The lattice of normal reflection subgroups of an irreducible reflection group

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Abstract

The reflection subgroups of a reflection group have a natural lattice structure given by the reflections that they contain. By considering the conjugation action orbits of the reflection subgroups for a given root line, we are able to give an essentially combinatorial way to calculate the lattice of all the normal reflection subgroups of a given (finite irreducible) reflection group, and natural generators for them. Moreover, we observe that every complex reflection group is a normal subgroup of the unique maximal reflection group which shares its collineation group. Hence, we are able to present the Shephard-Todd classification of the complex reflection groups as a collection of maximal reflection groups, together with appropriate (collineation preserving) normal reflection subgroups.

We investigate the quotients by the normal reflection subgroups, which are known to be reflection groups. We also consider the action of the collineation group on some appropriate small systems of lines, and how these results extend to quaternionic reflection groups. Some novel techniques are introduced, including the notion of a "hidden reflection", a combinatorial-geometric description of the reflection subgroups and the size of their conjugacy class, and the role played by the abelianisation of the reflection group.

Key Words: complex reflection groups, quaternionic reflection groups, collineation groups, normal reflection subgroups, roots of reflections, lattices of subgroups, parabolic subgroups, reflections, hidden reflections, maximal reflection groups. SICs (equiangular lines), Shephard-Todd classification, abelianisation of a group, dominos

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1 Introduction

A reflection group G is a group which is generated by reflections (see [Coh76], [Coh80], [BMR98] [LT09], [Bro10], [Pop23]). Therefore, its reflection subgroups (subgroups which are reflection groups) naturally form a lattice with the order relation given by inclusion of the reflections in the subgroups, and maximal element G and minimal element 1 (generated by the empty set of reflections). Since the intersection of two reflection subgroups may not be a reflection group, the meet in this lattice may or may not be the intersection of the subgroups, i.e., it may not be a sublattice of the lattice of all subgroups of G (though the inclusions and the join are the same).

We are interested in the (finite irreducible) complex and quaternionic reflection groups, which were classified by Shephard and Todd [ST54] and Cohen [Coh80]. These groups have all been implemented in computer algebra systems such as Magma, and so their reflection subgroups can be calculated [Tay12], [DPR13], [RS26]. Two classes of reflection subgroups have been studied in particular:

- The parabolic subgroups, i.e., those which fix pointwise a subset of the vector space that they act on. It was proved that these are reflection groups by Steinberg [Ste64] in the complex case, and by [BST23] in the quaternionic case. The nontrivial parabolic subgroups of G are not normal and are of lower rank (Proposition 5.1).
- The normal reflection subgroups. In [BBR02] (see [AW23]), it was shown that the quotient of a complex reflection group by a normal reflection subgroup gives a reflection group. If G is a primitive complex reflection group, then its nontrivial normal reflection subgroups are irreducible and hence of full rank (Theorem 2.1).

In this paper, we study the lattice of normal reflection subgroups of a finite complex or quaternionic irreducible reflection group. Key properties of a reflection group G that we consider are

- The collineation group, i.e., the action of G on lines.
- The reflection subgroups of G for a given root, i.e., the maximal rank one parabolic subgroups, up to conjugacy in G.

This information is a little more than knowing the order of the reflection group and its centre, and how many reflections of each order it contains – properties which are usually tabulated (explicitly or implicitly) in lists of reflection groups. From this information, we are led to our main insights:

- The lattice of normal reflection subgroups is completely determined by the orbits of the reflection subgroups for the roots. This provides a natural indexing for the subgroups, which can be calculated, together with generators, in an essentially combinatorial way (Lemma 2.1, Theorem 2.1).
- Every reflection group is a normal subgroup of a unique maximal reflection group which shares its collineation group. Consequently, the classification of reflection groups can be described as the maximal reflection groups, together with their collineation preserving normal reflection subgroups (Lemma 3.1, Theorem 3.1).

Much of the paper is devoted to calculating the maximal reflection groups and their collineation preserving normal reflection subgroups. This splits into the complex case (easier, with more normal reflection subgroups) and the quaternionic case (a little more involved, with fewer normal reflection subgroups). Within these cases there are the primitive groups (finitely many of them) and the imprimitive groups (infinite families).

We will assume some basic familiarity with reflection groups, and in particular the Shephard-Todd classification of the complex reflection groups into three infinite families of imprimitive groups $G(m,p,d)$ and the primitive groups $G_4, G_5, \ldots, G_{37}$ (see [LT09]).

2 Reflections and their orbits under conjugation

A **reflection** g on $\mathbb{C}^d$ or $\mathbb{H}^d$ is an invertible linear map which pointwise fixes a subspace of dimension $d-1$ (a hyperplane), is not the identity, and hence is characterised by

$$\operatorname{rank}(I-g)=1. \tag{2.1}$$

It is immediate from (2.1) that

- Conjugation within the general linear group preserves reflections.

Thus, we may assume, without loss of generality, that finite reflection groups are unitary. A unitary reflection can be specified by the hyperplane that if fixes, or its orthogonal complement, which is called the **root line**. Any nonzero vector a in the root line is called a **root** (**vector**). A (unitary) reflection g is therefore determined by a root vector a and its action on the root line, i.e.,

$$ga = a\xi,$$

where ξ is a unit scalar in $\mathbb{C}$ or $\mathbb{H}$, respectively. For reflections on $\mathbb{H}^d$, we treat $\mathbb{H}^d$ as a right vector space ($\mathbb{H}$-module), and (in both cases) the above reflection is given by the formula

$$g = r_{a,\xi} := I - \frac{a(1-\xi)a^*}{\langle a,a\rangle}, \tag{2.2}$$

with (a,ξ) called a **root** for the reflection. Clearly a reflection $g = r_{a,\xi}$ has order n if and only if ξ is a primitive n-th root of unity, i.e., n is the smallest power with $\xi^n = 1$.

Throughout, G will be a finite reflection group. Associated with a reflection in G, its root line, or a root a for it, is the subgroup of all reflections in G with that root line (and the identity), i.e.,

$$R_a = R_{a,G} := \{r_{a,\xi} : r_{a,\xi} \in G\}, \tag{2.3}$$

which we call the **reflection subgroup** for the root a, reflection $r_{a,\xi}$, root line $a\mathbb{H}$, etc. These are the maximal parabolic subgroups of rank 1 (they fix the hyperplane $a^\perp$),

We also define the following subgroups of the unit scalars under multiplication

$$H_a = H_{a,G} := \{\lambda \in U(\mathbb{H}) : r_{a,\lambda} \in G\}, \qquad K_a = K_{a,G} := \{\lambda \in U(\mathbb{H}) : a\lambda \in Ga\}, \tag{2.4}$$

where H_a is a normal subgroup of K_a. We observe that $H_a \to R_a : \lambda \mapsto r_{a,\lambda}$ is an isomorphism, and so the elements of R_a can be indexed

$$R_a = R_{a,G} = \{r_{a,\lambda} : \lambda \in H_{a,G}\}. \tag{2.5}$$

The (unitary) matrices of the group G act on the reflection subgroups for the roots via conjugation

$$g r_{a,\xi} g^{-1} = r_{ga,\xi} \quad \Longrightarrow \quad R_a^g := g R_a g^{-1} = R_{ga}, \qquad g \in G. \tag{2.6}$$

A reflection group G is therefore uniquely determined by the conjugacy classes

$$R_a^G := \{R_a^g : g \in G\}.$$

of the reflection subgroups for its roots (root lines). We observe that

- If H is a reflection subgroup of G, and a is a root of a reflection in G, then $R_{a,H}$ is a subgroup of $R_{a,G}$, possibly the identity (when H has no reflections with root a).
- Moreover, if $H = N$ is normal in G, then

  $$R_{a,N} \subset N \quad \Longrightarrow \quad g(R_{a,N})g^{-1} \subset gNg^{-1} = N, \qquad g \in G,$$

  so that the reflections $R_{a,N}^G$ are in N, where $\hat{R}_a = R_{a,N}$ is a subgroup of $R_a = R_{a,G}$.

This structure characterises the normal reflection subgroups.

Lemma 2.1 *Let G be a reflection group, and $R_{a_1}^G, \ldots, R_{a_m}^G$ be the orbits of the reflection subgroups for its roots. Then the normal reflection subgroups N of G are given by*

$$N = \langle \cup_j \hat{R}_{a_j}^G \rangle, \qquad \text{where each } \hat{R}_{a_j} \text{ is a subgroup of } R_{a_j}, \tag{2.7}$$

and these form a sublattice of the lattice of reflection subgroups, with the meet and join given by

$$\langle \cup_j \hat{R}_{a_j}^G \rangle \wedge \langle \cup_j \tilde{R}_{a_j}^G \rangle = \langle \cup_j (\hat{R}_{a_j} \cap \tilde{R}_{a_j})^G \rangle, \tag{2.8}$$

$$\langle \cup_j \hat{R}_{a_j}^G \rangle \vee \langle \cup_j \tilde{R}_{a_j}^G \rangle = \langle \cup_j \hat{R}_{a_j}^G \rangle \langle \cup_j \tilde{R}_{a_j}^G \rangle = \langle \cup_j (\hat{R}_{a_j} \tilde{R}_{a_j})^G \rangle. \tag{2.9}$$

Moreover, the following subsets form sublattices

(i) *The subgroups with a given collineation group.*

(ii) *The primitive subgroups.*

(iii) *The imprimitive subgroups.*

(iv) *The reducible subgroups.*

(v) *The subgroups of a given rank (or a sequence of successive ranks).*

Proof: Clearly $N = \langle \cup_j \hat{R}_{a_j}^G \rangle$ is a normal reflection subgroup, since it is generated by a set of reflections which are closed under conjugation by G, and we observed that all normal reflection subgroups must have this form.

Since the reflections in $\langle \cup_j \hat{R}_{a_j}^G \rangle \wedge \langle \cup_j \tilde{R}_{a_j}^G \rangle$ are $\cup_j (\hat{R}_{a_j} \cap \tilde{R}_{a_j})^G$, we obtain the formula for the meet. For the join, we first observe that since $\hat{H} = \langle \cup_j \hat{R}_{a_j}^G \rangle$ and $\tilde{H} = \langle \cup_j \tilde{R}_{a_j}^G \rangle$ are normal subgroups of G, so is their product. Thus it suffices to show that their product is the reflection subgroup

$$H = \langle \cup_j (\hat{R}_{a_j} \tilde{R}_{a_j})^G \rangle.$$

Since $\hat{R}_{a_j}$ and $\tilde{R}_{a_j}$ are subgroups of $\hat{R}_{a_j} \tilde{R}_{a_j}$, we have that $\hat{H}\tilde{H} \subset H$. On the other hand, $\hat{H}\tilde{H}$ contains the reflections which generate H, so that $\hat{H}\tilde{H} = H$.

The given subsets form sublattices since the property defining them increases or decreases as one moves down the lattice, e.g., the collineation group of a subgroup (see §3.1) is contained within the collineation group of the group, and the rank of a subgroup is less than or equal the rank of the group. □

In the most general setting, i.e., for quaternionic reflection groups, we index the normal subgroups of G by the reflection subgroups that define them, i.e.,

$$N = \langle \cup_j \hat{R}_{a_j}^G \rangle = G^{(\hat{R}_{a_1}, \ldots, \hat{R}_{a_m})}. \tag{2.10}$$

We will say that $\hat{R}_{a_j}^G$ is a **split orbit** of N if it is not an N-orbit, i.e.,

$$\hat{R}_{a_j}^N \neq \hat{R}_{a_j}^G. \tag{2.11}$$

At the outset, it is not clear whether a normal reflection subgroup N can have more than one label, i.e., come from different choices of subgroups $\{\hat{R}_{a_j}\}$. We will see that normal reflection subgroups of complex reflection groups do have a unique label, whereas for quaternionic reflection groups this is not always the case (Examples 10.1 and 10.3).

If G is a complex reflection group, then $R_{a_j} \cong H_{a_j} \subset \mathbb{C}^*$ is a cyclic group of order k_j, and so R_{a_j} and its subgroups can be indexed by their orders α_j, which divide k_j, i.e.,

$$N = \langle \cup_j \hat{R}_{a_j}^G \rangle = G^{(\alpha_1, \ldots, \alpha_m)}, \qquad \alpha_j := |\hat{R}_{a_j}|, \quad \alpha_j \;\Big|\; k_j, \tag{2.12}$$

where

$$\hat{R}_{a_j} = (R_{a_j})^{k_j/\alpha_j} = \{g^{k_j/\alpha_j} : g \in R_{a_j}\} = \{g \in R_{a_j} : g^{\alpha_j} = 1\}. \tag{2.13}$$

The lattice operations (2.8) and (2.9), then take the following simple and insightful form

$$G^\alpha \wedge G^\beta = G^{\gcd(\alpha,\beta)}, \qquad G^\alpha \vee G^\beta = G^\alpha G^\beta = G^{\mathrm{lcm}(\alpha,\beta)}, \tag{2.14}$$

where the gcd (greatest common divisor) and lcm (least common multiple) are taken coordinatewise. The uniqueness of the label $(\alpha_1, \ldots, \alpha_m)$ of (2.12) follows from the facts

- If G is a complex reflection group generated by the reflections $\mathcal{R}$, then any reflection in G is conjugate to a power of a reflection in $\mathcal{R}$ (Lemma 4.11 (iii) [Coh76]).

- Each $\hat{R}_{a_j}$ is a group, and hence is closed under taking powers.

In effect, the reflections in G^α are precisely $\cup_j \hat{R}^G_{a_j}$ (see Example 7.1 for details). The first fact is not true for quaternionic reflection groups (see Examples 10.1 and 10.3).

Therefore, for complex reflection groups, we have a very simple description of the lattice of normal reflection subgroups.

Theorem 2.1 *Let G be an irreducible complex reflection group, and $R^G_{a_1}, \ldots R^G_{a_m}$, with $k_j = |R_{a_j}|$, be the orbits of of the reflection subgroups for its roots. Then the lattice of normal reflection subgroups of G is isomorphic to the lattice of divisors of $(k_1, \ldots, k_m)$, with the operations (2.14). In particular, there are*

$$\sigma_0(k_1)\sigma_0(k_2)\cdots\sigma_0(k_m)$$

normal reflection subgroups, where $\sigma_0(n)$ is the number of divisors of n.

Moreover, if G is primitive, then every nontrivial normal subgroup is irreducible.

Proof: We have observed that the reflections in $N = G^{(\alpha_1,\ldots,\alpha_m)}$ of (2.12) are precisely $\cup_j \hat{R}^G_{a_j}$, and so each normal reflection subgroup has a unique label. Thus we have the lattice isomorphism claimed.

If N is a normal subgroup of $G \subset GL(\mathbb{C}^d)$, then it follows from Clifford's theorem that $\mathbb{C}^d$ decomposes into a direct sum of N-invariant subspaces which are conjugate to each other under the action of G. Thus if G is primitive, then there must be a single summand, i.e., N must be irreducible. □

To describe the indexing (2.7) of the lattice in Lemma 2.1 (and in Theorem 2.1), it suffices to know the reflection subgroups $R_{a_j} \cong H_{a_j}$ up to isomorphism. To capture this, and the number of conjugates n_j of R_{a_j} in G, we will say that the reflection group G has **reflection orbits** or **reflection type**

$$n_1 H_{a_1}, \ldots, n_m H_{a_m}, \tag{2.15}$$

where H_{a_j} can be written as an abstract group, or as a subgroup of $U(\mathbb{H})$. The orbit size n_j equals the size of the orbit of the root line given by a_j under the action of G.

The number of reflections of each possible order on each of the m orbits is easily calculated from the reflection type, e.g., if G is a complex reflection group, then its type is given by cyclic groups

$$n_1 C_{k_1}, \ldots, n_m C_{k_m}, \tag{2.16}$$

and

- The number of root lines (reflection hyperplanes) for G is $n_1 + \cdots + n_m$. This is equal to the sum of the codegrees of G plus its rank.
- The number of reflections of order $b \mid k_j$ on the $n_j C_{k_j}$ orbit is $n_j \varphi(b)$.
- The number of reflections of order b, and the total number of reflections are

$$\sum_{\substack{1\le j\le m \\ b\mid k_j}} n_j\varphi(b), \qquad \sum_{1\le j\le m} n_j(k_j-1).$$

The reflections in $G^{(\alpha_1,\ldots,\alpha_m)}$ can be counted in the same way, i.e.,

$$\#\text{ reflections in } G^{(\alpha_1,\ldots,\alpha_m)} = \sum_{1\le j\le m} n_j(\alpha_j-1). \tag{2.17}$$

The reflection type gives more information than listing the number of reflections of each order (which is commonly done), e.g., the number of orbits. For example, the primitive Shephard-Todd groups G_{12} and G_{13} have 12 and 18 reflections of order 2, which is often written 12^{12} and 12^{18}, and reflection types $12C_2$ (one orbit) and $6C_2, 12C_2$ (two orbits).

We give now give a simple illustrative example.

Example 2.1 *Consider the imprimitive irreducible reflection group* $G := G(2,1,2)$, *which is generated by the reflections*

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

This real reflection group is conjugate to $G(4,4,2)$, *and is isomorphic to the dihedral group of order* 8 *(see [Wal26a]). It has four reflections of order* 2, *which lie in two orbits*

$$\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\}.$$

and so G *has reflection type* $2C_2, 2C_2$, *with the two reflection subgroup orbits being*

$$\left\{\left\langle\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right\rangle\right\}, \qquad \left\{\left\langle\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\rangle,\left\langle\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\rangle\right\}. \tag{2.18}$$

Hence G *has four normal reflection subgroups, with the two proper ones being of order* 4

$$\begin{tikzcd}[column sep=small]
& G^{(2,2)} = G \arrow[dl, dash] \arrow[dr, dash] & \\
G^{(2,1)} = \left\langle\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right\rangle \arrow[dr, dash] & & G^{(1,2)} = \left\langle\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\rangle \arrow[dl, dash] \\
& G = G^{(1,1)} = 1 &
\end{tikzcd}$$

We observe that $G^{(2,1)} \ne 1$ *is reducible, and so the condition that* G *be primitive in the second part of Theorem 2.1 is necessary.*

There are five other subgroups of G, *i.e., the normal subgroup of order* 4 *given by*

$$\left\langle\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\rangle,$$

which is not a reflection group, and the four reflection subgroups of (2.18) of order two (which are not normal), for the roots $(0,1),(1,0),(1,-1),(1,1)$ *of* G, *respectively.*

3 Collineations and maximal reflection groups

We now consider how groups of matrices, such as reflection groups, act on lines, i.e., one-dimensional subspaces. This leads to the notion of a "collineation group" [Bli17].

3.1 Collineations

Let $G \subset U(\mathbb{C}^d)$ be a group of complex unitary matrices (the quaternionic case will be considered later). Then the action of $g \in G$ on a (complex) line $\mathbb{C}a$ ($a \in \mathbb{C}^d$, $a \neq 0$) is the same as that of any nonzero scalar multiple λg of g, i.e.,

$$\lambda g(\mathbb{C}a) = \mathbb{C}\lambda g a = \mathbb{C}ga = g(\mathbb{C}a). \tag{3.19}$$

The group G factored by the (unit) scalar matrices in G is called the **collineation group** of G, and a matrix defined up to a scalar multiple is called a **collineation** matrix. We will denote the collineation group by G_{coll}.

In view of (3.19), there is a natural faithful action of a collineation group on the set of lines in $\mathbb{C}^d$ (see Lemma 4.1 for an example of a faithful action on a finite set of lines). By Schur's lemma, we have

- If G is irreducible, then its (normal) subgroup of scalar matrices is its centre $Z(G)$, and so its collineation group is $G_{\mathrm{coll}} = G/Z(G)$.

Collineation groups with the same collineation matrices (matrices up to a scalar multiple) are considered to be the same. In the interests of giving a canonical form, there are two approaches that have been taken:

- Consider subgroups of $U(\mathbb{C}^d)$ that contain all the unit scalar matrices (and factor by these).

- Consider subgroups of the special unitary group $SU(\mathbb{C}^d)$ that contain the scalar matrices of determinant 1, i.e., those given by the d-th roots of unity (and factor by these).

In light of the latter (every matrix can be scaled to have determinant 1), the classification of finite collineation groups follows from that of the finite subgroups of $SU(\mathbb{C}^d)$.

For groups $G \subset U(\mathbb{H}^d)$ of quaternionic matrices, the noncommutativity of $\mathbb{H}$ implies that the analogue of (3.19), i.e.,

$$\lambda g(\mathbb{H}a) = g(\mathbb{H}a), \tag{3.20}$$

does not hold, unless λ is real, i.e., $\lambda = \pm 1$. Since the centre of $M_d(\mathbb{H})$ is the real scalar matrices, we can define collineation groups and collineations as before, where the scalar matrices are taken to be real. We observe that under the correspondence with (complex) symplectic matrices given by

$$\begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} \in M_{2d}(\mathbb{C}) \quad \iff \quad A + Bj \in M_d(\mathbb{H}), \tag{3.21}$$

the only scalar symplectic unitary matrices are precisely the real ones, i.e., $\pm I$.

3.2 Hidden reflections and maximal reflection groups

We will say that a complex collineation matrix is a **reflection** if some scalar multiple of it is a reflection, and that a matrix is a **hidden reflection** if some scalar multiple of it is a reflection, but it is not a reflection. In other words:

- The collineation given by a matrix is a reflection if and only if the matrix is reflection or a hidden reflection.

Reflections and hidden reflections are easily recognised by their spectral structure:

- A matrix $g \in U(\mathbb{C}^d)$ is a reflection or a hidden reflection if and only if it has exactly two eigenvalues λ_1 and λ_2 with multiplicities 1 and $d-1$. It is a reflection when $\lambda_2 = 1$ (or $\lambda_1 = \lambda_2 = 1$, when $d = 2$).
- For $d = 2$, there are two scalar multiples of a hidden reflection that give a reflection.
- For $d \geq 3$, there is a unique scalar multiple of a hidden reflection g which is a reflection, i.e., $\lambda_2^{-1} g$, which we call the **reflection given by** g. Furthermore, if g has finite order, then λ_2^{-1} is a root of unity.

We can now make a key definition:

Definition 3.1 *The* **maximal reflection group** *for a complex reflection group G is the reflection group generated by its reflections and the reflections given by its hidden reflections.*

We will also say that a reflection group is **maximal** if it equals its maximal reflection group, i.e., it contains no hidden reflections. The identity subgroup is (technically) a maximal reflection group. We will not fuss over this, e.g., presenting the lattice of maximal reflection subgroups without it. A key observation is the following:

- If g is a hidden reflection of a complex reflection group G, then the reflection group generated by G and the reflection $\lambda_2^{-1} g$ is the group generated by the reflections in G and the scalar matrix $\lambda_2 I$, and so it has the same collineation group as G.

This implies that the number of reflections plus the number of hidden reflections of G equals the number of reflections in its maximal reflection group.

Example 3.1 *The Shephard-Todd group $G_{12} \subset U(\mathbb{C}^2)$ of order* 48 *(with* 12 *reflections) has maximal reflection group G_{11} of order* 576*, which has* 46 *reflections. Therefore, G_{12} has* 12 *reflections and* 34 *hidden reflections.*

For reflection groups of rank 2, every nonscalar matrix is either a reflection or a hidden reflection, and so the maximal reflection groups can be identified by counting their reflections.

