unless $q = 1$, $p \neq \infty$

To obtain the sharpness of (5.7), Schmidt determines the extremal functions for (5.3) (when $q > 1$) by considering the conditions for equality in (5.4), namely

$$D\xi \geq 0 \ \text{ on } (0, \sigma) \cup (\sigma', 1), \qquad D\xi \leq 0 \ \text{ on } (\sigma, \sigma_0) \cup (\sigma_0, \sigma'),$$

and in Hölder's inequality (5.5), that

$$|\eta \circ \xi|^{q'} \|D\xi\|_q^q = |D\xi|^q \|\eta \circ \xi\|_{q'}^{q'} \quad \text{a.e.}$$

This leads to the conclusion that

$$\sigma = \frac{1}{4}, \qquad \sigma_0 = \frac{1}{2}, \qquad \sigma' = \frac{3}{4},$$

and the corresponding extremal function $S := S_{p,q}$ satisfies the differential equation

$$|DS| = 4I_{p,q}(1 - |S|^p)^{1/q}, \tag{5.9}$$

where $I_{p,q}$ is given by (2.18). (The notation $\chi = 1/(4I_{p,q})$ is used in Schmidt's paper). From (2.17) we observe that

$$S = E_{p,q}(4\,\cdot) \ \text{ on } [0, 1/4].$$

Similar considerations (taking account of the sign of S, DS) for the intervals $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ show that

$$S = E^*_{p,q}(4\,\cdot) \ \text{ on } [0, 1],$$

where

$$E^*_{p,q} : \mathbb{R} \to [-1, 1]$$

is the extension of $E_{p,q}$ to a 4-periodic function determined by the conditions that

$$\begin{aligned} &E^*_{p,q}(x) = -E^*_{p,q}(-x) && \text{(it is odd)}, \\ &E^*_{p,q}(1 + x) = E^*_{p,q}(1 - x) && \text{(it is even about 1)}. \end{aligned} \tag{5.10}$$

For $p = \infty$ and $q = \infty$ also define $E^*_{p,q}$ using (2.19) and (5.10). In the case $p = 1$, $q = 2$, when $E_{p,q}$ is a quadratic, the extension is (a shift of) the *quadratic Euler spline* $\mathcal{E}_2$ defined by

$$\mathcal{E}_2 := E^*_{1,2}(\cdot + 1),$$

which is an extremal for certain *Landau–Kolmogorov inequalities.*

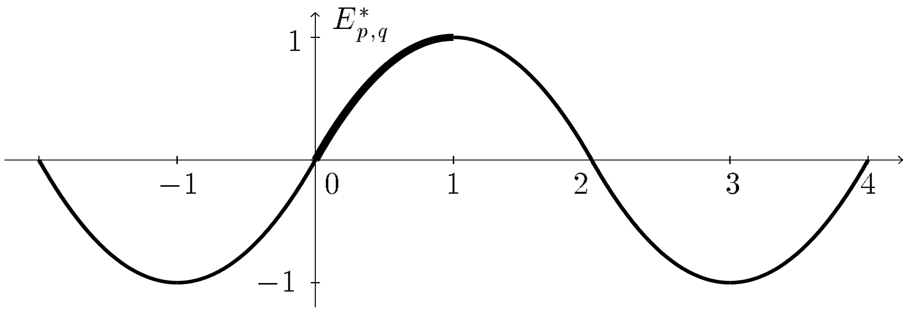

Fig 5.3. $E^*_{p,q}$ the 'symmetrisation' of $E_{p,q}$ (thick line). This example is $p = 1$, $q = 2$ (a shift of the Euler spline $\mathcal{E}_2$).

Thus, (by considering the limit cases $p \to \infty$ and $q \to \infty$), it follows that for $0 < p \leq \infty$, $1 < q \leq \infty$ or $p = \infty$, $q = 1$ equality holds in (5.3) if and only if f is a scalar multiple of

$$E^*_{p,q}(4 \cdot -t)|_{[0,1]}, \qquad t \in \mathbb{R},$$

and there is strict inequality for $q = 1$, $p \neq \infty$.

