The quaternionic reflection groups with imprimitive complexifications and the finite subgroups of $GL(\mathbb{C}^2)$

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Abstract

We give a transparent development of the quaternionic reflection groups which have imprimitive complexifications, including a description of their reflections and the corresponding normal reflection subgroups. They are seen to be of rank one or rank two, and are obtained from minimal complex reflection groups by the addition of one quaternionic reflection and a complex scalar matrix of finite order. In the case of the rank two groups, we use our techniques to consider the broader question of what are the minimal irreducible subgroups of $GL(\mathbb{C}^2)$ with a given collineation group. This leads to a complete classification of the finite irreducible subgroups of $GL(\mathbb{C}^2)$ in terms of the minimal groups, with the groups giving rise to quaternionic reflection groups located within it.

Key Words: complex reflection groups, quaternionic reflection groups, imprimitive complexification, lattice of normal reflection subgroups, reflections, hidden reflections, collineation groups, roots of reflections, Shephard-Todd classification

AMS (MOS) Subject Classifications: primary 05B30, 15B33, 20C25, 20G20, 51M05, 51M20, secondary 15B57, 51E99, 51M15, 65D30.

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1 Introduction

A group G of complex matrices in $M_{2d}(\mathbb{C})$ gives rise to a group of quaternionic matrices in $M_d(\mathbb{H})$ if it is conjugate to a group of matrices of the **symplectic form**

$$[A+Bj]_{\mathbb{C}} := \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} \in M_{2d}(\mathbb{C}) \quad \Longleftrightarrow \quad A+Bj \in M_d(\mathbb{H}). \tag{1.1}$$

A finite matrix group over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is a **reflection group** if it is generated by reflections (matrices which pointwise fix a hyperplane). Here we consider the quaternionic reflection groups whose symplectic matrices have the block monomial symplectic forms

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \Longleftrightarrow A, \qquad \begin{pmatrix} 0 & -B \\ \overline{B} & 0 \end{pmatrix} \Longleftrightarrow Bj, \qquad A, B \in M_d(\mathbb{C}), \tag{1.2}$$

i.e., the complexification has a system of imprimitivity consisting of two d-dimensional complex subspaces. Cohen [Coh80] shows that these are the only possible complex system of imprimitivity for a primitive quaternionic reflection group, and that the block monomial form of (1.2) can be obtained after a suitable change of variables. In the classification of Schmitt [Sch23] Fig. 1.2.1, the groups we consider are the "symplectically irreducible complex irreducible symplectically primitive complex imprimitive symplectic reflection groups".

We will see from an elementary argument (Lemma 2.1) that reflections of the type Bj are highly restricted. As a consequence (the reflections of the type A being well understood), all of the quaternionic reflection groups with matrices of the form (1.2) can be written

$$\langle G, \zeta, s \rangle, \qquad \mathrm{rank}(G) = 1, 2, \qquad s = \begin{pmatrix} j \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \quad \text{(respectively)},$$

where G is a complex reflection group, ζ is a complex scalar matrix, and the s above is a quaternionic reflection of the form Bj. From this observation, we proceed as follows:

- We determine the minimal collineation preserving reflection subgroups G in $GL(\mathbb{C}^2)$.
- We consider the structure of the reflection groups $\mathcal{H}_a(G)$ obtained from them by adding s and a scalar matrix ζ of finite order (Lemma 2.3), and determine when a given group can be obtained from different pairs (G, ζ), thereby classifying all such reflection groups (Theorems 2.1 and 2.2).
- We undertake the corresponding classification of the finite subgroups of $GL(\mathbb{C}^2)$ in terms of the minimal subgroups with a given collineation group, choosing the parameters so that no groups are repeated (Theorems 3.1, 3.2, and 3.3).
- We then investigate the lattice of normal reflection subgroups of the quaternionic reflection groups $\mathcal{H}_a(G)$ in terms of the indexing lattice of [Wal26a]. By way of comparison with the complex reflection groups where the indexing is 1–1, it is seen that this is often not 1–1 (Theorem 4.3). Indeed, there are quaternionic reflection groups which do not require at least one reflection of each order to generate it.

We give numerous examples, tables and figures which summarise the algebraic and geometric structures of the complex and quaternionic groups that we consider.

2 Quaternionic reflections which have imprimitive complexifications

We assume, without loss of generality (since finite subgroups of $GL(\mathbb{C}^2)$ are conjugate to unitary ones), that the matrices A and Bj of (1.2) are unitary, which by the calculation

$$(Bj)(Bj)^* = Bjj^*B^* = BB^*,$$

is equivalent to the complex matrices A and B being unitary. Reflections of the type A are simply complex reflections. We assume basic familiarity with complex reflection groups (e.g., [ST54], [Coh76], [OT92], [BMM99], [LT09], [Bro10], also [BST23], [RS26]),

We now determine the reflections of the type Bj, by using

$$jz = \overline{z}j, \qquad \forall z \in \mathbb{C}. \tag{2.3}$$

Lemma 2.1 *A matrix Bj, $B \in M_d(\mathbb{C})$, is a unitary reflection in precisely the cases*

(i) *$d = 1$ and Bj is of order 4 given by $Bj = \big(\xi j\big)$, $\xi \in \mathbb{C}$, $|\xi| = 1$.*

(ii) *$d = 2$ and Bj is of order 2 given by the monomial matrix*

$$Bj = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi j \\ -\xi j & 0 \end{pmatrix}, \qquad \xi \in \mathbb{C}, \quad |\xi| = 1. \tag{2.4}$$

Proof: We first observe that

$$(Bj)^{2m+1} = (BjBj)^m Bj = (-B\overline{B})^m Bj \neq I,$$

so that no matrix Bj can have an odd order. The reflection r in $U(\mathbb{H}^d)$ with root $a \in \mathbb{H}^d$ and $ra = a\lambda$, $\lambda \in \mathbb{H}$, $|\lambda| = 1$, is given by the formula

$$r_{a,\lambda} = I - a(1-\lambda)a^* = Bj, \qquad B = (I - a(1-\lambda)a^*)(-j), \qquad \langle a, a\rangle = 1.$$

The condition for this $B \in M_d(\mathbb{H})$ to be complex is that its j and k parts are zero, i.e.,

$$B + iBi - jBj + kBk = 0, \qquad B + iBi + jBj - kBk = 0.$$

Now

$$B+iBi = 2I(-j)+a(1-\lambda)a^*j-2ia(1-\lambda)a^*k = 0 \quad \Longleftrightarrow \quad 2I = a(1-\lambda)a^*-ia(1-\lambda)a^*i,$$

$$jBj-kBk = 2I(j)-ja(1-\lambda)a^*-ka(1-\lambda)a^*i = 0 \quad \Longleftrightarrow \quad 2I = a(1-\lambda)a^*-ia(1-\lambda)a^*i,$$

so that B is complex if and only if

$$2I = a(1-\lambda)a^* - ia(1-\lambda)a^*i. \tag{2.5}$$

Let $a = u + vj$, $u, v \in \mathbb{C}^d$, so that $\langle a, a\rangle = \langle u, u\rangle + \langle v, v\rangle = 1$, and $\lambda = \alpha + \beta j$, $\alpha, \beta \in \mathbb{C}$. Then repeatedly using (2.3) gives

$$a(1-\lambda)a^* = \{(1-\alpha)uu^* + \overline{(1-\alpha)}vv^* - \beta uv^* + \overline{\beta}vu^*\}$$

$$+\{-(1-\alpha)uv^T+\overline{(1-\alpha)}vu^T-\beta uu^T-\overline{\beta}vv^T\}j, \tag{2.6}$$

so that (2.5) divided by 2 becomes

$$\begin{aligned} I &= (1-\alpha)uu^*+\overline{(1-\alpha)}vv^*-\beta uv^*+\overline{\beta}vu^* \\ &= \{(1-\alpha)u+\overline{\beta}v\}u^*+\{\overline{(1-\alpha)}v-\beta u\}v^*. \end{aligned} \tag{2.7}$$

We observe the right hand side of (2.7) has rank at most 2, so we can only have $d=1,2$. Taking the trace of (2.7) gives

$$d=(1-\alpha)\langle u,u\rangle+\overline{(1-\alpha)}\langle v,v\rangle-\beta\langle v,u\rangle+\overline{\beta}\langle u,v\rangle,$$

and since $-\beta\langle v,u\rangle+\overline{\beta}\langle u,v\rangle$ is pure imaginary, taking the real part of the above gives

$$d=\mathrm{Re}(1-\alpha)(\langle u,u\rangle+\langle v,v\rangle)=1-\mathrm{Re}(\alpha) \quad\Longrightarrow\quad \mathrm{Re}(\alpha)=1-d.$$

For $d=1$, ξ is unitary if and only if $|\xi|=1$, and the calculation

$$(\xi j)^2=\xi j\xi j=\xi\overline{\xi}j^2=-1,$$

shows that $Bj=r_{1,\xi j}=\xi j$ is a reflection of order 4.

For $d=2$, we have $\mathrm{Re}(\alpha)=1-d=-1$, so that $\lambda=\alpha=-1$ and $\beta=0$. By (2.7), we have $I=2(uu^*+vv^*)$, so that u and v are orthogonal and of length $\frac{1}{\sqrt{2}}$. Hence by (2.6), the reflection is given by

$$r_{a,-1}=I-2aa^*=2(uv^T-vu^T)j=2\begin{pmatrix}0 & u_1v_2-v_1u_2\\ -(u_1v_2-v_1u_2) & 0\end{pmatrix}j,$$

and so is monomial. The following choice for the root vector a gives

$$u=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix},\quad v=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\\xi\end{pmatrix}\quad\Longrightarrow\quad a=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\xi j\end{pmatrix}\quad\Longrightarrow\quad r_{a,-1}=\begin{pmatrix}0 & \xi j\\ -\xi j & 0\end{pmatrix}.$$

and so we obtain all the reflections asserted. □

Remark 2.1 *A unitary matrix Bj has order 2 if and only if*

$$(Bj)^2=BjBj=B\overline{B}j^2=-B(B^T)^*=I \quad\Longleftrightarrow\quad -B=B^T,$$

i.e., B is skew symmetric, and so for $d=2$ all elements Bj of order 2 are reflections.

It is possible to construct matrices Bj of order $2m$. For example, one may take a unitary B of the form

$$B=\xi\begin{pmatrix}a & \overline{b}\\ -b & \overline{a}\end{pmatrix},\qquad |\xi|=1.$$

A simple calculation shows that $\det((Bj)^2)=\det(-B\overline{B})=1$, so the eigenvalues of $-B\overline{B}$ (of order m) must be ζ, $\overline{\zeta}$, with ζ a primitive m-th root of unity. By considering

$$\mathrm{trace}(-B\overline{b})=\zeta+\overline{\zeta},$$

we obtain the following sufficient condition for Bj to have order $2m$

$$\mathrm{Re}(b) = \cos\frac{\pi}{m},$$

and as a particular case, we may take

$$B_m = \xi \begin{pmatrix} \sin\frac{\pi}{m} & \cos\frac{\pi}{m} \\ -\cos\frac{\pi}{m} & \sin\frac{\pi}{m} \end{pmatrix}, \qquad m = 1, 2, \ldots.$$

This gives

$$B_1 j = \begin{pmatrix} 0 & \xi j \\ -\xi j & 0 \end{pmatrix}, \quad B_2 j = \xi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} j, \quad B_3 j = \xi \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} j, \quad B_4 j = \frac{\xi}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} j,$$

of orders 2, 4, 6 and 8. There are many more such matrices that can obtained by starting with B of a more general form, e.g.,

$$Bj = \frac{\xi}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ (order 4)}, \qquad Bj = \frac{\xi}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \text{ (order 6)}.$$

The ξj for the reflections of the type (2.4) in a finite monomial group form what is called a *reflection system* in [Wal25]. In particular, they are closed under the operation $a \circ b := ab^{-1}a$, which in this case is

$$\xi j \circ \omega j = \xi j (\omega j)^{-1} \xi j = \xi j j^* \overline{\omega} \xi j = \xi \overline{\omega} \xi = (\xi \circ \omega) j,$$

i.e., the complex factors ξ are closed under $a \circ b$. It is always possible to apply a change of variables to a monomial group so that a reflection system transforms to one containing any given unit quaternion (in the formal definition this is taken to be 1). In our situation, this implies that if there is reflection of type Bj in a group, then we can take it to be s, e.g., by the explicit change of variables

$$\begin{pmatrix} j & 0 \\ 0 & \xi j \end{pmatrix} \begin{pmatrix} 0 & \xi j \\ -\xi j & 0 \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & \xi j \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\overline{\xi} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -j & 0 \\ 0 & -\xi j \end{pmatrix} = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}.$$

It then follows that all rank two quaternionic reflection groups with with imprimitive complexifications have the form:

$$\langle G, \zeta_m, s \rangle, \qquad \zeta_m := e^{\frac{2\pi i}{m}}, \quad s := \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \tag{2.8}$$

where G is a complex reflection group, and the reflections of type (2.4) in the group are given by ξs with $\xi \in \langle Z(G), \zeta_m \rangle$ (we recall by Schur's lemma that if G is irreducible then its centre consists of scalar matrices). Before classifying all such groups and determining their reflections, there are two technical points to consider:

1. If a complex unitary change of variables UGU^{-1} is applied to G, then does one still get the same group?

2. Does $\langle G, \zeta_m, s \rangle$ give a finite group, and what is its structure?

For the first point, we calculate that

$$UsU^{-1} = U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} jU^* = U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^T j = \det(U)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} j = \det(U)s,$$

so that s is fixed by the change of variables U if and only if $\det(U) = 1$. Therefore, with $\hat{U} := \alpha U$, $\alpha := 1/\sqrt{\det(U)}$, we have $\det(\hat{U}) = 1$, and hence

$$\hat{U}\langle G, \zeta_m, s\rangle \hat{U}^{-1} = \langle (\alpha U)G(\alpha U)^{-1}, \hat{U}\zeta_m\hat{U}^{-1}, \hat{U}s\hat{U}^{-1}\rangle = \langle UGU^{-1}, \zeta_m, s\rangle,$$

i.e., the same quaternionic reflection group is obtained.

For the second point, we first consider how a matrix $A \in GL(\mathbb{C}^2)$ is conjugated by s

$$A^s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^s = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} = \left\{ \det(\overline{A}) \frac{1}{\det(\overline{A})} \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix} \right\}^T = \det(\overline{A})(\overline{A}^{-1})^T,$$

with $\det(A^s) = \overline{\det(A)}$. Hence, for $A = g$ a unitary matrix, we have

$$g^s = sgs^{-1} = \det(g^*)g, \qquad g \in U(\mathbb{C}^2). \tag{2.9}$$

Thus $\langle G, \zeta_m, s\rangle$ gives a finite group

$$\langle G, \zeta_m, s\rangle = \langle \hat{G}, \zeta_m, s\rangle, \qquad \hat{G} := \langle G, G^s\rangle = \langle G, \det(G)\rangle, \tag{2.10}$$

with its structure (multiplication) determined by (2.9). By (2.9), $\hat{G} = \langle G, G^s\rangle$ is a finite complex reflection group (generated by the reflections in G and G^s), which is closed under conjugation by s. Thus, it suffices to consider only the complex reflection groups that are closed under conjugation by s. This is (for better or worse) most of them.