Proposition 3.1 *An irreducible reflection group $G \subset U(\mathbb{C}^2)$ is a maximal reflection group if and only if it has $2(|G|/|Z(G)|-1)$ reflections.*

Proof: Since a maximal reflection group has no hidden reflections, every nonscalar matrix in G must be a reflection, with each nontrivial collineation in $G/Z(G)$ giving exactly two reflections. □

Example 3.2 *All the reflection groups in $U(\mathbb{C}^1)$ are maximal. They are given by the m-th roots of unity, i.e., the cyclic group of order m, and we will use the notation*

$$C_m = G(m, 1, 1).$$

The reducible reflection groups in $U(\mathbb{C}^2)$ must be a product of cyclic groups. When C_1 is a factor, one obtains a group generated by a single reflection of order m, which we also denote by C_m, as we have done for the reflection type. The maximal reflection group for $C_a \times C_b$ is

$$C_m \times C_m, \qquad m := \operatorname{lcm}(a, b).$$

These are all the maximal reducible reflection groups in $U(\mathbb{C}^2)$.

Lemma 3.1 *Every complex reflection group G is a normal (collineation preserving) subgroup of its maximal reflection group.*

Proof: We have already observed that a reflection group G is a subgroup of its maximal reflection group, which has the same collineation group. Since the maximal reflection group of G has elements of the form λg, where λ is a nonzero scalar and $g \in G$, the calculation

$$(\lambda g)G(\lambda g)^{-1} = (\lambda\lambda^{-1})gGg^{-1} = G,$$

shows that G is normal in its maximal reflection group. $\square$

Theorem 3.1 *(Classification) The irreducible complex reflection groups are the collineation group preserving normal reflection subgroups of the maximal irreducible reflection groups.*

Proof: In view of Lemma 3.1, it suffices to observe that a complex reflection group is irreducible if and only its maximal reflection group is, since these groups have the same matrices up to a scalar. $\square$

To put Theorems 2.1 and 3.1 into full effect, we will go through the Shephard-Todd classification and

- Determine the (primitive and imprimitive) maximal irreducible complex reflection groups.
- Find all the collineation preserving normal reflection subgroups of the maximal reflection groups (these have the same primitivity).
- Explore how the indexing of (2.12) relates to the structure of these normal reflection subgroups and their quotients.

Before doing this, we consider the normal reflection subgroups of G_{11} in great detail, including showing how Theorem 2.1 allows for the systematic calculation of all normal reflection subgroups, and natural generators for them.

Remark 3.1 *The collineation group seems to play less of a role for the quaternionic reflection groups, e.g., all the primitive quaternionic reflection groups have centre $\langle -1 \rangle$ (see Table 5), and hence they have no hidden reflections.*

4 The Shephard-Todd group G_{11} and the SIC

Here we consider the maximal reflection group $G_{11} \subset U(\mathbb{C}^2)$, its subgroups, and their generators and action on a set of four equiangular lines.

4.1 Generators for G_{11} and equiangular lines

A set of four equiangular lines in $\mathbb{C}^2$ (known as a SIC) is given by an orbit of a (fiducial) vector/line $v \in \mathbb{C}^2$, e.g.,

$$v = \begin{pmatrix} 1-\sqrt{3} \\ 1-i \end{pmatrix} \in \mathbb{C}^2, \tag{4.22}$$

under the action of the Heisenberg group, which is generated by

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (cyclic shift)}, \qquad \Omega := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (modulation)}.$$

This is the reflection group $G(2,1,2)$ of Example 2.1. Since $S\Omega = -\Omega S$, its collineation group is the Klein four-group $C_2 \times C_2$. The normaliser of the Heisenberg group (as a collineation group) is the Clifford group, which is generated by S, Ω and the matrices

$$F := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ (discrete Fourier transform)}, \qquad R := \begin{pmatrix} 1 & 0 \\ 0 & i^3 \end{pmatrix}.$$

These notions generalise to a SIC of d^2 equiangular lines in $\mathbb{C}^d$ (see [Wal18]).

The matrices S, Ω, F, R are reflections. The reflection group that they generate has a hidden reflection RF of order 3, and so is not maximal (it is G_9). We define the Zauner matrix, which is a reflection of order 3, which fixes a fiducial vector, by

$$Z := e^{\frac{2\pi i}{24}} RF = \frac{1}{4} \begin{pmatrix} 1+\sqrt{3}+(\sqrt{3}-1)i & 1+\sqrt{3}+(\sqrt{3}-1)i \\ \sqrt{3}-1-(\sqrt{3}+1)i & 1-\sqrt{3}+(\sqrt{3}+1)i \end{pmatrix}.$$

The reflection group generated by F, R, Z (which contains S, Ω) is the maximal reflection group G_{11}. Indeed, it has the same presentation as that given in [LT09], which has generators

$$r_1 = Z^2 = Z^{-1}, \qquad r_3 = R^2 F R^{-2}, \qquad r_4 = R^3 = R^{-1}.$$

A calculation shows that the reflection type (reflection orbits) of $G_{11} = \langle F, R, Z \rangle$ is

$$12\langle F \rangle, 6\langle R \rangle, 8\langle Z \rangle = 12C_2,\, 6C_4, 8C_3, \tag{4.23}$$

i.e., $G_{11} = G_{11}^{(2,4,3)}$, and G_{11} has 12 normal reflection subgroups (by Theorem 2.1). The reflection count of Proposition 3.1 shows that $G = G_{11}$ is a maximal reflection group, i.e.,

$$\#\text{ reflections in } G = 12(2-1)+6(4-1)+8(3-1) = 46 = 2\Big(\frac{576}{24}-1\Big) = 2\Big(\frac{|G|}{|Z(G)|}-1\Big).$$

Here the centre $Z(G)$ has order 24, and so the collineation group has order $\frac{|G|}{|Z(G)|} = 24$.

In our computations, we will at times require more than one element of maximal order from a reflection orbit. These can be easily obtained by conjugation of a given representative with reflections from different orbits, e.g.,

$$F^G = \{gFg^{-1} : g \in G_{11}\} = \{gFg^{-1} : g \in \langle R, Z\rangle\}.$$

We now decribe the collineation group. Let v be the fiducial vector of (4.22), which is a root of the reflection Z. The orbit of v under the Heisenberg group gives the four equiangular lines $v_1 = v$, $v_2 = Sv$, $v_3 = \Omega v$, $v_4 = S\Omega v$, and the orthogonal complement of these vectors/lines in $\mathbb{C}^2$ gives a second such SIC. Thus we obtain eight lines given by the vectors

$$v_1, v_2, v_3, v_4 = \begin{pmatrix} 1-\sqrt{3} \\ 1-i \end{pmatrix}, \begin{pmatrix} 1-i \\ 1-\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1-\sqrt{3} \\ i-1 \end{pmatrix}, \begin{pmatrix} 1-i \\ 1-\sqrt{3} \end{pmatrix},$$

$$v_1^\perp, v_2^\perp, v_3^\perp, v_4^\perp = \begin{pmatrix} 1+i \\ \sqrt{3}-1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}-1 \\ 1+i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1-\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1-\sqrt{3} \\ 1+i \end{pmatrix}.$$

The permutation action of the generators F, R, Z for G_{11} (the Clifford group) on these lines is (with the obvious notation)

$$F: \ (1\,4^\perp)(2\,2^\perp)(3\,3^\perp)(4\,1^\perp), \qquad R: \ (1\,2^\perp\,3\,4^\perp)(2\,3^\perp\,4\,1^\perp), \qquad Z: \ (2\,3\,4)(2^\perp\,3^\perp\,4^\perp).$$

Thus, G_{11} (the Clifford group) acts on the four pairs of orthogonal lines $\{v_j, v_j^\perp\}$ via

$$F: \ (1\,4), \qquad R: \ (1\,2\,3\,4), \qquad Z: \ (2\,3\,4). \tag{4.24}$$

Lemma 4.1 *There is a surjective homomorphism $\Psi : G_{11} \to S_4$ given by*

$$F \mapsto (1\,4), \qquad R \mapsto (1\,2\,3\,4), \qquad Z \mapsto (2\,3\,4), \tag{4.25}$$

whose quotient is an isomorphism from the collineation group to S_4. In particular, the collineation group of a subgroup $H \subset G_{11}$ is isomorphic to $\Psi(H) \subset S_4$.

Proof: Since the permutations of (4.24) generate the symmetric group S_4, which is of order 24, and a scalar matrix acts on a pair of orthogonal lines as the identity, the map of (4.25) gives a surjective homomorphism $G_{11} \to S_4$, with kernel containing the scalar matrices in G_{11}. Since $Z(G_{11})$ consists of scalar matrices, we obtain the isomorphism $G_{11}/Z(G_{11}) = G_{11}/\ker\Psi \cong S_4$. □

Lemma 4.1 gives the following geometric description of the 46 reflections in G_{11}. These correspond to the 23 nontrivial collineations in S_4.

- The 12 reflections of order 2 conjugate to F act like $(1\,2)$, i.e., they fix two of the pairs of orthogonal lines and permute the other two (six collineations in total).
- The 12 reflections of order 4 conjugate to R or R^3 act like $(1\,2\,3\,4)$, i.e., they cycle through the four pairs of orthogonal lines.
- The 6 reflections of order 2 conjugate to R^2 act like $(1\,2)(3\,4)$, i.e., the permute two pairs of pairs of orthogonal lines.
- The 16 reflections of order 3 conjugate to Z or Z^2 act like $(1\,2\,3)$, i.e., they fix one pair of orthogonal lines and cycle through the remaining three.

4.2 Generators for the normal reflection subgroups

We now use $G_{11}^{(2,4,3)}$ to illustrate a general method for finding natural generating sets for the normal reflection subgroups of a reflection group $G = G^{(k_1,\ldots,k_m)}$, with reflection orbits $R^G_{a_1},\ldots,R^G_{a_m}$, and reflection type

$$n_1C_{k_1},\ldots,n_mC_{k_m}.$$

By (2.11), (2.13), the j-th orbit of a normal reflection subgroup $H = G^{(\alpha_1,\ldots,\alpha_m)}$ is split if

$$((R_{a_j})^{k_j/\alpha_j})^H \neq ((R_{a_j})^{k_j/\alpha_j})^G. \tag{4.26}$$

Our earlier observation

- If G is a complex reflection group generated by the reflections $\mathcal{R}$, then any reflection in G is conjugate to a power of a reflection in $\mathcal{R}$ (Lemma 4.11 (iii) [Coh76]).

implies that a generating set $\mathcal{R} = \mathcal{R}^{(k_1,\ldots,k_m)}$ of reflections for $G = G^{(k_1,\ldots,k_m)}$ must contain at least one reflection (of highest order) from each of the m reflection orbits (see Lemma 7.1), i.e., $\mathcal{R}$ can be partitioned

$$\mathcal{R} = \mathcal{R}_{a_1}\cup\cdots\cup\mathcal{R}_{a_m}, \qquad \mathcal{R}_{a_j} = \mathcal{R}^{(k_1,\ldots,k_m)}_{a_j} := \mathcal{R}\cap R^G_{a_j}, \tag{4.27}$$

where $|\mathcal{R}_{a_j}|\geq 1$. Our choice of generators F, R, Z consisting of one reflection of maximal order from each of the three orbits is therefore a minimal generating set of reflections. In this particular case, any such choice gives a minimal generating set (see Theorem 5.2).

It is also quite possible that more than one reflection may need to be taken from a reflection orbit to obtain a generating set, e.g., $G = G_8$ has a single reflection orbit $6C_4$, and so at least two reflections must be taken from it to obtain a generating set (it can in fact be generated by two reflections of order 4, but not all such pairs).

Given a generating set $\mathcal{R}$ of reflections for $G = G^{(k_1,\ldots,k_m)}$ of the form (4.27), the reflections

$$\mathcal{D}^{(\alpha_1,\ldots,\alpha_m)}_{\mathcal{R},a_j} := \{g^{k_j/\alpha_j} : g\in\mathcal{R}^{(k_1,\ldots,k_m)}_{a_j}\}, \qquad \alpha_j\neq 1, \tag{4.28}$$

are reflections of maximal order in $R^G_{a_j}\cap G^{(\alpha_1,\ldots,\alpha_m)}$, and we call them the reflections **inherited** from $\mathcal{R}$. The inherited reflections show the inclusion $G^{(\alpha_1,\ldots,\alpha_m)}\subset G^{(k_1,\ldots,k_m)}$ and come for free. We will investigate to what extent they are useful for constructing a nice generating set of reflections for $G^{(\alpha_1,\ldots,\alpha_m)}$.

We now calculate the reflection subgroups of G_{11} and minimal sets of reflections which generate them. We then summarise the general algorithm for such calculations.

Example 4.1 *(Normal reflection subgroups of G_{11}). We have already observed that there are twelve normal reflection subgroups of $G_{11} = G_{11}^{(2,4,3)} = \langle F,R,Z\rangle$ (see the indices in Figure 1). Those which are maximal subgroups of G_{11} (in the lattice) are generated by the inherited reflections, and have no split orbits. They (and their reflection types) are*

$$\begin{aligned} G_{15} = G_{11}^{(2,2,3)} &= \langle F,R^2,Z\rangle, &\quad& 12\langle F\rangle, 6\langle R^2\rangle, 8\langle Z\rangle = 12C_2, 6C_2, 8C_3 \\ G_9 = G_{11}^{(2,4,1)} &= \langle F,R,Z^3\rangle = \langle F,R\rangle, && 12\langle F\rangle, 6\langle R\rangle, 8\langle Z^3\rangle = 12C_2, 6C_4, 8C_1, \\ G_{10} = G_{11}^{(1,4,3)} &= \langle F^2,R,Z\rangle = \langle R,Z\rangle, && 12\langle F^2\rangle, 6\langle R\rangle, 8\langle Z\rangle = 12C_1, 6C_4, 8C_3. \end{aligned}$$

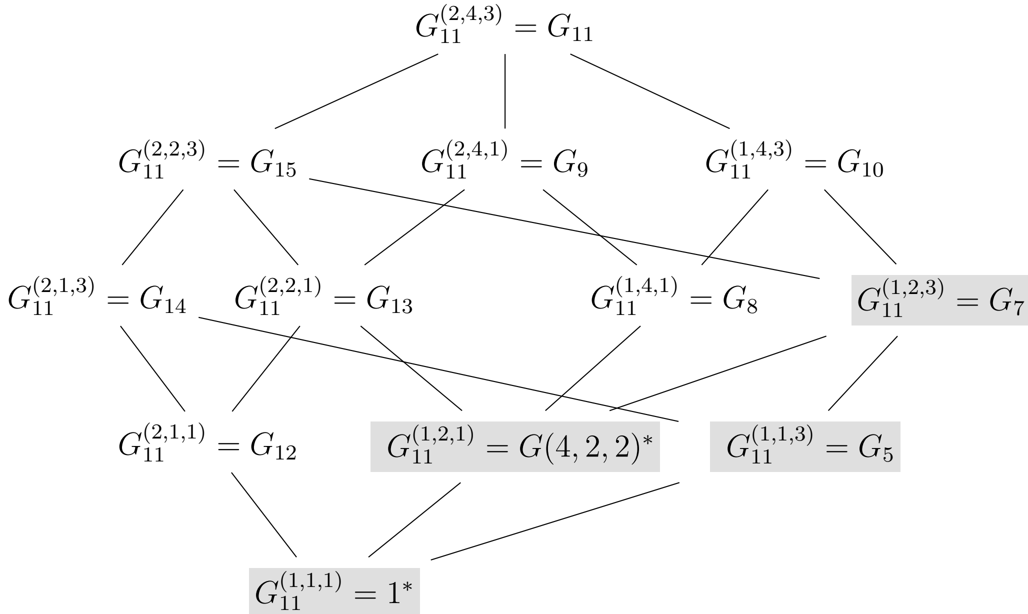


Figure 1: The lattice of the 12 normal reflection subgroups of G_{11}. Those in grey have a smaller collineation group, and the meets with a * are not the intersection of the groups. Note that all the nontrivial subgroups above are irreducible (as per Theorem 2.1), with one being imprimitive, i.e., $G(4,2,2)$. The groups not in grey form the sublattice of the collineation preserving subgroups of G_{11}, with those in grey giving its lower boundary.

Here we have given the reflection type in a transparent way. When the identity subgroup of R_{a_j} is selected, i.e., $\alpha_j = 1$, the corresponding orbit disappears, i.e., no reflections are taken from it, and henceforth we won't record it in the reflection type, e.g., we will write

$$G_9 = G_{11}^{(2,4,1)} = \langle F, R\rangle, \qquad 12\langle F\rangle, 6\langle R\rangle = 12C_2, 6C_4.$$

The next level of normal reflection subgroups in the lattice are

$$\begin{aligned}
G_{14} = G_{11}^{(2,1,3)} &= \langle F, Z\rangle, && 12\langle F\rangle, 8\langle Z\rangle = 12C_2, 8C_3,\\
G_{13} = G_{11}^{(2,2,1)} &= \langle F, F^Z, R^2\rangle, && 12\langle F\rangle, 6\langle R^2\rangle = 12C_2, 6C_2,\\
G_8 = G_{11}^{(1,4,1)} &= \langle R, R^F\rangle, && 6\langle R\rangle = 6C_4,\\
G_7 = G_{11}^{(1,2,3)} &= \langle R^2, Z, Z^R\rangle, && 6\langle R^2\rangle, 4\langle Z\rangle, 4\langle Z^R\rangle = 6C_2, 4C_3, 4C_3.
\end{aligned}$$

In addition to the inherited reflections, the last three require a second reflection from one orbit, i.e.,

$$F^Z = ZFZ^{-1}, \qquad R^F = FRF^{-1}, \qquad Z^R = RZR^{-1}.$$

The first three have an orbit disappear, and the last G_7 has a split orbit, i.e., its reflection type has an extra orbit obtained by "splitting" $8\langle Z\rangle = 8C_3$ into two G_7-orbits $4C_3, 4C_3$. The latter fact means that we can apply Theorem 2.1 to G_7 to find normal reflection subgroups of it which are not normal in G_{11}, namely G_4 and G_6 (see Figure 4)

It is now a convenient time to consider the collineation group, whose size decreases as we move down the lattice. By Lemma 4.1, $\Psi(G_{11}) = S_4$, *and*

$$\Psi(G_{14}) = \langle \Psi(F), \Psi(Z) \rangle = \langle (1\,4), (2\,3\,4) \rangle = S_4, \quad \Psi(G_{13}) = \langle (1\,4), (1\,3), (1\,3)(2\,4) \rangle = S_4,$$

$$\Psi(G_8) = \langle (1\,2\,3\,4), (1\,4\,2\,3) \rangle = S_4, \quad \Psi(G_7) = \langle (1\,3)(2\,4), (2\,3\,4), (1\,2\,3) \rangle = A_4.$$

Thus G_{11} *is the maximal reflection group for all the normal reflection subgroups found so far, i.e., they are collineation preserving subgroups, except for* G_7. *It follows that* G_7 *is a maximal reflection group by Proposition 3.1, i.e., it has* $2(|A_4|-1) = 22$ *reflections.*

The next level of the lattice gives the remaining nontrivial normal reflection subgroups

$$\begin{aligned} G_{12} = G_{11}^{(2,1,1)} &= \langle F, F^Z, F^{Z^2} \rangle, & & 12\langle F \rangle = 12C_2, \\ G_5 = G_{11}^{(1,1,3)} &= \langle Z, Z^R \rangle, & & 4\langle Z \rangle, 4\langle Z^R \rangle = 4C_3, 4C_3, \\ G(4,2,2) = G_{11}^{(1,2,1)} &= \langle R^2, (R^2)^Z, (R^2)^{Z^2} \rangle, & & 2\langle R^2 \rangle, 2\langle (R^2)^Z \rangle, 2\langle (R^2)^{Z^2} \rangle = 2C_2, 2C_2, 2C_2, \end{aligned}$$

with collineation groups

$$\Psi(G_{12}) = \langle (1\,4), (1\,3), (1\,2) \rangle = S_4, \qquad \Psi(G_5) = \langle (2\,3\,4), (1\,2\,3) \rangle = A_4,$$

$$\Psi(G(4,2,2)) = \langle (1\,3)(2\,4), (1\,2)(3\,4), (1\,4)(2\,3) \rangle = C_2 \times C_2.$$

Here the subgroups which do not preserve the collineation group have split orbits, i.e., G_5 *has* $8\langle Z \rangle$ *split into two orbits, and* $G(4,2,2)$ *has* $6\langle R^2 \rangle$ *split into three orbits.*

The above example suggests that for a nontrivial normal reflection subgroup of a complex reflection group, the following are equivalent:

- There is a split orbit.
- The collineation group is not preserved.
- The number of reflections plus the number of hidden reflections is not preserved.

This is indeed true in general, and follows from the action of the collineation group on the root lines (reflection subgroups). Moreover, the first occurrence of a split orbit gives a maximal reflection group.

Lemma 4.2 *Let* G *be an irreducible reflection group. Then there is a natural action of the collineation group of* G *on the reflection subgroups for the roots given by*

$$g \cdot R_a = g R_a g^{-1} = R_{ga}, \qquad \forall g \in G.$$

By construction, this action is transitive on each of the reflection orbits R_a^G.

Proof: Suppose without loss of generality that G is unitary, and λ is a unit scalar in the collineation group ($\lambda = \pm 1$ when G is quaternionic). Then from (2.2), we calculate

$$r_{a,\xi}^{\lambda g} = \lambda g \Big(I - \frac{a(1-\xi)a^*}{\langle a, a \rangle} \Big) (\lambda g)^{-1} = I - \frac{\lambda g a (1-\xi) a^* (\lambda g)^*}{\langle a, a \rangle} = r_{\lambda g a, \xi} = r_{ga,\xi}, \quad g \in G,$$

so that $\lambda g \cdot R_a = g \cdot R_a = R_{ga}$, and the collineation group of G acts on the reflection subgroups (and the root lines). □

Theorem 4.1 *Let G be a complex reflection group, and $N = G^{(\hat{R}_{a_1},\ldots,\hat{R}_{a_m})}$ be a normal reflection subgroup of G. Then*

(i) *The collineation groups of N and G are different if and only if some $\hat{R}^G_{a_j}$ is a split orbit, i.e., $\hat{R}^N_{a_j} \neq \hat{R}^G_{a_j}$.*

(ii) *If N is in the lower boundary of the sublattice of the collineation preserving normal reflection subgroups, i.e., there is a normal reflection subgroup immediately above it with the same collineation group as G, then N is a maximal reflection group.*

Proof: Some $\hat{R}^G_{a_j}$ is a split orbit of N if and only the (collineation) action of N on $\hat{R}^G_{a_j}$ is not transitive, i.e., is different from the (collineation) action of G (which is transitive), and so the collineation groups for N and G must be different.

If N has a split orbit, then we may add to it scalar matrices from G so that its hidden reflections become reflections, thereby obtaining N' the maximal reflection group for N as a normal reflection subgroup of G. Since N' is not collineation preserving and $N \subset N'$, we must have $N' = N$ when N is in the lower boundary of the collineation preserving subgroups, i.e., N must be a maximal reflection group in this case. □

In other words, N is a collineation preserving subgroup of G if and only if it has no split orbits, i.e.,

$$\hat{R}^N_{a_j} = \hat{R}^G_{a_j}, \qquad \forall j,$$

and as we work down the lattice of normal reflection subgroups as soon as we hit one which is not collineation preserving, it is a maximal reflection group.

Later we will show the collineation preserving normal reflection subgroups $N = G^\alpha$ of G are determined by the order of their abelianisation $(G^\alpha)_{\rm ab}$ (Corollary 7.1).

We now give a prototypical example of calculating a minimal set of generators when more than one element from an orbit is required.