An immediate consequence of Schmidt's inequality 5.1 is the following.

Corollary([MLV96:Prop.6.8,p431]) 5.11. *Let $1 \leq p \leq q \leq \infty$. Then for all $f \in W^1_q[-\pi, \pi]$ satisfying*

$$f(-\pi) = f(\pi) = 0 \qquad (f \text{ is periodic})$$

and

$$f(t + \pi) = -f(t) \tag{5.12}$$

there is the sharp inequality

$$\|f\|_p \leq \frac{1}{4} C(p, q)\, (2\pi)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.13}$$

where $C(p, q)$ is defined by (2.13).

Proof: The condition (5.12) implies that

$$\max f + \min f = 0,$$

and so, by Schmidt's inequality 5.1, the inequality (5.13) holds. The sharpness follows since the functions

$$E^*_{p,q}(\frac{4}{2\pi} \cdot)|_{[-pi,\pi]}$$

giving equality (or near equality as $q \to 1$) in (5.3) satisfy the condition (5.12).

Other Schmidt inequalities

Schmidt indicates that the argument just outlined can be modified to obtain other inequalities. The first of these is the following.

Theorem([Sc40:(13),p304]) 5.14. *Let $0 < p \le \infty$, $1 \le q \le \infty$. Then, for all $f \in W_q^1[a,b]$ satisfying*

$$f(a) = f(b) = 0, \tag{5.15}$$

there is the sharp inequality

$$\|f\|_p \le \frac{1}{2} C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.16}$$

where $C(p,q)$ is defined by (2.13).

This result is stated for $1 \le p, q \le \infty$ with the value of $C(p,q)$ given incorrectly (due to a slight error in its proof) in Fink [F74:p408] (see the constant $D(1,1,p,q)$), and is stated correctly for $1 \le p < \infty$, $1 < q < \infty$ (together with the extremal functions) but without proof in Talenti [Ta76:p357] (there it is mentioned as a 1-dimensional analogue of a Sobolev inequality for functions in $W_q^1(\mathbb{R}^m)$). Neither author makes reference to Schmidt [Sc40].

In Schmidt's statement of Theorem 5.14, the condition (5.15) is given as

$$f(a) = f(b), \qquad f \text{ has (at least) one zero on } [a,b].$$

It's proof differs from that of 5.1 in that f is normalised to obtain ξ with

$$\max \xi = 1$$

and

$$\xi(0) = 0, \quad \xi(\sigma) = 1, \quad \xi(1) = 0,$$

where $0 < \sigma < 1$. This leads to

$$F = \int_0^\sigma (\eta \circ \xi) D\xi = -\int_\sigma^1 (\eta \circ \xi) D\xi,$$

giving (instead of (5.4))

$$2F \le \int_0^1 (\eta \circ \xi)|D\xi|, \tag{5.17}$$

then as before (with the condition for equality in (5.17) giving $\sigma = 1/2$, and the constant 4 in (5.9) being replaced by 2).

Equality holds in (5.16) if and only if $0 < p \le \infty$, $1 < q \le \infty$ or $p = \infty$, $q = 1$ and f is a scalar multiple of

$$E^*_{p,q}(2\frac{\cdot - a}{b-a})|_{[a,b]}.$$

It is not difficult to see that a variation of the 'elementary argument' where f is split by (3.1) into functions g and h with

$$g(a) = g(\theta) = h(\theta) = h(b) = 0$$

can be used, together with Theorem 5.14, to compute the best constant in the inequality

$$\|f\|_p \leq \frac{1}{2} C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.18}$$

where $f \in W_q^1$ satisfies

$$f(a) = f(\theta) = f(b).$$

More generally, by using similar variations of the 'elementary argument' and induction, it is (in principle) possible to compute the best constant in (5.18) where $f \in W_q^1$ satisfies

$$f(\theta_1) = f(\theta_2) = \cdots = f(\theta_n) = 0,$$

for some choice

$$a \leq \theta_1 < \theta_2 < \cdots < \theta_n \leq b,$$

together with the extremal functions, which exist except when $q = 1$, $p \neq \infty$, and are constructed from $E_{p,q}$ – in a similar manner to those for (5.16). A typical example of such an inequality is the upper bound of the n-width $s_n(B_q^1, L_p)$ given in Theorem 5.21.