The **collineation group** associated with a matrix group G is the group considered up to multiplication by a scalars [Bli17], i.e., for irreducible groups $G_{\text{coll}} = G/Z(G)$. We recall from [Wal26a] that each primitive reflection group in $U(\mathbb{C}^2)$ is a collineation preserving subgroup of the maximal primitive reflection groups

$$G_7 = \langle R^2, Z, Z^R\rangle, \qquad G_{11} = \langle F, R, Z\rangle, \qquad G_{19} = \langle R^2, Z, M\rangle,$$

as given in Figures 1, 2, 3, where

$$F := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R := \begin{pmatrix} 1 & 0 \\ 0 & i^3 \end{pmatrix}, \qquad Z := e^{\frac{2\pi i}{24}} RF, \tag{2.11}$$

$$M := \frac{e^{-\frac{2\pi i}{5}}}{2}\begin{pmatrix} \tau^{-1} - \tau i & -1 \\ 1 & \tau^{-1} + \tau i \end{pmatrix}, \qquad \tau = \frac{1+\sqrt{5}}{2}, \tag{2.12}$$

are reflections of orders $2, 4, 3, 5$ (respectively).

Lemma 2.2 *Let $G \subset U(\mathbb{C}^2)$ be an irreducible complex reflection group. Then G is closed under conjugation by s, i.e., $\hat{G} = G = G^s$, equivalently $\det(G) \subset G$, in all cases except*

$$\hat{G}_4 = G_5, \qquad \hat{G}_6 = G_7,$$
$$\hat{G}(m, p, 2) = G(2m, 2p, 2), \quad m \text{ odd}.$$

Proof: In view of (2.9), G is closed under conjugation by s if and only if

$$\det(g^*) = \det(g^{-1}) = \det(g)^{-1} \in G, \qquad \forall g \in G.$$

Since $g \mapsto \det(g)$ is a homomorphism, it suffices to check whether (or not) $\det(g) \in G$ for all g from a generating set of reflections. If this is not case, then

$$\det(g) = gsg^{-1}s \notin Z(G),$$

for some reflection $g \in G$.

We first consider the primitive reflection groups, each of which is a collineation preserving subgroup of the maximal primitive reflection groups

$$G_7 = \langle R^2, Z, Z^R \rangle, \qquad G_{11} = \langle F, R, Z \rangle, \qquad G_{19} = \langle R^2, Z, M \rangle,$$

as given in Figures 1, 2, 3. It follows from

$$\det(F) = -1, \qquad \det(R) = -i, \qquad \det(Z) = \zeta_3, \qquad \det(M) = \zeta_5^{-1},$$

and the scalar matrices in these subgroups (the centre of these subgroups, see Table 1), that they are all closed under conjugation by s, except for G_4 and G_6, which each appear twice as normal subgroups of G_7. i.e.,

$$G_4 = \langle Z, Z^S \rangle, \langle Z^R, Z^{SR} \rangle, \qquad G_6 = \langle R^2, Z \rangle, \langle R^2, Z^R \rangle.$$

In this case

$$\zeta_3 = \det(Z) = \det(Z^R) = ZsZ^{-1}s \notin Z(G_4), Z(G_6), \qquad Z(G_4) = \langle -1 \rangle, \quad Z(G_6) = \langle i \rangle,$$

and since $\det(R^2) = -1$, we have

$$\langle G_4, s \rangle = \langle G_4, \zeta_3, s \rangle = \langle G_5, s \rangle = \langle \hat{G}_5, s \rangle, \quad \langle G_6, s \rangle = \langle G_6, \zeta_3, s \rangle = \langle G_7, s \rangle = \langle \hat{G}_7, s \rangle.$$

Hence $\hat{G}_4 = G_5$, $\hat{G}_6 = G_7$ (in each case).

The calculations for the imprimitive reflection groups

$$G(m,p,2) = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m \\ \overline{\zeta_m} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{m}{p}} & 0 \\ 0 & 1 \end{pmatrix} \rangle, \qquad Z(G(m,p,2)) = \langle \zeta_{\gcd(p,2)\frac{m}{p}} \rangle, \quad (2.13)$$

are similar. The determinants of the above generators are -1, -1 and $\zeta_{\frac{m}{p}} \in \langle \zeta_{\gcd(p,2)\frac{m}{p}} \rangle$. Thus $G(m,p,2)$ is closed under conjugation except for when m is odd, in which case

$$\begin{aligned}
\langle G(m,p,2), s \rangle &= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m \\ \overline{\zeta_m} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{m}{p}} & 0 \\ 0 & 1 \end{pmatrix}, -1, s \rangle \\
&= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\zeta_m \\ -\overline{\zeta_m} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{m}{p}} & 0 \\ 0 & 1 \end{pmatrix}, s \rangle \\
&= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_{2m} \\ \overline{\zeta_{2m}} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{2m}{2p}} & 0 \\ 0 & 1 \end{pmatrix}, s \rangle \\
&= \langle G(2m, 2p, 2), s \rangle,
\end{aligned}$$

which gives $\hat{G}(m,p,2) = G(2m,2p,2)$. □

Table 1: The reflection types and collineation groups of the primitive complex reflection groups of rank 2 (with generators) as given in [Wal26a] (also see Tables C.1, C.2 [OT92]).

ST	Order	Reflection type	$\lvert Z(G)\rvert$	Reflections	G_{coll}	Generators
G_4	24	$4C_3$	2	3^8	A_4	Z, Z^S or Z^R, Z^{SR}
G_5	72	$4C_3, 4C_3$	6	3^{16}	A_4	Z, Z^R
G_6	48	$6C_2, 4C_3$	4	$2^6 3^8$	A_4	R^2, Z or R^2, Z^R
G_7	144	$6C_2, 4C_3, 4C_3$	12	$2^6 3^{16}$	A_4	R^2, Z, Z^R
G_8	96	$6C_4$	4	$2^6 4^{12}$	S_4	R, R^F
G_9	192	$12C_2, 6C_4$	8	$2^{18} 4^{12}$	S_4	F, R
G_{10}	288	$6C_4, 8C_3$	12	$2^6 3^{16} 4^{12}$	S_4	R, Z
G_{11}	576	$12C_2, 6C_4, 8C_3$	24	$2^{18} 3^{16} 4^{12}$	S_4	F, R, Z
G_{12}	48	$12C_2$	2	2^{12}	S_4	F, F^Z, F^{Z^2}
G_{13}	96	$12C_2, 6C_2$	4	2^{18}	S_4	F, F^Z, R^2
G_{14}	144	$12C_2, 8C_3$	6	$2^{12} 3^{16}$	S_4	F, Z
G_{15}	288	$12C_2, 6C_2, 8C_3$	12	$2^{18} 3^{16}$	S_4	F, R^2, Z
G_{16}	600	$12C_5$	10	5^{48}	A_5	M, M^{R^2}
G_{17}	1200	$30C_2, 12C_5$	20	$2^{30} 5^{48}$	A_5	R^2, M
G_{18}	1800	$20C_3, 12C_5$	30	$3^{40} 5^{48}$	A_5	Z, M
G_{19}	3600	$30C_2, 20C_3, 12C_5$	60	$2^{30} 3^{40} 5^{48}$	A_5	R^2, Z, M
G_{20}	360	$20C_3$	6	3^{40}	A_5	Z, Z^M
G_{21}	720	$30C_2, 20C_3$	12	$2^{30} 3^{40}$	A_5	R^2, Z^M
G_{22}	240	$30C_2$	4	2^{30}	A_5	$R^2, (R^2)^Z, (R^2)^M$

Remark 2.2 *In the notation of [Coh80], the group $\langle G, s\rangle$ for a finite group of unitary matrices closed under conjugation, i.e., with $\hat{G} = G$ or equivalently* $\det(g) \in G$, $\forall g \in G$, *is given by*

$$E(G) = \textit{the complexification of } \langle G, s\rangle.$$

The $E(H)$ given for $H \subset GL(\mathbb{C}^2)$ with $\det(h) \in H$, $\forall h \in H$, *consists of matrices in the symplectic form (1.1) if and only if the matrices in H are unitary. Indeed*

$$E(H) := \langle \hat{h}, [s]_{\mathbb{C}} : h \in H\rangle, \qquad \hat{h} := \begin{pmatrix} h & 0 \\ 0 & (h^T)^{-1} \end{pmatrix}.$$

To put this in the symplectic form, let $U \in GL(\mathbb{C}^2)$ be a change of variables for which $G = UHU^{-1}$ is a group of unitary matrices and $\det(U) = 1$ *(this preserves the condition* $\det(h) \in H$, $\forall h \in H$*). Then*

$$\hat{U}\hat{h}\hat{U}^{-1} = \begin{pmatrix} UhU^{-1} & 0 \\ 0 & (U^T)^{-1}(h^T)^{-1}U^T \end{pmatrix} = [UhU^{-1}]_{\mathbb{C}},$$

since

$$\overline{UhU^{-1}} = ((UhU^{-1})^T)^* = ((U^{-1})^T h^T U^T)^{-1} = (U^T)^{-1}(h^T)^{-1}U^T.$$

Similarly,

$$\hat{U}[s]_{\mathbb{C}}\hat{U}^{-1} = \begin{pmatrix} 0 & -USU^T \\ (U^T)^{-1}SU^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -S \\ S & 0 \end{pmatrix} = [s]_{\mathbb{C}}, \qquad S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

since

$$U^TSU = \det(U)S = S, \qquad (U^{-1})^TSU^{-1} = \det(U^{-1})S = \det(U)^{-1}S = S,$$

and we have

$$\hat{U}(E(H))\hat{U}^{-1} = \langle \hat{U}\hat{H}\hat{U}^{-1}, \hat{U}[s]_{\mathbb{C}}\hat{U}^{-1}\rangle = \langle [G]_{\mathbb{C}}, [s]_{\mathbb{C}}\rangle = [\langle G, s\rangle]_{\mathbb{C}},$$

where $G = UHU^{-1}$ is unitary.

Let $G \subset U(\mathbb{C}^2)$ be an irreducible finite group. Then by Schur's lemma, the group of scalar matrices in G is its centre $Z(G)$. We may add to G the reflection s and a complex scalar matrix ζ_m of order m, which is equivalent to adding the reflections s and ζs, to obtain the following class of finite irreducible quaternionic groups

$$\mathcal{H}_a(G) := \langle G, \zeta_{|Z(\hat{G})|a}, s\rangle, \qquad a \geq 1. \tag{2.14}$$

Here the scalar added has the form $\zeta_{|Z(\hat{G})|a}$ since $\hat{G} \subset \langle G, s\rangle$ already contains $\zeta_{|Z(\hat{G})|}$, and so we have

$$\mathcal{H}_1(G) = \langle G, s\rangle, \qquad \mathcal{H}_a(G) = \mathcal{H}_a(\hat{G}),$$

with

$$\mathcal{H}_b(H) \subset \mathcal{H}_a(G), \qquad b \mid a, \quad H \subset G. \tag{2.15}$$

In view of the observation (2.8), all of the rank two quaternionic reflection groups with imprimitive complexifications have the form (2.14). They have the following structure.

Lemma 2.3 *Let $G \subset U(\mathbb{C}^2)$ be an irreducible finite group. Then $\mathcal{H}_a(G)$ is the (internal) semidirect product*

$$\mathcal{H}_a(G) = \langle \hat{G}, \zeta_{|Z(\hat{G})|a}\rangle \rtimes \langle s\rangle, \qquad |\mathcal{H}_a(G)| = 2|\hat{G}|a, \tag{2.16}$$

where the multiplication is given by

$$s^2 = 1, \qquad sgs = \det(g^{-1})g, \qquad \forall g \in \langle \hat{G}, \zeta_{|Z(\hat{G})|a}\rangle. \tag{2.17}$$

Moreover, $\mathcal{H}_a(G)$ is a quaternionic reflection group if and only if $\langle \zeta_{|Z(\hat{G})|a}\rangle G$ contains a set of reflections which generate a collineation preserving subgroup of G.

Proof: Since $\hat{G}$ is closed under conjugation by s, and

$$s\xi s^{-1} = \overline{\xi} = \xi^{-1}, \qquad \xi \in \mathbb{C},\ |\xi| = 1,$$

$N = \langle \hat{G}, \zeta_{|Z(\hat{G})|a} \rangle$ is a normal reflection subgroup of $\mathcal{H}_a(G)$. Since N consists of complex matrices, $\mathcal{H}_a(G) = N\langle s \rangle$ with $N \cap \langle s \rangle = 1$, i.e., we have the asserted semidirect product. Since the scalar matrices in $\hat{G}$ and $\mathcal{H}_a(G)$ are $Z(\hat{G})$ and $\langle \zeta_{|Z(\hat{G})|a} \rangle$, we have

$$|\mathcal{H}_a(G)| = |\langle s \rangle| \cdot |N| = 2 \cdot \frac{|\hat{G}|}{|Z(\hat{G})|} |Z(\hat{G})|a = 2|\hat{G}|a.$$

Since $s^2 = 1$, the conjugation formula (2.9) establishes (2.17) for $g \in \hat{G}$, and for ξg, we use this and the multiplicativity of the determinant function to obtain (2.17), i.e.,

$$\begin{aligned} s(\xi g)s = (s\xi s)(sgs) &= \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}^{-1} \det(g^{-1})g \\ &= \det\Big(\begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}^{-1}\Big)\xi \det(g^{-1})g = \det((\xi g)^{-1})(\xi g). \end{aligned}$$

The largest reflection subgroup contained in $\mathcal{H}_a(G)$ is generated by the reflections in $\langle \zeta_{|Z(\hat{G})|a} \rangle G$ and s, and this can equal $\mathcal{H}_a(G)$ if only if the complex reflections generate a collineation preserving subgroup of G. □

A finite subgroup $G \subset GL(\mathbb{C}^2)$ is said to be **minimal** (for its collineation group) if there is no proper subgroup H of G for which $G = \langle \zeta_m \rangle H$ for same scalar ζ_m. Clearly, all the finite irreducible subgroups of $GL(\mathbb{C}^2)$ are of the form $\langle \zeta_m \rangle H$, for some minimal group H (see [NvdPT08]), and in particular the groups $\langle G, \zeta_m \rangle$ giving rise to quaternionic reflection groups $\langle G, \zeta_m, s \rangle$ have this form. The basic mechanics of our development relies on the following fact:

- For every $g \in U(\mathbb{C}^2)$ of finite order which is not a scalar matrix, two scalar multiples of it give a reflection (a matrix with eigenvalue 1) of finite order.

If such a matrix g is not a reflection, then we call it a **hidden reflection** (see [Wal26a]), since it may be multiplied by a scalar to obtain/reveal a reflection. Thus for any finite irreducible group H (minimal or not), adding certain scalars to it will give a group of the form $\langle G, \zeta \rangle$, G a reflection group, with the same collineation group as H, i.e., $G/Z(G) = H/Z(H)$. In this way, every quaternionic reflection group of the type (2.14) can be written

$$\mathcal{H}_a(G) = \langle G, \zeta_{|Z(\hat{G})|a}, s \rangle,$$

where either

1. G is a reflection group which is minimal (has no proper collineation preserving reflection subgroups), and there is no restriction on a.