Example 4.2 *Consider $G = G_{13} = \langle F, F^Z, R^2\rangle$, which has reflection type $12\langle F\rangle, 6\langle R^2\rangle$, and order 96. The group H generated by the six reflections in the $6\langle R^2\rangle$ orbit, i.e.,*

$$\Bigl\{\pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\Bigr\}, \tag{4.29}$$

has order 16, and the group generated by adding any additional reflection from the $12\langle F\rangle$ orbit has order 32. Hence at least two reflections must be taken from the $12\langle F\rangle$ orbit to obtain a generating set of reflections for G_{13}. The action of H on the $12\langle F\rangle$ orbit gives three orbits of size four, i.e.,

$$\begin{gathered}\Bigl\{\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}, \pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\Bigr\}, \quad \Bigl\{\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}, \pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix}\Bigr\},\\ \Bigl\{\pm\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1+i\\ 1-i & 0\end{pmatrix}, \pm\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1-i\\ 1+i & 0\end{pmatrix}\Bigr\}.\end{gathered} \tag{4.30}$$

It can easily be verified that the minimal generating sets for G_{13} are given by

- *One reflection from the* $6\langle R^2\rangle$ *orbit (4.29).*
- *Two reflections from different* H*-orbits (4.30) of the* $12\langle F\rangle$ *orbit.*

The action of the group of order 48 *generated by the reflections in the* $12\langle F\rangle$ *orbit on the* $6\langle R^2\rangle$ *orbit gives a single orbit.*

In summary, minimal sets of generating reflections can be calculated as follows:

Basic algorithm for calculating the normal reflection subgroups of $G^{(k_1,\ldots,k_m)}$

- Find a minimal set of reflections that generate $G = G^{(k_1,\ldots,k_m)}$, by taking at least one element of maximal order from each of its reflection orbits $R^G_{a_j}$, and then additional maximal order reflections from the $\langle \cup_{\ell\neq j} R^G_{a_\ell}\rangle$-orbits of $R^G_{a_j}$, as required (in Example 4.2 there was at most one element taken from each of these orbits).
- For a normal reflection subgroup $H = G^{(\alpha_1,\ldots,\alpha_m)}$ of G which is a subgroup of some $K = G^{(\beta_1,\ldots,\beta_m)}$ in the lattice, take the reflections inherited from a minimal generating set of reflections for K, adding additional reflections of maximal order from the reflection orbits of K, as required (see above). In particular, additional reflections will be required when a reflection orbit of K splits into more parts than the number of inherited reflections from K that are in it.

We have not been completely prescriptive above, as calculations can easily be done by hand, and there is an art to it: one would like generators that reflect the lattice structure and are obviously related to the original ones. If calculating all the normal subgroups, then one would work down the lattice considering the inherited reflections for all the reflection subgroups K immediately above a given H.

4.3 All the reflection subgroups of G_{11}

We will now calculate all the reflection subgroups of G_{11} by identifying its set of maximal reflection subgroups, and then working down from these.

Example 4.3 *(Reflection subgroups of* G_{11}*) The maximal reflection group* G_{11} *has* 135 *subgroups (up to conjugacy), of which* 38 *are reflection subgroups, with* 12 *being normal, i.e.,*

$$1,\ G_5,\ G_7,\ G_8,\ G_9,\ G_{10},\ G_{11},\ G_{12},\ G_{13},\ G_{14},\ G_{15},\ G(4,2,2). \tag{4.31}$$

The remaining 26 *non-normal reflection subgroups must be normal subgroups in their maximal reflection group (since they are contained in the maximal reflection group* G_{11}*).*

The following normal reflection subgroups of the maximal reflection group G_7 *are not normal in* G_{11}

$$G_4,\ G_6, \tag{4.32}$$

and from the normal maximal reflection subgroup $G(4,2,2)$ *of (4.31), we obtain*

$$B_2,\ C_2\times C_2, \qquad \text{where } B_2 = G(2,1,2)\cong G(4,4,2). \tag{4.33}$$

On the lattice of Figure 11 ($p = 2$), these appear as $G(2,1,2)$, $G(2,1,2)$, $G(4,4,2)$, and $C_2 \times C_2$, $G(2,2,2)$, $G(2,2,2)$, respectively, with each triple being conjugate in G_{11}. The reflection group $C_2 \times C_2$ above is also maximal, and its normal subgroup C_2 corresponds to a reflection orbit. We will consider this later with the other such cases.

The remaining non-normal maximal reflection subgroups of G_{11} are contained in the non-normal maximal reflection subgroups $G(6,2,2)$ and $G(8,2,2)$.

The seven nontrivial normal reflection subgroups of $G(6,2,2)$ give new reflection subgroups of G_{11} (see Figure 11, $p = 3$), i.e.,

$$G(6,2,2),\ G(3,1,2),\ G(3,1,2),\ G(6,6,2),\ C_3 \times C_3,\ G(3,3,2),\ G(3,3,2). \tag{4.34}$$

Here the repeated groups $G(3,1,2)$ and $G(3,3,2)$ are not conjugate in G_{11}. The reflection group $C_3 \times C_3$ above is a maximal reflection group, and we will consider its normal reflection subgroups later.

We now consider the maximal subgroup $G(8,2,2)$. This appears as the reflection subgroup of G_{11} generated by the monomial reflections (and hence is imprimitive), i.e.,

$$G(8,2,2) = H = H^{(4,2,2)} = \langle R, S, F^{Z^2} \rangle, \qquad 2\langle R \rangle, 4\langle S \rangle, 4\langle F^{Z^2} \rangle.$$

This has three conjugates in G_{11} (and so is not normal). Ten of the twelve normal reflection subgroups of $G(8,2,2)$ give new reflection subgroups of G_{11} (see Fig. 2), i.e.,

$$\begin{aligned} &G(8,2,2),\ G(8,4,2),\ G(4,1,2),\ G(4,1,2),\ G(8,8,2), \\ &G(4,2,2),\ C_4 \times C_4,\ B_2,\ B_2,\ C_2 \times C_2. \end{aligned} \tag{4.35}$$

Here $G(4,2,2)$ appears twice as a subgroup of $G(8,2,2)$, including once as a normal subgroup of G_{11}, which was already counted in (4.31), and B_2 appears twice as $G(4,4,2)$. The subgroup $G(4,2,2)$ above is a maximal reflection group, and its normal reflection subgroups give no new reflection subgroups of G_{11}.

Finally, we consider the reflection subgroups of the maximal reflection groups

$$C_2 \times C_2 = \langle F, -F \rangle, \qquad C_3 \times C_3 = \langle Z, e^{\frac{2\pi i}{3}} Z \rangle, \qquad C_4 \times C_4 = \langle R, iR \rangle \supset \langle R^2, -R^2 \rangle = C_2 \times C_2,$$

of (4.33), (4.34) and (4.35). These are (see Example 3.2)

$$C_2 = \langle F \rangle, \qquad C_3 = \langle Z \rangle, \qquad C_2 \times C_4 = \langle R^2, iR \rangle, \quad C_4 = \langle R \rangle, \quad C_2 = \langle R^2 \rangle, \tag{4.36}$$

where $C_2 = \langle R^2 \rangle$ is not parabolic. Thus, the $38 = 12 + 2 + 2 + 7 + 10 + 5$ reflection subgroups of G_{11} are given by (4.31), (4.32), … (4.36).

Example 4.4 *It is easy to calculate minimal generating sets of reflections for the* 12 *normal reflection subgroups of*

$$G(8,2,2) = H = H^{(4,2,2)} = \langle R, S, F^{Z^2} \rangle, \qquad 2\langle R \rangle, 4\langle S \rangle, 4\langle F^{Z^2} \rangle,$$

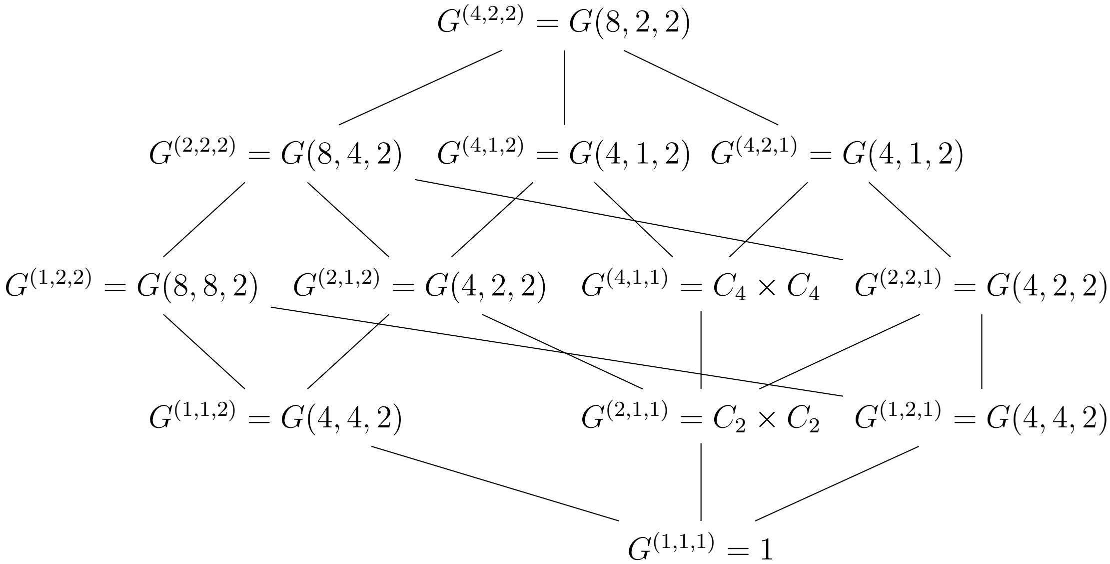


Figure 2: The lattice of normal reflection subgroups of $G^{(4,2,2)} = G(8,2,2) \subset G_{11}$, $2C_4, 4C_2, 4C_2$, where $C_2 \times C_2 = G(2,1,2)$, $G(4,4,2) = B_2$.

which give (4.35). Those which preserve the collineation group are (see Fig. 2)

$$\begin{aligned}
G(8,4,2) = H^{(2,2,2)} &= \langle R^2, S, F^{Z^2} \rangle, && 2\langle R^2 \rangle, 4\langle S \rangle, 4\langle F^{Z^2} \rangle,\\
G(4,1,2) = H^{(4,1,2)} &= \langle R, F^{Z^2} \rangle, && 2\langle R \rangle, 4\langle F^{Z^2} \rangle,\\
G(4,1,2) = H^{(4,2,1)} &= \langle R, S \rangle, && 2\langle R \rangle, 4\langle S \rangle,\\
G(8,8,2) = H^{(1,2,2)} &= \langle S, F^{Z^2} \rangle, && 4\langle S \rangle, 4\langle F^{Z^2} \rangle.
\end{aligned}$$

The remaining nontrivial subgroups have split orbits and smaller collineation groups

$$\begin{aligned}
G(4,2,2) = H^{(2,2,1)} &= \langle R^2, S, S^R \rangle, && 2\langle R^2 \rangle, 2\langle S \rangle, 2\langle S^R \rangle,\\
G(4,2,2) = H^{(2,1,2)} &= \langle R^2, F^{Z^2}, F^{RZ^2} \rangle, && 2\langle R^2 \rangle, 2\langle F^{Z^2} \rangle, 2\langle F^{RZ^2} \rangle,\\
C_4 \times C_4 = H^{(4,1,1)} &= \langle R, R^S \rangle, && \langle R \rangle, \langle R^S \rangle,\\
G(4,4,2) = H^{(1,2,1)} &= \langle S, S^R \rangle, && 2\langle S \rangle, 2\langle S^R \rangle,\\
G(4,4,2) = H^{(1,1,2)} &= \langle F^{Z^2}, F^{RZ^2} \rangle, && 2\langle F^{Z^2} \rangle, 2\langle F^{RZ^2} \rangle,\\
C_2 \times C_2 = H^{(2,1,1)} &= \langle R^2, (R^2)^S \rangle, && \langle R^2 \rangle, \langle (R^2)^S \rangle.
\end{aligned}$$

We observe that the normal reflection subgroups $C_3 \times C_3$ and $C_2 \times C_2, C_4 \times C_4$ of the imprimitive reflection groups $G(6,2,2)$ and $G(8,2,2)$, respectively, are not irreducible, as is allowed by Theorem 2.1.

In summary, all the reflection subgroups of $G_{11} = \langle F, R, Z \rangle$ can be calculated by knowing that

$$G(8,2,2) = \langle R, S, F^{Z^2} \rangle, \qquad G(6,2,2) = \langle Z, F^{Z^2 R^2}, F^{RZ^2} \rangle$$

are reflection subgroups of G_{11} which are not normal and are maximal reflection groups. The reflection subgroups of G_{11} can then be found by calculating the normal reflection subgroups of G_{11}, $G(8,2,2)$, $G(6,2,2)$, moving down the lattice to the point where another maximal reflection group is found, i.e., a split orbit (see Figure 3), and then applying this process to that maximal reflection group.

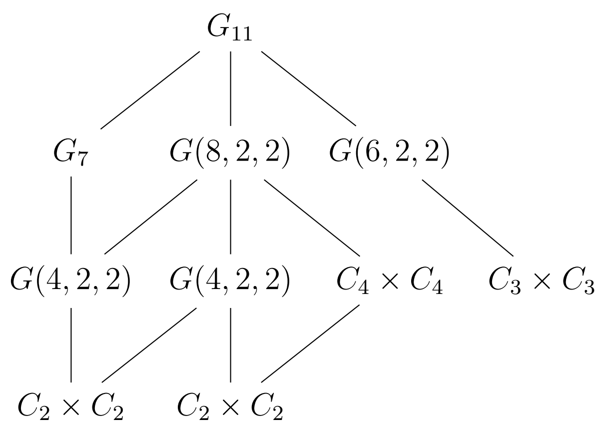


Figure 3: The lattice of the maximal reflection subgroups of G_{11}, up to conjugacy ($G(8,2,2)$ and $G(6,2,2)$ are not normal, and have three and four conjugates). Here $C_3 \times C_3 = \langle Z, e^{\frac{2\pi i}{3}} Z \rangle$, $C_4 \times C_4 = \langle R, iR \rangle \supset C_2 \times C_2 = \langle R^2, -R^2 \rangle$, $C_2 \times C_2 = \langle F, -F \rangle$. See Table 1 for the reflection type, collineation group, and generators for each subgroup.

Table 1: The maximal reflection subgroups of G_{11} as in the lattice of Figure 3.

ST	Rank	Order	Reflection type	K_a	Reflections	G_{coll}	Generators
G_{11}	2	576	$12C_2, 6C_4, 8C_3$	C_{24}	$2^{18}3^{16}4^{12}$	S_4	F, R, Z
G_7	2	144	$6C_2, 4C_3, 4C_3$	C_{12}	$2^6 3^{16}$	A_4	R^2, Z, Z^R
$G(8,2,2)$	2	64	$2C_4, 4C_2, 4C_2$	C_8	$2^{10}4^4$	D_4	R, S, F^{Z^2}
$G(6,2,2)$	2	36	$2C_3, 3C_2, 3C_2$	C_6	$2^6 3^4$	D_3	Z^S, F, F^{Z^2}
$G(4,2,2)$	2	16	$2C_2, 2C_2, 2C_2$	C_4	2^6	$C_2 \times C_2$	R^2, S, S^R
$G(4,2,2)$	2	16	$2C_2, 2C_2, 2C_2$	C_4	2^6	$C_2 \times C_2$	R^2, F^{Z^2}, F^{SZ^2}
$C_4 \times C_4$	2	16	C_4, C_4	C_4	$2^2 4^4$	C_4	R, iR
$C_3 \times C_3$	2	9	C_3, C_3	C_3	3^4	C_3	$Z, e^{\frac{2\pi i}{3}} Z$
$C_2 \times C_2$	2	4	C_2, C_2	C_2	2^2	C_2	$F, -F$
$C_2 \times C_2$	2	4	C_2, C_2	C_2	2^2	C_2	$R^2, -R^2$

5 The normal reflection subgroups of the primitive complex reflection groups

We are now in a position to easily identify the remaining primitive complex reflection groups which are maximal reflection groups, and thereby complete their classification.

First we have G_7 and its normal reflection subgroups (with generators).

Example 5.1 *(Normal reflection subgroups of G_7). There are eight normal reflection subgroups (see the indices in Figure 4) of the maximal reflection group*

$$G_7 = G_7^{(2,3,3)} = \langle R^2, Z, Z^R \rangle, \qquad 6\langle R^2\rangle, 4\langle Z\rangle, 4\langle Z^R\rangle = 6C_2, 4C_3, 4C_3,$$

which has collineation group A_4. The first level down in the lattice, we have

$$\begin{aligned} G_5 = G_7^{(1,3,3)} &= \langle Z, Z^R \rangle, & 4\langle Z\rangle, 4\langle Z^R\rangle &= 4C_3, 4C_3, \\ G_6 = G_7^{(2,1,3)} &= \langle R^2, Z^R \rangle, & 6\langle R^2\rangle, 4\langle Z^R\rangle &= 6C_2, 4C_3, \\ G_6 = G_7^{(2,3,1)} &= \langle R^2, Z \rangle, & 6\langle R^2\rangle, 4\langle Z\rangle &= 6C_2, 4C_3. \end{aligned}$$

At the next level, we have

$$\begin{aligned} G_4 = G_7^{(1,1,3)} &= \langle Z^R, (Z^R)^S \rangle, & 4\langle Z^R\rangle &= 4C_3, \\ G(4,2,2) = G_7^{(2,1,1)} &= \langle R^2, S, S^R \rangle, & 2\langle R^2\rangle, 2\langle S\rangle, 2\langle S^R\rangle &= 2C_2, 2C_2, 2C_2, \quad \textit{(split orbit)} \\ G_4 = G_7^{(1,3,1)} &= \langle Z, Z^S \rangle, & 4\langle Z\rangle &= 4C_3. \end{aligned}$$

The subgroup $G(4,2,2)$ is the only one above with a split orbit, and hence a smaller collineation group, i.e., $C_2 \times C_2$.

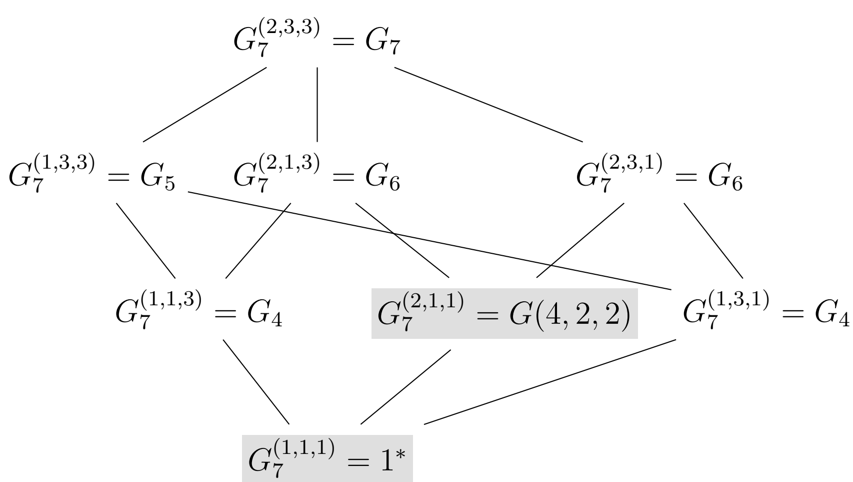


Figure 4: The lattice of normal reflection subgroups of $G_7 = G_7^{(2,3,3)}$ (those in grey have smaller collineation groups, i.e., split orbits). We observe that G_6 and G_4 appear twice as normal subgroups.

We now consider the maximal reflection group G_{19}. Define a reflection of order 5 by

$$M := \left(-\tfrac{\tau}{4} + \tfrac{\sqrt{1-\tau^2/4}}{2} i\right) \begin{pmatrix} -\tau + i & 1-\tau \\ \tau - 1 & -\tau - i \end{pmatrix}, \qquad \tau = \frac{1+\sqrt{5}}{2}.$$

and (in the notation of [LT09], $r = R^2$, $r_1 = Z^2$, $r_5 = M$)

$$G_{19} = G_{19}^{(2,3,5)} = \langle R^2, Z, M \rangle.$$

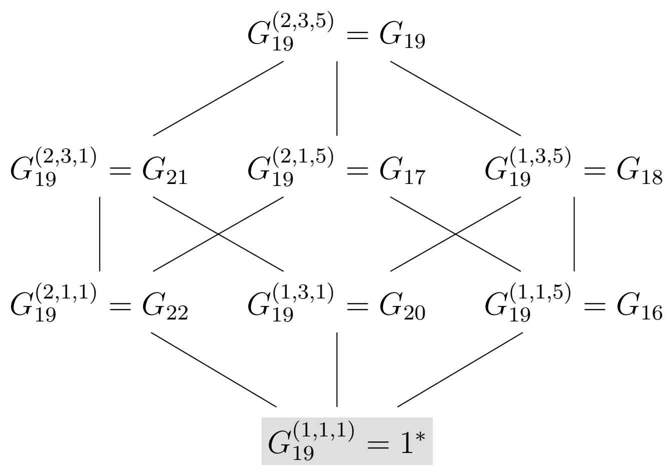


Figure 5: The lattice of normal reflection subgroups for G_{19}. Here the intersection $G_{22} \cap G_{20} \cap G_{16}$ is a collineation preserving normal subgroup of G_{19}, with centre $\langle -1 \rangle$.