Remarks on Lemma 2.14

A third result mentioned by Schmidt [Sc40:(20),p306] is the inequality that: for $f \in W_q^1$ with $f(a) = 0$,

$$\|f\|_p \leq C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.19}$$

which was stated earlier as Lemma 2.14.

The special case of this result when

$$p = q = 2k \quad \text{an even integer}$$

was given by Hardy and Littlewood [HL32:Th.5] (see also [HLP34:256,p182]) using what they describe as a proof of 'type C' (depends essentially on the calculus of variations). They refer to the extremal functions $E_{2k,2k}$ (for $[a,b] = [0,1]$) as *hyperelliptic curves* and compute

$$C(2k,2k) = \left(\frac{1}{2k-1}\right)^{1/2k} \left(\frac{2k}{\pi} \sin \frac{\pi}{2k}\right).$$

This example motivated Schmidt to give his proof of 'type A' (strictly elementary). To quote Hardy and Littlewood [HL32] (and [HLP34]): (their) "proof is of 'type C' and (in view of the difficulty of calculating the slope-function) is might be difficult to construct a much more elementary proof" (either of 'type B' or simpler still of 'type A'). There is further discussion of the results of Schmidt in Levin and Stechkin's supplement [LS48] to the Russian edition of Hardy, Littlewood and Pólya's book on inequalities [HLP48].

For $1 \leq p,q \leq \infty$ Lemma 2.14 is given in [F74:p407]. There $C(p,q)$ is denoted by $C(1,1,p,q)$ and it is given incorrectly (due to a slight error in the proof). Fink was unaware of the earlier result of Schmidt.

The statement of Lemma 2.14 given by Schmidt is infact slightly stronger. It asserts the sharp inequality in (5.19) where the condition (2.15) is replaced by

$$f \text{ has (at least) one zero on } [a, b]$$

(and the discussion of the extremals shows the sharpness occurs only when the zero is at a or b). A quantative form of this result is given by our Theorem 4.9.

The Hölder inequality argument, if given, would involve the normalisation of f to obtain ξ with

$$\xi(\sigma) = 0, \quad \xi(1) = 1,$$

where $0 \le \sigma < 1$ (if $\xi(0) = \xi(1) = 0$, then (5.16) holds). This leads to

$$F = \int_\sigma^1 (\eta \circ \xi) D\xi$$

giving

$$F \le \int_0^1 (\eta \circ \xi)|D\xi| \tag{5.20}$$

with the remainder of the argument as before (with $\sigma = 0$ necessary for equality in (5.20), which leads to (2.17) for the extremal). In particular, (as has already been discussed) equality holds in (5.19) if and only if $0 < p \le \infty$, $1 < q \le \infty$ or $p = \infty$, $q = 1$ and f is a scalar multiple of

$$E_{p,q}\left(\frac{\cdot - a}{b - a}\right).$$

n-widths

The constant $C(p, q)$ naturally occurs in the computation of the n-widths of the set

$$B_q^1 := \{f \in W_q^1 : \|Df\|_q \le 1\}$$

in L_p, since for $1 \le p \le q \le \infty$ an optimal linear operator of rank n is given by Lagrange interpolation by piecewise constants (cf (1.2)).