2. G is a minimal group, and a is such that $\mathcal{H}_a(G)$ is a reflection group.

We begin with the first, our initial approach, ultimately adopting the second in some cases where it is more convenient. The collineation groups $G/Z(G)$, i.e., the irreducible finite subgroups of $PGL(\mathbb{C}^2)$ were classified by Klein [Kle56]. As abstract groups they are

$$D_n, \quad n \geq 2 \quad \text{(dihedral group of order } 2n\text{)}, \qquad A_4, \qquad S_4, \qquad A_5.$$

A way to give a standardised/canonical group of matrices with a given collineation group, which is akin to revealing reflections, is to take any such matrix group and replace each of its elements by the two scalar multiples which have determinant 1, thereby obtaining the finite irreducible subgroups of $SU(\mathbb{C}^2)$ (of twice the order), which we denote by

$$\mathsf{D}_n, \quad n \geq 2 \quad \text{(binary dihedral group of order } 4n\text{)}, \qquad \mathsf{T}, \qquad \mathsf{O}, \qquad \mathsf{I}.$$

We now consider the primitive groups $\mathsf{T}, \mathsf{O}, \mathsf{I}$ which are called the binary tetrahedral, octahedral and icosahedral groups (of orders $24, 48, 120$). In [Wal26a], it was observed that any reflection group is a normal subgroup of the reflection group that is obtained by revealing all its hidden reflections, the so called **maximal reflection group** for the underlying collineation group. This is the reflection group obtained by revealing all the hidden reflections of the reflection-free groups $\mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I}$, respectively. It is convenient to denote all the groups that can be obtained from a group G by adding scalars by

$$M_a(G) := \langle G, \zeta_{|Z(G)|a} \rangle, \qquad a \geq 1, \qquad M_1(G) = G, \quad |M_a(G)| = |G| \cdot a. \tag{2.18}$$

Each group is different, with

$$M_b(H) \subset M_a(G), \quad b \mid a, \ H \subset G, \qquad \mathcal{H}_a(G) = \langle M_a(G), s \rangle, \quad \text{when } \hat{G} = G.$$

The lattices of the collineation preserving normal reflection subgroups of the maximal reflection groups were described in [Wal26a] by taking subsets of the reflection orbits to move down the lattice, i.e., hiding reflections without changing the collineation groups. Conversely, one can move up the lattice by adding scalars to reveal reflections. The scalars that must be added to move to a reflection group above is simply its centre.

We now consider A_4 (equivalently T), which has maximal reflection group G_7. This case is slightly more technical since $\hat{G}_4 = G_5, \hat{G}_6 = G_7$. The minimal reflection subgroups in the reflection subgroup lattice of Figure 1 are G_4 which appears twice. This is also a minimal group. Applying $G \mapsto \hat{G}$ to the lattice gives the relevant subgroups for the groups $\mathcal{H}_a(G)$. There is one minimal reflection subgroup G_5, which is not a minimal group, since T is a subgroup with $G_5 = M_3(\mathsf{T}) = \langle \zeta_6 \rangle \mathsf{T}$. Starting with G_5, we therefore have the quaternionic reflection groups

$$\mathcal{T}_1(G_5) := \mathcal{H}_a(G_5), \qquad a \geq 1. \tag{2.19}$$

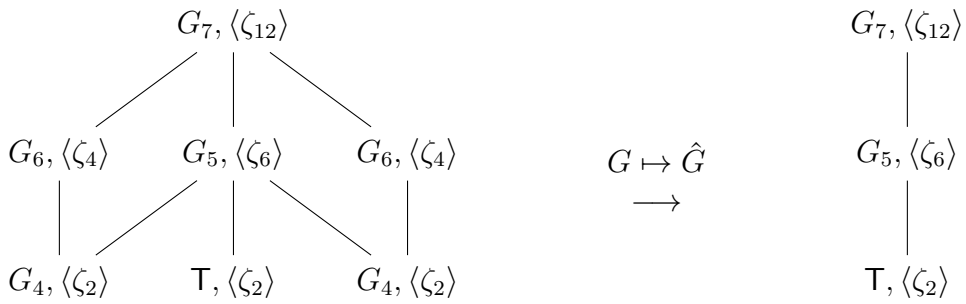


Figure 1: The lattice of the collineation preserving normal reflection subgroups of G_7 and their centres, and its image under $G \mapsto \hat{G}$. The remaining collineation preserving normal subgroups are the reflection free subgroup T, also depicted, and $\langle \zeta_4 \rangle \mathsf{T}$.

We now consider S_4 (equivalently O), which has maximal reflection group G_{11}. Here there are two minimal reflection subgroups G_{12} and G_8, see Figure 2, which are also minimal groups. The corresponding families of quaternionic reflection groups are

$$\mathcal{O}_a(G_{12}) := \mathcal{H}_a(G_{12}), \quad a \geq 1, \qquad \mathcal{O}_b(G_8) := \mathcal{H}_b(G_8), \quad b \geq 1. \tag{2.20}$$

These cross over (first intersect) at

$$\mathcal{O}_4(G_{12}) = \langle \zeta_8 \rangle G_{12} = G_9 = \langle \zeta_4 \rangle G_8 = \mathcal{O}_2(G_8). \tag{2.21}$$

The presentation in [Coh80] consists of the following subfamily of $\mathcal{O}_a(G_{12})$

$$\mathcal{H}_{2a}(\mathsf{O}) = \langle \zeta_{4a} \rangle \mathsf{O} = \mathcal{H}_a(G_{13}) = \mathcal{H}_{2a}(G_{12}) = \langle \zeta_{4a} \rangle G_{12} = \mathcal{O}_{2a}(G_{12}), \qquad a \geq 1,$$

and a family denoted by [Coh80], [Tay25], [Sch23] (respectively) as

$$\begin{aligned} &E\Big((\mu_{4d}|\mu_{2d}; \mathsf{O}|\mathsf{T})\Big) && (d \text{ an odd multiple of 1 or 2}), \\ &E(\mathcal{C}_d \boxdot_2 \mathcal{O}) && (d \text{ a multiple of 4 but not divisible by 16}), \\ &E(\mathsf{OT}_{2d}) && (d \text{ not divisible by 4}). \end{aligned} \tag{2.22}$$

The family (2.22) will be seen to be the reflection groups for which the intersection of the underlying complex group with $SU(\mathbb{C}^2)$ is T (as opposed to O).

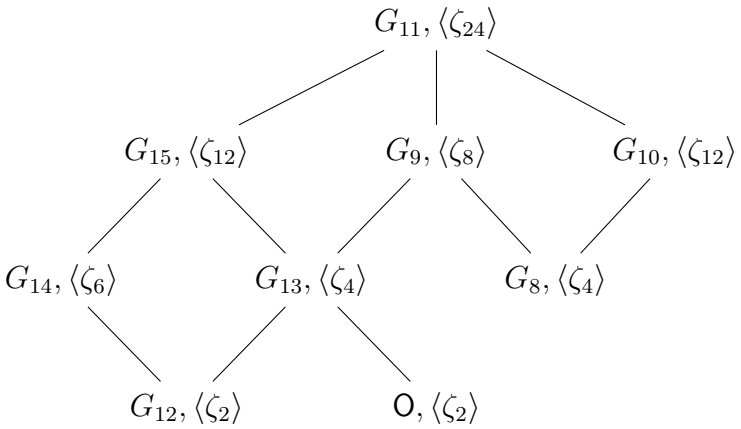


Figure 2: The lattice of the eight collineation preserving normal reflection subgroups of G_{11} and their centres. The remaining normal collineation preserving subgroups are the reflection free subgroup O, also depicted, and $\langle \zeta_6 \rangle \mathsf{O}$, for which $\mathsf{O} \subset \langle \zeta_6 \rangle \mathsf{O} \subset G_{15}$.

For A_5 (equivalently I), the maximal reflection group is G_{19}. There are three minimal reflection subgroups G_{22}, G_{20} and G_{16}, see Figure 3, none of which is a minimal group since

$$G_{22} = \langle \zeta_4 \rangle \mathsf{I}, \qquad G_{20} = \langle \zeta_6 \rangle \mathsf{I}, \qquad G_{16} = \langle \zeta_{10} \rangle \mathsf{I}. \tag{2.23}$$

This leads to three infinite (and intersecting) families

$$\mathcal{H}_a(G_{22}), \quad a \geq 1, \qquad \mathcal{H}_b(G_{20}), \quad b \geq 1, \qquad \mathcal{H}_c(G_{16}), \quad c \geq 1, \tag{2.24}$$

which, in view of (2.23), are most easily described by

$$\mathcal{I}_a(\mathsf{I}) := \mathcal{H}_a(\mathsf{I}), \qquad \text{for appropriate choices of } a \geq 1.$$

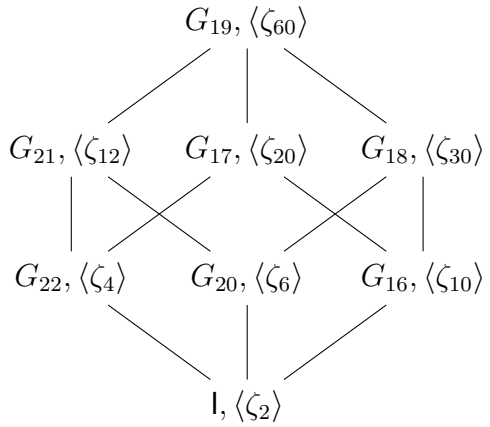


Figure 3: The lattice of the collineation preserving normal reflection subgroups of G_{19} and their centres, and the collineation preserving normal subgroup $\mathsf{I} = G_{22} \cap G_{20} \cap G_{16}$.

Recently, it has been shown that a primitive complex reflection group can have a quaternionic system of imprimitivity, and therefore be conjugate to an imprimitive quaternionic reflection group. The only cases are G_{12}, G_{13} and G_{22} (see Theorem 4.2 of [Wal26b] and Theorem 7.3 of [Tay25]). It is possible that adding the quaternionic reflection s to these groups might give a quaternionic reflection group which shares this system of imprimitivity (and hence is imprimitive). Indeed, this is exactly what happens.

Lemma 2.4 *Let G be a rank two primitive complex reflection group. The quaterionic reflection group $\mathcal{H}_a(G)$ is imprimitive in precisely the three cases*

$$\begin{aligned} \mathcal{O}_1(G_{12}) &= \langle G_{12}, s \rangle = G_{\mathcal{O}}(L^{\mathcal{O}}_{14}, 1), \\ \mathcal{O}_2(G_{12}) &= \langle G_{13}, s \rangle = G_{\mathcal{O}}(L^{\mathcal{O}}_{20}, C_2), \\ \mathcal{I}_2(\mathsf{I}) &= \langle G_{22}, s \rangle = G_{\mathcal{I}}(L^{\mathcal{I}}_{32}, C_2), \end{aligned} \tag{2.25}$$

where the latter notation in terms of reflection systems is from [Wal25].

Proof: The quaternionic system of imprimitivity for each of $G = G_{12}, G_{13}, G_{22}$ (as presented here) is given by (see [Wal26b])

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -zj \\ zj & -1 \end{pmatrix}, \qquad z \in \mathbb{C}, \quad |z| = 1, \tag{2.26}$$

where the normalisation of U ensures that $U^2 = 1$. For $\langle G, s \rangle$ to remain imprimitive, U must conjugate s to a monomial matrix. We calculate

$$UsU^{-1} = \frac{1}{2} \begin{pmatrix} z - \overline{z} & -z^2 j - j \\ j - z^2 j & z - \overline{z} \end{pmatrix},$$

which is monomial if and only if $z = \pm 1$, and so $\langle G_{12}, s\rangle, \langle G_{13}, s\rangle, \langle G_{22}, s\rangle$ are imprimitive.

To see if there are additional imprimitive groups $\mathcal{H}_a(G)$, we consider which complex scalar matrices ξ are conjugated to monomial matrices by U. We have

$$U\xi U^{-1} = \begin{pmatrix} \xi + \overline{\xi} & \overline{\xi} zj - \xi zj \\ \overline{\xi} zj - \xi zj & \xi + \overline{\xi} \end{pmatrix},$$

which is monomial if and only if $\overline{\xi} = \pm\xi$, i.e., ξ is a 4-th root of unity. Since $G_{12} \subset G_{13}$ and $Z(G_{13}) = Z(G_{22}) = \langle \zeta_4 \rangle$, adding any additional complex scalar matrices to the above imprimitive groups gives a primitive group, and so there are not additional imprimitive groups of the form $\mathcal{H}_a(G)$. □

We are now in a position to describe all the primitive quaternionic reflection groups with imprimitive complexifications (without any groups repeated).

Theorem 2.1 *(Classification) The primitive quaternionic reflection groups which have imprimitive complexifications are of rank one, given by the dicyclic (binary dihedral) groups*

$$\mathcal{D}_n = \mathcal{H}_n(C_2) = \langle \zeta_{2n}, j\rangle, \qquad n \geq 2, \tag{2.27}$$

or of rank two, and have types $\mathcal{T}, \mathcal{O}, \mathcal{I}$ as follows

(i) $\mathcal{T}_a(G_5) = \mathcal{H}_a(G_5) = \langle Z, Z^R, \zeta_{6a}, s\rangle$, $a \geq 1$.

(ii) $\mathcal{O}_a(G_{12}) = \mathcal{H}_a(G_{12}) = \langle F, F^Z, F^{Z^2}, \zeta_{2a}, s\rangle$, $a \geq 3$,
$\mathcal{O}_b(G_8) = \mathcal{H}_b(G_8) = \langle R, R^F, \zeta_{4b}, s\rangle$, $b \geq 1$, *b is odd.*

(iii) $\mathcal{I}_a(\mathsf{I}) = \mathcal{H}_a(\mathsf{I}) = \langle iR^2, \zeta_3 Z, \zeta_5^3 M, \zeta_{2a}, s\rangle$, $a \geq 3$, *where* $2 \mid a$, $3 \mid a$ *or* $5 \mid a$.

The structure of these groups is given by Lemma 2.3, and their inclusions by (2.15).

Proof: The rank one complex reflection groups are $G = C_n = \langle \zeta_n \rangle \subset \mathbb{C}^*$, $n \neq 1$. With $s = j$, we have $-1 = j^2 = s^2 \in \mathcal{H}_a(G)$, so that (2.27) gives all the possible groups (not withstanding the fact that C_p, p a prime, are the minimal reflection groups).

We have already given all the possible rank two groups, and so it remains to give generators (from Table 1), and to restrict the parameters in such a way that no group is repeated (up to conjugacy), and to then exclude the imprimitive groups of (2.25).

For A_4 there is a single family (2.19), and we need no restrictions on $a \geq 1$ (the groups all have different orders). In [Coh80], the groups are given as $\langle \zeta_{6a} \rangle \mathsf{T} = \mathcal{T}_a(G_5) = \mathcal{H}_a(\mathsf{T})$.

For S_4 there are two families (2.20). We will take the $\mathcal{O}_a(G_{12})$ family (of groups with different orders) without any restriction on $a \geq 1$ (it could be done the other way round, but this is simplest and closest to the presentation in [Coh80]). As soon as $b \geq 1$ becomes even, $\mathcal{O}_b(G_8)$ becomes equal to an $\mathcal{O}_a(G_{12})$ group (otherwise not), with (2.21) being the first instance. This is because $\mathcal{O}_b(G_8)$ contains $\langle \zeta_8 \rangle$, and so (see Figure 2) we have that

$$\mathcal{O}_b(G_8) = \langle G_8, \zeta_{4b}, s\rangle = \langle G_9, \zeta_{4b}, s\rangle = \langle G_{12}, \zeta_{2(2b)}, s\rangle = \mathcal{O}_{2b}(G_{12}), \qquad b \text{ even}.$$

For b odd, $\mathcal{O}_b(G_8)$ does not contain $\langle \zeta_8 \rangle$, and hence is not an $\mathcal{O}_a(G_{12})$ group.

For A_5 there are three families (2.24). We could impose conditions on $a, b, c \geq 1$ so that a given group appears only once. However, it is much neater to start with I (as does [Coh80]) which sits below the three minimal reflection groups G_{22}, G_{20}, G_{16} (see (2.23)) and then impose a condition on the parameter a of $\mathcal{H}_a(\mathsf{I})$ which insures that it contains one of these minimal groups, i.e., $2 \mid a$, $3 \mid a$ or $5 \mid a$ (respectively). Since there is single parameter $a \geq 1$, this ensures that no groups are repeated.