Example 5.2 *(Normal reflection subgroups of G_{19}). There are eight normal reflection subgroups (see the indices in Figure 5) of the maximal reflection group*

$$G_{19} = G_{19}^{(2,3,5)} = \langle R^2, Z, M \rangle, \qquad 30\langle R^2 \rangle, 20\langle Z \rangle, 12\langle M \rangle = 30C_2, 20C_3, 12C_3,$$

with no split orbits and no repeated subgroups. The common collineation group of the nontrivial normal reflection subgroups is A_5. The first level down in the lattice, we have

$$\begin{aligned} G_{18} = G_{19}^{(1,3,5)} &= \langle Z, M \rangle, & 20\langle Z \rangle, 12\langle M \rangle &= 20C_3, 12C_5, \\ G_{17} = G_{19}^{(2,1,5)} &= \langle R^2, M \rangle, & 30\langle R^2 \rangle, 12\langle M \rangle &= 30C_2, 12C_5, \\ G_{21} = G_{19}^{(2,3,1)} &= \langle R^2, Z^M \rangle, & 30\langle R^2 \rangle, 12\langle Z^M \rangle &= 30C_2, 20C_3. \end{aligned}$$

We observe that the reflections $Z \in G_{11}$ and $Z^M \notin G_{11}$ of order 3, play different roles, since

$$Z \in G_{21} = \langle R^2, Z^M \rangle, \qquad \langle R^2, Z \rangle = G_6. \tag{5.37}$$

At the next level, we have

$$\begin{aligned} G_{16} = G_{19}^{(1,1,5)} &= \langle M, M^{R^2} \rangle, 12\langle M \rangle = 12C_5, \\ G_{20} = G_{19}^{(1,3,1)} &= \langle Z, Z^M \rangle, \qquad 20\langle Z \rangle = 20C_3, \\ G_{22} = G_{19}^{(2,1,1)} &= \langle R^2, (R^2)^Z, (R^2)^M \rangle, \qquad 30\langle R^2 \rangle = 30C_2, \end{aligned}$$

with the identity group below that. The intersection $G_{16} \cap G_{20} \cap G_{22}$ is a normal collineation preserving subgroup of G_{19} with order 120 *(which is not a reflection group).*

The observation (5.37) that G_6 is a necessarily not normal subgroup of G_{21} suggests other such subgroup relationships between the normal subgroups of G_7, G_{11} and G_{19}. We observe that the reflection types of these groups are

$$G_6: \ 6C_2, 4C_3, \qquad G_{21}: \ 30C_2, 20C_3,$$

and so such an inclusion could hold, but not as a normal subgroup. By considering such possibilities, it is easy to determine all such subgroup relationships (see Figure 6). We first observe that since S_4 is not a subgroup of A_5 (by Lagrange's theorem), the groups with maximal reflection group G_{11} (collineation group S_4) and with maximal reflection group G_{19} (collineation group A_5) cannot be subgroups of each other. Therefore, only a subgroup of G_7 could be contained in a collineation preserving subgroup of G_{11} or G_{19}. A basic set of such inclusions (for the generators of Table 2), which implies all others, is

$$G_7 = G_7 \cap G_{21} \subset G_{21} \qquad (G_7 \text{ has five conjugates in } G_{21} \text{ and } G_{19}),$$

$$G_5 = G_5 \cap G_{20} \subset G_{20} \qquad (G_5 \text{ has five conjugates in } G_{20} \text{ and } G_{19}).$$

In particular, the first inclusion gives our initial observation: $G_6 \subset G_7 \subset G_{21}$, and we observe that

$$G_{11} \cap G_{19} = G_7.$$

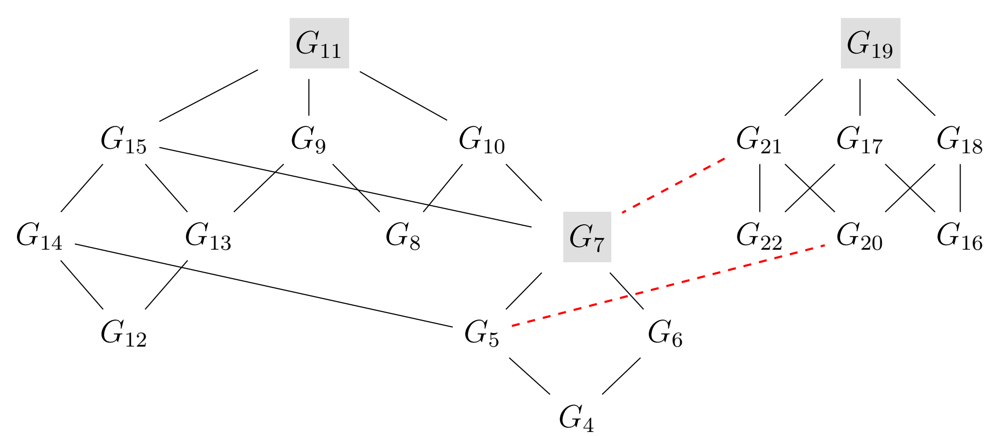


Figure 6: The inclusions between the primitive complex reflection groups $G_4, \ldots, G_{22}$. The lines indicate normal subgroups of G_{11}, G_7, G_{19}, and the dashed lines indicate the further inclusions $G_7 \subset G_{21}$ and $G_5 \subset G_{20}$, with G_7 and G_5 having five conjugates in those subgroups of G_{19} that contain them (and hence not being normal).

The remaining primitive complex reflection groups have reflections of order 2, 3 (which are not composite), and so only those with more than one reflection orbit can have normal subgroups. These groups are G_{26} and G_{28} (see Table 3), and the corresponding normal subgroups are given in Figure 7.

$$\begin{tikzcd}[column sep=small]
& G_{26}^{(2,3)} = G_{26} \arrow[dl, dash] \arrow[dr, dash] & \\
G_{26}^{(1,3)} = G_{25} \arrow[dr, dash] & & G_{26}^{(2,1)} = G(3,3,3) \arrow[dl, dash] \\
& G_{26}^{(1,1)} = 1^* &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& G_{28}^{(2,2)} = G_{28} \arrow[dl, dash] \arrow[dr, dash] & \\
G_{28}^{(1,2)} = G(2,2,4) \arrow[dr, dash] & & G_{28}^{(2,1)} = G(2,2,4) \arrow[dl, dash] \\
& G_{28}^{(1,1)} = 1^* &
\end{tikzcd}$$

Figure 7: The normal reflection subgroup lattices for G_{26} (rank 3) and G_{28} (rank 4). Here G_{25} is a collineation preserving subgroup of (its maximal reflection group)G_{26}.

We now collect our results to characterise the primitive complex reflection groups which are maximal reflection groups.

Theorem 5.1 *The primitive complex reflection groups which are maximal reflection groups are*

$$G_7,\ G_{11},\ G_{19},\ G_{23},\ G_{24},\ G_{26}, \ldots, G_{37}, \tag{5.38}$$

with the remaining ones being proper normal subgroups of the maximal reflection groups that contain them. The only other nontrivial normal reflection subgroups of the 17 *groups of (5.38) are the following three full rank imprimitive complex reflection groups*

$$G(4,2,2) \lhd G_7, \quad G(4,2,2) \lhd G_{11}, \quad G(3,3,3) \lhd G_{26}, \quad G(2,2,4) \lhd G_{28}, \tag{5.39}$$

which are precisely the three (imprimitive) complex reflection groups with more than one system of imprimitivity.

Proof: We have already observed which primitive complex reflection groups are normal subgroups of their maximal reflection group.

We further observe the primitive reflection groups associated with a given maximal reflection group are uniquely determined by the order of their collineation group. Indeed, the only cases where the collineation group of maximal reflection groups have equal orders is for G_{19} and G_{23} (order 60, isomorphic to A_5), G_{32} and G_{33} (order 25920, isomorphic), with all these groups having different ranks.

It is a simple observation that the three imprimitive complex reflection groups of (5.39) give all the complex reflection groups with multiple systems of imprimitivity, e.g., Theorem 2.16 of [LT09] gives these groups together with $G(2,1,2) \cong G(4,4,2)$, which is a real reflection group (see Example 2.1). We also note that the last three of (5.39) appear as Exercises 2.8, 2.6, 2.9 (respectively) of [LT09]. $\square$

By considering the collineation groups and reflection types, we came across the following observations (compare the first with the Remark 3.1).

Example 5.3 *The group G_{35} is the only primitive complex reflection group with trivial centre, i.e., it is an ordinary representation of its abstract group, or, equivalently equals its collineation group.*

Example 5.4 *The* 19 *primitive complex reflection groups (of* 34*)*

$$G_4,\ G_8,\ G_{12},\ G_{16},\ G_{20},\ G_{22},\dots,G_{25},\ G_{27},\ G_{29},\dots,G_{37} \tag{5.40}$$

have no nontrivial normal reflection subgroups, by virtue of each having a single reflection orbit corresponding to a reflection of prime order (either 2, 3 *or* 5*), see Figure 8.*

$$\begin{tikzcd}[row sep=large] G^{(m)} = G_k \arrow[d, dash] \\ G^{(1)} = 1 \end{tikzcd} \qquad \begin{tikzcd}[row sep=large] G^{(4)} = G_8 \arrow[d, dash] \\ G^{(2)} = G(4,2,2) \arrow[d, dash] \\ G^{(1)} = 1 \end{tikzcd}$$

Figure 8: The lattice of normal reflection subgroups for those G_k of (5.40) with a single orbit of reflections of prime order $m \in \{2,3,5\}$, and of G_8 a single $6C_4$ orbit (reflections of orders 2 and 4), with the normal subgroup $G(4,2,2)$ having the orbit split into three.

Example 5.5 *In addition to the inclusions of Figure 6, the groups G_{27} (rank* 3*) and G_{31} (rank* 4*) contain reflection subgroups of the same rank which are not normal and are maximal reflection groups (see Figure 9).*

$$\begin{tikzcd} G_{20} \arrow[d, dash] & G_{21} \arrow[d, dash] & G_{27} \arrow[d, dash] & & G_{31} \arrow[dl, dash] \arrow[dr, dash] & \\ G_5 & G_7 & G_{23} & G_{28} & & G_{29} \end{tikzcd}$$

Figure 9: Some incidental inclusions, i.e., those not occurring as normal subgroups.

Since the complex reflection groups of Example 5.4 which have one orbit are not cyclic groups, a minimal generating set of the reflections for them will require at least two reflections from the orbit. At the other extreme, we will say that a reflection group is **nicely generated** if its minimal generating sets of reflections are precisely those obtained by taking a reflection from each orbit. A rank d complex reflection group has a minimal generating set of d or $d+1$ reflections, with those generated by d reflections said to be **well-generated**.

Theorem 5.2 *The nicely generated primitive complex reflection groups are precisely*

$$G_7,\ G_{11},\ G_{19},\quad \textit{(three generators)},\qquad G_6,\ G_{10},\ G_{18},\quad \textit{(two generators)}.$$

The groups G_6, G_{10}, G_{18} are also well generated (giving starry complex polygons).

Proof: A minimal generating set of reflections for a complex reflection group must contain such a set of reflections (see Lemma 7.1), and it is easily verified that all such subsets generate the group. □

Table 2: The reflection types and collineation groups of the primitive complex reflection groups of rank 2 (with generators).

ST	Order	Reflection type	$\lvert Z(G)\rvert$	K_a	Reflections	G_{coll}	Generators
G_4	24	$4C_3$	2	C_6	3^8	A_4	Z, Z^S or Z^R, Z^{SR}
G_5	72	$4C_3, 4C_3$	6	C_6	3^{16}	A_4	Z, Z^R
G_6	48	$6C_2, 4C_3$	4	C_4, C_{12}	$2^6 3^8$	A_4	R^2, Z or R^2, Z^R
G_7	144	$6C_2, 4C_3, 4C_3$	12	C_{12}	$2^6 3^{16}$	A_4	R^2, Z, Z^R
G_8	96	$6C_4$	4	C_4	$2^6 4^{12}$	S_4	R, R^F
G_9	192	$12C_2, 6C_4$	8	C_8	$2^{18} 4^{12}$	S_4	F, R
G_{10}	288	$6C_4, 8C_3$	12	C_{12}	$2^6 3^{16} 4^{12}$	S_4	R, Z
G_{11}	576	$12C_2, 6C_4, 8C_3$	24	C_{24}	$2^{18} 3^{16} 4^{12}$	S_4	F, R, Z
G_{12}	48	$12C_2$	2	C_2	2^{12}	S_4	F, F^Z, F^{Z^2}
G_{13}	96	$12C_2, 6C_2$	4	C_4, C_8	2^{18}	S_4	F, F^Z, R^2
G_{14}	144	$12C_2, 8C_3$	6	C_6	$2^{12} 3^{16}$	S_4	F, Z
G_{15}	288	$12C_2, 6C_2, 8C_3$	12	C_{12}, C_{24}, C_{12}	$2^{18} 3^{16}$	S_4	F, R^2, Z
G_{16}	600	$12C_5$	10	C_{10}	5^{48}	A_5	M, M^{R^2}
G_{17}	1200	$30C_2, 12C_5$	20	C_{20}	$2^{30} 5^{48}$	A_5	R^2, M
G_{18}	1800	$20C_3, 12C_5$	30	C_{30}	$3^{40} 5^{48}$	A_5	Z, M
G_{19}	3600	$30C_2, 20C_3, 12C_5$	60	C_{60}	$2^{30} 3^{40} 5^{48}$	A_5	R^2, Z, M
G_{20}	360	$20C_3$	6	C_6	3^{40}	A_5	Z, Z^M
G_{21}	720	$30C_2, 20C_3$	12	C_{12}	$2^{30} 3^{40}$	A_5	R^2, Z^M
G_{22}	240	$30C_2$	4	C_4	2^{30}	A_5	$R^2, (R^2)^Z, (R^2)^M$

Table 3: The reflection types of the primitive complex reflection groups of rank $d \geq 3$.

ST	Rank	Order	Reflection type	$\lvert Z(G)\rvert$	K_a	Reflections	$\lvert G_{\text{coll}}\rvert$ (G_{coll} simple)
G_{23}	3	120	$15C_2$	2	C_2	2^{15}	60 (simple)
G_{24}	3	336	$21C_2$	2	C_2	2^{21}	168 (simple)
G_{25}	3	648	$12C_3$	3	C_6	3^{24}	216
G_{26}	3	1296	$9C_2, 12C_3$	6	C_6	$2^9 3^{24}$	216
G_{27}	3	2160	$45C_2$	6	C_6	2^{45}	360 (simple)
G_{28}	4	1152	$12C_2, 12C_2$	2	C_2	2^{24}	576
G_{29}	4	7680	$40C_2$	4	C_4	2^{40}	1920
G_{30}	4	14400	$60C_2$	2	C_2	2^{60}	7200
G_{31}	4	46080	$60C_2$	4	C_4	2^{60}	11520
G_{32}	4	155520	$40C_3$	6	C_6	3^{80}	25920 (simple)
G_{33}	5	51840	$45C_2$	2	C_6	2^{45}	25920 (simple)
G_{34}	6	39191040	$126C_2$	6	C_6	2^{126}	6531840
G_{35}	6	51840	$36C_2$	1	C_2	2^{36}	51840
G_{36}	7	2903040	$63C_2$	2	C_2	2^{63}	1451520 (simple)
G_{37}	8	696729600	$120C_2$	2	C_2	2^{120}	348364800 (simple)

The nontrivial normal reflection subgroups of the primitive reflection groups are primitive (Theorem 2.1), and hence of full rank. We now show that the subgroups of irreducible reflection groups which are not of full rank are not normal.

A subgroup of a reflection group is **parabolic** if it is the pointwise stabiliser of some set of points (and hence of their linear span). The stabilisers of larger subspaces give smaller groups, with the identity stabilising the whole space. It is a classical result of Steinberg [Ste64] that parabolic subgroups of complex reflection groups are reflection groups, and this extends to the quaternionic reflection groups [BST23]. Clearly

- The parabolic subgroups of G have a smaller rank (by definition).
- The reflection subgroups R_a for a given root (which pointwise fix $a^\perp$) are the maximal parabolic subgroups of rank one.
- The smallest nontrivial subgroups of the R_a are (by Steinberg's theorem) the minimal nontrivial parabolic subgroups of G.

Proposition 5.1 *A parabolic subgroup of an (irreducible) reflection group is not normal unless it is the trivial subgroup.*

Proof: Let H be a parabolic subgroup of an irreducible reflection group G acting on V, so that there exists a nonzero $v \in V$ with

$$hv = v, \qquad \forall h \in H.$$

Suppose that H is normal in G. Then

$$\begin{aligned} g^{-1}hgv = v, \quad \forall g \in G \quad &\implies \quad h(gv) = gv, \quad \forall g \in G \\ &\implies \quad H \text{ pointwise fixes } \operatorname{span}\{gv\}_{g \in G}. \end{aligned}$$

Since G is irreducible, i.e., $\operatorname{span}\{gv\}_{g\in G} = V$, this implies that H pointwise fixes V, and hence is the identity group (since it acts faithfully on V). $\square$

We observe that the condition G be a reflection group is not required in the above argument. In particular, we have the following.

Example 5.6 *The reflection subgroups R_a of an irreducible reflection group G are not normal, and hence the reflection orbits R_a^G contain more than one point.*

If G is reducible, then R_a^G can have a single point, e.g., take G to be the reducible subgroups $C_4 \times C_4$ and $C_2 \times C_2$ of $G(8, 2, 2)$ in Example 4.4.

Before considering the normal subgroups of the imprimitive reflection groups, we use our examples for primitive groups to introduce a geometric description of the reflection type and its relationship with the abelianisation of a reflection group.

6 A geometric description of the reflection type

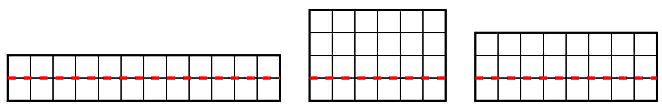
For a complex reflection group G with reflection type

$$n_1 C_{k_1}, \dots n_m C_{k_m},$$

we can visualise the reflection orbit $R_{a_j}^G$, of type $n_j C_{k_j}$, as a $k_j \times n_j$ matrix (block)

$$[R_{a_j}, R_{a_j}^{g_2}, \dots, R_{a_j}^{g_{n_j}}],$$

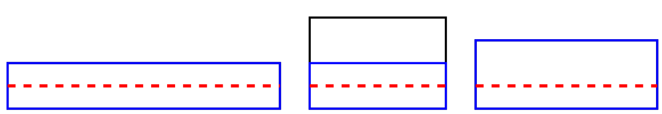
where $g_1 = 1, g_2, \dots, g_{n_j}$ is a transversal of G/R_{a_j}. The i-th $k_j \times 1$ column contains the k_j elements of the cyclic group $R_{a_j}^{g_i}$, $1 \le i \le n_j$, which are $k_j - 1$ reflections and the identity, which we give some fixed ordering with the identity at the bottom (below the dotted red line), e.g., for $G = G_{11} = G^{(2,4,3)}$, of reflection type $12C_2, 6C_4, 8C_3$, we have

$G = G_{11}$  $12C_2, 6C_4, 8C_3$.

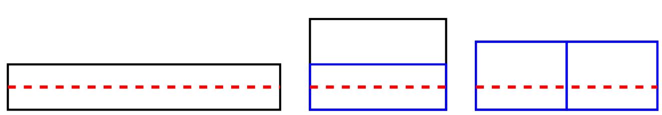
We could remove the squares below the red line, but prefer to leave them, so that each column gives all the elements of a copy of the group C_{k_j}. We have

- Each small square above the red line corresponds to a reflection in the orbit, with those below being the identity, e.g, G_{11} has $12 \cdot 1 + 6 \cdot 3 + 8 \cdot 2 = 46$ reflections.
- The conjugation action of G acts transitively on the columns of each block, giving the reflection orbit.
- Taking the G-orbit of a subgroup $\hat{R}_a$ if R_a corresponds to removing rows of the corresponding block (with G continuing to act transitively on the subcolumns).

Each reflection subgroup H of G corresponds its set of reflections, which we will indicate by enclosing them in a blue box (if there are reflections taken from the orbit), with the small boxes corresponding to individual elements not printed for clarity. Moreover, we will indicate the H-orbits orbits of the reflection subgroups (columns) by sub-boxes if they do not equal the G-orbits, e.g., for a split orbit of a normal reflection subgroup. For example, for $G^{(2,2,3)} = G_{15}$, we have

$G^{(2,2,3)} = G_{15}$  $12C_2, 6C_2, 8C_3$.

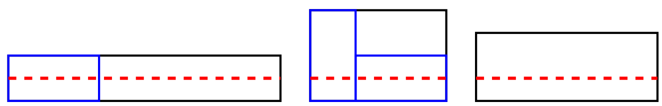
and for $G^{(1,2,3)} = G_7$, which has a split orbit, we have

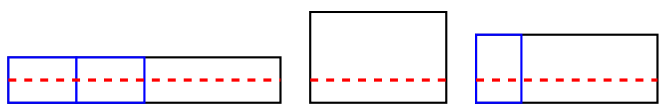
$G^{(1,2,3)} = G_7$  $6C_2, 4C_3, 4C_3$ (split orbit).

The normal reflection subgroups of G are characterised by

- The union of the blue boxes in each orbit is a single box extending across the whole block, i.e., the reflections are obtained by removing rows of the block (i.e., taking a subgroup of R_{a_j}).
- Moreover, any partitioning of the large blue box is into blue boxes of the same size (length and height) corresponds to the splitting of the orbit.

In light of the above, we now consider the orbit structure of the reflection subgroups which are not normal. For the maximal reflection subgroups $G(8,2,2)$ and $G(6,2,2)$ of G_{11} (see Figure 3), which are not normal, we have

$G(8,2,2)$  $4C_2, 2C_4, 4C_2$ (3 conjugates)

$G(6,2,2)$  $3C_2, 3C_2, 2C_3$ (4 conjugates)

Since the conjugation action is transitive on columns (and subcolumns), it is clear that these groups are not normal. Moreover, since $|G_{11}|/|G(8,2,2)| = 9$, by the orbit size theorem, $G(8,2,2)$ must have 3 or 9 conjugates in G_{11}, with the different images of the blue boxes giving different conjugates. It so happens, that the images of the $4C_2$ boxes and of the $2C_4$ boxes are disjoint, and we conclude that there are 3 conjugates of $G(8,2,2)$. These are uniquely determined by the images of the $4C_2$ reflections inside the original $12C_2$ orbit, which must partition it (and similarly for the $2C_4$ orbit). Similarly, $G(6,2,2)$ has 4 conjugates. This leads to the following notion.

We will call a subset of a reflection orbit of size n_j a **domino** if

- It consists of columns $[\hat{R}_{a_j}^{g_1}, \ldots, \hat{R}_{a_j}^{g_\ell}]$ for some fixed nontrivial subgroup $\hat{R}_{a_j}$ of R_{a_j}.
- Its G-orbit partitions $\hat{R}_{a_j}^G$, equivalently, the G-orbit of $[R_{a_j}^{g_1}, \ldots, R_{a_j}^{g_\ell}]$ partitions $R_{a_j}^G$.
- The reflection group that it generates contains no additional reflections.

The number of conjugates of a reflection subgroup constructed using dominos is relatively easy to calculate, since conjugation permutes the set of dominos, i.e., conjugate dominos have no overlap (by definition). Every reflection orbit has dominos of lengths n_j and 1, and we have

- A reflection subgroup is normal (has one conjugate) if and only if its reflections consist of dominos of length n_j.
- The (parabolic) reflection groups of rank one obtained by taking a single domino $[\hat{R}_{a_j}]$ of length 1 are not normal, and have n_j conjugates.

For G_{11} there are also dominos of other lengths, and these are counted up to conjugation. As with reflection orbits, we will say that an $\alpha_j \times \ell$ domino $[\hat{R}_{a_j}^{g_1}, \ldots, \hat{R}_{a_j}^{g_\ell}]$, $\alpha_j = |\hat{R}_{a_j}|$, has **reflection type** ℓC_{α_j}, and that it is **unique** if its G-orbit gives the only partition of $\hat{R}_{a_j}^G$ into dominos of that size.

Lemma 6.1 *The complex reflection group G_{11}, of type $12C_2, 6C_4, 8C_3$, has dominos of the following sizes (reflection types)*

$$\begin{aligned} 12C_2: &\quad C_2, \quad 2C_2 \text{ (6 dominos)}, \quad 3C_2 \text{ (4 dominos)}, \quad 4C_2 \text{ (3 dominos)}, \quad 12C_2, \\ 6C_4: &\quad C_4,\, C_2, \quad 2C_4,\, 2C_2, \text{ (3 dominos)}, \quad 6C_4,\, 6C_2, \\ 8C_3: &\quad C_3, \quad 2C_3, \text{ (4 dominos)}, \quad 4C_3, \text{ (2 dominos)}, \quad 8C_3, \end{aligned}$$

and these are unique, except for the $3C_2$ dominos of which there are two (partitions).