Recall (see, e.g., Pinkus [P85]), the following definitions. Let A be a subset of a normed linear space X. The **Kolmogorov n-width** of A in X is

$$d_n(A, X) := \inf_{X_n} \sup_{a \in A} \inf_{x \in X_n} \|a - x\|$$

where the infimum is taken over all n-dimensional subspaces X_n of X. The **linear n-width** of A in X is

$$\delta_n(A, X) := \inf_{P_n} \sup_{a \in A} \|a - P_n a\|$$

where the infimum is taken over all (continuous) linear operators $P_n : X \to X$ of rank n. The **Gel'fand n-width** of A in X is

$$d^n(A, X) := \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|$$

where the infimum is taken over all subspaces L^n of codimension n. In the case when these widths are all equal, we use the notation

$$s_n(A, X) := d_n(A, X) = \delta_n(A, X) = d^n(A, X)$$

to denote their common value. With this notation, the following is known.

Theorem ([T70]) 5.21. *Let $[a,b]=[0,1]$. Then, for $1 \le p \le q \le \infty$*

$$s_n(B_q^1, L_p) = \frac{1}{2}C(p,q)\frac{1}{n}, \qquad n = 1,2,3,\dots. \tag{5.22}$$

The space F_n consisting of step functions with break points

$$\frac{1}{n},\ \frac{2}{n},\dots,\frac{n-1}{n}$$

is an optimal n-dimensional subspace, and the operator L_n of Lagrange interpolation from F_n at the points

$$\frac{1}{2n},\ \frac{3}{2n},\ \frac{5}{2n}\dots\frac{2n-1}{2n} \quad \textit{(the midpoints of each step)}$$

is an optimal linear operator of rank n.

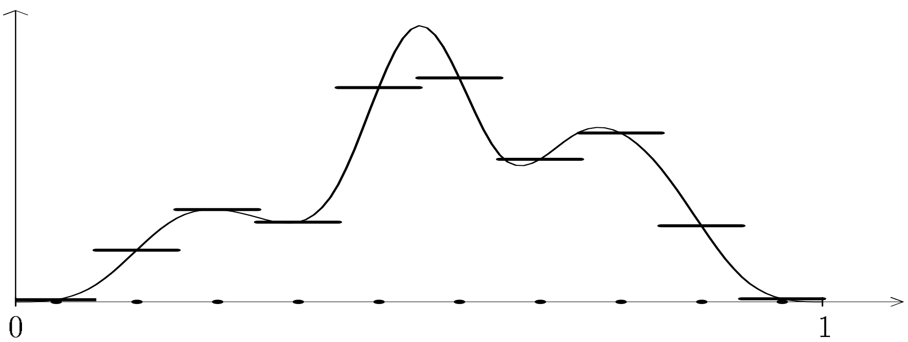

Fig 5.4. Example of the Lagrange interpolant $L_n f$ (thick) to f from the space of step functions F_n (n=10).

Since the linear n-width $\delta_n(B_q^1, L_p)$ is the largest of the three, we need only show the upper bound (5.22) for it. This is done as follows. Firstly, suppose that $1 \le p \le q < \infty$. On each of the intervals

$$[(i-1)/n, i/n], \qquad i = 1,\dots,n$$

the function $f - L_n f$ is zero at the midpoint (where $L_n f$ interpolates f) and has derivative Df (since $L_n f$ is piecewise constant). Thus, from Theorem 4.9, with θ the midpoint, one obtains

$$\int_{[(i-1)/n,i/n]} |f - L_n f|^p \le \left\{ \frac{1}{2}C(p,q)\left(\frac{1}{n}\right)^{1+\frac{1}{p}-\frac{1}{q}} \left(\int_{[(i-1)/n,i/n]} |Df|^q\right)^{\frac{1}{q}} \right\}^p.$$

Summing over i leads to

$$\|f - L_n f\|_p \le \frac{1}{2}C(p,q)\left(\frac{1}{n}\right)^{1+\frac{1}{p}-\frac{1}{q}} \|(\gamma_i)_{i=1}^n\|_{\ell_p},$$

where

$$\gamma_i := \left(\int_{[(i-1)/n, i/n]} |Df|^q \right)^{1/q}.$$

By Hölder's inequality

$$\|(\gamma_i)\|_{\ell_p} \leq n^{\frac{1}{p}-\frac{1}{q}} \|(\gamma_i)\|_{\ell_q} = n^{\frac{1}{p}-\frac{1}{q}} \|Df\|_q,$$

giving the upper bound

$$\|f - L_n f\|_p \leq \frac{1}{2} C(p,q) \frac{1}{n} \|Df\|_q, \tag{5.23}$$

which (by the usual continuity argument) also holds when q (and p) becomes infinite. The above argument also works if $[0,1]$ is split into $2n$ intervals of length $1/(2n)$ on which $f - L_n f$ is zero at one endpoint.