Finally, we further restrict the parameters for $\mathcal{O}_a(G_{12})$ and $\mathcal{I}_a(\mathsf{I})$ to $a \geq 3$ to exclude the imprimitive groups of (2.25). □

This gives the groups in Theorem 3.6 of [Coh76] (which are seen to be distinct), in addition to their structure and inclusions. In the Theorem 3.6 of [Coh76] the imprimitive groups (2.25) are listed as primitive groups.

Our presentation of the O groups differs the most from those developed in (2.22), which are not of the $\mathcal{H}_a(G)$ form. For Taylor's presentation in (2.22), we observe that

$$d = 4a \text{ but } 4a \text{ not divisible by } 16 \quad \Longleftrightarrow \quad d = 4a \text{ with } a \text{ not divisible by } 4,$$

so that

$$E\Big((\mu_{4d}|\mu_{2d}; \mathsf{O}|\mathsf{T})\Big) = E(\mathcal{C}_{4d} \boxdot_2 \mathcal{O}) = E(\mathsf{OT}_{2d}) \qquad (d \text{ an odd multiple of 1 or 2}),$$

with these groups appearing in Theorem 2.1 as (symplectic groups)

$$E(\mathsf{OT}_{2a}) = \mathcal{O}_a(G_{12}), \quad a \text{ odd}, \qquad E(\mathsf{OT}_{4b}) = \mathcal{O}_b(G_8), \quad b \text{ odd}. \tag{2.28}$$

Example 2.1 *In Section 4.1.1 of [Sch23], various inclusions between the "primitive" quaternionic reflection groups $\mathcal{H}_a(G)$ of Theorem 2.1 were given. These are instances of (2.15), as illustrated in Figure 5.*

The Table 4.2.2 of [Sch23] gives a list of groups $M_a(G)$ for $G = G_0$ a primitive rank two complex reflection group with $\hat{G} = G$ (i.e., $G \neq G_4, G_6$) which contain no reflections other than those in G. As an example, the $\mu_4\mathsf{O}$ and OT_6 entries are

$$G_0 = \mu_4\mathsf{O} = G_{13} = M_2(G_{12}), \qquad \mu_d\mathsf{O} = M_{\frac{d}{2}}(G_{12}), \quad d \in \{4, 20, 28\},$$
$$G_0 = \mathsf{OT}_6 = G_{14} = M_3(G_{12}), \qquad \mathsf{OT}_d = M_{\frac{d}{2}}(G_{12}), \quad d \in \{6, 18, 30\}.$$

For $G_0 = G_{13}$, the groups are $\mu_d\mathsf{O} = M_{\frac{d}{4}}(G_{13}) = M_{\frac{d}{2}}(G_{12})$, where $d = 4a$ and a is not divisible by 2 or 3 (see Figure 2), i.e.,

$$\mu_d\mathsf{O} = M_{\frac{d}{2}}(G_{12}), \quad d \in \{4a : \gcd(a, 2) = \gcd(a, 3) = 1\} = \{4, 20, 28, 44, 52, 68, 76, \ldots\}$$

and, similarly, for $G_0 = G_{14}$ the groups are

$$\mathsf{OT}_d = M_{\frac{d}{2}}(G_{12}), \quad d \in \{6a : \gcd(a, 2) = 1\} = \{6, 18, 30, 42, 54, 66, 78, 90, 102, \ldots\}.$$

We finish our investigation for the rank two primitive complex reflection groups G by giving the lattice showing their inclusions (Figure 4), and the corresponding lattice of quaternionic reflection groups $\mathcal{H}_1(G)$ that evolve from them (Figure 5).

We observe that it is somewhat by chance that the maximal reflection groups that are obtained from the hidden reflections of $\mathsf{G} = \mathsf{T}, \mathsf{O}, \mathsf{I}$ do contain G. Indeed, T and O are minimal collineation preserving normal subgroups of G_7 and G_{11}, but they are not below all of the collineation preserving normal reflection subgroups.

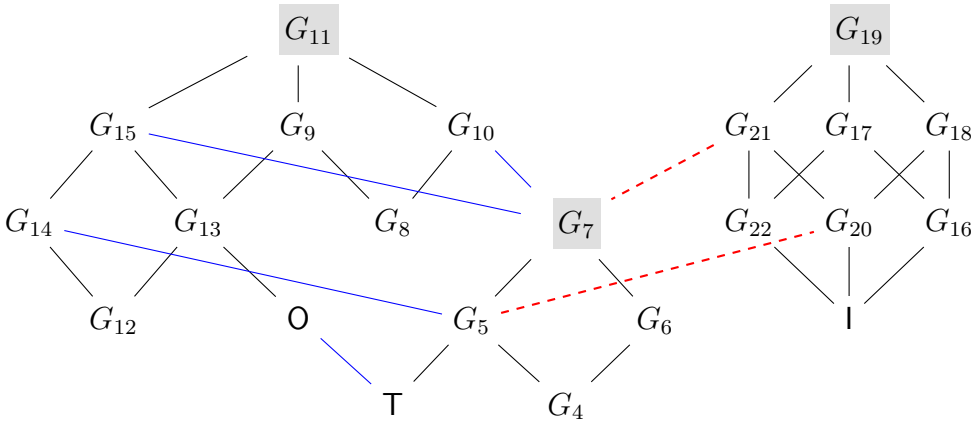


Figure 4: The inclusions between the rank two primitive complex reflection groups, with $\mathsf{T}, \mathsf{O}, \mathsf{I}$ included. The lines indicate normal reflection subgroups of the maximal reflection groups G_{11}, G_7, G_{19}, with those in blue not being collineation preserving. The red dashed lines are inclusions which are neither normal or collineation preserving.

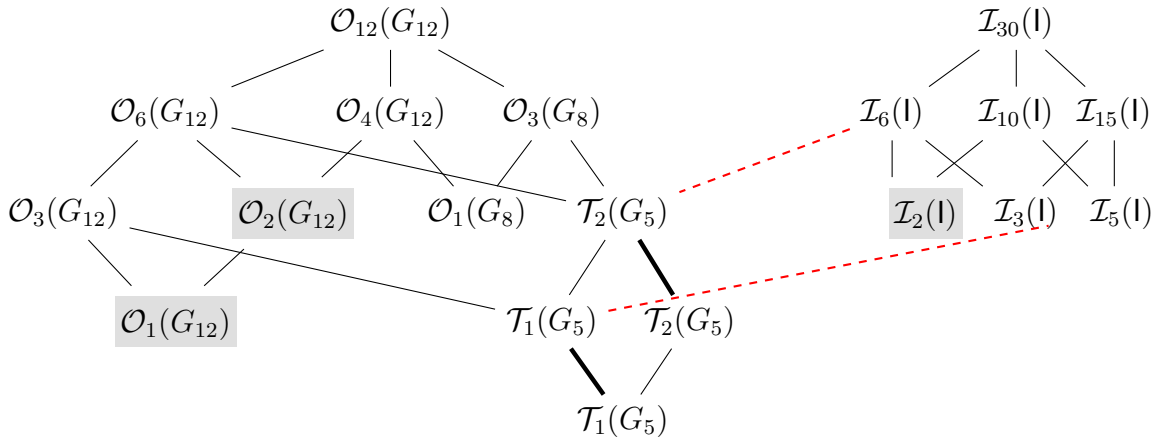


Figure 5: The inclusions between the quaternionic reflection groups of the form $\mathcal{H}_1(G)$ corresponding to those of Figure 4 for the primitive complex reflection groups G (using the notation of Theorem 2.1). Note that the groups in grey are imprimitive, and the thick black lines indicate groups that are equal.

We now consider the quaternionic reflection groups obtained from D_n (equivalently D_n). The corresponding complex reflection groups are imprimitive, given by

$$G(m,p,2) = \mathcal{G}(\{1,\zeta_m\},\{\zeta_m^p\}) = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m \\ \zeta_m^{-1} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_m^p & 0 \\ 0 & 1 \end{pmatrix} \rangle, \qquad p \mid m,$$

and so the corresponding quaternionic reflection groups are also imprimitive. In view of Lemma 2.2, we may restrict ourselves to m even.

The collineation group of $G(m,p,2)$ has order

$$\frac{|G(m,p,2)|}{|Z(G(m,p,2))|} = \frac{2m^2/p}{(m/p)\gcd(p,2)} = \frac{2m}{\gcd(p,2)} \quad \Longrightarrow \quad n = \frac{m}{\gcd(p,2)}. \tag{2.29}$$

The groups of interest are therefore:

$$G_{a,m,p} := \mathcal{H}_a(G(m,p,2)) = \langle G(m,p,2), \zeta_{\frac{m}{p}\gcd(p,2)a}, s\rangle, \qquad (m \text{ even}). \tag{2.30}$$

These have

$$|G_{a,m,p}| = 2|G(m,p,2)|a = \frac{4m^2a}{p}, \qquad |Z(G_{a,m,p})| = \frac{m}{p}\gcd(p,2)a, \tag{2.31}$$

and, since the number of reflections in a complex reflection group is the sum of its degrees minus its rank, adding these reflections to those induced by the complex scalars gives

$$\#\text{ of reflections in } G_{a,m,p} = \left(m + \frac{2m}{p} - 2\right) + \frac{m}{p}\gcd(p,2)a. \tag{2.32}$$

To understand the groups $G_{a,m,p}$, we now consider the groups $G(n,a,b,r)$ of [Wal25], which consist of monomial matrices (thereby being imprimitive) with entries in the binary dihedral group $\mathcal{D}_n := \langle \omega, j\rangle$, $\omega := \zeta_{2n}$, $n \geq 2$, and indices $[n,a,b,r] \in \Lambda_n$, where

$$\Lambda_n = \bigcup_{(a,b)\in\Omega_n} \{[n,a,b,\tfrac{n}{ab}]\} \cup \bigcup_{\substack{(a,b)\in\Omega_n \\ ab \text{ is odd}}} \{[n,a,b,\tfrac{2n}{ab}]\}, \tag{2.33}$$

$$\Omega_n := \{(a,b) : 1 \leq a \leq b \leq n,\ a \mid n,\ b \mid n,\ \gcd(a,b) = 1\}. \tag{2.34}$$

With the definition of $\mathcal{G}$ being self evident, we calculate

$$\begin{aligned}
G(n,a,b,r) &= \mathcal{G}(\{1,\omega^a,j,\omega^b j\},\{\omega^{\frac{2n}{r}}\}) \\
&= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^a \\ \omega^{-a} & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^b j \\ -\omega^b j & 0 \end{pmatrix}, \begin{pmatrix} \omega^{\frac{2n}{r}} & 0 \\ 0 & 1 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^a \\ \omega^{-a} & 0 \end{pmatrix}, \begin{pmatrix} \omega^{\frac{2n}{r}} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega^b & 0 \\ 0 & \omega^b \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_{\frac{2n}{a}} \\ \zeta_{\frac{2n}{a}}^{-1} & 0 \end{pmatrix}, \begin{pmatrix} (\zeta_{\frac{2n}{a}})^{\frac{2n}{ar}} & 0 \\ 0 & 1 \end{pmatrix}, \zeta_{\frac{2n}{b}}, s \rangle \\
&= \langle G(\frac{2n}{a}, \frac{2n}{ar}, 2), \zeta_{\frac{2n}{b}}, s\rangle, \qquad [n,a,b,r] \in \Lambda_n.
\end{aligned} \tag{2.35}$$

The centre of the complex reflection group above has order

$$\begin{aligned}
|Z(G(\frac{2n}{a}, \frac{2n}{ar}, 2))| &= \frac{\frac{2n}{a}}{\frac{2n}{ar}}\gcd(2, \frac{2n}{ar}) = r\gcd(2,\frac{2n}{ar}) = r\begin{cases} \gcd(2,2b), & r = \frac{n}{ab}; \\ \gcd(2,b), & r = \frac{2n}{ab},\ ab \text{ is odd}; \end{cases} \\
&= \frac{2n}{ab},
\end{aligned}$$

and since $(\frac{2n}{b})/(\frac{2n}{ab}) = a$, we have

$$G(n,a,b,r) = \mathcal{H}_a(G(\frac{2n}{a},\frac{2n}{ar},2)) = G_{a,\frac{2n}{a},\frac{2n}{ar}}, \tag{2.36}$$

i.e., each $G(n,a,b,r)$ is a $G_{a,m,p}$ group. As we would expect from previous examples, there may be several groups $G_{a,m,p}$ that are conjugate to each other.

By (2.29), the corresponding "n" of the collineation group D_n for $G(\frac{2n}{a},\frac{2n}{ar},2)$ is

$$\frac{\frac{2n}{a}}{\gcd(\frac{2n}{ar},2)} = \begin{cases} \frac{n}{a}, & r=\frac{n}{ab}; \\ \frac{2n}{a}, & r=\frac{2n}{ab} \end{cases} = rb. \tag{2.37}$$

Now $G(n,a,b,r)$ and $G(n,b,a,r)$ are conjugate to each other (the choice $a \le b$ in the indexing Λ_n of (2.33) was made to avoid this repetition). Indeed, with

$$U := \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix},$$

we calculate

$$\begin{aligned} UG(n,a,b,r)U^{-1} &= U\mathcal{G}(\{1,\omega^a,j,\omega^b j\},\{\omega^{\frac{2n}{r}}\})U^{-1} \\ &= \mathcal{G}(\{-j,-\omega^a j,1,\omega^b\},\{\omega^{\frac{2n}{r}}\}) = G(n,b,a,r), \end{aligned}$$

with the last equality holding because

$$\left\{\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}^2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}^2 = -1 \quad \Longrightarrow \quad -1 \in G(n,a,b,r),$$

and U is monomial, since $G(n,a,b,r)$ has no additional systems of imprimitivity [Wal26b]. Thus, we have

$$U\mathcal{H}_a(G(\frac{2n}{a},\frac{2n}{ar},2))U^{-1} = \mathcal{H}_b(G(\frac{2n}{b},\frac{2n}{br},2)), \tag{2.38}$$

i.e., for $a \ne b$, $G(n,a,b,r)$ comes from a second complex reflection group $G(\frac{2n}{b},\frac{2n}{br},2)$ (with a different collineation group). We observe that, by (2.37), the numbers of complex reflections and reflections of the type ξs in the conjugate groups $G(n,a,b,r) = G_{a,\frac{2n}{a},\frac{2n}{ar}}$ and $G(n,b,a,r) = G_{b,\frac{2n}{b},\frac{2n}{br}}$ are

$$\frac{2n}{a}+2r-2, \quad \frac{2n}{b}, \qquad \frac{2n}{b}+2r-2, \quad \frac{2n}{a},$$

respectively, and the total number of reflections in each group is the same, as it must be. However, conjugation does not simply interchange the reflections of the two types (as one might expect). It therefore seems, that deriving the groups $G(n,a,b,r)$ by identifying when the groups $G_{a,m,p}$ are conjugate, using (2.31) and (2.32), may be quite involved.