Proof: By direct computation in Magma. □

Example 6.1 *Taking a single domino gives a reflection subgroup of the same type. The one-domino subgroups given by the dominos in Lemma 6.1 are*

$$\begin{aligned} 12C_2: &\quad C_2, \quad C_2 \times C_2, \quad G(3,3,2), \quad B_2 = G(4,4,2), \quad G_{12}, \\ 6C_4: &\quad C_4,\, C_2, \quad C_4 \times C_4,\, C_2 \times C_2, \quad G_8,\, G(4,2,2), \\ 8C_3: &\quad C_3, \quad C_3 \times C_3, \quad G_4, \quad G_5, \end{aligned}$$

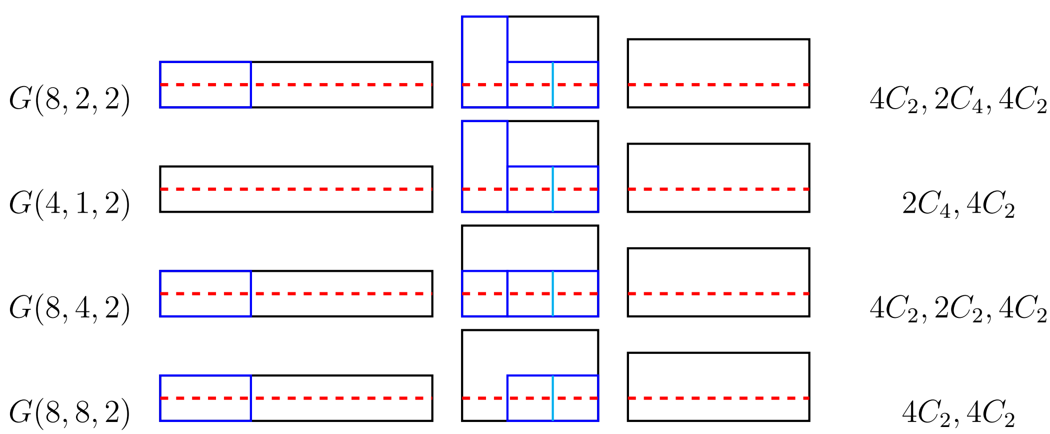
and their number of conjugates is the number dominos in the partition, i.e., n_j divided by the domino length. Each appears once (up to conjugacy), except for $G(3,3,2)$ (the domino is not unique) which occurs twice, see (4.34).

It turns out that the reflection orbit of $G(8,2,2)$ which is not a domino, but a disjoint union of two dominos, is an exceptional case.

Theorem 6.1 *Only four of the 38 reflection subgroups of G_{11} have reflection orbits which do not consist of (single) dominos, namely $G(8,2,2)$ and its subgroups $G(4,1,2)$, $G(8,4,2)$, $G(8,8,2)$. These subgroups all have a common reflection orbit which is the disjoint union of two $2C_2$ dominos sitting inside the $6C_4$ reflection orbit of G_{11}, with all the other reflection orbits being dominos, and they each have three conjugates in G_{11}.*

Proof: By direct computation in Magma, or by using the generators of Table 2. □

Example 6.2 *The reflection types of the four reflection subgroups of G_{11} in Theorem 6.1, which have a $4C_2$ orbit consisting of a disjoint union of two $2C_2$ (square) dominos.*



Corollary 6.1 *The G_{11}-orbit of every reflection orbit of the reflection subgroups of G_{11} partitions the G_{11} reflection orbit it is contained in, except for the $4C_2$ reflection orbit of $G(4,1,2)$ and one of the $4C_2$ reflection orbits of $G(8,2,2)$, $G(8,4,2)$ and $G(8,8,2)$.*

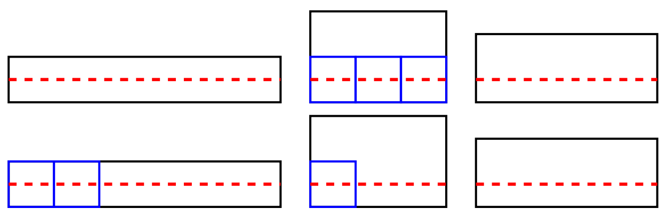
The reflection subgroups can be constructed from and understood using domino decompositions of the reflection orbits. We will not develop the entire theory here, but instead give some indicative examples.

The reflection subgroups are defined by their reflection orbits, and their occurrences and number of conjugacies by the decomposition into dominos (which is unique), e.g., $G(4,2,2)$ occurs twice as a subgroup of G_{11}, with one occurrence being normal.

Example 6.3 *The decompositions of the reflections for the two occurrences of the subgroup $G(4,2,2)$ of G_{11}, of reflection type $2C_2, 2C_2, 2C_2$, into dominos is as follows.*

$G(4,2,2)$ *(normal, split orbit)*

$G(4,2,2)$ *(3 conjugates)*



We observe the first copy above is normal and the second is not.

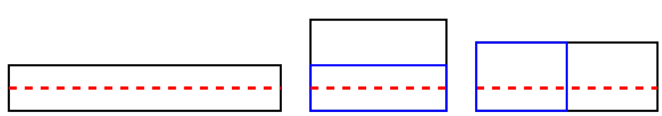
To construct reflection subgroups, we can begin with a domino in some reflection orbit and consider the domino structure of those reflections (from the other orbits) which stabilise it (this is most often a single domino). The stabiliser of a domino is generally not a reflection group, but is does contain unique reflection subgroup generated by its reflections.

Example 6.4 *For the dominos of length 1 given by the reflections for a single root on the reflection orbits $12C_2, 6C_4, 8C_3$ of G_{11}, the stabiliser groups have orders $48, 96, 72$, and contain $2, 6, 4$ reflections on the given orbit (those for the root and its conjuguate). These stablising reflections give the $2C_m$ dominos (of length 2), which correspond to the reflection subgroups $C_m \times C_m$.*

Example 6.5 *The dominos of length n_j are fixed by all reflections, and so each union of them gives a different group, i.e., the normal reflection subgroups.*

Example 6.6 *There is just one reflection subgroup of G_{11} given by a single domino which has two conjugates, namely the $4C_3$ domino which gives G_4. Hence, any other reflection subgroups with two conjugates would be given by the $4C_3$ domino together with dominos of maximum length from the other orbits which stabilise it. There is one such case, namely the $4C_3$ domino and the $6C_2$ domino, which gives G_6.*

G_6 *(2 conjugates)*



Here the reflections which stabilise the $4C_3$ domino are the $8C_3$ reflection orbit it belongs to and the $6C_2$ domino inside the $6C_4$ orbit, a union of dominos of maximum length, which therefore generate a normal reflection subgroup.

7 The abelianisation of a reflection group and its reflection type

Here we consider the close relationship between the reflection type of a complex reflection group G and its abelianisation G_{ab} ([Bec11] has a very general development including the relationship with braid groups and Stanley-Springer's Theorem). As a consequence, we explain why the normal reflection subgroups G^α of G are all different (Example 7.1).

Let $G' = [G,G]$ be the (derived) commutator subgroup of a finite group G. Then the **abelianisation** of G is the surjective homomorphism (or its image)

$$G \to \frac{G}{[G,G]},$$

where $G_{\mathrm{ab}} := \frac{G}{[G,G]}$ is the largest quotient of G by a normal subgroup which is abelian.

Lemma 7.1 *If $G = G^{(k_1,\dots,k_m)}$ is complex reflection group, then its abelianisation maps reflections to reflections of the same order, with each reflection orbit $n_j C_{k_j}$ mapping onto the internal direct summand C_{k_j} of*

$$G_{\mathrm{ab}} = \frac{G}{[G,G]} = C_{k_1} \times \cdots \times C_{k_m}.$$

i.e., the abelianisation of $G^{(k_1,\dots,k_m)}$ is $C_{k_1} \times \cdots \times C_{k_m}$. As a consequence, a generating set of reflections for G must contain at least one reflection from each reflection orbit, and $[G,G]$ is a reflection free normal subgroup of G.

Proof: Consider the group homomorphism (linear character) given by taking the determinant

$$\det : G \to \mathbb{C}^*,$$

which maps reflections to elements of the same order, which are not in the kernel of det. We observe that $\det(G)$ is abelian, and hence $\det|_G$ factors through G_{ab}.

Since the abelianisation maps the conjugacy class of an element to a single element, the image of a sequence of reflections $r_j \in n_j C_{k_j}$ of order k_j, gives commuting elements of orders $k_1, \dots, k_m$ which generate $\frac{G}{[G,G]}$, which is therefore equal to $C_{k_1} \times \cdots \times C_{k_m}$.

By taking the natural action of $C_{k_1} \times \cdots \times C_{k_m}$ on $\mathbb{C}^m$, the images of the reflections can be viewed as reflections. Indeed, if $G = C_{k_1} \times \cdots \times C_{k_m}$ then $G = G^{(k_1,\dots,k_m)}$. $\square$

Example 7.1 *For a complex reflection group $G = G^{(k_1,\dots,k_m)}$, the image of the generating reflections for the normal reflection subgroup $N = G^{(\alpha)} = G^{(\alpha_1,\dots,\alpha_m)}$ of (2.12) under the abelianisation of G gives the subgroup $C_{\alpha_1} \times \cdots \times C_{\alpha_m}$ of $G_{\mathrm{ab}} = C_{k_1} \times \cdots \times C_{k_m}$, so that different choices of α give different normal subgroups G^α.*

The reflections in G^α are precisely $((R_{a_j})^{k_j/\alpha_j})^G$, and so the subgroup $C_{\alpha_1} \times \cdots \times C_{\alpha_m}$ of G_{ab} above is the abelianisation of G^α if and only if G^α has no split orbits, i.e., there is equality in

$$|(G^\alpha)_{\mathrm{ab}}| \geq \alpha_1 \alpha_2 \cdots \alpha_m.$$

Thus we may add another equivalence to Theorem 4.1.

Corollary 7.1 *A normal reflection subgroup $N = G^\alpha$ of a complex reflection group $G = G^{(k_1,\ldots,k_m)}$ is collineation preserving (has no split orbits) if and only of*

$$|(G^\alpha)_{\rm ab}| = \alpha_1\alpha_2\cdots\alpha_m. \tag{7.41}$$

Proof: We have observed in Example 7.1, that $\hat{R}^G_{a_j}$ of (2.12) is split orbit if and only if its image under the abelianisation of G is not cyclic, i.e., the image is $C_{\alpha_j}^{\ell_j}$, where ℓ_j is the number of N-orbits that $\hat{R}^G_{a_j}$ splits into. Indeed,

$$(G^\alpha)_{\rm ab} = C_{\alpha_1}^{\ell_1}\times\cdots\times C_{\alpha_m}^{\ell_m} \quad\implies\quad |(G^\alpha)_{\rm ab}| = {\alpha_1}^{\ell_1}{\alpha_2}^{\ell_2}\cdots{\alpha_m}^{\ell_m} \ge \alpha_1\alpha_2\cdots\alpha_m,$$

where there is equality in $|(G^\alpha)_{\rm ab}| \ge \alpha_1\alpha_2\cdots\alpha_m$ above if and only if $\ell_1 = \ldots \ell_m = 1$, i.e., there is no split orbit. □

As an example of the mechanics of Corollary 7.1, we have

$$G_7 = G_{11}^\alpha = G_{11}^{(1,2,3)} \lhd G_{11} = G_{11}^{(2,4,3)}, \qquad G_7 = G_7^{(2,3,3)},$$

so that

$$(G_7)_{\rm ab} = C_2\times C_3\times C_3 \ne C_2\times C_3 = C_{\alpha_1}\times\cdots\times C_{\alpha_m},$$

which gives

$$|(G_7)_{\rm ab}| = 18 > 6 = \alpha_1\alpha_2\cdots\alpha_m,$$

and hence G_7 is not a collineation preserving normal subgroup of G_{11}.

Example 7.2 *Let $G = G^{(k_1,\ldots,k_m)}$ be a complex reflection group. If G has more than one reflection orbit, i.e., $m \ge 2$, then the intersection of the reflection subgroups generated by each orbit, i.e.,*

$$H = H_G := G^{(k_1,1,\ldots,1)} \cap G^{(1,k_2,1,\ldots,1)} \cap\cdots\cap G^{(1,\ldots,1,k_m)}, \tag{7.42}$$

is a reflection free normal subgroup of $[G,G]$ and of G. This follows from the fact the abelianisation of G maps each of the above reflection subgroups to the corresponding internal direct summand of $G_{\rm ab} = C_{k_1}\times\cdots\times C_{k_m}$, and hence to 1, so that H is contained in the kernel of the abelianisation map, which is $[G,G]$. Moreover, H is an intersection of normal subgroups of G, and hence is normal in G (and therefore in $[G,G]$).

Typically, H is not the identity for G nonabelian, thereby giving an example where the intersection of normal reflection subgroups is not a reflection group, e.g., for G_{11}, the subgroups generated by the orbits are

$$G_{12} = \langle F^{G_{11}}\rangle = \langle F, F^Z, F^{Z^2}\rangle, \quad G_8 = \langle R^{G_{11}}\rangle = \langle R, R^F\rangle, \quad G_5 = \langle Z^{G_{11}}\rangle = \langle Z, Z^R\rangle,$$

and the intersection of any two of them gives H, which has order 24. It is easily verified that H is a collineation preserving subgroup of G_7 (22 reflections, 0 hidden reflections), which has 22 hidden reflections (and 0 reflections), with $H = [G_{11}, G_{11}]$, and hence

$$\frac{G_{11}}{H} = C_2\times C_4\times C_3.$$

It may be that H is a proper subgroup of $[G,G]$, thereby giving a nonabelian quotient, e.g., for the primitive reflection groups $G = G_{13}, G_{15}, G_{26}, G_{28}$, we have

$$\begin{aligned}
G_{13}: &\quad G/H \cong G(6,6,2), \quad G/[G,G] \cong C_2 \times C_2, \\
G_{15}: &\quad G/H \cong G(6,2,2), \quad G/[G,G] \cong C_2 \times C_2 \times C_3, \\
G_{26}: &\quad G/H \cong C_2 \times G_4, \quad G/[G,G] \cong C_2 \times C_3, \\
G_{28}: &\quad G/H \cong C_2 \times G(3,1,2), \quad G/[G,G] \cong C_2 \times C_2.
\end{aligned}$$

The quotients G/H above were identified with reflection groups as abstract groups, which suggests the normal subgroups H may be examples of the "bon sous-groupes distingués" of [BBR02] discussed in Section 9.

Interestingly, there is a collineation preserving reflection-free normal subgroup of G_{11} of order 48 (with 46 hidden reflections), which contains the group H of (7.42). We have no pursued such "reflection-free reflection groups" (though there are other cases, e.g., for G_{19} and G_{33}).

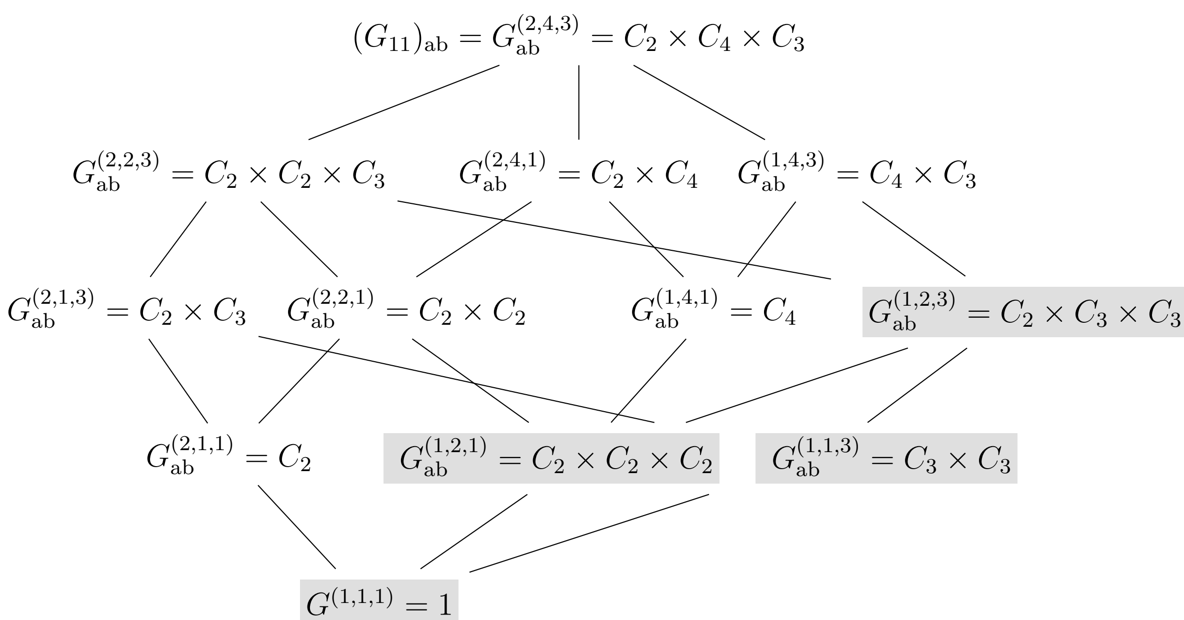


Figure 10: The abelianisations of the normal reflection subgroups of G_{11} placed in the lattice of Figure 1. Note the subgroups $G^\alpha \neq 1$ which are not collineation preserving (i.e., with split orbits) in grey, which are characterised by the inequality $|G_{\mathrm{ab}}^\alpha| > \alpha_1\alpha_2\alpha_3$.

For the quaternionic reflection groups, the Lemma 7.1 does not hold. Indeed, the abelianisation may be trivial.

Example 7.3 *The primitive quaternion reflection groups $G = O_1, O_2, P_2, R, S_3$ of [Coh80] have $G_{\mathrm{ab}} = 1$ (see Table 5). Groups such as these which equal their commutator, i.e., $G = [G,G]$, and hence have no nontrivial abelian quotients, are said to be* **perfect**. *The given reflection groups are not simple, though the collineation groups for O_1, O_2, R are.*

8 The imprimitive complex reflection groups

We now seek to identify the imprimitive complex reflection groups which are maximal reflection groups, and their normal reflection subgroups (which are imprimitive, but not necessarily irreducible). We first determine the reflection type of

$$G(m,p,d), \qquad p \,\Big|\, m,$$

which acts on $\mathbb{C}^d$. Let $\omega := e^{\frac{2\pi i}{m}}$, a primitive m-th root of unity. We recall that the the reflections in $G(m,p,d)$ have two types, i.e., those with root e_j, the j-th standard basis vector, which gives the diagonal reflections

$$r_{e_j,\xi} = \operatorname{diag}(1,\ldots,1,\xi,1,\ldots,1), \qquad \xi^{\frac{m}{p}} = 1, \tag{8.43}$$

where ξ is a power of ω^p, i.e., is an $(\frac{m}{p})$-th root of unity, and those of order 2 with roots $\zeta e_j - e_k$, $\zeta^m = 1$, in the $\frac{1}{2}d(d-1)$ subspaces $\operatorname{span}\{e_j, e_k\}$, $j \neq k$, for which (2.2) gives

$$r_{\zeta e_j - e_k, -1}\Big|_{\operatorname{span}\{e_j,e_k\}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} \zeta \\ -1 \end{pmatrix}\begin{pmatrix} \overline{\zeta} & -1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta \\ \overline{\zeta} & 1 \end{pmatrix}, \qquad \zeta^m = 1, \tag{8.44}$$

i.e., ζ is an m-th root of unity.

Lemma 8.1 *The rank 2 (imprimitive) reflection group $G(m,p,2)$ can have one, two or three orbits of reflections, as follows*

$$\begin{aligned} m = p: &\qquad mC_2 \quad (p \text{ odd}), &&\qquad \frac{m}{2}C_2, \frac{m}{2}C_2 \quad (p \text{ even}), \\ m \neq p: &\qquad mC_2, 2C_{\frac{m}{p}} \quad (p \text{ odd}), &&\qquad \frac{m}{2}C_2, \frac{m}{2}C_2, 2C_{\frac{m}{p}} \quad (p \text{ even}), \end{aligned}$$

and the rank $d \geq 3$ reflection group $G(m,p,d)$ can have one or two orbits of reflections, as follows

$$\begin{aligned} m = p: &\qquad \frac{1}{2}d(d-1)mC_2, \\ m \neq p: &\qquad \frac{1}{2}d(d-1)mC_2, dC_{\frac{m}{p}}. \end{aligned}$$

In summary, we have

$$G(m,p,d)_{\mathrm{ab}} = C_2 \times C_{\gcd(2,p)} \times C_{\frac{m}{p}}, \tag{8.45}$$

Proof: It suffices to find which reflection orbit the reflections of (8.43) and (8.44) lie on. We observe that $g_{jk} := r_{e_j - e_k, -1} \in G(m,p,d)$ is the permutation matrix which exchanges e_j and e_k, and so

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \quad \Longrightarrow \quad g_{jk}^{-1} r_{e_j,\xi} g_{jk} = r_{e_k,\xi},$$

and hence there is a single orbit of the d reflection subgroups $C_{\frac{m}{p}}$ $(m \neq p)$ for the diagonal reflections, and these reflections generate the reducible reflection subgroup $(C_{\frac{m}{p}})^d$.

For the $\binom{d}{2}m$ reflections (8.44) of order 2, we first consider the case $d = 2$. Now

$$\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & \overline{\xi}\\ \xi & 0\end{pmatrix}, \quad \begin{pmatrix}0 & \zeta\\ \overline{\zeta} & 0\end{pmatrix}^{-1}\begin{pmatrix}0 & \overline{\xi}\\ \xi & 0\end{pmatrix}\begin{pmatrix}0 & \zeta\\ \overline{\zeta} & 0\end{pmatrix} = \begin{pmatrix}0 & \zeta^2\xi\\ \overline{\zeta}^2\overline{\xi} & 0\end{pmatrix},$$

so that the orbit of $r_{e_1-e_2,-1}$ is

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{G(m,p,2)} = \Big\{ \begin{pmatrix}0 & \zeta\\ \overline{\zeta} & 0\end{pmatrix} : \zeta = \omega^{2j+pk}, j, k \in \mathbb{Z}_m \Big\},$$

which is all the reflections of type (8.44), unless p is even (and so m is even), in which case there are two orbits, i.e.,

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{G(m,p,2)} = \Big\{ \begin{pmatrix}0 & \omega^{2j}\\ \overline{\omega}^{2j} & 0\end{pmatrix} \Big\}_{1\le j\le \frac{m}{2}}, \qquad \begin{pmatrix}0 & \omega\\ \overline{\omega} & 0\end{pmatrix}^{G(m,p,2)} = \Big\{ \begin{pmatrix}0 & \omega^{2j-1}\\ \overline{\omega}^{2j-1} & 0\end{pmatrix} \Big\}_{1\le j\le \frac{m}{2}}.$$

For $d \geq 3$ and any value of p, let $V := \operatorname{span}\{e_1, e_2, e_3\}$ and

$$g := (r_{e_2-e_3,-1})(r_{\zeta e_2-e_3,-1})(r_{e_1-e_2,-1}), \quad \zeta^m = 1, \qquad g\Big|_V = \begin{pmatrix}0 & 1 & 0\\ \overline{\zeta} & 0 & 0\\ 0 & 0 & \zeta\end{pmatrix}.$$

Then

$$g\Big|_V (r_{\zeta e_1-e_2,-1}) g^{-1}\Big|_V = g|_V \begin{pmatrix}0 & \zeta & 0\\ \overline{\zeta} & 0 & 0\\ 0 & 0 & 1\end{pmatrix} g^{-1}|_V = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = r_{e_1-e_2,-1},$$

so that $r_{\zeta e_1-e_2,-1}$ is in the orbit of $r_{e_1-e_2,-1}$. Since all the permutation matrices are contained in $G(m,p,d)$, $r_{\zeta e_1-e_2,-1}$ can be conjugated to any $r_{\zeta e_j-e_k,-1}$, and so there is a single orbit of reflections of the type (8.44). $\square$

We observe that for $m = p$, we have $C_{\frac{m}{p}} = 1$, which contains no reflections, and so the separate treatment of the cases $m = p$ and $m \neq p$ in Lemma 8.1 is simply for clarity.