The sharp bound (5.23) can also be obtained using inequality (5.14) and a variation of the 'elementary argument', instead of the above argument (of Tikhomirov). This alternative approach also extends to the case $q < p$.

A simple proof of the lower bound based on *Borsuk's antipodality theorem* was given by Markovoz [M72:Th.2]. For a general form of Markavoz's lower bound together with its application to Theorem 5.21 see Lorentz, Golitscheik and Makovoz [MLV96:Th.5.1].

Further developements of this circle of ideas can be found in, e.g., Buslaev and Tikhomirov [BT92], Buslaev and Yashina [BY94] (numerical computations), and Tikhomirov [T94] (multivariate generalisations). $\blacksquare$

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6. Some details not included in the paper

1. The simplification of

$$f(A^*) = \frac{(1-\theta)^{-\beta(1-\alpha)/\alpha}\theta^\beta + \theta^{-\beta(1-\alpha)/\alpha}(1-\theta)^\beta}{(\theta^{-\beta/\alpha} + (1-\theta)^{-\beta/\alpha})^{1-\alpha}}$$

was done as follows.

Let

$$a := \theta^\beta, \qquad b := (1-\theta)^\beta.$$

Then

$$\begin{aligned} f(A^*) &= \frac{\frac{a}{b^{\frac{1-\alpha}{\alpha}}} + \frac{b}{a^{\frac{1-\alpha}{\alpha}}}}{\left(\frac{1}{a^{1/\alpha}} + \frac{1}{b^{1/\alpha}}\right)^{1-\alpha}} \\ &= \frac{\left(\frac{a^{1/\alpha}+b^{1/\alpha}}{a^{\frac{1-\alpha}{\alpha}} b^{\frac{1-\alpha}{\alpha}}}\right)}{\left(\frac{a^{1/\alpha}+b^{1/\alpha}}{a^{1/\alpha}b^{1/\alpha}}\right)^{(1-\alpha)}} \\ &= (a^{1/\alpha} + b^{1/\alpha})^\alpha. \end{aligned}$$

This implies

$$W(p,q,\theta) = (a^{1/\alpha} + b^{1/\alpha})^{\alpha/p} = (\theta^{\beta/\alpha} + (1-\theta)^{\beta/\alpha})^{\alpha/p}.$$

Since

$$\beta/\alpha = 1 + \frac{pq}{q-p}, \qquad \alpha/p = \frac{1}{p} - \frac{1}{q}$$

$$W(p,q,\theta) = \left(\theta^{1+\frac{pq}{q-p}} + (1-\theta)^{1+\frac{pq}{q-p}}\right)^{\frac{1}{p}-\frac{1}{q}}.$$

2. Solution of (2.17) for $p = q = 2$. By $\Gamma(1/2) = \sqrt{\pi}$, see Jones [J93], we calculate

$$I_{2,2} = \frac{\Gamma(1/2)\Gamma(1/2)}{2\Gamma(1)} = \frac{\pi}{2}.$$

So the differential equation (2.17) becomes

$$\frac{Df}{\sqrt{1-f^2}} = \frac{\pi}{2},$$

which integrates to

$$\sin(^{-1} f(x)) = \frac{\pi}{2}x,$$

giving

$$E_{2,2}(x) = \sin(\frac{\pi}{2}x).$$

3. An integral. By making the substitution $u = y^p$, one computes

$$\begin{aligned}\int_0^1 (1-y^p)^\alpha &= \frac{1}{p}\int_0^1 (1-u)^\alpha u^{1/p-1}\,du \\ &= \frac{1}{p}\frac{\Gamma(\alpha+1)\Gamma(1/p)}{\Gamma(\alpha+1+1/p)} = \frac{\Gamma(\alpha+1)\Gamma(1+1/p)}{\Gamma(\alpha+1+1/p)}, \qquad \alpha+1>0,\ 1+1/p>0.\end{aligned}$$