The classification of [Wal25] also gave two "exceptional" groups of monomial matrices with entries in $\mathcal{D}_n$, namely

$$\begin{aligned} G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_n) &= \mathcal{G}(\{1,\omega,j,\omega j\},\{\omega,j\}), \\ G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_{n/2}) &= \mathcal{G}(\{1,\omega,j,\omega j\},\{\omega^2,j\}), \qquad (n \text{ is even}). \end{aligned}$$

These have common generators

$$\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&\omega\\\overline{\omega}&0\end{pmatrix},\begin{pmatrix}0&j\\-j&0\end{pmatrix},\begin{pmatrix}0&\omega j\\-\omega j&0\end{pmatrix},\quad\begin{pmatrix}j&0\\0&1\end{pmatrix},$$

equivalently

$$\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&\omega\\\overline{\omega}&0\end{pmatrix},\begin{pmatrix}\omega&0\\0&\omega\end{pmatrix},\quad\begin{pmatrix}0&j\\-j&0\end{pmatrix},\quad\begin{pmatrix}j&0\\0&1\end{pmatrix},$$

together with (respectively)

$$\begin{pmatrix}\omega&0\\0&1\end{pmatrix},\qquad\begin{pmatrix}\omega^2&0\\0&1\end{pmatrix}\quad(n\text{ is even}).$$

The first two generators together with the above reflections (respectively) give

$$G(2n,1,2),\qquad G(2n,2,2),\quad(n\text{ is even}).$$

Since the centre of these imprimitive complex reflection groups is $\langle\omega\rangle$, we conclude that the exceptional groups are obtained by adding two quaternionic reflections (the last not having a block monomial symplectic form), i.e.,

$$G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_n)=\langle G(2n,1,2),s,\begin{pmatrix}j&0\\0&1\end{pmatrix}\rangle=\langle G(n,1,1,2n),\begin{pmatrix}j&0\\0&1\end{pmatrix}\rangle,$$
$$G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_{n/2})=\langle G(2n,2,2),s,\begin{pmatrix}j&0\\0&1\end{pmatrix}\rangle=\langle G(n,1,1,n),\begin{pmatrix}j&0\\0&1\end{pmatrix}\rangle,\quad(n\text{ is even}).$$

In summary, the groups $G(n,a,b,r)$ have an imprimitive complexification with the block monomial form (i.e., a system of imprimitivity with two subspaces), whereas the imprimitive quaternionic reflection groups $G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_n)$ and $G_{\mathcal{D}_n}(\mathcal{D}_n,\mathcal{D}_{n/2})$ (n even) have an imprimitive complexification (a system of imprimitivity with four subspaces) which is not block monomial.

In Example 5.2 of [Wal26b], it was shown that the only groups $G(n,a,b,r)$ which are conjugate to complex reflection groups are $G(n,1,n,1)\cong_{\mathbb{H}}G(2n,2,2)$, and accordingly, we modify the indexing to $\Lambda_n^*:=\Lambda_n\setminus\{[n,1,n,1]\}$. By combining a number of results and observations, we obtain the following classification.

Theorem 2.2 *(Classification) The rank two imprimitive quaternionic reflection groups which have block monomial complexifications, i.e., their complexification has a system of imprimitivity of two subspaces (in addition to one of four subspaces), are given by*

$$G(n,a,b,r)=\mathcal{H}_a(G(\frac{2n}{a},\frac{2n}{ar},2))\cong_{\mathbb{H}}\mathcal{H}_b(G(\frac{2n}{b},\frac{2n}{br},2)),\qquad[n,a,b,r]\in\Lambda_n^*,\tag{2.39}$$

and

$$\mathcal{O}_1(G_{12})=\langle G_{12},s\rangle,\qquad\mathcal{O}_2(G_{12})=\langle G_{13},s\rangle,\qquad\mathcal{I}_2(\mathsf{I})=\langle G_{22},s\rangle.\tag{2.40}$$

Proof: The classification of the imprimitive quaternionic reflection groups obtained from D_n in terms of the $G(n,a,b,r)$ was given in [Wal25] (which corrects that of [Coh80]), and we have discussed the exclusion of those which are conjugate to complex reflection groups. The alternative formulas were derived in (2.35) and (2.38). The three groups $\mathcal{H}_a(G)$, with G primitive, which are imprimitive quaternionic reflection groups (with block monomial complexifications) were identified in Lemma 2.4. □

3 The finite subgroups of $GL(\mathbb{C}^2)$

In this section, we use our techniques to find all the minimal finite irreducible subgroups $H \subset GL(\mathbb{C}^2)$ with a given collineation group, from which all of the finite irreducible subgroups with this collineation group can be obtained explicitly in form of (2.18), i.e.,

$$M_a(H) = \langle H, \zeta_{|Z(H)|a} \rangle, \qquad a \geq 1, \qquad M_1(H) = H, \quad |M_a(H)| = |H| \cdot a.$$

We then restrict the parameters so that no group is repeated (up to conjugation), and identify all of those which are reflection groups. We will essentially follow[NvdPT08].

The elements of a general finite irreducible group H with collineation group

$$\frac{H}{Z(H)} = \frac{\mathsf{G}}{Z(\mathsf{G})} = \frac{\mathsf{G}}{\langle -1 \rangle}, \qquad \mathsf{G} = \mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I} \quad (\frac{\mathsf{G}}{\langle -1 \rangle} = D_n, A_4, S_4, A_5),$$

can be written as scalar multiples λg of the elements $g \in \mathsf{G}$. These scalars form a finite group

$$\begin{aligned} \mathrm{mult}(H) &:= \{\lambda \in \mathbb{C}^* : \lambda g \in H, g \in \mathsf{G}\} \\ &= \{\pm\sqrt{\det(h)} : h \in H\}, \quad \text{when } -1 \in Z(H). \end{aligned} \tag{3.41}$$

It obvious that $\mathrm{mult}(H)$ is a finite group, and it contains the centre $Z(H)$ of H, since if $\xi \in Z(H)$, then $\xi = \xi I$, $I \in \mathsf{G}$. The second formula follows from the fact that each $h \in H$ can be written as a scalar multiple of an element of G in exactly two possible ways, i.e.,

$$h = \pm\sqrt{\det h}\Big(\frac{h}{\pm\sqrt{\det h}}\Big), \qquad \frac{h}{\pm\sqrt{\det h}} \in \mathsf{G},$$

and it appears in both ways if and only if $-1 \in Z(H)$. Further, $\mathrm{mult}(H)$ is generated by the scalars $\pm\sqrt{\det h}$ that it contains for h in any generating set for H.

Understanding $\mathrm{mult}(H) \subset \mathbb{C}^*$ will help us understand the possible choices for H. There is a group homomorphism onto $\mathrm{mult}(H)/Z(H)$ (essentially the h of [NvdPT08]) given by

$$\Theta_H : \mathsf{G} \to \frac{\mathrm{mult}(H)}{Z(H)} : g \mapsto \lambda Z(H), \qquad \lambda g \in H. \tag{3.42}$$

Elementary calculations show that Θ_H is well defined and is a homomorphism. Indeed

$$\lambda_1 g, \lambda_2 g \in H \quad \Longrightarrow \quad \lambda_1\lambda_2^{-1} \in H \quad \Longrightarrow \quad \lambda_1\lambda_2^{-1} \in Z(H) \quad \Longrightarrow \quad \lambda_1 Z(H) = \lambda_2 Z(H),$$

$$\lambda_1 g_1, \lambda_2 g_2 \in H \quad \Longrightarrow \quad \lambda_1\lambda_2 g_1 g_2 \in H \quad \Longrightarrow \quad \Theta_H(g_1)\Theta_H(g_2) = \lambda_1\lambda_2 Z(H) = \Theta_H(g_1 g_2).$$

The kernel of Θ_H is the set of elements in H with determinant 1, i.e.,

$$\mathrm{spec}(H) := \ker(\Theta_H) = \{g \in \mathsf{G} : \lambda g \in H, \forall \lambda \in Z(H)\} = H \cap SL(\mathbb{C}^2), \tag{3.43}$$

and

$$\frac{\mathrm{mult}(H)}{Z(H)} \cong \frac{\mathsf{G}}{\mathrm{spec}(H)}. \tag{3.44}$$

Since the image of Θ_H of (3.42) is abelian, Θ_H must factor through $\mathsf{G}/[\mathsf{G},\mathsf{G}]$, which is given by

$$\frac{\mathsf{D}_n}{[\mathsf{D}_n,\mathsf{D}_n]} = \frac{\mathsf{D}_n}{\mathsf{C}_n} = C_4, \quad \frac{\mathsf{T}}{[\mathsf{T},\mathsf{T}]} = \frac{\mathsf{G}}{\mathsf{D}_2} = C_3, \quad \frac{\mathsf{O}}{[\mathsf{O},\mathsf{O}]} = \frac{\mathsf{O}}{\mathsf{T}} = C_2, \quad \frac{\mathsf{I}}{[\mathsf{I},\mathsf{I}]} = 1. \tag{3.45}$$

Thus $\mathrm{mult}(H)/Z(H)$ is (isomorphic to) a subgroup of $\mathsf{G}/[\mathsf{G},\mathsf{G}]$, and we do have the implied inclusion

$$[\mathsf{G},\mathsf{G}] \subset \mathrm{spec}(H) \subset \mathsf{G}. \tag{3.46}$$

If $\mathrm{mult}(H)/Z(H) = 1$, i.e., $\mathrm{mult}(H) = Z(H)$, then

$$H = M_a(\mathsf{G}) = \langle \zeta_{2a} \rangle \mathsf{G}, \qquad a \geq 1, \tag{3.47}$$

with $\mathsf{G} = M_1(\mathsf{G}) = \langle \zeta_2 \rangle \mathsf{G}$ being minimal. This accounts for all groups H for $\mathsf{G} = \mathsf{I}$.

Before treating the cases where the $\mathrm{mult}(H)/Z(H)$ is not trivial, we observe that generators for any G can be obtained by taking any set of reflections which generate a reflection group with the same collineation group (see Table 1), and then multiplying each of them by a scalar so that it lies in G, i.e., has determinant 1. In this way, we will choose a generating set for G whose image under Θ_H has the fewest number of nonidentity elements as is possible. We observe that the reflections F, R, Z, M of (2.11) and (2.12) can be "hidden" in $SU(\mathbb{C}^2)$ (which has no reflections) as iF and

$$\sqrt{i}R = \begin{pmatrix} \sqrt{i} & 0 \\ 0 & \sqrt{i}^{-1} \end{pmatrix}, \quad \zeta_3 Z = \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \quad \zeta_5 M = \frac{1}{2}\begin{pmatrix} \tau^{-1}-\tau i & -1 \\ 1 & \tau^{-1}+\tau i \end{pmatrix}, \tag{3.48}$$

and that we have

$$\mathsf{T} = \langle iR^2, \zeta_3 Z \rangle \subset \mathsf{O} = \langle \sqrt{i}R, \zeta_3 Z \rangle = \langle iF, \zeta_3 Z \rangle.$$

For $G = \mathsf{T} = \langle iR^2, \zeta_3 Z \rangle$, there could be (and are) groups with $|\,\mathrm{mult}(H)| = 3|Z(H)|$. To see what these are, we suppose, without loss of generality, that

$$|Z(H)| = 3^r \cdot 2k, \quad \gcd(3,k) = 1, \qquad r \geq 0.$$

To have $|\,\mathrm{mult}(H)| = 3 \cdot 3^r \cdot 2k$, the group H must have an element of T of order divisible by 3, multiplied by $\zeta_{3^{r+1}}$ (multiplying an element of G of order not divisible by 3 would introduce the scalar $\zeta_{3^{r+1}}$ to the centre of H). The only such elements are two conjugacy classes of elements of order three given by $\zeta_3 Z$ and $(\zeta_3 Z)^2$, and two of elements of order six given by $-\zeta_3 Z$ and $-(\zeta_3 Z)^2$. Since $-1 \in \mathsf{T}$ and T is a normal subgroup of O with $\zeta_3 Z$ and $(\zeta_3 Z)^2$ conjugate in O, it suffices to choose any of the four elements above, and we take $\zeta_3 Z$. For H to have collineation group A_4, it must contain a scalar multiple of $iR^2 \in \mathsf{T}$, and the scalar multiplier cannot have an order divisible by 3^{r+1} (which would introduce $\zeta_{3^{r+1}}$ to the centre of H), and so this scalar belongs to $Z(H)$. Therefore, such an H contains the group

$$\langle \zeta_{3^r \cdot 2k}, iR^2, \zeta_{3^{r+1}}(\zeta_3 Z) \rangle = \langle \zeta_{2k} \rangle \langle iR^2, \zeta_{3^{r+1}}(\zeta_3 Z) \rangle, \quad (\text{since } (\zeta_{3^{r+1}}(\zeta_3 Z))^3 = \zeta_{3^r}).$$

Thus the minimal groups with $\mathrm{mult}(H)/Z(H) = C_3$ are

$$H_r^{A_4} := \langle iR^2, \zeta_{3^{r+1}}(\zeta_3 Z)\rangle, \qquad |H_r^{A_4}| = 3^r|\mathsf{T}| = 3^r \cdot 24, \qquad r \geq 0. \tag{3.49}$$

Of these, only one could be a reflection group based on there being a minimal collineation preserving normal reflection subgroup of G_7 with the same order and centre, which it is, i.e.,

$$H_0^{A_4} = \langle iR^2, \zeta_3^2 Z\rangle = \langle \zeta_3^2 Z, (iR^2)\zeta_3^2 Z(iR^2)^{-1}\rangle = G_4 \neq \mathsf{T}. \tag{3.50}$$

For $\mathsf{G} = \mathsf{O} = \langle iF, \zeta_3 Z\rangle = \langle \sqrt{i}R, \zeta_3 Z\rangle$, there are groups with $|\,\mathrm{mult}(H)| = 2|Z(H)|$. These have

$$|Z(H)| = 2^r \cdot 2k = 2^{r+1}k, \quad \gcd(2,k) = 1, \qquad r \geq 0.$$

Since $|\mathsf{O}| = 2^4 \cdot 3$, the previous reasoning implies that H must contain $\zeta_{2^r \cdot 2k}$ and $\zeta_3 Z$. To have an element of order 2^{r+2} in $\mathrm{mult}(H)$, H must also contain an element $\zeta_{2^{r+2}}g$, $g \in \mathsf{O}$, where order of g is a power of 2 and $\langle g, \zeta_3 Z\rangle = \mathsf{O}$. The only possibilities for g up to conjugation in O are iF (order 4) and $\pm\sqrt{i}R$ (order 8). We first consider iF, i.e., that H contains

$$H_{r,k} := \langle \zeta_{2^r \cdot 2k}, \zeta_{2^{r+2}}(iF), \zeta_3 Z\rangle.$$

For $r \geq 1$, we have

$$(\zeta_{2^{r+2}}(iF))^{2+2^{r+1}} = \zeta_{2^{r+1}} \quad \Longrightarrow \quad H_{r,k} = \langle \zeta_k\rangle\langle \zeta_{2^{r+2}}(iF), \zeta_3 Z\rangle, \quad r \geq 1.$$

The above formula for $H_{r,k}$ also holds for $r = 0$, since in this case

$$\zeta_{2^{r+1}} = -1 \in \langle \zeta_{2^{r+2}}(iF), \zeta_3 Z\rangle = \langle -F, \zeta_3 Z\rangle.$$

The relation

$$\zeta_3 Z = (\sqrt{i}R)(iF),$$

implies that

$$\langle \zeta_{2^{r+2}}(iF), \zeta_3 Z\rangle = \langle \zeta_{2^{r+2}}(\sqrt{i}R), \zeta_3 Z\rangle.$$

Thus the minimal groups with $\mathrm{mult}(H)/Z(H) = C_2$ are

$$\begin{aligned} H_r^{S_4} &:= \langle \zeta_{2^{r+2}}(iF), \zeta_3 Z\rangle \\ &= \langle \zeta_{2^{r+2}}(\sqrt{i}R), \zeta_3 Z\rangle, \qquad |H_r^{S_4}| = 2^r \cdot |\mathsf{O}| = 2^r \cdot 48, \qquad r \geq 0. \end{aligned} \tag{3.51}$$

By examination of the minimal reflection groups for S_4, we have

$$H_0^{S_4} = \langle i(iF), \zeta_3 Z\rangle = G_{12} \neq \mathsf{O}, \quad |H_0^{S_4}| = |\mathsf{O}|, \qquad H_1^{S_4} = \langle i\sqrt{i}F, \zeta_3 Z\rangle = G_8. \tag{3.52}$$

Remark 3.1 *Our group* $\mathrm{mult}(H)$ *of (3.41) is in the same spirit as the Schur multiplier, but it is not same concept. Indeed, the Schur multiplier group of* A_4, S_4, A_5 *and* D_n *(*n *even) is* C_2*, and it is trivial for* D_n *(*n *odd). We can recognise the Schur covering groups as minimal subgroups, e.g.,* T *and* G_4 *are the unique Schur cover of* A_4*,* O *and* G_{12} *nonisomorphic Schur covers of* S_4*, and* I *is the unique Schur cover of* A_5*.*

We can now prove the following.