Example 8.1 *From Lemma 8.1, we may write $G(m,p,d) = G^{(k_1,\ldots,k_j)}$, $j \in \{1,2,3\}$, and therefore construct the lattice of its normal reflection subgroups (see, e.g., Figures 2, 11). For the six cases, in the order presented, we have*

$$G^{(2)}, \quad G^{(2,2)}, \qquad G^{(2,\frac{m}{p})}, \quad G^{(2,2,\frac{m}{p})}, \qquad G^{(2)}, \quad G^{(2,\frac{m}{p})}.$$

In this way, many familiar facts about $G(m,p,d)$ and its normal reflection subgroups can be obtained and understood in a transparent way, e.g., from the exercises of [LT09], we have

- *$G(m,p,d) \lhd G(m,1,d)$ (Exer. 2.5).*

- *$G(m,p,2) \lhd G(2m,2,2)$ (Exer. 2.7).*
- *For $d \geq 3$, the reflections in $G(m,m,d)$ form a single conjugacy class (Exer. 2.14).*
- *Every reflection in $G(m,m,d)$ has order 2 if and only $m = p$ or $2p$ (Exer. 2.15).*

From the first two cases, we have that $G(m,m,2)$, the dihedral group of order $2m$, $m > 2$, has a (nontrivial) normal reflection subgroup if and only if m is even, i.e., $G(\frac{m}{2}, \frac{m}{2}, 2)$ which occurs twice as a normal subgroup.

Example 8.2 *We have $G(1,1,n) = S_n$, which has rank $n-1$. Its reflections $r_{e_j - e_k}$ act as transpositions (odd permutations) on $e_1, \ldots, e_n$. Since $G(1,1,n) = G^{(2)}$ it has no normal reflection subgroups. We observe that for $n \geq 5$ the normal subgroups of S_n are the alternating group A_n which is not a reflection group, since it contains no reflections.*

Example 8.3 *The imprimitive reflection group $G(m,p,d)$ has a unique maximal abelian normal reflection subgroup $(C_{\frac{m}{p}})^d$, i.e., the one generated by its diagonal reflections.*

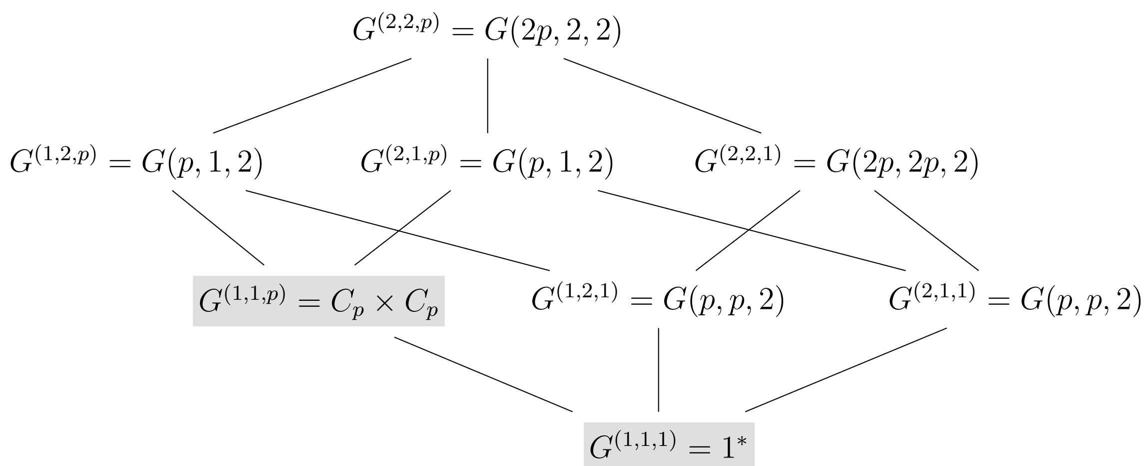


Figure 11: The lattice of normal reflection subgroups of $G^{(2,2,p)} = G(2p,2,2)$, for p prime (which is maximal). Note the repeated normal subgroups $G(p,1,2)$ and $G(p,p,2)$.

We now seek those $G(m,p,d)$ which are maximal reflection groups. By the celebrated Chevalley-Shephard-Todd theorem, a finite group $G \subset GL(\mathbb{C}^n)$ of rank n is a complex reflection group if and only if its ring of invariants is a polynomial ring, i.e., there exist algebraically independent homogeneous polynomials $p_1, \ldots, p_n$ of degrees $d_1, \ldots, d_n$ for which $\mathbb{C}[p_1, \ldots, p_n]$ is the ring of polynomial invariants $\mathbb{C}[x_1, \ldots, x_n]^G$. The fundamental invariants $d_1, \ldots, d_n$ are called the **degrees** of the reflection group G.

The centre of an irreducible complex reflection group G is a group of scalar matrices, and hence is cyclic, of order

$$|Z(G)| = \gcd(d_1, \ldots, d_n), \tag{8.46}$$

$$\begin{tikzcd}[column sep=small]
 & G^{(2,\frac{m}{p})} = G(m,p,d) \arrow[dl, dash] \arrow[dr, dash] & \\
G^{(1,\frac{m}{p})} = (C_{\frac{m}{p}})^d \arrow[dr, dash] & & G^{(2,\frac{m}{q})} = G(m,q,d) \arrow[dl, dash] \\
 & G^{(2,\frac{m}{p})} = (C_{\frac{m}{q}})^d &
\end{tikzcd}$$

Figure 12: A sublattice of normal reflection subgroups of $G^{(2,\frac{m}{p})} = G(m,p,d)$, $d \geq 3$.

where $d_1, \ldots, d_n$ are the degrees of G, and

$$|G| = d_2 d_2 \cdots d_n.$$

For $G = G(m,p,d)$, the degrees are

$$m, 2m, \ldots, (d-1)m, \;\; dm/p,$$

which gives

$$|Z(G(m,p,d))| = \gcd(m, d\frac{m}{p}) = \gcd(p\frac{m}{p}, d\frac{m}{p}) = \frac{m}{p}\gcd(p,d).$$

Since $G(m,p,d)$ is irreducible for $(m,p,d) \neq (2,2,2)$, its collineation group has order

$$\frac{|G(m,p,d)|}{|Z(G(m,p,d))|} = \frac{d!m^{d-1}(m/p)}{\gcd(p,d)(m/p)} = \frac{d!m^{d-1}}{\gcd(p,d)},$$

and we obtain in all cases (including $(m,p,d) = (2,2,2)$ and $d = 1$) that

$$|G(m,p,d)_{\mathrm{coll}}| = \frac{d!m^{d-1}}{\gcd(p,d)}, \tag{8.47}$$

which gives the collineation preserving inclusion

$$G(m,p,d) \lhd G(m, \gcd(p,d), d). \tag{8.48}$$

In view of (8.48), the candidate maximum reflection subgroups are

$$G(ka,k,d), \qquad k \,\Big|\, d, \quad a = 1,2,3,\ldots. \tag{8.49}$$

For $d = 2$, we have the collineation preserving inclusion (see Exercise 2.7 of [LT09])

$$G(m,1,2) \lhd G(2m,2,2), \tag{8.50}$$

and so taking $k = 1$ in (8.49) does not give a maximal reflection group.

For $d = 2$ (two degrees), the number of reflections is $d_1 + d_2 - 2$, and so we can use Proposition 3.1 to determine when a rank 2 complex reflection group is a maximal reflection group, i.e., when

$$d_1 + d_2 - 2 = 2\Big(\frac{d_1 d_2}{\gcd(d_1, d_2)} - 1\Big) \quad \Longleftrightarrow \quad d_1 = d_2.$$

For primitive groups, we recover the maximal reflection groups G_7, G_{11}, G_{19}, and for imprimitive groups we have the following.

Example 8.4 *The degrees of $G(m, p, 2)$ are equal, giving a maximal reflection group, if and only if*

$$m = 2\frac{m}{p} \quad \Longleftrightarrow \quad p = 2.$$

Since $p \,\big|\, m$, m must be even, and so

$$G(2a, 2, 2), \qquad a = 2, 3, 4, \ldots$$

are all the maximal imprimitive reflection groups of rank 2. Here we exclude the case $a = 1$, which gives a maximal reflection group $G(2, 2, 2)$ which is reducible.

From (8.48), we have

$$G(m, p, 2) \lhd G(m, \gcd(p, 2), 2) \qquad \textit{(collineation preserving)},$$

and this and (8.50) give the collineation preserving inclusions

$$\begin{aligned}
&G(m, p, 2) \lhd G(m, 1, 2) \lhd G(2m, 2, 2) && \textit{(m odd)},\\
&G(m, p, 2) \lhd G(m, 2, 2) && \textit{(m even, p even)},\\
&G(m, p, 2) \lhd G(m, 1, 2) \lhd G(2m, 2, 2) && \textit{(m even, p odd)}.
\end{aligned}$$

Therefore the maximal reflection group which contains $G(m, p, 2)$ as a normal subgroup is as follows

$$G(m, p, 2) \lhd G(\frac{2m}{\gcd(m, p, 2)}, 2, 2) \qquad \textit{(maximal reflection group)}. \tag{8.51}$$

We now present the general case.

Theorem 8.1 *The imprimitive complex reflection groups which are maximal reflection groups are*

$$\begin{aligned}
&G(2a, 2, 2), \qquad d = 2, \quad a = 2, 3, 4, \ldots,\\
&G(ka, k, d), \qquad d \geq 3, \quad k \,\big|\, d, \quad a = 1, 2, 3, \ldots,
\end{aligned} \tag{8.52}$$

with the remaining ones being proper normal subgroups of the maximal reflection groups that contain them, i.e.,

$$\begin{aligned}
&G(m, p, 2) \lhd G(\frac{2m}{\gcd(m, p, 2)}, 2, 2), \qquad d = 2,\\
&G(m, p, d) \lhd G(m, \gcd(p, d), d), \qquad d \geq 3.
\end{aligned} \tag{8.53}$$

Proof: We have already considered the case $d = 2$ in Example 8.4. We now consider the case $d \geq 3$, which is simpler because the reflection orbits are not as complicated.

If some of the candidate groups in (8.49) are not maximal reflection groups, then they appear as a collineation preserving normal subgroups of a larger such group, i.e., there is an analogue of (8.50). Therefore, we now consider normal reflection subgroups of the groups $G(ka, k, d)$ of (8.49), as given by the reflection types of Lemma 8.1.

For $a = 1$ ($m = p$), we have $G(k, k, d) = G^{(2)}$, which has no nontrivial normal reflection subgroups. For $a \geq 2$ ($m \neq p$), we have

$$G(ka, k, d) = G^{(2,a)},$$

giving the normal reflection subgroups

$$G^{(1,a)}, \qquad\qquad G^{(2,\frac{a}{r})} = G(ka, kr, d), \quad r \,\Big|\, a.$$

The subgroup $G^{(1,a)}$ is generated by the diagonal reflections in $G(ka, k, d)$, and hence is reducible. By (8.47), the subgroup $G^{(2,\frac{a}{r})} = G(ka, kr, d) = G(kr(\frac{a}{r}), kr, d)$ is collineation preserving if and only if

$$d!\frac{(ka)^{d-1}}{\gcd(k,d)} = d!\frac{(ka)^{d-1}}{\gcd(kr,d)} \qquad \Longleftrightarrow \qquad \gcd(k,d) = \gcd(kr,d).$$

For $G(ka, kr, d)$ to have the form of (8.52), we must have $kr \,\Big|\, d$, and so

$$\gcd(k,d) = \gcd(kr,d) \quad \Longrightarrow \quad k = kr \quad \Longrightarrow \quad r = 1,$$

and so none of the proper normal reflection subgroups of $G(ka, k, d)$ are collineation preserving, i.e., (8.52) gives all the maximal reflection groups.

The maximal reflection group for $G(m, p, 2)$ was given in Example 8.4. For $d \geq 3$, we observe that (8.48) gives the desired maximal reflection group, since $p \,\Big|\, m$ gives

$$G(m,p,d) \lhd G(m, \gcd(p,d), d) = G(k\frac{m}{\gcd(p,d)}, k, d), \qquad k = \gcd(p,d).$$

i.e., since $k \,\Big|\, d$, the larger group in the collineation preserving inclusion above is a maximal reflection group. $\square$

Table 4: The reflection types of the imprimitive complex reflection groups.

ST	Rank	Order	Reflection type	$\lvert Z(G)\rvert$	$\lvert G_{\text{coll}}\rvert$
$G(1,1,d),\ d \geq 2$	$d-1$	$d!$	$\frac{1}{2}d(d-1)C_2$	1	$d!$
$G(m,m,2),\ m$ odd, $m \neq 1$	2	$2m$	mC_2	1	$2m$
$G(m,m,2),\ m$ even, $m \neq 2$	2	$2m$	$\frac{m}{2}C_2,\ \frac{m}{2}C_2$	2	m
$G(m,m,2),\ p$ odd, $p \neq m$	2	$2m$	$mC_2, 2C_{\frac{m}{p}}$	1	$2m$
$G(m,p,2),\ p$ even, $p \neq m$	2	$2m$	$\frac{m}{2}C_2,\ \frac{m}{2}C_2, 2C_{\frac{m}{p}}$	2	m
$G(m,m,d),\ d \geq 3$	d	$d!m^{d-1}$	$\frac{1}{2}d(d-1)mC_2$	$\gcd(m,d)$	$\frac{d!m^{d-1}}{\gcd(m,d)}$
$G(m,p,d),\ d \geq 3, m \neq p$	d	$d!\frac{m^d}{p}$	$\frac{1}{2}d(d-1)mC_2, 2C_{\frac{m}{p}}$	$\frac{m}{p}\gcd(p,d)$	$\frac{d!m^{d-1}}{\gcd(p,d)}$

The maximal reflection groups of (8.52) have types

$$G(2a,2,2)=G^{(2,2,a)},\qquad G(ka,k,d)=G^{(2,a)},$$

and hence have $4\sigma_0(a)$ and $2\sigma_0(a)$ normal reflection subgroups, respectively.

Corollary 8.1 *The complex reflection group $G(m,p,d)$, $d\geq 3$, is a maximal reflection group if and only $p \mid d$.*

Proof: Since $p \mid m$, we have $G(m,p,d)=G(p(\frac{m}{p}),p,d)$. □

Example 8.5 *As simple example of (8.53), we have the collineation preserving inclusion*

$$G(4,2,3)\lhd G(4,\gcd(2,3),3)=G(4,1,3),$$

which shows that the condition $k \mid d$ is necessary to obtain a maximal reflection group.

9 The quotient by a normal reflection subgroup

Given that we have calculated the normal reflection subgroups of the complex reflection groups of rank d, it is now natural to consider the quotients. These were shown to be reflection groups (acting on an appropriate subspace of $\mathbb{C}^d$) by [BBR02], with the quotients (mostly) calculated in [AW23]. We now give some details.

For a complex reflection group G acting on $V=\mathbb{C}^d$, [BBR02] characterised those "nice" normal subgroups ("bon sous-groupes distingués") N for which the quotient group G/N acts faithfully on the tangent space to the quotient variety V/N at the point 0 as a reflection group, and that such groups include the normal reflection subgroups. For N a normal reflection subgroup, [AW20] (proof of Theorem 1.2) effectively show that this action is given by the action of G on the span of the subset of the basic N-invariant polynomials $f_1,\ldots,f_d$ with $R_G(f_j)=0$, where R_G the Reynolds operator for G, i.e.,

$$R_G(f):=\frac{1}{|G|}\sum_{g\in G} f\circ g.$$

For example, if $N=1$, basic N-invariant polynomials are $x_1,\ldots,x_d$, and $R_G(x_j)=0$ is equivalent to $\sum_{g\in G} g=0$, which holds for G irreducible, in which case the action of G on $x_1,\ldots,x_d$ is given by its action on $(x_1,\ldots,x_d)$. We now consider some examples.

Basic invariant polynomials for $G(m,p,d)$ are given by

$$\sum_{j=1}^d x_j^{mk},\quad 1\leq k\leq d-1,\qquad (x_1x_2\cdots x_d)^{\frac{m}{p}},\tag{9.54}$$

which have degrees $m,2m,\ldots,(d-1)m$, and $d\frac{m}{p}$. The "power sum" polynomials above can be replaced by $e_k(x_1^m,\ldots,x_d^m)$, where e_k is the k-th elementary symmetric function (see [KM25] for a discussion about "good basic invariants").

Example 9.1 *Consider the primitive complex reflection group $G = G_{26}$ (order 1296) and its normal reflection subgroup $N = G(3,3,3)$ (order 54), which are given by*

$$G = G_{26} = \langle r_1, r_2, P_{(12)} \rangle, \qquad N = G(3,3,3) = \langle r_3, P_{(12)}, P_{(23)} \rangle,$$

where

$$r_1 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_2 = \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & \overline{\omega} & \overline{\omega} \\ \overline{\omega} & \omega & \overline{\omega} \\ \overline{\omega} & \overline{\omega} & \omega \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 & \omega & 0 \\ \overline{\omega} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \omega := e^{\frac{2\pi i}{3}},$$

$$P_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By (9.54), invariant polynomials for N (of degrees $3, 6, 3$) are given by

$$x^3 + y^3 + z^3, \qquad x^6 + y^6 + z^6, \qquad xyz.$$

Applying the Reynolds operator for G to these gives

$$R_G(x^3 + y^3 + z^3) = 0, \qquad R_G(xyz) = 0,$$

$$R_G(x^6 + y^6 + z^6) = \frac{1}{6}\Big(x^6 + y^6 + z^6 - 10(x^3y^3 + x^3z^3 + y^3z^3)\Big),$$

and so the N-invariant polynomial of degree 6 can be replaced by a G-invariant one (on which G and hence G/N acts trivially). On the span of the remaining N-invariant polynomials G/N acts faithfully as a reflection group (i.e., reflections map to reflections). Let's consider the action of the inverse of the generators for G. For r_1, we have

$$x^3 + y^3 + z^3 \mapsto (\omega x)^3 + y^3 + z^3 = x^3 + y^3 + z^3, \quad xyz \mapsto (\omega x)yz = \omega(xyz),$$

and for r_2, we have

$$x^3 + y^3 + z^3 \mapsto \frac{i}{\sqrt{3}}\Big\{x^3 + y^3 + z^3 + 6\overline{\omega}xyz\Big\}, \quad xyz \mapsto \frac{i}{3\sqrt{3}}\Big\{\overline{\omega}(x^3 + y^3 + z^3) - 3\omega xyz\Big\},$$

with the action of $P_{(12)}$ being trivial. Hence the faithful action of G/N on

$$V = \operatorname{span}_{\mathbb{C}}\{x^3 + y^3 + z^3, xyz\}$$

gives the complex reflection group generated by

$$[r_1] = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \qquad [r_2] = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega}/3 \\ 6\overline{\omega} & -\omega \end{pmatrix},$$

which is easily seen to be the imprimitive complex reflection group G_4 (order 24).

Example 9.2 *The normal reflection subgroups of G_7 and their invariant polynomials are as follows:*

$$\begin{aligned}
G^{(1,1,1)} = 1 :& \qquad x,\ y, \\
G^{(1,1,3)} = G_4 :& \qquad x^4 - 2\sqrt{3}ix^2y^2 + y^4,\ x^5y - xy^5, \\
G^{(1,3,1)} = G_4 :& \qquad x^4 + 2\sqrt{3}ix^2y^2 + y^4,\ x^5y - xy^5, \\
G^{(2,1,1)} = G(4,2,2) :& \qquad x^4 + y^4,\ x^2y^2, \\
G^{(1,3,3)} = G_5 :& \qquad x^5y - xy^5,\ (x^4 \pm 2\sqrt{3}ix^2y^2 + y^4)^3, \\
G^{(2,1,3)} = G_6 :& \qquad x^4 - 2\sqrt{3}ix^2y^2 + y^4,\ (x^5y - xy^5)^2, \\
G^{(2,3,1)} = G_6 :& \qquad x^4 + 2\sqrt{3}ix^2y^2 + y^4,\ (x^5y - xy^5)^2, \\
G^{(2,3,3)} = G_7 :& \qquad (x^4 \pm 2\sqrt{3}ix^2y^2 + y^4)^3, \quad (x^5y - xy^5)^2,
\end{aligned}$$

These were obtained by applying the Reynolds operator to homogeneous polynomials (of the appropriate degree) in the invariant polynomials for subgroups, starting with the monomials (for the trivial subgroup 1). We observe the following linear relations

$$\begin{aligned}
&(x^4 + 2\sqrt{3}ix^2y^2 + y^4)^3 + (x^4 - 2\sqrt{3}ix^2y^2 + y^4)^3 = 2(x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}), \\
&(x^4 + 2\sqrt{3}ix^2y^2 + y^4)^3 - (x^4 - 2\sqrt{3}ix^2y^2 + y^4)^3 = 12\sqrt{3}i(x^5y - xy^5)^2,
\end{aligned}$$

and so the first invariant polynomial for G_7 given above can be replaced by one with integer coefficients, if so desired. Calculating the action of the generators for G_7 on the invariant polynomials given for G_4 gives the following

$$\begin{aligned}
G^{(1,1,3)} = G_4 :& \qquad [R^2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ [Z] = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix},\ [Z^R] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
G^{(1,3,1)} = G_4 :& \qquad [R^2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ [Z] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ [Z^R] = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

so that

$$\frac{G_7}{G_4} = C_2 \times C_3 \qquad \text{(in both cases).}$$

Example 9.3 *We consider $N = G(4,2,2) \lhd G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{13}, G_{15} \subset G_{11}$. A set of basic invariant polynomials for $N = G(4,2,2)$ is*

$$x^4 + y^4, \quad x^2y^2.$$

With respect to this basis, the action of the generators for G_{11} on $\operatorname{span}_{\mathbb{C}}\{x^4 + y^4, x^2y^2\}$ is given by

$$[F] = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & -\frac{1}{2} \end{pmatrix}, \quad [R] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [Z] = \begin{pmatrix} -\frac{1}{2}\overline{\omega} & \frac{1}{4}\overline{\omega} \\ -3\overline{\omega} & -\frac{1}{2}\overline{\omega} \end{pmatrix}.$$

These matrices are reflections (not all unitary), and it is not obvious that they generate an imprimitive reflection group. A basis of eigenvectors for $[Z]$ is given by

$$B := \begin{pmatrix} 1 & -1 \\ 2\sqrt{3}i & 2\sqrt{3}i \end{pmatrix},$$

and in this basis we have

$$[F]_B = \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}, \quad [R]_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad [Z]_B = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix},$$

where $[g]_B = B^{-1}[g]B$. *Thus*

$$\frac{G_{11}}{G(4,2,2)} \cong \langle \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \rangle = G(6,2,2),$$

with the other quotients giving reflection subgroups of $G(6,2,2)$, *i.e.,*

$$\frac{G_{15}}{G(4,2,2)} \cong \langle [F]_B, [R^2]_B, [Z]_B \rangle = \langle \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \rangle \cong G(3,1,2),$$
$$\frac{G_9}{G(4,2,2)} \cong \langle [F]_B, [R]_B \rangle = \langle \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \rangle \cong G(6,6,2),$$
$$\frac{G_{10}}{G(4,2,2)} \cong \langle [R]_B, [Z]_B \rangle = \langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \rangle \cong G(3,1,2),$$
$$\frac{G_{13}}{G(4,2,2)} \cong \langle [F]_B, [F^Z]_B, [R^2]_B \rangle = \langle \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \cong G(3,3,2) \cong G(1,1,3),$$
$$\frac{G_8}{G(4,2,2)} \cong \langle [R]_B, [R^F]_B \rangle = \langle \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\overline{\omega} \\ -\omega & 0 \end{pmatrix} \rangle \cong G(3,3,2) \cong G(1,1,3),$$
$$\frac{G_7}{G(4,2,2)} \cong \langle [R^2]_B, [Z]_B, [Z^R]_B \rangle = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \rangle \cong C_3 \times C_3,$$
$$\frac{G_6}{G(4,2,2)} \cong \langle [R^2]_B, [Z]_B \rangle = \langle \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \rangle, \langle [R^2]_B, [Z^R]_B \rangle = \langle \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \rangle \cong C_3.$$

From the reflection orbits of Lemma 8.1, we obtain the following.