4. $E_{p,q}$ is the p-th power of the inverse of an incomplete Beta function. Let $I(p,q)$ be the **(normalised) incomplete Beta** function (see, e.g., [EMOT53])

$$I(p,q) : [0,1] \to [0,1] : x \mapsto B_x(p,q)/B_1(p,q),$$

where

$$B_x(p,q) := \int_0^x t^{p-1}(1-t)^{q-1}\,dt,$$

and $I^{-1}(p,q)$ be the inverse function. Then

$$E_{p,q} = (I^{-1}(1/p, 1/q'))^p.$$

5.

$$\begin{aligned}C(q,p,1) &= \left(\frac{\pi}{2}\right)(1-\frac{1}{q}+\frac{1}{p})^{1/q-1/p}\left(\frac{1}{2\pi}\right)^{1/q-1/p}(q')^{\frac{1}{q}}p^{-1/p}\frac{\Gamma(\frac{1}{p}+\frac{1}{q'})}{\Gamma(1+\frac{1}{p})\Gamma(\frac{1}{q'})} \\ &= \frac{1}{4}(2\pi)^{1+\frac{1}{p}-\frac{1}{q}}\frac{(\frac{1}{p}+\frac{1}{q'})^{-\frac{1}{p}-(1-\frac{1}{q})}}{(\frac{1}{p})^{-\frac{1}{p}}(\frac{1}{q'})^{\frac{-1}{q'}}}\frac{\Gamma(1+\frac{1}{p}+\frac{1}{q'})}{\Gamma(1+\frac{1}{p})\Gamma(1+\frac{1}{q'})} \\ &= \frac{1}{4}C(p,q)\,(2\pi)^{1+1/p-1/q}\end{aligned}$$

6.

$$\begin{aligned}C(2k,2k) &= \left(\frac{1}{2k}\right)^{\frac{1}{2k}}\left(\frac{2k-1}{2k}\right)^{1-\frac{1}{2k}}\frac{\Gamma(2)}{\Gamma(1+\frac{1}{2k})\Gamma(1+1-\frac{1}{2k})} \\ &= \left(\frac{1}{2k-1}\right)^{\frac{1}{2k}} 2k\frac{1/(2k)}{\Gamma(1+\frac{1}{2k})}\frac{1-1/(2k)}{\Gamma(1+1-\frac{1}{2k})} \\ &= \left(\frac{1}{2k-1}\right)^{\frac{1}{2k}}\frac{2k}{\Gamma(\frac{1}{2k})\Gamma(1-\frac{1}{2k})} \\ &= \left(\frac{1}{2k-1}\right)^{\frac{1}{2k}}\left(\frac{2k}{\pi}\sin\frac{\pi}{2k}\right)\end{aligned}$$

using the fact that

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$$

7. Check (4.5) satisfies the Euler-Lagrange equation.

8. Details for showing

$$\sigma = \frac{1}{4}, \qquad \sigma_0 = \frac{1}{2}, \qquad \sigma' = \frac{3}{4}.$$

Since

$$\frac{1}{|DS|} = \chi\, \eta(|S|)^{-\frac{1}{q-1}},$$

one has

$$\sigma = \int_0^\sigma dt = \int_0^1 \frac{dt}{dS}\, dS = \chi \int_0^1 \eta(|S|)^{-\frac{1}{q-1}}\, dS = \sigma_0 - \sigma.$$

Similarly,

$$\sigma' - \sigma_0 = 1 - \sigma' = \chi \int_{-1}^0 \eta(|S|)^{-\frac{1}{q-1}}\, dS.$$

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