Theorem 3.1 *(Classification) The minimal finite primitive subgroups of $GL(\mathbb{C}^2)$ for the possible collineation groups A_4, S_4, A_5 are (up to conjugation)*

(i) A_4: $\quad \mathsf{T} = \langle iR^2, \zeta_3 Z\rangle$ *and* $H_r^{A_4} = \langle iR^2, \zeta_{3^{r+1}}(\zeta_3 Z)\rangle$, $r \geq 0$.

(ii) S_4: $\quad \mathsf{O} = \langle \sqrt{i}R, \zeta_3 Z\rangle$ *and* $H_r^{S_4} = \langle \zeta_{2^{r+2}}(\sqrt{i}R), \zeta_3 Z\rangle$, $r \geq 0$.

(iii) A_5: $\quad \mathsf{I} = \langle iR^2, \zeta_5 M\rangle$.

with the generators given by (3.48). Exactly three of these are reflection groups, namely

$$H_0^{A_4} = G_4, \qquad H_0^{S_4} = G_{12}, \qquad H_1^{S_4} = G_8. \tag{3.53}$$

Basic properties of these minimal groups are collected in Table 2.

Proof: We have established the possible minimal subgroups (3.49), (3.51), in addition to the minimal groups $\mathsf{T}, \mathsf{O}, \mathsf{I}$. It remains to show that these are indeed minimal.

A collineation preserving subgroup H of $H_r^{A_4}$ must contain a scalar multiple of iR^2 and one of $\zeta_3 Z$. Since iR^2 has order 2, and -1 is commutator in T, and hence is a commutator in H, we have (after taking appropriate powers) that $iR^2 \in H$. The only scalar multiples of $\zeta_3 Z$ in $H_r^{A_4}$ are those by the scalars in $\zeta_{3^{r+1}}\langle\zeta_{3^r}\rangle$, and so $\zeta_{3^{r+1}}\lambda \in H$, for $\lambda \in \langle\zeta_{3^r}\rangle$. Thus H contains $\langle iR^2, \zeta_{3^{r+1}}\lambda(\zeta_3)Z\rangle = H_r^{A_4}$, and $H_r^{A_4}$ is minimal.

Now suppose that H is a collineation preserving subgroup of $H_r^{S_4}$, and so contains scalar multiples of $\sqrt{i}R$ and $\zeta_3 Z$. The scalar multiples of $\sqrt{i}R$ in $H_r^{S_4}$ are those by scalars $\zeta_{2^{r+2}}\lambda$, $\lambda \in Z(H_r^{S_4}) = \langle\zeta_{2^{r+1}}\rangle$, and $\zeta_3 Z$ appears as $\lambda(\zeta_3 Z)$ (a power of which equals $\zeta_3 Z$). Thus H contains $\langle\zeta_{2^{r+2}}\lambda\sqrt{i}R, \zeta_3 Z\rangle = \langle\zeta_{2^{r+2}}\sqrt{i}R, \zeta_3 Z\rangle = H_r^{S_4}$, and so $H_r^{S_4}$ is minimal.

The minimal groups which are reflection groups are given in (3.50) and (3.51). □

In [NvdPT08], it is claimed that the groups O and $H_0^{S_4} = G_{12}$ of order 48 are conjugate, and so there is a missing group in (4) (c) of their Theorem 4.

We now consider the dihedral groups D_n, $n \geq 2$, for which the binary dihedral group is monomial (imprimitive), and given by

$$\mathsf{D}_n = \langle A, B\rangle, \qquad A = \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \left\{\text{or } \ B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right\}. \tag{3.54}$$

For H a minimal group with collineation group D_n, $\mathrm{mult}(H)/Z(H)$ is a subgroup of C_4, with the trivial subgroup giving $H = \mathsf{D}_n$. We now suppose that $\mathrm{mult}(H)/Z(H)$ is not trivial. Since the collineation group of H is D_n, it must contain a subgroup

$$H_{\lambda,\mu} := \langle \lambda A, \mu B\rangle, \qquad \lambda, \mu \in \mathrm{mult}(H).$$

We observe that D_n is also generated by B and AB, which have the same algebraic structure within the group. In particular, $B^2 = (AB)^2 = -1$, so that

$$(\mu B)^2 = -\mu^2, \quad (\lambda A \mu B)^2 = -\lambda^2\mu^2 \quad \Longrightarrow \quad \langle\lambda^2, -\mu^2\rangle \subset Z(H) \subset \langle\lambda, \mu\rangle = \mathrm{mult}(H_{\lambda,\mu}).$$

By the symmetry in the generators B and AB, we may assume, without loss of generality, that μ is a $2^{r+2}\cdot k$ primitive root of unity, where $\gcd(2, k) = 1$, $r \geq -1$. Now

$$(\mu B)^{2^{r+2}} = \hat{\zeta}_k, \quad r \geq 0, \qquad (\mu B)^{2(k+1)} = (\zeta_{2k} B)^{2(k+1)} = \hat{\zeta}_k, \quad r = -1,$$

with $\hat{\zeta}_k$ a primitive k root of unity, so that

$$\langle \zeta_k \rangle \langle \lambda A, \mu^k B \rangle \subset \langle \lambda A, \mu B \rangle \subset H.$$

By the minimality of H, we must have $k = 1$, and μ is a primitive 2^{r+2} root of unity. By the same reasoning for AB, $\lambda\mu$ is is a primitive 2^{r+2} root of unity, and hence $\lambda = (\lambda\mu)\mu^{-1}$ is a 2^s root of unity, where $0 \le s \le r+2$. For $r = -1$, we have $\mu = -1$, $\lambda = \pm 1$, so that $H_{\lambda,\mu} = \langle \pm A, -B \rangle = \mathsf{D}_n$, since $Z(\mathsf{D}_n) = \langle -1 \rangle$, and so we must have $r \ge 0$.

The 2^{r+2} primitive roots of unity are obtained by multiplying any given one μ by the 2^{r+1} roots of unity. Now $-\mu^2 \in Z(H)$, and so $\zeta_{2^{r+1}}$ must be in the centre of H, except for possibly when $r = 0$ (i.e., $\mu = \pm i$). Hence for $r \neq 0$, the choices for λ and μ can be reduced to $\mu = \zeta_{2^{r+2}}$ (i.e., any primitive root can be taken) and $\lambda = 1$ or $\lambda = \zeta_{2^{r+2}}$. Accordingly, we define groups

$$H_r^{D_n} := \langle A, \zeta_{2^{r+2}} B \rangle, \qquad K_r^{D_n} := \langle \zeta_{2^{r+2}} A, \zeta_{2^{r+2}} B \rangle, \qquad r \ge 0, \tag{3.55}$$

where for $r = 0$ there may be other groups. We note that from our discussion

$$Z(H) = \langle \zeta_{2^{r+1}} \rangle, \quad \mathrm{mult}(H) = \langle \zeta_{2^{r+2}} \rangle, \qquad H = H_r^{D_n}, K_r^{D_n}, \qquad r \ge 1, \tag{3.56}$$

and that these groups will be seen to be minimal.

We now consider all the possible groups for $r = 0$, i.e., $\langle \lambda A, \mu B \rangle$, $\lambda, \mu \in \{1, -1, i, -i\}$. We have the conjugation

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \qquad PAP^{-1} = A, \quad PBP^{-1} = -B \quad \text{(for either choice of } B\text{)},$$

and so the following pairs of groups are conjugate

$$H_0^{D_n} = \langle A, iB \rangle, \langle A, -iB \rangle, \qquad K_0^{D_n} = \langle iA, iB \rangle, \langle iA, -iB \rangle,$$

$$\langle -A, iB \rangle, \langle -A, -iB \rangle, \qquad \langle -iA, iB \rangle, \langle -iA, -iB \rangle.$$

They are in fact all equal, except for $\langle -A, iB \rangle$ and $\langle -A, -iB \rangle$ when n is odd.

We now suppose that n is odd. Since $(iB)^2 = 1$, the $2n$ elements of $H = \langle -A, iB \rangle$ are

$$\{1, -A, A^2, -A^3, \ldots, -A^{2n-1}\}\{1, iB\},$$

so that $Z(H) = 1$, $\mathrm{mult}(H) = \langle i \rangle$, i.e., $\mathrm{mult}(H)/Z(H) = C_4$. Since $Z(H) = 1$, this group is minimal and equal to D_n. Moreover, (by considering orders) we have

$$\langle -1 \rangle D_n = \langle -1 \rangle \langle -A, iB \rangle = \langle A, iB \rangle = H_0^{D_n}, \qquad \langle i \rangle D_n = \langle i \rangle \langle -A, iB \rangle = K_0^{D_n}, \langle -iA, iB \rangle,$$

and so there is just one minimal group $D_n = \langle -A, iB \rangle$, which is a real reflection group.

We now suppose that n is even. Then

$$\begin{aligned} (\pm A)^n = -1 \quad &\Longrightarrow \quad H_0^{D_n} = \langle A, iB \rangle = \langle -A, iB \rangle = \langle A, -iB \rangle = \langle -A, -iB \rangle, \\ (\pm iB)^2 = -1 \quad &\Longrightarrow \quad K_0^{D_n} = \langle iA, iB \rangle = \langle -iA, iB \rangle = \langle iA, -iB \rangle = \langle -iA, -iB \rangle, \end{aligned}$$

and we will see that $H_0^{D_n}$ and $K_0^{D_n}$ are minimal groups.

Theorem 3.2 *(Classification) The minimal finite imprimitive irreducible subgroups of $GL(\mathbb{C}^2)$ for the possible collineation groups D_n, $n \geq 2$, are (up to conjugation)*

(i) $\mathsf{D}_n = \langle A, B\rangle$ *(binary dihedral group).*

(ii) $D_n = \langle -A, iB\rangle$ *(dihedral group) and* $H_r^{D_n} = \langle A, \zeta_{2^{r+2}}B\rangle$, $r \geq 1$, *for n odd.*

(iii) $H_r^{D_n} = \langle A, \zeta_{2^{r+2}}B\rangle$ *and* $K_r^{D_n} = \langle \zeta_{2^{r+2}}A, \zeta_{2^{r+2}}B\rangle$, $r \geq 0$, *for n even.*

These groups are distinct, except for when $n = 2$, $H_r^{D_2}$ and $K_r^{D_2}$ are conjugate. Of these groups, the following are reflection groups

$$\begin{aligned} D_n &= G(n, n, 2), && n \text{ odd}, \\ K_r^{D_n} &= G(n, b, 2), && n \text{ even},\ n = 2^{r+1}b,\ \gcd(2, b) = 1, \\ H_0^{D_n} &= G(2n, 2n, 2), && n \text{ even}. \end{aligned} \tag{3.57}$$

Basic properties of these minimal groups are collected in Table 2.

Proof: We have shown that D_n and D_n (for n odd) are minimal groups. It therefore remains to determine whether the additional possible minimal groups of (3.56) and $H_0^{D_n}$ and $K_0^{D_n}$ (for n even) are minimal, and whether there are any conjugacies between them.

For $r = 0$ and n even, we have $A^n = (\zeta_{2^{r+2}}A)^n = -1$, so that the centre of $H_r^{D_n}$ and $K_r^{D_n}$ is $\langle\zeta_{2^{r+1}}\rangle$, in this case, as well as for $r \geq 1$. Now suppose that $r \geq 1$ or $r = 0$ and n is even, and let H be a collineation preserving subgroup of $H_r^{D_n}$ or $K_r^{D_n}$. Then H must contain a scalar multiple of the generators, i.e.,

$$\lambda A, \quad \lambda\zeta_{2^{r+2}}A \quad \text{(respectively)}, \qquad \mu\zeta_{2^{r+2}}B, \qquad \lambda, \mu \in \langle\zeta_{2^{r+1}}\rangle.$$

Since $(\mu\zeta_{2^{r+2}}B)^2 = -\mu\zeta_{2^{r+1}}$ generates $\langle\zeta_{2^{r+1}}\rangle$, for $r \geq 1$, and $(\lambda A)^n = (\lambda\zeta_{2^{r+2}}A)^n = -1$ generates $\langle\zeta_{2^{r+1}}\rangle$, for $r = 0$ and n even, these elements generate the corresponding groups $H_r^{D_n}$ and $K_r^{D_n}$ (respectively), which are therefore minimal.

For n odd and $r \geq 1$, we have

$$(\zeta_{2^{r+2}}A)^n = \zeta_{2^{r+2}}^n(-1),$$

which is a primitive 2^{r+2} root of unity, so that $H_r^{D_n}$ is contained within $K_r^{D_n}$, and, since they are both minimal groups, we have $H_r^{D_n} = K_r^{D_n}$, which gives (ii).

For n even and $r \geq 0$, we have (by examination and the fact $|H| = 2|\operatorname{spec}(H)|$) that

$$\operatorname{spec}(H_r^{D_n}) = \langle A\rangle = \mathsf{C}_{2n}, \qquad \operatorname{spec}(K_r^{D_n}) = \langle A^2, AB\rangle \cong \mathsf{D}_{n/2}.$$

Since the determinant of a matrix is preserved under conjugation and D_n is not cyclic for $n \neq 2$, we conclude that $H_r^{D_n}$ and $K_r^{D_n}$ are not conjugate for $n \neq 2$. For $n = 2$, $A^2 = -1$, and we have $\langle A\rangle \cong C_4 \cong \langle A^2, AB\rangle = \langle AB\rangle$. It is easy to verify that $H_r^{D_2}$ and $K_r^{D_2}$ are conjugate. If we take

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \left\{\text{or} \quad P = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\right\},$$

for the choices of A and B in (3.54), then

$$PAP^{-1} = AB, \qquad PBP^{-1} = -B.$$

Since -1 in the centre of both $H_r^{D_2}$ and $K_r^{D_2}$, we therefore have

$$P(K_r^{D_2})P^{-1} = P\langle \zeta_{2^{r+2}}A, -\zeta_{2^{r+2}}B\rangle P^{-1} = \langle \zeta_{2^{r+2}}AB, \zeta_{2^{r+2}}B\rangle = \langle A, \zeta_{2^{r+2}}B\rangle = H_r^{D_2}.$$

The above two cases for n give (iii).