Example 9.4 *For* m *odd,* $G(m,m,2) = G^{(2)}$ *has no normal reflection subgroups, and for* m *even, we have* $G(m,m,2) = G^{(2,2)}$, $G^{(1,2)} \cong G^{(2,1)} \cong G(\frac{m}{2}, \frac{m}{2}, 2)$, *and*

$$\frac{G^{(2,2)}}{G^{(1,2)}}, \frac{G^{(2,2)}}{G^{(2,1)}} = \frac{G(m,m,2)}{G(\frac{m}{2}, \frac{m}{2}, 2)} \cong C_2, \qquad (m \text{ even}). \tag{9.55}$$

Now suppose that $p \mid m$, $p \neq m$, *and* $\frac{m}{q} \mid \frac{m}{p}$, *i.e.,* $p \mid q$ *and* $q \mid m$.

For p *odd (two orbits), we have* $G(m,p,2) = G^{(2,\frac{m}{p})}$, *and*

$$\frac{G^{(2,\frac{m}{p})}}{G^{(1,\frac{m}{q})}} = \frac{G(m,p,2)}{C_{\frac{m}{q}} \times C_{\frac{m}{q}}} \cong G(q,p,2), \qquad (p \text{ odd}), \tag{9.56}$$

$$\frac{G^{(2,\frac{m}{p})}}{G^{(2,\frac{m}{q})}} = \frac{G(m,p,2)}{G(m,q,2)} \cong C_{\frac{q}{p}}, \qquad (p \text{ odd}). \tag{9.57}$$

For p even (three orbits), we have $G(m,p,2) = G^{(2,2,\frac{m}{p})}$, and

$$\frac{G^{(2,2,\frac{m}{p})}}{G^{(1,1,\frac{m}{q})}} = \frac{G(m,p,2)}{C_{\frac{m}{q}} \times C_{\frac{m}{q}}} \cong G(q,p,2), \qquad (p \text{ even}), \tag{9.58}$$

$$\frac{G^{(2,2,\frac{m}{p})}}{G^{(2,2,\frac{m}{q})}} = \frac{G(m,p,2)}{G(m,q,2)} \cong C_{\frac{q}{p}}, \qquad (p \text{ even}), \tag{9.59}$$

$$\frac{G^{(2,2,\frac{m}{p})}}{G^{(1,2,\frac{m}{q})}}, \frac{G^{(2,2,\frac{m}{p})}}{G^{(2,1,\frac{m}{q})}} = \frac{G(m,p,2)}{G(\frac{m}{2},\frac{q}{2},2)} \cong C_2 \times C_{\frac{q}{p}}, \qquad (p \text{ even}). \tag{9.60}$$

By collecting cases with different orbit structures, one obtains

$$\frac{G(m,p,2)}{C_{\frac{m}{q}} \times C_{\frac{m}{q}}} \cong G(q,p,2), \qquad \frac{G(m,p,2)}{G(m,q,2)} \cong C_{\frac{q}{p}}, \qquad \frac{G(m,p,2)}{G(\frac{m}{2},\frac{q}{2},2)} \cong C_2 \times C_{\frac{q}{p}} \quad (p \text{ even}). \tag{9.61}$$

which are (2.a), (2.b), (2.c) of Theorem 6.1 of [AW23].

Example 9.5 *For $d \geq 3$, $G(m,m,d) = G^{(2)}$ has no normal reflection subgroups, and for $G(m,p,d) = G^{(2,\frac{m}{p})}$, $p \neq m$, we have*

$$\frac{G^{(2,\frac{m}{p})}}{G^{(1,\frac{m}{q})}} = \frac{G(m,p,d)}{(C_{\frac{m}{q}})^d} \cong G(q,p,d), \qquad d \geq 3, \tag{9.62}$$

$$\frac{G^{(2,\frac{m}{p})}}{G^{(2,\frac{m}{q})}} = \frac{G(m,p,d)}{G(m,q,d)} \cong C_{\frac{q}{p}}, \qquad d \geq 3, \tag{9.63}$$

where $p \,\Big|\, q$ and $q \,\Big|\, m$.

Example 9.6 *The complex reflection group $G = G_{28}$ is generated by the reflections*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Invariant polynomials for $N = G(2,2,4)$ are

$$x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad x_1^4 + x_2^4 + x_3^4 + x_4^4, \quad x_1^6 + x_2^6 + x_3^6 + x_4^6, \quad x_1x_2x_3x_4,$$

with the first also being G-invariant, and for the third $R_G(x_1^6 + x_2^6 + x_3^6 + x_4^6) \neq 0$. Thus, we consider the action of G on $x_1^4 + x_2^4 + x_3^4 + x_4^4$ and $x_1x_2x_3x_4$. These polynomials are fixed by the permutation matrices, and the action of the diagonal reflection is

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \mapsto x_1^4 + x_2^4 + x_3^4 + x_4^4, \quad x_1x_2x_3x_4 \mapsto -x_1x_2x_3x_4.$$

The final (non-monomial) generating reflection maps as follows

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \mapsto -\frac{1}{2}(x_1^4 + x_2^4 + x_3^4 + x_4^4) + 6(x_1x_2x_3x_4) + \frac{3}{4}(x_1^2 + x_2^2 + x_3^2)^2,$$

$$x_1x_2x_3x_4 \mapsto \frac{1}{8}(x_1^4 + x_2^4 + x_3^4 + x_4^4) + \frac{1}{2}(x_1x_2x_3x_4) - \frac{1}{16}(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2,$$

so that, in this case, G does not map the span of the given basic invariant polynomials for N with $R_G(f) = 0$ to itself, and so instead we must take the image of this space under the action of G. In this way, the action of the last two reflections on the span of $(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$, $x_1^4 + x_2^4 + x_3^4 + x_4^4$ and $x_1x_2x_3x_4$ is given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & \frac{3}{4} & -\frac{1}{16} \\ 0 & -\frac{1}{2} & \frac{1}{8} \\ 0 & 6 & \frac{1}{2} \end{pmatrix},$$

which generate a rank two reflection group, namely $G/N \cong G(3,3,2) \cong G(1,1,3) \cong S_3$.

Let $G = G^{(k_1,\ldots,k_m)}$ be a complex reflection group, $N = G^\alpha = G^{(\alpha_1,\ldots,\alpha_m)}$ be a normal reflection subgroup, and $r_{a_j} \in R_{a_j}^G$ be reflection of (maximal) order k_j. Then

- $r_{a_j}N$ is a reflection in G/N.
- The order of $r_{a_j}N$ is k_j/α_j.

The latter follows from the fact $r_{a_j}^m \in R_{a_j}^G$ and $R_{a_j}^G \cap R_{a_j}^N = \langle r_{a_j}^{k_j/\alpha_j}\rangle$, which gives

$$(r_{a_j}N)^m = r_{a_j}^m N = N \iff r_{a_j}^m \subset \langle r_{a_j}^{k_j/\alpha_j}\rangle \iff (k_j/\alpha_j) \Bigm| m.$$

Moreover, $G \to G/N$ maps reflection orbits to reflection orbits, and so Lemma 7.1 gives

$$\Big(\frac{G}{N}\Big)_{\mathrm{ab}} = \Big(\frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}}\Big)_{\mathrm{ab}} \cong C_{\frac{k_1}{\alpha_1}} \times \cdots \times C_{\frac{k_m}{\alpha_m}}. \tag{9.64}$$

From Example 9.3, we collect the nonabelian quotients by $G(4,2,2)$.

$$\begin{aligned} &\frac{G_{11}}{G(4,2,2)} \cong G(6,2,2), \quad \frac{G_{15}}{G(4,2,2)} \cong G(3,1,2), \quad \frac{G_9}{G(4,2,2)} \cong G(6,6,2), \\ &\frac{G_{10}}{G(4,2,2)} \cong G(3,1,2), \quad \frac{G_{13}}{G(4,2,2)} \cong G(1,1,3), \quad \frac{G_8}{G(4,2,2)} \cong G(1,1,3). \end{aligned} \tag{9.65}$$

We can now describe all quotients by normal reflection subgroups.

Theorem 9.1 *Let $G = G^{(k_1,\ldots,k_m)}$ be a complex reflection group and $N = G^{(\alpha_1,\ldots,\alpha_m)} \neq 1$ be a normal reflection subgroup, then the reflection group G/N is abelian, i.e., $[G,G] \lhd N$, given by*

$$\frac{G}{N} = \frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}} \cong C_{\frac{k_1}{\alpha_1}} \times \cdots \times C_{\frac{k_m}{\alpha_m}}, \tag{9.66}$$

except for the following cases

(a) $\dfrac{G(m,p,d)}{(C_{\frac{m}{q}})^d} \cong G(m,q,d)$*, $d \geq 2$, where $p \bigm| q$ and $q \bigm| m$.*

(b) *$G = G_8, G_9, G_{10}, G_{11}, G_{13}, G_{15}$ and $N = G(4,2,2)$, with G/N given by (9.65).*

(c) $\dfrac{G_{26}}{G(3,3,3)} \cong G_4$ *(the only case where G/N is primitive).*

(d) $\dfrac{G_{28}}{G(2,2,4)} \cong G(3,3,2) \cong G(1,1,3) \cong S_3$.

We observe that the normal subgroups $G(4,2,2)$, $G(3,3,3)$, $G(2,2,4)$ above are precisely the imprimitive complex reflection groups with more than one system imprimitivity.

Moreover, if N is a collineation preserving subgroup of G then G/N is cyclic, which is equivalent to

$$\frac{|G^{(k_1,\ldots,k_m)}|}{|G^{(\alpha_1,\ldots,\alpha_m)}|} = \operatorname{lcm}(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m}). \tag{9.67}$$

Proof: By the definition of the abelianisation and (9.64), we have that

$$\frac{G}{N} = \frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}} \to \Big(\frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}}\Big)_{\text{ab}} \cong C_{\frac{k_1}{\alpha_1}} \times \cdots \times C_{\frac{k_m}{\alpha_m}}$$

is a surjective homomorphism, which is injective, i.e., G/N is abelian and given by (9.66), if and only if the group order is preserved, i.e.,

$$\frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}} \text{ is abelian} \quad \iff \quad \frac{|G^{(k_1,\ldots,k_m)}|}{|G^{(\alpha_1,\ldots,\alpha_m)}|} = \frac{k_1k_2\cdots k_m}{\alpha_1\alpha_2\cdots\alpha_m}. \tag{9.68}$$

It is easy to verify when (9.68) holds from the reflection group orders and reflection types (see Example 9.7), which leads to the exceptions (a), (b), (c), (d). The nonabelian quotients are calculated in Examples 9.1, 9.2, 9.3, 9.4, 9.5.

If N is collineation preserving, then N is irreducible, so that $G = \langle\zeta\rangle N$, where ζ is a primitive $|Z(G)|/|Z(N)| = |G|/|N|$ root of unity, and $G/N \cong \langle\zeta\rangle$ is cyclic, so we must have

$$C_{\frac{k_1}{\alpha_1}} \times \cdots \times C_{\frac{k_m}{\alpha_m}} \text{ is cyclic} \quad \iff \quad \frac{k_1k_2\cdots k_m}{\alpha_1\alpha_2\cdots\alpha_m} = \operatorname{lcm}(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m}). \tag{9.69}$$

Further, since G/N is abelian, (9.68) gives

$$\frac{G^{(k_1,\ldots,k_m)}}{G^{(\alpha_1,\ldots,\alpha_m)}} \text{ is abelian} \quad \iff \quad \frac{|G^{(k_1,\ldots,k_m)}|}{|G^{(\alpha_1,\ldots,\alpha_m)}|} = \frac{k_1k_2\cdots k_m}{\alpha_1\alpha_2\cdots\alpha_m} = \operatorname{lcm}(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m}),$$

which is (9.67). Conversely, if (9.67) holds, then we must have equality in

$$\frac{|G^{(k_1,\ldots,k_m)}|}{|G^{(\alpha_1,\ldots,\alpha_m)}|} \geq \frac{k_1k_2\cdots k_m}{\alpha_1\alpha_2\cdots\alpha_m} = \operatorname{lcm}(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m}),$$

so that (9.68) gives that G/N is abelian, and (9.69) that it is cyclic. $\square$

From the proof of Theorem 9.1, we have the inequalities

$$\frac{|G^{(k_1,\ldots,k_m)}|}{|G^{(\alpha_1,\ldots,\alpha_m)}|} \ge \frac{k_1k_2\cdots k_m}{\alpha_1\alpha_2\cdots\alpha_m} \ge \operatorname{lcm}(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m}), \tag{9.70}$$

where there is equality in the first if and only if G/N is abelian, and equality in the second if and only if G/N is cyclic.

Example 9.7 *Here we illustrate the cases of equality in (9.70) for $G^{(2,2,\frac{m}{p})} = G(m,p,2)$, p even (three orbits), of Example 9.4. From the group orders, we have*

$$\frac{|G^{(2,2,\frac{m}{p})}|}{|G^{(1,1,\frac{m}{q})}|} = \frac{2m(m/p)}{(m/q)^2} = 2q\frac{q}{p} > \frac{2\cdot 2\cdot(m/p)}{1\cdot 1\cdot(m/q)} = 4\frac{q}{p},\ q\ne 2 \quad\Longrightarrow\quad G/N \textit{ is not abelian},$$

$$\frac{|G^{(2,2,\frac{m}{p})}|}{|G^{(2,2,\frac{m}{q})}|} = \frac{2m(m/p)}{2m(m/q)} = \frac{q}{p} = \operatorname{lcm}\Big(\frac{2}{2},\frac{2}{2},\frac{m/p}{m/q}\Big) \quad\Longrightarrow\quad G/N \textit{ is cyclic},$$

$$\frac{|G^{(2,2,\frac{m}{p})}|}{|G^{(1,2,\frac{m}{q})}|} = \frac{2m(m/p)}{2(m/2)\frac{m/2}{q/2}} = 2\frac{q}{p} = \frac{2\cdot 2\cdot(m/p)}{1\cdot 2\cdot(m/q)} \quad\Longrightarrow\quad G/N \textit{ is abelian.}$$

For $q=2$ in the first case above, there is equality in the strict inequality, and we obtain an abelian quotient, i.e.,

$$\frac{G^{(2,2,\frac{m}{2})}}{G^{(1,1,\frac{m}{2})}} = \frac{G(m,2,2)}{C_{\frac{m}{2}}\times C_{\frac{m}{2}}} = C_2\times C_2.$$

For the last case above, $G/N = C_2\times C_{\frac{q}{p}}$, and we can check (9.67) using

$$\operatorname{lcm}\Big(\frac{2}{1},\frac{2}{2},\frac{m/p}{m/q}\Big) = \begin{cases} 2\frac{q}{p}, & \frac{q}{p} \textit{ odd},\\ \frac{q}{p}, & \frac{q}{p} \textit{ even};\end{cases}$$

to conclude that G/N is cyclic when $\frac{q}{p}$ is odd.

It is natural to consider to what extent (9.67) determines the collineation preserving normal subgroups.

Example 9.8 *If G/N is cyclic, i.e., (9.67) holds, then N is collineation preserving, except for the cases*

$$\frac{G_{10}}{G_5}\cong C_4,\quad \frac{G_{10}}{G_7}\cong C_2,\quad \frac{G_{14}}{G_5}\cong C_2,\quad \frac{G_{15}}{G_7}\cong C_2,$$

$$\frac{G(m,p,2)}{G(\frac{m}{2},\frac{q}{2},2)}\cong C_{2\frac{q}{p}}\quad (\tfrac{p}{2}\textit{ even},\ \tfrac{q}{p}\textit{ odd}),\qquad \frac{G(m,p,d)}{G(m,q,d)}\cong C_{\frac{q}{p}},\quad \gcd(p,d)\ne\gcd(q,d).$$

This follows by direct calculation of the size of the collineation groups of G and N.

Heuristically, the normal reflection subgroups $N = G^{(\alpha_1,\ldots,g_m)} \neq 1$ of $G =^{(k_1,\ldots,k_m)}$ have a hierarchy of "size" given by

$$\text{primitive,} \quad \text{imprimitive,} \quad \text{reducible (abelian)} \qquad \text{(large to small)}, \tag{9.71}$$

with the quotients G/N being of opposite size. For example, we have

- If N is primitive, then $G/N \cong C_{\frac{k_1}{\alpha_1}} \times \cdots \times C_{\frac{k_m}{\alpha_m}}$ is reducible.
- If N is imprimitive, then G/N is imprimitive or reducible, except for the one case $\dfrac{G_{26}}{G(3,3,3)} \cong G_4$.
- If N is reducible, then G and G/N are imprimitive, i.e., $\dfrac{G(m,p,d)}{(C_{\frac{m}{q}})^d} \cong G(m,q,d)$.

In the same vein, we also have

- If N is a collineation preserving subgroup of G, then G/N is reducible (cyclic).

We now consider when a normal reflection subgroup appears multiple times.

For a complex reflection group $G = G^{(k_1,\ldots,k_m)}$ of reflection type $n_1C_{k_1},\ldots,n_mC_{k_m}$, we consider the subgroup of permutations in S_m which preserve the reflection type, i.e., the stabiliser of $(k_1,\ldots,k_m)$ and $(n_1,\ldots,n_m)$. This group $\mathrm{Stab}_{S_m}(k,n)$ acts on the normal reflection subgroups $N = G^{(\alpha_1,\ldots,\alpha_m)}$, with the orbits being natural candidates for isomorphic normal subgroups. We now consider the cases when this group is nontrivial.

Example 9.9 *We consider those reflection groups $G = G^{(k_1,\ldots,k_m)}$ for which $\mathrm{Stab}_{S_m}(k,n)$ is nontrivial, and determine the corresponding orbits containing more than one normal reflection subgroup. For the primitive complex reflection groups, there are three cases*

$$\begin{aligned}
G_5 = G^{(3,3)}: &\qquad G^{(1,3)}, G^{(3,1)} \cong G_4 \lhd G_5,\\
G_7 = G^{(2,3,3)}: &\qquad G^{(2,1,3)}, G^{(2,3,1)} \cong G_6 \lhd G_7,\\
&\qquad G^{(1,1,3)}, G^{(1,3,1)} \cong G_4 \lhd G_7,\\
G_{28} = G^{(2,2)}: &\qquad G^{(1,2)}, G^{(2,1)} \cong G(2,2,4) \lhd G_{28}.
\end{aligned}$$

As an example, for $G_7 = G^{(2,3,3)}$ above, we have $\mathrm{Stab}_{S_3}((2,3,3),(6,4,4)) = \{(\,),(2\,3)\}$.

For the imprimitive complex reflection groups $G(m,p,d)$, by Example 9.4, there are three reflection orbits of the same type if and only if $\frac{m}{2} = 2$, $\frac{m}{p} = 2$, which gives

$$G(4,2,2) = G^{(2,2,2)}: \qquad G^{(1,2,2)}, G^{(2,1,2)}, G^{(2,2,1)} \cong G(4,4,2) \cong G(2,1,2) \lhd G(4,2,2),$$

$$G(4,2,2) = G^{(2,2,2)}: \qquad G^{(1,1,2)}, G^{(1,2,1)}, G^{(2,1,1)} \cong G(2,2,2) \cong C_2 \times C_2 \lhd G(4,2,2),$$

and there are two when

$$G(m,p,2) = G^{(2,2,\frac{m}{p})},\ p \text{ even}: \qquad G^{(1,2,\frac{m}{q})}, G^{(2,1,\frac{m}{q})} \cong G(\frac{m}{2},\frac{q}{2},2) \lhd G(m,p,2),$$

where $p \mid q$ and $q \mid m$. There are no other cases.

We now show that Example 9.9 gives all the cases where there is a repeated normal reflection subgroup.

Theorem 9.2 *The normal reflection subgroups $G^\alpha = G^{(\alpha_1,\ldots,\alpha_m)}$ and $G^\beta = G^{(\beta_1,\ldots,\beta_m)}$ of the complex reflection group $G = G^{(k_1,\ldots,k_m)}$ are isomorphic if and only if $\beta = \sigma\alpha$ for some permutation $\sigma \in S_m$ which fixes $(k_1,\ldots,k_m)$.*

The cases where a reflection group appears multiple times as a normal subgroup are

(i) $G_4 \lhd G_5$, $\quad G_6 \lhd G_7$, $\quad G_4 \lhd G_7$, $\quad G(2,2,4) \lhd G_{28}$ *(each appears twice).*

(ii) $G(4,4,2) \cong G(2,1,2) \lhd G(4,2,2)$, $\quad G(2,2,2) \cong C_2 \times C_2 \lhd G(4,2,2)$ *(three times).*

(iii) $G(\frac{m}{2},\frac{q}{2},2) \lhd G(m,p,2) \neq G(4,2,2)$, $\quad p$ *even,* $p \,\Big|\, q$, $q \,\Big|\, m$ *(each appears twice).*

Proof: For normal reflection subgroups to be isomorphic, equivalently conjugate in the general linear group (having the same character), they must have the same reflection type, i.e., one must have $\beta = \sigma\alpha = (\alpha_{\sigma j})$, for some $\sigma \in S_m$ where

$$n_j C_{\alpha_j} = n_{\sigma j} C_{\alpha_{\sigma j}} \quad \Longrightarrow \quad n_{\sigma j} = n_j, \quad \alpha_j = \alpha_{\sigma j}.$$

There are just two cases where some k_j is composite: G_{11} and its normal subgroups G_8, G_9, G_{10}, and $G(m,p,d)$ for $\frac{m}{p}$ composite. In each of these cases, the orbit length n_j appears only once, and so $n_{\sigma j} = n_j$ gives $\sigma j = j$ and hence $k_{\sigma j} = k_j$.

If k_j is prime, then we can have either

$$\alpha_j = k_j \quad \Longrightarrow \quad k_j = \alpha_j = \alpha_{\sigma j} = k_{\sigma j},$$

or $\alpha_j = 1$, in which case there is no restriction on σj other than $n_{\sigma j} = n_j$, which implies that $k_j = k_{\sigma j}$ (by inspection). Hence $\sigma \in \mathrm{Stab}_{S_m}(k,n)$, and therefore all the repeated normal subgroups are given by Example 9.9. $\square$

This gives the Theorem 6.2 of [AW23], with (1),(2) corresponding to (ii), (3) to (iii) and (4),(5) to (i), except for the following cases which are missing

$$G_6 \lhd G_7, \quad G_4 \lhd G_7, \quad \text{(each appears twice)}.$$

One might hope that the lattice of reflection groups obtained from that of the normal reflection subgroups of G by replacing N by G/N (see Figure 13) might give an "upside down" lattice of reflection subgroups of G, as the orders of $|G/N|$ would allow. However, this appears not to be the case, as G_7 has no reflection subgroups (normal or not) of order 6, and so the quotient $C_2 \times C_3 = G_7/G_4$ cannot be on such a sublattice. There is an inversion symmetry of the lattice of normal subgroups given by the correspondence

$$G^\alpha = G^{(\alpha_1,\ldots,\alpha_m)} \longleftrightarrow G^{k/\alpha} = G^{(\frac{k_1}{\alpha_1},\ldots,\frac{k_m}{\alpha_m})}. \tag{9.72}$$

This sets up a correspondence (duality) between normal reflection subgroups, e.g.,

G_7 : $\quad G_5 \longleftrightarrow G(4,2,2)$, $\quad G_4 \longleftrightarrow G_6$,

G_{11} : $\quad G_{15} \longleftrightarrow G(4,2,2)$, $\quad G_9 \longleftrightarrow G_5$, $\quad G_{10} \longleftrightarrow G_{12}$, $\quad G_{14} \longleftrightarrow G_8$, $\quad G_{13} \longleftrightarrow G_7$,

G_{19} : $\quad G_{21} \longleftrightarrow G_{16}$, $\quad G_9 \longleftrightarrow G_5$, $\quad G_{13} \longleftrightarrow G_7$.

This duality flips the group's location in the ordering (9.71), e.g., for the pairs above $|G^\alpha| \cdot |G^{k/\alpha}|$ is constant, except for the pair $G_{15} \longleftrightarrow G(4,2,2)$, which is the only case where one of the groups is not the intersection of those above it in the lattice, namely $G(4,2,2)$. Similarly, by (7.41), $|(G^\alpha)_{\text{ab}}| \cdot |(G^{k/\alpha})_{\text{ab}}|$ is constant, equal to $|G_{\text{ab}}|$, when the groups are collineation preserving (have no split orbits).

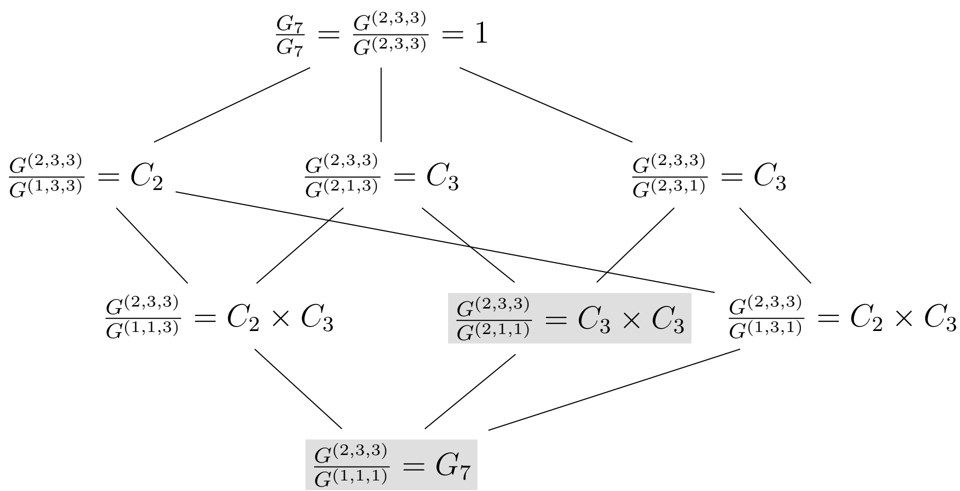


Figure 13: The quotient reflection groups G/N for the normal reflection subgroups $N = G^\alpha$ of $G = G_7 = G^{(2,3,3)}$, arranged as in the lattice of Figure 4. These all happen to be abelian, and hence equal to the abelianisation of G/N, for $N \neq 1$.

10 Quaternionic reflection groups

Here we consider some examples of the normal reflection subgroups of the quaternionic reflection groups. The main observations are

- The labels (2.10) giving the normal reflection subgroups may not be unique (see Examples 10.1 and 10.3), as they are for complex reflection groups (Theorem 2.1). Indeed, the reflections on one orbit may generate reflections on a different orbit.
- The stabiliser of a reflection subgroup R_a may not pointwise stabilise it, as is the case for complex reflection groups (Theorem 10.1).
- The abelianisation (which can be trivial) gives limited insight into the reflection subgroups for the roots, especially for primitive reflection groups (see Table 5).

We have not calculated all the possible multiple labels, and hence all the normal reflection subgroups, though this has been done for the primitive quaternionic reflection groups.

Example 10.1 *We consider the primitive quaternionic reflection groups of type P in [Coh80], which are given by (see [BW26])*

$$P_1 := \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+j & -1+j \\ -1+j & 1+j \end{pmatrix} \rangle, \qquad |P_1| = 320,$$
$$P_2 := \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+j & -1+j \\ -1+j & 1+j \end{pmatrix} \rangle, \qquad |P_2| = 1920,$$
$$P_3 := \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rangle, \qquad |P_3| = 3840.$$

with $P_1 \subset P_2 \subset P_3$ The group P_3 has 50 reflections of order two and 60 of order four. The reflections of order four are all conjugate (and also conjugate in P_2), and correspond to 10 root lines, giving a $10Q_8$ orbit (containing 10 reflections of order two), where

$$Q_8 = \{1, i, j, k, -1, -i, -j, -k\},$$

The remaining 40 reflections of order two form a single $40C_2$ orbit. The quaternion group Q_8 is Hamiltonian, i.e., is nonabelian with all of its six subgroups being normal. Hence, there are 12 labels of the form (2.10) giving normal subgroups of $G = G^{(C_2,Q_8)} = P_3$. The subgroups corresponding to these labels are

$$P_3 = G^{(C_2,Q_8)} = G^{(C_2,\langle i\rangle)} = G^{(C_2,\langle j\rangle)} = G^{(C_2,\langle k\rangle)} = G^{(C_2,\langle -1\rangle)} = G^{(C_2,1)} \quad \text{(6 labels)},$$
$$P_2 = G^{(1,Q_8)} = G^{(1,\langle i\rangle)} = G^{(1,\langle j\rangle)} = G^{(1,\langle k\rangle)} \quad \text{(4 labels)},$$
$$K = G^{(1,\langle -1\rangle)} \quad \text{(1 label)},$$
$$1 = G^{(1,1)} \quad \text{(1 label)},$$

where $K \subset P_1$ is the imprimitive quaternionic reflection group

$$K := \langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \rangle, \qquad |K| = 32.$$

Thus, P_3 has four normal reflection subgroups.

Since P_2 has a single $10Q_8$ orbit, it has, similarly, three normal reflection subgroups, i.e., 1, K, P_2 (4 labels). Since $G = G^{(C_4)} = P_1$ has a single $10C_4$ orbit, it has three normal reflection subgroups P_1, $G^{(C_2)} = K$, $G^{(1)} = 1$, with unique labels.

The reflection type of the imprimitive reflection group K is $2C_2, 2C_2, 2C_2, 2C_2, 2C_2$, and so there are $32 = 2^5$ labels for $K = K^{(2,2,2,2,2)}$ (here we adopt the notation (2.10) used in the complex case, since the reflection type consists only of cyclic groups). Let n_α be the number of 2's in the index α for a normal reflection subgroup K^α of K. The group K has six labels, for $n_\alpha = 5, 4$, i.e.,

$$K = K^{(2,2,2,2,2)} = K^{(1,2,2,2,2)} = K^{(2,1,2,2,2)} = K^{(2,2,1,2,2)} = K^{(2,2,2,1,2)} = K^{(2,2,2,2,1)} \quad \text{(6 labels)},$$

and the remaining 26 labels give distinct normal subgroups, namely

$$K^\alpha, \quad n_\alpha = 3, \quad \text{(10 isomorphic groups)}, \quad K^\alpha, \quad n_\alpha = 2, \quad \text{(10 isomorphic groups)},$$
$$K^\alpha, \quad n_\alpha = 1, \quad \text{(5 isomorphic groups)}, \quad K^{(1,1,1,1,1)} = 1 \quad (n_\alpha = 0).$$

Thus K has 27 normal reflection subgroups. We note that K has 68 normal subgroups and five subgroups which are not normal (the reflection subgroups R_a for its roots).

We observe that

- There need not be a unique label (2.10) for quaternionic reflection groups, with the number of labels being an upper bound for the number of normal reflection subgroups. Indeed, there is a lattice homomorphism from the lattice of labels onto the lattice of normal reflection subgroups.
- In particular, some quaternionic reflection groups can be generated by a set of reflections which does not contain at least one reflection from each orbit, i.e., Lemma 7.1 does not hold for quaternionic reflection groups.

The analogue of the abelianisation of Lemma 7.1 for a quaternionic reflection group G with reflection type $n_1 H_{a_1}, n_2 H_{a_2}, \ldots, n_m H_{a_m}$ would be the existence of a surjective homomorphism

$$G \to H_{a_1} \times H_{a_2} \times \ldots \times H_{a_m}, \tag{10.73}$$

which implies that a generating set of reflections for G must contain a reflection from each reflection orbit. By considering the normal subgroups of G, it is easy see that this is not possible for the groups of Example 10.1.

Example 10.2 *For the quaternionic reflection groups G in Example 10.1, the groups $H_{a_1} \times \ldots \times H_{a_m}$ of (10.73), the possible orders of nontrivial homomorphic images of G, and the abelianisations of G are*

$$\begin{array}{llll} C_2 \times Q_8, & Q_8, & C_4, & C_2 \times C_2 \times C_2 \times C_2 \times C_2, \\ 2, 120, 1920, & 60, 960, & 2, 10, 160, & 2, 4, 8, 16, \\ (P_3)_{\mathrm{ab}} = C_2, & (P_2)_{\mathrm{ab}} = 1, & (P_1)_{\mathrm{ab}} = C_2, & K_{\mathrm{ab}} = C_2 \times C_2 \times C_2 \times C_2, \end{array}$$

and so we conclude that there is no surjective homomorphism of the form (10.73) for any of these groups. We observe, however, that the abelianisation of K maps each of its five $2C_2$ reflection orbits to a different subgroup of order 2 (of which there are 15). Thus, we can conclude that a set of reflections which generates K must contain a reflection from four of its five reflection orbits. In [BW26], it is observed that all such sets of four reflections do generate K, which leads to the six labels for K in Example 10.1.

The reflection groups P_1, P_2, P_3, K have centre $\langle -1 \rangle$, and hence have no hidden reflections and have different collineation groups. In particular, we observe that

- The normal reflection subgroup P_2 of P_3 has reflection type $10Q_8$ and so does not have a split orbit, despite it not being collineation preserving.
  Therefore, Theorem 4.1 does not hold for quaternionic reflection groups.

There are also multiple labels for the quaternionic reflection groups of type O.

Example 10.3 *We consider the primitive quaternionic reflection groups of type O given in [Coh80]. In [Wal24], these are given by*

$$H_{120} = O_1 = \langle r_1, r_2, r_3 \rangle, \quad H_{720} = O_2 = \langle r_1, r_2, r_4 \rangle, \quad H_{1440} = O_3 = \langle r_1, r_2, r_5 \rangle,$$

where the first index is the order of the group, and the reflections $r_1, \ldots, r_5$ *are*

$$r_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}, \quad r_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} i & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}} i & \frac{1}{2} + \frac{1}{2\sqrt{3}} i \end{pmatrix}, \quad r_3 = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{3}} j & \frac{\sqrt{2}}{\sqrt{3}} k \\ \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{6}} k & \frac{1}{\sqrt{3}} j \end{pmatrix},$$

$$r_4 = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2} j & 0 \\ 0 & 1 \end{pmatrix}, \quad r_5 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} k \\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} k & 0 \end{pmatrix}.$$

Since H_{1440} *has reflection type* $20C_3, 30C_2$*, it has normal reflection subgroups*

$$H_{1440} = G^{(C_2,C_3)} = G^{(C_2,1)} \quad \text{(2 labels)}, \qquad H_{720} = G^{(1,C_3)}, \qquad 1 = G^{(1,1)}.$$

The reflection types of H_{720} *and* H_{120} *are* $20C_3$ *and* $10C_3$*, so they have no nontrivial normal reflection subgroups. These groups have centre* $\langle -1 \rangle$*, which is their only other normal subgroup, and so, in particular,* H_{720} *is not a collineation preserving normal subgroup of* H_{1440} *(which does not have a split orbit). The abelianisations of these groups are*

$$(H_{120})_{\mathrm{ab}} = 1, \quad (H_{720})_{\mathrm{ab}} = 1, \quad (H_{1440})_{\mathrm{ab}} = C_2.$$

These abelianisations map r_1, r_2, r_3, r_4 *to the identity and* r_5 *to the element of order* 2*.*

Unlike the complex case, for a quaternionic reflection group G, it is possible to have a proper subgroup $\hat{R}_{a_j}$ of the reflection subgroup R_{a_j} for a root a_j, with

$$\hat{R}_{a_j}^G = R_{a_j}^G, \qquad \hat{R}_{a_j} \neq R_{a_j}, \tag{10.74}$$

which leads to multiple labels.

We now explain this phenomenon in terms of the conjugation action of G on R_{a_j}.

Theorem 10.1 *Let* G *be a reflection group,* R_a *be the reflection subgroup for a root* a*, and* $\operatorname{Stab}_G(R_a) = \{g \in G : gR_a g^{-1} = R_a\}$ *be the stabiliser of* R_a*. Then*

(i) *If* G *is a complex reflection group or* $|R_a| = 2$*, then the stabiliser of* R_a *pointwise stabilises* R_a*, i.e.,* $g r_{a,\xi} g^{-1} = r_{a,\xi}$*,* $\forall r_{a,\xi} \in R_a$*,* $\forall g \in \operatorname{Stab}_G(R_a)$*.*

(ii) *If* G *is a quaternionic reflection group and* $|R_a| \neq 2$*, then there may be reflections in* R_a *of order greater than* 2 *which are not fixed by the stabiliser of* R_a*.*

As an example of (ii)*, all the reflections of order* 4 *in* P_1, P_2, P_3 *are conjugate, and all the reflections of order* 3 *in* $H_{120}, H_{720}, H_{1440}$ *are conjugate.*

Proof: If $g \in G$ conjugates the reflection $r_{a,\xi}$ to one with the same root, then (2.6) gives $ga = a\beta$, for some unit scalar $\beta \in K_a$, where K_a is defined in (2.4), and

$$g r_{a,\xi} g^{-1} = r_{ga,\xi} = r_{a\beta,\xi} = r_{a,\beta\xi\beta^{-1}},$$

which equals $r_{a,\xi}$ when G is complex, since in this case $\beta\xi\beta^{-1} = \xi$, or when $\xi = -1$, i.e., $r_{a,\xi}$ has order 2. For G a quaternionic reflection group and $\xi \neq -1$, it may be that $\beta\xi\beta^{-1} \neq \xi$, which gives a different reflection of the same order for the root a. We observe that $r_{a,\xi}$ and $r_{a,\xi}^{-1} = r_{a,\xi^{-1}}$ are reflections in R_a of the same order, and that they are different if and only if their order is not 2.

The Examples 10.1 and 10.3 give the listed examples, where $r_{a,\xi} \neq r_{a,\xi}^{-1}$ (of orders 4 and 3, respectively) are conjugate by some $g \in \operatorname{Stab}_G(R_a)$. □

The geometric description of the reflection type in terms of dominos (see Section 6) can be complicated for quaternionic reflection groups.

Example 10.4 *Let G be P_2 or P_3, which have a $10Q_8$ orbit. Let $R_a = \{r_{a,\xi}\}_{\xi\in Q_8}$ be the reflection subgroup for a given root a. Since this is not pointwise stabilised by* $\mathrm{Stab}_G(R_a)$*, it is natural to think of it as a set, rather than as a column (ordered set). The corresponding orbit gives an 8×10 matrix $[R_a^{g_1},\ldots,R_a^{g_{10}}]$, which partitions the 70 reflections by their roots. Now let $\hat{R}_a$ be the subgroup $R_{a,\langle i\rangle} = \{r_{a,\xi}\}_{\xi\in\langle i\rangle}$. The orbit of this "column of size 4" gives a 4×30 matrix, which includes three columns $R_{a,\langle i\rangle}, R_{a,\langle j\rangle}, R_{a,\langle k\rangle}$ for the root a, and similarly for the other roots. The total number of reflections in this matrix is $90 = 3\cdot 30$, with each reflection $r_{a,-1}$ appearing three times. Thus $\hat{R}_a = R_{a,\langle i\rangle}$ should be thought of as a domino with some caution.*

The reflection types and attributes of the remaining primitive quaternionic reflection groups of types Q, R, S, T, U are tabulated in Table 5. We give a quick overview.

Example 10.5 *The primitive quaternionic reflection groups*

$$Q,\ R\ \textit{(rank 3)},\qquad S_1,\ S_2,\ S_3\ \textit{(rank 4)},\qquad T, U\ \textit{(rank 5)}\ ,$$

each have a single n_1C_2 orbit, and hence have no nontrivial normal reflection subgroups. In this way, they can be thought of as being "close to simple". They all have centre $\langle -1\rangle$, which gives a third normal subgroup. In the cases where the collineation group $G/\langle -1\rangle$ is simple, we list it in the final column of Table 5. Moreover, all the proper normal subgroups are reflection free. In particular, Q and U have a collineation preserving normal subgroup of index 2 (with all the reflections being hidden), which is isomorphic to the collineation group (which is $PUS_3(3)$ and $PSU_5(2)$, respectively).

We have not yet calculated all the normal reflection subgroups of the imprimitive quaternionic reflection groups. However, we have included the imprimitive group K in Table 5, which seems to be indicative, as for the imprimitive quaternionic reflection groups $G(n,a,b,r)$ of [Wal25], for which $K = G(2,1,1,2)$, we have

$$G(n,a,b,r)_{\mathrm{ab}} = \begin{cases} C_2\times C_2, & \frac{n}{a}, \frac{n}{b}, r \text{ are odd;}\\ C_2\times C_2\times C_2\times C_2, & \frac{n}{a}, \frac{n}{b}, r \text{ are even, } r\neq \frac{2n}{ab};\\ C_2\times C_2\times C_2, & \text{otherwise.}\end{cases}$$

We observe that for the particular case

$$G(n,1,n,1) \cong_{\mathbb{H}} G(2n,n,2),$$

we have the abelianisation given by (8.45), i.e.,

$$G(2n,n,d)_{\mathrm{ab}} = C_2\times C_2\times C_{\gcd(2,n)}.$$

In all the cases calculated so far, it appears that the abelianisation of a quaternionic reflection group is an elementary 2-group.

Table 5: The primitive quaternionic reflection groups G (with the imprimitive group K for comparision), giving the reflection type, abelianisation, centre, with the last column giving the number of normal reflection subgroups and the number of normal subgroups (respectively). All the normal subgroups are reflection free except for G. For the groups Q and U there is index 2 collineation preserving reflection free simple normal subgroup.

Rank	G	Order	Reflection type	G_{ab}	$Z(G)$	$\#\triangleleft$
2	O_1	120	$10C_3$	1	$\langle -1\rangle$	2, 3 A_5
2	O_2	720	$20C_3$	1	$\langle -1\rangle$	2, 3 S_6
2	O_3	1440	$20C_3, 30C_2$	C_2	$\langle -1\rangle$	3, 4
2	P_1	320	$10C_4$	C_2	$\langle -1\rangle$	3, 5
2	P_2	1920	$10Q_8$	1	$\langle -1\rangle$	3, 4
2	P_3	3840	$10Q_8, 40C_2$	C_2	$\langle -1\rangle$	4, 5
3	Q	12096	$63C_2$	C_2	$\langle -1\rangle$	2, 4 $PSU_3(3)$
3	R	1209600	$315C_2$	1	$\langle -1\rangle$	2, 3 J_2
4	S_1	6912	$36C_2$	C_2	$\langle -1\rangle$	2, 10
4	S_2	82944	$72C_2$	C_2	$\langle -1\rangle$	2, 7
4	S_3	3317760	$180C_2$	1	$\langle -1\rangle$	2, 4
5	T	2592000	$180C_2$	C_2	$\langle -1\rangle$	2, 5
5	U	27371520	$165C_2$	C_2	$\langle -1\rangle$	2, 4 $PSU_5(2)$
2	K	32	$2C_2, 2C_2, 2C_2, 2C_2, 2C_2$	$(C_2)^4$	$\langle -1\rangle$	27, 68

11 Concluding remarks

Reflection groups G are typically generated by a small number of the reflections that they contain. Thus, to find reflection subgroups naively, one could take a small set of generators from the set of all reflections. The method given here involves taking a larger, but structured subset of generators, i.e., the reflection orbits $\hat{R}^G_{a_j}$ and more generally "dominos", then selecting a smaller subset of generators (when desired). This gives a simple method to find the normal reflection subgroups and their properties (Lemma 2.1).

A key observation is that every reflection group is a normal collineation preserving subgroup of its maximal reflection group, and so it suffices to determine the maximal reflection groups (Theorem 3.1). By combining the Theorems 5.1 and 8.1, we have:

Theorem 11.1 *The maximal complex reflection groups are the primitive groups*

$$G_7,\ G_{11},\ G_{19},\ G_{23},\ G_{24},\ G_{26},\ldots,G_{37},$$

and the imprimitive groups

$$\begin{aligned} &G(2a,2,2), && d=2, \quad a=2,3,4,\ldots, \\ &G(ka,k,d), && d\geq 3, \quad k \,\big|\, d, \quad a=1,2,3,\ldots, \end{aligned}$$

with each of these groups corresponding to a unique collineation group.

Given their prominence, one might develop an indexing for the 17 maximal primitive complex reflection groups G which reflects their properties, e.g., $M_{d,|G|}^{(k_1,\ldots,k_m)}$, where d is the rank of G, $|G|$ its order, and $n_1 C_{k_1}, \ldots, n_m C_{k_m}$ is its reflection type, i.e.,

$$M_{2,144}^{(2,3,3)} = G_7, \quad M_{2,576}^{(2,4,3)} = G_{11}, \quad M_{2,3600}^{(2,3,5)} = G_{19}, \quad \ldots \quad M_{8,696729600}^{(2)} = G_{37}. \qquad (11.75)$$

The rank and order uniquely defines a maximal primitive complex reflection group.

We gave two ways to calculate all the reflection subgroups (illustrated in the most complicated case of G_{11}), i.e., as normal subgroups of the reflection subgroups which are maximal reflection groups (Example 4.3), and using "dominos" (Section 6), which has the advantage that reflection subgroups and their conjugates can be visualised. Other results for complex reflection subgroups include:

- The splitting of the reflection orbit of a normal reflection subgroup corresponds to a change of the collineation group (Theorem 4.1, Corollary 7.1).
- The abelianisation corresponds to the reflection type (Lemma 7.1).
- The quotients by the normal reflection subgroups (which are reflection groups) were calculated (Theorem 9.1).

The (normal) reflection subgroups of the quaternionic reflection groups were also investigated up to the point where the differences in their structural properties from those of the complex reflection groups became apparent. In particular:

- Every quaternionic reflection group appears to be maximal.
- The abelianisation has a weaker correspondence with the reflection type.
- The reflections from one reflection orbit can generate reflections on another orbit, and so it is not necessary to pick a generating reflection from each orbit.
- The stabiliser of a reflection subgroup R_a may not pointwise stabilise it.

Heuristically, quaternionic reflection groups are less abelian than complex reflection groups (with K being an exception), and so they have fewer normal subgroups. The Lemma 2.1 provides a systematic way to calculate all the normal reflection subgroups of the quaternionic reflection groups and to quantify the various subsets of reflections that generate them (see [Wal26b]).

One could also take the view that the multiple labels (generating sets of reflections) that would give different normal reflection subgroups in the complex case, but give just a single normal reflection subgroup in the quaternionic case (because of the high noncommutatively of matrices over the quaternions), results in there being fewer normal reflection subgroups for quaternionic reflection groups (than might be expected).

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