We now consider which of these groups are reflection groups. For an imprimitive reflection group $G(m,p,2)$ to have collineation group D_n (of order $2n$), we must have

$$\frac{|G(m,p,2)|}{|Z(G(m,p,2))|} = \frac{2m}{\gcd(p,2)} = 2n,$$

which gives two cases: p is odd and $m = n$, or p is even and $m = 2n$. Given the collineation preserving inclusion

$$G(m,p,2) \lhd G(m, \gcd(p,2), 2),$$

we have, for the cases p odd and p even, the possible minimal reflection groups (based on their orders being the minimal possible)

$$\begin{array}{ll} G(n,n,2), & n \text{ odd}, \\ G(n,b,2), & n \text{ even},\ n = 2^{r+1}b,\ \gcd(2,b) = 1, \\ G(2n,2n,2), & n \geq 2. \end{array}$$

These are easily seen to give the minimal groups in (3.57), and we will see later that there are no more cases (Corollary 3.2), i.e., no other $G(m,p,2)$ are minimal groups. □

For $n \neq 2$ even, $\mathsf{D}_n := \langle A, B\rangle$ has two normal subgroups $\langle A^2, B\rangle, \langle A^2, AB\rangle \cong \mathsf{D}_n$, and we observe from the proof that it is the second of these that appears as $\mathrm{spec}(K_r^{D_n})$. We also observe that for $n = 2$, we have

$$H_0^{D_2} = G(4,4,2) \cong G(2,1,2) = K_0^{D_2}.$$

Remark 3.2 *For n odd, [NvdPT08] define a single family of groups*

$$H_k := \langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \zeta_{2^k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle, \qquad k \geq 1.$$

These relate to our presentation as follows. Since $(\zeta_{2^{r+2}}B)^2 = -\zeta_{2^{r+1}}$, for $r \geq 1$, we have

$$H_r^{D_n} = \langle \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \zeta_{2^{r+2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle = \langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \zeta_{2^{r+2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle = H_{r+2}, \quad r \geq 1 \quad (n \text{ odd}),$$

and

$$H_2 = \langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle = \langle \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle = \langle A, B\rangle = \mathsf{D}_n,$$

$$H_1 = \langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle = \langle A^2, iB\rangle \cong D_n.$$

Table 2: The minimal finite subgroups H of $GL(\mathbb{C}^2)$ corresponding to the collineation groups A_4, S_4, A_5 (primitive) and D_n, $n \geq 2$ (imprimitive). All appear just once, except for $H_r^{D_2} \cong K_r^{D_2}$ (here $\mathsf{C}_4 \cong \mathsf{D}_1$).

$H/Z(H)$	H	$\lvert H\rvert$	$\lvert Z(H)\rvert$	$\frac{\text{mult}(H)}{Z(H)}$	G	$\text{spec}(H)$
A_4	T	24	2	1	T	T
A_4	$H_r^{A_4}$, $r \geq 0$	$3^r \cdot 24$	$3^r \cdot 2$	C_3	T	D_2
S_4	O	48	2	1	O	O
S_4	$H_r^{S_4}$, $r \geq 0$	$2^r \cdot 48$	$2^r \cdot 2$	C_2	O	T
A_5	I	120	2	1	I	I
D_n, n odd	D_n	$4n$	2	1	D_n	D_n
	D_n	$2n$	1	C_4	D_n	C_n
	$H_r^{D_n}$, $r \geq 1$	$2^r \cdot 4n$	$2^r \cdot 2$	C_2	D_n	C_{2n}
D_n, n even	D_n	$4n$	2	1	D_n	D_n
	$H_r^{D_n}$, $r \geq 0$	$2^r \cdot 4n$	$2^r \cdot 2$	C_2	D_n	C_{2n}
	$K_r^{D_n}$, $r \geq 0$	$2^r \cdot 4n$	$2^r \cdot 2$	C_2	D_n	$\mathsf{D}_{n/2}$

We now seek to describe all the finite (collineation preserving) subgroups of $GL(\mathbb{C}^2)$ uniquely in terms of the minimal groups. For a given minimal group H, all the subgroups obtained from it by adding additional scalars, i.e.,

$$M_a(H) = \langle H, \zeta_{|Z(H)|a}, \rangle, \qquad a \geq 1,$$

have different orders, and hence are different groups. For A_5, there is just one minimal group I, and this is the end of the story. In the other cases, we need to determine when $M_a(H) = M_b(K)$ for different minimal groups H and K, and then restrict the parameters so that a given group is counted exactly once. We begin with an example.

Example 3.1 *The reflection group G_{13} can be obtained from minimal groups in two ways, i.e.,*

$$G_{13} = \langle \zeta_4 \rangle G_{12} = \langle \zeta_4 \rangle \mathsf{O} \quad \Longleftrightarrow \quad M_2(H_0^{S_4}) = M_2(\mathsf{O}), \quad G_{12} = H_0^{S_4}.$$

Here, we observe that $\text{mult}(G)/Z(G)$ *can reduce as we add scalars to a minimal group*

$$\frac{\text{mult}(M_a(\mathsf{O}))}{Z(M_a(\mathsf{O}))} = \frac{\langle \zeta_{2a} \rangle}{\langle \zeta_{2a} \rangle} = 1, \qquad \frac{\text{mult}(G_{12})}{Z(G_{12})} = \frac{\langle i \rangle}{\langle -1 \rangle} = C_2, \quad \frac{\text{mult}(M_2(G_{12}))}{Z(M_2(G_{12}))} = \frac{\langle i \rangle}{\langle i \rangle} = 1,$$

and, equivalently, by (3.44), $\text{spec}(G)$ *can increase*

$$\text{spec}(\mathsf{O}) = \text{spec}(M_a(\mathsf{O})) = \mathsf{O}, \qquad \text{spec}(G_{12}) = \mathsf{T}, \quad \text{spec}(M_2(G_{12})) = \mathsf{O}.$$

In the our classification of the finite irreducible subgroups of $GL(\mathbb{C}^2)$ below, we will choose (when a choice must be made) the minimal group H giving a subgroup G, so that

$$\frac{\mathrm{mult}(G)}{Z(G)} = \frac{\mathrm{mult}(H)}{Z(H)} \cong \frac{\mathsf{G}}{\mathrm{spec}(H)},$$

i.e., G_{13} would appear as $M_2(\mathsf{O})$, rather than as $M_2(H_0^{S_4})$.

It is convenient to formalise the observation of (3.47), and give a direct proof which does not utilise the isomorphism (3.43).

Lemma 3.1 *Let G be a finite irreducible subgroup of $GL(\mathbb{C}^2)$. Then*

$$\frac{\mathrm{mult}(G)}{Z(G)} = 1 \quad \Longleftrightarrow \quad G = M_a(\mathsf{G}), \quad a \geq 1, \ \mathsf{G} = \mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I}.$$

Proof: Suppose that $\frac{\mathrm{mult}(G)}{Z(G)} = 1$, i.e., $\mathrm{mult}(G) = Z(G)$, and that $G = M_a(H)$ for some minimal group H. If $H \neq D_n$, for n odd, then $-1 \in Z(H) \subset Z(G) = \mathrm{mult}(G)$, so that $\langle -1 \rangle \mathsf{G} = \mathsf{G} \subset G$, and $G = M_a(\mathsf{G})$, for $a = \frac{1}{2}|Z(G)| = \frac{1}{2}|\,\mathrm{mult}(G)|$. For the only other possible case, since $\mathrm{mult}(D_n) = \langle i \rangle$, for n odd, we have

$$\frac{\mathrm{mult}(M_a(D_n))}{Z(M_a(D_n))} = \frac{\langle \zeta_a \rangle \langle i \rangle}{\langle \zeta_a \rangle} = 1 \quad \Longleftrightarrow \quad a = 4m \quad \Longleftrightarrow \quad M_a(D_n) = M_{2m}(\mathsf{D}_n). \quad (3.58)$$

The converse follows from $\mathrm{mult}(M_a(\mathsf{G})) = \langle \zeta_{2a} \rangle \langle -1 \rangle = \langle \zeta_{2a} \rangle = Z(M_a(\mathsf{G}))$. □

Theorem 3.3 *(Classification) The finite irreducible subgroups of $GL(\mathbb{C}^2)$ are precisely the primitive groups (for A_4, S_4, A_5)*

(i) $M_a(\mathsf{T}), \quad a \geq 1, \qquad M_c(H_r^{A_4}), \quad c \geq 1, \ \gcd(3, c) = 1, \ r \geq 0.$

(ii) $M_a(\mathsf{O}), \quad a \geq 1, \qquad M_b(H_r^{S_4}), \quad b \geq 1, \ \gcd(2, b) = 1, \ r \geq 0.$

(iii) $M_a(\mathsf{I}), \quad a \geq 1.$

and the imprimitive groups (for D_n, $n \geq 2$)

(i) $M_a(\mathsf{D}_n), \quad a \geq 1$ *(binary dihedral group).*

(ii) *For n odd:* $M_b(D_n)$ *and* $M_b(H_r^{D_n}), \quad b \geq 1, \ \gcd(2, b) = 1, \ r \geq 2.$

(iii) *For n even:* $M_b(H_r^{D_n})$ *and* $M_b(K_r^{D_n}), \quad b \geq 1, \ \gcd(2, b) = 1, \ r \geq 1.$

These groups are distinct, except for when $n = 2$, $M_b(H_r^{D_2})$ and $M_b(K_r^{D_2})$ are conjugate. Basic properties of these groups are given in Table 3.

Proof: We consider the cases in order, observing first that $G = M_a(\mathsf{G})$, $a \geq 1$, are groups of different orders, with $\mathrm{mult}(G)/Z(G) = \mathrm{mult}(\mathsf{G})/Z(\mathsf{G}) = 1$. We start by taking each of these groups.

We also note that if G contains (generators for) a collineation preserving subgroup H which is a minimal group, then $G = M_a(H)$, for $a = |Z(G)|/|Z(H)|$.

We recall that $\zeta_{3^r} = (\zeta_{3^{r+1}}(\zeta_3 Z))^3 \in H_r^{A_4}$ and $\zeta_{3^{r+1}} \notin H_r^{A_4}$, so that

$$\begin{aligned} 3 \mid c \quad &\iff \quad \zeta_{3^{r+1}} \in M_c(H_r^{A_4}) \\ &\iff \quad \zeta_3 Z = \zeta_{3^{r+1}}^{-1}(\zeta_{3^{r+1}}(\zeta_3 Z)) \in M_c(H_r^{A_4}) \\ &\iff \quad \mathsf{T} \subset M_c(H_r^{A_4}) \\ &\iff \quad M_c(H_r^{A_4}) = M_{3^r c}(\mathsf{T}). \end{aligned}$$

Thus, we require the condition $\gcd(c, 3) = 1$ to obtain additional groups $G = M_c(H_r^{A_4})$, which satisfy

$$\frac{\mathrm{mult}(G)}{Z(G)} = \frac{Z(M_c(H_r^{A_4}))\,\mathrm{mult}(H_r^{A_4})}{Z(M_c(H_r^{A_4}))} = \frac{\langle \zeta_{3^r \cdot 2c} \rangle \langle \zeta_{3^{r+1} \cdot 2} \rangle}{\langle \zeta_{3^r \cdot 2c} \rangle} = \frac{\langle \zeta_{3^{r+1} \cdot 2c} \rangle}{\zeta_{3^r \cdot 2c}} = C_3.$$

This condition also ensures that the groups obtained have different orders, and hence are not conjugate to each other, i.e., if $\gcd(c, 3) = \gcd(c', 3) = 1$, then

$$|M_c(H_r^{A_4})| = |M_{c'}(H_s^{A_4})| \quad \iff \quad 3^r \cdot 24c = 3^s \cdot 24c' \quad \iff \quad r = s, \quad c = c'.$$

Thus, we obtain (i). Similarly, $\zeta_{2^{r+1}} \in H_r^{S_4}$, $\zeta_{2^{r+2}} \notin H_r^{S_4}$, which gives

$$\begin{aligned} 2 \mid b \quad &\iff \quad \zeta_{2^{r+1}} \in M_b(H_r^{S_4}) \quad \iff \quad \sqrt{i} R \in M_b(H_r^{S_4}) \quad \iff \quad \mathsf{O} \in M_b(H_r^{S_4}) \\ &\iff \quad M_b(H_r^{S_4}) = M_{2^r b}(\mathsf{O}), \end{aligned}$$

giving additional groups $G = M_b(H_r^{S_4})$ for $\gcd(b, 2) = 1$, with $\mathrm{mult}(G)/Z(G) = C_2$, and different orders, since

$$|M_b(H_r^{S_4})| = |M_{b'}(H_s^{S_4})| \quad \iff \quad 2^r \cdot 48b = 2^s \cdot 48b' \quad \iff \quad r = s, \quad b = b'.$$

For the groups $M_a(H)$ for D_n, we will use Lemma 3.1 to obtain the restriction on a giving additional groups, and then show they are all different. For $H = H_r^{D_n}, K_r^{D_n}$, we have

$$\frac{\mathrm{mult}(M_a(H))}{Z(M_a(H))} = \frac{Z(M_a(H))\,\mathrm{mult}(H)}{Z(M_a(H))} = \frac{\langle \zeta_{2^{r+1}a} \rangle \langle \zeta_{2^{r+2}} \rangle}{\langle \zeta_{2^{r+1}a} \rangle} = \begin{cases} C_2, & \gcd(2, a) = 1; \\ 1, & 2 \mid a, \end{cases}$$

so that we require that a be odd. The corresponding additional groups $M_b(H_r^{D_n})$, $\gcd(2, b) = 1$, have different orders, and so are different groups.

For n even, there are also groups $M_b(K_r^{D_n})$, $\gcd(2, b) = 1$, each different from each other, which we claim are different from the corresponding group $M_b(H_r^{D_n})$ of the same order. To distinguish these groups, we observe that

$$A \in \mathsf{C}_{2n} = \mathrm{spec}(H_r^{D_n}) \subset \mathrm{spec}(M_b(H_r^{D_n})), \quad A \notin \mathsf{D}_{n/2} = \mathrm{spec}(K_r^{D_n}),$$

and, since $K_r^{D_n} = \langle \zeta_{2^{r+2}} A, \zeta_{2^{r+2}} B \rangle$, we have

$$A \in M_b(K_r^{D_n}) \quad \iff \quad \zeta_{2^{r+2}} \in Z(M_b(K_r^{D_n})) \quad \iff \quad 2^{r+2} \mid 2^r \cdot 2b \quad \iff \quad \gcd(b, 2) = 2,$$

i.e., the groups $M_b(H_r^{D_n})$ and $M_b(K_r^{D_n})$ are not conjugate when $\gcd(b,2)=1$, with their $\mathrm{spec}(G)$ subgroups given by C_{2n} and $\mathsf{D}_{n/2}$ (which differ for $n \neq 2$).

Finally, for n odd, we have the groups $M_a(D_n)$ to consider. By (3.58), the condition $\gcd(4,a)=4$ gives $M_a(D_n) = M_{a/2}(\mathsf{D}_n)$, so we may suppose that $\gcd(4,a) = 1, 2$. In these cases, we have

$$\frac{\mathrm{mult}(M_a(D_n))}{Z(M_a(D_n))} = \frac{\langle \zeta_a \rangle \langle i \rangle}{\langle \zeta_a \rangle} = \begin{cases} C_4, & \gcd(4,a)=1; \\ C_2, & \gcd(4,a)=2. \end{cases}$$

For $\gcd(4,a)=1$, i.e., a odd, we get additional groups G with $\mathrm{mult}(G)/Z(G) = C_4$. For $\gcd(4,a)=2$, i.e., a even with $a/2$ odd, using $A^n = -1$ and $Z(H_0^{D_n}) = \langle -1 \rangle$, we have

$$M_a(D_n) = \langle -A, iB, \zeta_a \rangle = \langle A, iB, \zeta_{a/2} \rangle = M_{a/2}(H_0^{D_n}),$$

so that there are no additional groups in this case. □

As we have shown, the classification of the finite subgroups of $GL(\mathbb{C}^2)$, equivalently of $U(\mathbb{C}^2)$, naturally follows from that of $PGL(\mathbb{C}^2)$, and it is described in [Coh76] (Theorem on page 393) as being "well-known". In addition to the presentation of [NvdPT08], we have found Theorem 2.2 of [FP04] (which considers minimal numbers of generators – these must contain generators for the collineation group) and Theorem 11.4 of [BFO21] (which considers the imprimitive/monomial subgroups of $GL(\mathbb{C}^p)$ for p prime). Other more general references include [DV64] (see the final section) and [Cox91].

The following table summarises the properties of the irreducible groups $G = M_a(H)$, with H minimal and $\mathrm{mult}(G)/Z(G) = \mathrm{mult}(H)/H$.

Table 3: The finite irreducible subgroups G of $GL(\mathbb{C}^2)$, in terms of the minimal groups. Here $a, b, c \geq 1$, with $\gcd(2,b)=1$, $\gcd(3,c)=1$, ($a,b,c=1$ give the minimal groups), and $\hat{R} = \sqrt{i}R$, $\hat{Z} = \zeta_3 Z$, $\hat{M} = \zeta_5 M$. The only repeated groups are $M_b(K_r^{D_2}) \cong M_b(K_r^{D_2})$.

G	$\lvert G\rvert$	$G/Z(G)$	$\mathrm{spec}(G)$	$\lvert Z(G)\rvert$	$\frac{\mathrm{mult}(G)}{Z(G)}$	generators of G
$M_a(\mathsf{T})$	$24a$	A_4	T	$2a$	1	$\hat{R}^2, \hat{Z}, \zeta_{2a}$
$M_c(H_r^{A_4})$, $r \geq 0$	$3^r \cdot 24c$	A_4	D_2	$3^r \cdot 2c$	C_3	$\hat{R}^2, \zeta_{3^{r+1}}\hat{Z}, \zeta_{3^r \cdot 2c}$
$M_a(\mathsf{O})$	$48a$	S_4	O	$2a$	1	$\hat{R}, \hat{Z}, \zeta_{2a}$
$M_b(H_r^{S_4})$, $r \geq 0$	$2^r \cdot 48b$	S_4	T	$2^r \cdot 2b$	C_2	$\zeta_{2^{r+2}}\hat{R}, \hat{Z}, \zeta_{2^r \cdot 2b}$
$M_a(\mathsf{I})$	$120a$	A_5	I	$2a$	1	$\hat{R}^2, \hat{M}, \zeta_{2a}$
$M_a(\mathsf{D}_n)$	$4na$	D_n	D_n	$2a$	1	A, B, ζ_{2a}
$M_b(H_r^{D_n})$, $r \geq 1$	$2^r \cdot 4nb$	D_n, n odd	C_{2n}	$2^r \cdot 2b$	C_2	$A, \zeta_{2^{r+2}}B, \zeta_{2^r \cdot 2b}$
$M_b(D_n)$	$2nb$	D_n, n odd	C_n	b	C_4	$-A, iB, \zeta_b$
$M_b(H_r^{D_n})$, $r \geq 0$	$2^r \cdot 4nb$	D_n, n even	C_{2n}	$2^r \cdot 2b$	C_2	$A, \zeta_{2^{r+2}}B, \zeta_{2^r \cdot 2b}$
$M_b(K_r^{D_n})$, $r \geq 0$	$2^r \cdot 4nb$	D_n, n even	$\mathsf{D}_{n/2}$	$2^r \cdot 2b$	C_2	$\zeta_{2^{r+2}}A, \zeta_{2^{r+2}}B, \zeta_{2^r \cdot 2b}$

The following simple example illustrates that for each collineation group "there are not many irreducible groups of a given order" – indeed there are either 1, 2, or 3 groups.

Example 3.2 *If G is subgroup of $GL(\mathbb{C}^2)$ with collineation group A_4, then $|G| = 3^r \cdot 24b$, $\gcd(3, b) = 1$, and there are two groups with this order, i.e.,*

$$M_{3^r b}(\mathsf{T}), \qquad M_b(H_r^{A_4}).$$

Since all reflection groups with collineation group A_4 (and also A_5) are obtained from just one minimal group, it follows that there are no reflection groups of the same order in these cases. Similarly, it is possible that there are two reflection groups with collineation group S_4 of the same order, and this does occur in two cases:

$$\begin{aligned} G_8 = M_1(H_1^{S_4}), \quad G_{13} = M_2(\mathsf{O}), \quad &\textit{(order 96)}, \\ G_{10} = M_3(H_1^{S_4}), \quad G_{15} = M_6(\mathsf{O}), \quad &\textit{(order 288)}. \end{aligned}$$

Motivated by this example, we now give an alternative indexing of the irreducible subgroups of $GL(\mathbb{C}^2)$ by their order and collineation group (this is essentially what was done in the theorem of page 393 in [Coh76]). Let

$$G = G(a, \mathsf{G}, \mathsf{H}), \qquad a \geq 1, \quad \mathsf{G} \in \{\mathsf{T}, \mathsf{O}, \mathsf{I}, \mathsf{D}_n\},$$

be the unique group from Theorem 3.3, with

- The collineation group of G is given by G
- $\mathsf{H} = \mathrm{spec}(G) \neq \mathsf{C}_n$.
- $|G| = |\mathsf{G}|a$, equivalently $|Z(G)| = 2a$.

This accounts for all groups from Theorem 3.3, except for $M_b(D_n)$ for nb odd, which we denote similarly by

$$G(b, \mathsf{D}_n, \mathsf{C}_n), \qquad (n \text{ odd}, b \text{ odd}).$$

Corollary 3.1 *(Classification by order) The finite irreducible subgroups G of $GL(\mathbb{C}^2)$ have two types*

(i) *The centre of G has odd order, i.e., $-1 \notin G$, equivalently $\gcd(|G|, 4) = 2$, and*

$$G = G(b, \mathsf{D}_n, \mathsf{C}_n) = \langle \zeta_b \rangle \times D_n, \quad |G| = 2nb, \quad |Z(G)| = b, \qquad \textit{(n odd, b odd)},$$

where $\mathrm{mult}(G)/Z(G) = \mathsf{D}_n/\mathsf{C}_n = C_4$.

(ii) *The centre of G has even order, i.e., $-1 \in G$, equivalently $\gcd(|G|, 4) = 4$, and*

$$G = G(a, \mathsf{G}, \mathsf{H}), \quad |G| = |\mathsf{G}|a, \quad |Z(G)| = 2a, \qquad \mathsf{G} = \mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I}, \quad \mathsf{G}/\mathsf{H} = 1, C_2, C_3.$$

The above groups are distinct, and their properties and presentation in terms of the minimal groups is given in Table 4. We note, in particular, that the minimal groups are

$$G(1, \mathsf{D}_n, \mathsf{C}_n) = D_n, \quad \textit{for } n \textit{ odd}, \qquad G(a, \mathsf{G}, \mathsf{H}), \quad a = |\mathsf{G}/\mathsf{H}|^r, \; r \geq 0 \; . \tag{3.59}$$

Proof: By examination of Table 4, every group appears exactly once, either as type (i) or (ii). Here the conjugate groups $M_b(H_r^{D_2})$ and $M_b(H_r^{D_2})$ appear as $G(a, \mathsf{D}_2, \mathsf{C}_4)$ and $G(a, \mathsf{D}_2, \mathsf{D}_1)$, which are same (since $\mathsf{C}_4 \cong \mathsf{D}_1$).

The minimal groups are

$$G(1,\mathsf{T},\mathsf{T}),\ G(3^r,\mathsf{T},\mathsf{D}_2), \qquad G(1,\mathsf{O},\mathsf{O}),\ G(2^r,\mathsf{O},\mathsf{T}), \qquad G(1,\mathsf{I},\mathsf{I}) \qquad r \geq 0,$$

$$\begin{array}{lllll} n \text{ odd:} & G(1,\mathsf{D}_n,\mathsf{D}_n), & G(2^r,\mathsf{D}_n,\mathsf{C}_{2n}),\ r \geq 1, & G(1,\mathsf{D}_n,\mathsf{C}_n), \\ n \text{ even:} & G(1,\mathsf{D}_n,\mathsf{D}_n), & G(2^r,\mathsf{D}_n,\mathsf{C}_{2n}),\ r \geq 0, & G(2^r,\mathsf{D}_n,\mathsf{D}_{n/2}),\ r \geq 0. \end{array}$$

and these can be summarised by (3.59). □

From Corollary 3.1, we observe that there are no irreducible subgroups of $GL(\mathbb{C}^2)$ of odd order, an instance of Frobenius's dimension divisibility theorem for representations.

Table 4: The finite irreducible subgroups $G = G(a, \mathsf{G}, \mathsf{H})$ of $GL(\mathbb{C}^2)$, as they evolve from the minimal groups. Here $a, b, c \geq 1$, with $\gcd(2, b) = 1$, $\gcd(3, c) = 1$. The notation of Cohen [Coh76] is included in the last column, for comparison. We observe that every family with $\mathsf{H} \neq \mathsf{G}$ contains infinitely many minimal subgroups.

$G = G(a, \mathsf{G}, \mathsf{H})$	G	$\lvert G\rvert$	$\lvert Z(G)\rvert$	G/H	Cohen's notation
$G(a,\mathsf{T},\mathsf{T})$	$M_a(\mathsf{T})$	$24a$	$2a$	1	$\mu_{2a}\mathsf{T}$
$G(a,\mathsf{T},\mathsf{D}_2)$	$M_c(H_r^{A_4}),\ a = 3^r c$	$24a$	$2a$	C_3	$(\mu_{6a}\lvert\mu_{2a};\mathsf{T}\lvert\mathsf{D}_2)$
$G(a,\mathsf{O},\mathsf{O})$	$M_a(\mathsf{O})$	$48a$	$2a$	1	$\mu_{2a}\mathsf{O}$
$G(a,\mathsf{O},\mathsf{T})$	$M_b(H_r^{S_4}),\ a = 2^r b$	$48a$	$2a$	C_2	$(\mu_{4a}\lvert\mu_{2a};\mathsf{O}\lvert\mathsf{T})$
$G(a,\mathsf{I},\mathsf{I})$	$M_a(\mathsf{I})$	$120a$	$2a$	1	$\mu_{2a}\mathsf{I}$
$G(a,\mathsf{D}_n,\mathsf{D}_n)$	$M_a(\mathsf{D}_n)$	$4na$	$2a$	1	$\mu_{2a}\mathsf{D}_n$
$G(a,\mathsf{D}_n,\mathsf{C}_{2n})$	$M_b(H_r^{D_n})$, $a = 2^r b$, a even, n odd $M_{2a}(D_n)$, a odd, n odd	$4na$	$2a$	C_2	$(\mu_{4a}\lvert\mu_{2a};\mathsf{D}_n\lvert\mathsf{C}_{2n})$
$G(b,\mathsf{D}_n,\mathsf{C}_n)$	$M_b(D_n)$, b odd, n odd	$2nb$	b	C_4	$(\mu_{4b}\lvert\mu_b;\mathsf{D}_n\lvert\mathsf{C}_n)$
$G(a,\mathsf{D}_n,\mathsf{C}_{2n})$	$M_b(H_r^{D_n})$, $a = 2^r b$, n even	$4na$	$2a$	C_2	$(\mu_{4a}\lvert\mu_{2a};\mathsf{D}_n\lvert\mathsf{C}_{2n})$
$G(a,\mathsf{D}_n,\mathsf{D}_{n/2})$	$M_b(K_r^{D_n})$, $a = 2^r b$, n even	$4na$	$2a$	C_2	$(\mu_{4a}\lvert\mu_{2a};\mathsf{D}_n\lvert\mathsf{D}_{n/2})$

For the $G(a, \mathsf{D}_n, \mathsf{C}_{2n})$ entry of Table 4, we note that $M_{2a}(D_n) = M_a(H_0^{D_n})$ for a odd and n odd, and so $G(a, \mathsf{D}_n, \mathsf{C}_{2n}) = M_b(H_r^{D_n})$, where $a = 2^r b$ and b odd, in both cases.

Remark 3.3 *This presentation is essentially that of the (unproved) theorem stated on page 393 of Cohen [Coh76], where the notation there can be seen to correspond to ours as follows*

$$G = G(a, \mathsf{G}, \mathsf{H}) \quad \Longleftrightarrow \quad (\mathrm{mult}(G)|Z(G); \mathsf{G}|\,\mathrm{spec}(G)).$$

There, $\mathrm{mult}(G)$*,* $Z(G)$*, and* $\mathrm{spec}(G)$ *are first defined abstractly as*

$$\mu_{wd} = \langle \zeta_{wd} \rangle = \mathrm{mult}(G), \qquad \mu_d = \langle \zeta_d \rangle = Z(G), \qquad \mathsf{H} = \mathrm{spec}(G) \lhd \mathsf{G},$$

an isomorphism $\varphi : \mathrm{mult}(G)/Z(G) \to \mathsf{G}/\mathsf{H}$ *is supposed to exist, and the groups are defined as*

$$\{\lambda g : \lambda \in \mathrm{mult}(G), g \in \mathsf{G}, \varphi(\lambda\langle\zeta_{|Z(G)|}\rangle) = g\mathsf{H}\}.$$

The conditions on wd, d and H *for such a group to exist, e.g.,* G/H *is cyclic, are given, and it is concluded the particular choice of the isomorphism* φ *does not affect the group, and so it is omitted in the notation. It is observed that every finite irreducible subgroup* G *of* $GL(\mathbb{C}^2)$ *has this form, where, for example,*

$$\langle\zeta_{wd}\rangle = G \cdot SL(\mathbb{C}^2) \cap Z(GL(\mathbb{C}^2)) = \{\lambda \in \mathbb{C}^* : \lambda \in G \cdot SL(\mathbb{C}^2)\}.$$

To illustrate the asserted correspondence, we show that $\mathrm{mult}(G) = \langle\zeta_{wd}\rangle$*. We have*

$$\lambda \in \mathrm{mult}(G) \quad \Longleftrightarrow \quad \lambda g \in G,\ g \in \mathsf{G} \quad \Longleftrightarrow \quad \lambda \in Gg^{-1},\ g \in \mathsf{G} \quad \Longrightarrow \quad \lambda \in G{\cdot}SL(\mathbb{C}^2).$$

For the converse of the last implication

$$\begin{aligned} \lambda \in G \cdot SL(\mathbb{C}^2) \quad &\Longrightarrow \quad \lambda = hU, \quad h \in G,\ U \in SL(\mathbb{C}^2) \\ &\Longrightarrow \quad \lambda = Gg^{-1}, \quad g = U^{-1} = \lambda^{-1}h \in \mathsf{G}. \end{aligned}$$

There are two small oversights in the list of groups given:

- *Groups* $G = G(a, \mathsf{D}_n, \mathsf{C}_n)$ *for* n *odd and* a *even, with* $\mathrm{mult}(G)/Z(G) = C_4$*, are listed. However, no such groups exist, since as soon as* a *becomes even (for* n *odd), the scalar* -1 *is in the group, which effectively then becomes* $G(a, \mathsf{D}_n, \mathsf{C}_{2n})$*, which has* $\mathrm{mult}(G)/Z(G) = C_2$*.*
- *It is not observed that the groups* $G(a, \mathsf{D}_2, \mathsf{C}_4)$ *and* $G(a, \mathsf{D}_2, \mathsf{D}_1)$ *are conjugate.*

It is relatively easy to identify "groups in the wild" using the $G(a, \mathsf{G}, \mathsf{H})$ indexing:

- For G a finite subgroup of $GL(\mathbb{C}^2)$ it is easily checked whether it is irreducible, e.g., the condition
$$\sum_{g \in G} \mathrm{trace}(g)\,\mathrm{trace}(g^{-1}) = |G|,$$
which can be calculated at the level of conjugacy classes if need be.
- The group G is determined by the collineation group $G/Z(G)$, and, in all by a few cases, just by the order of the collineation group.
- The group $\mathsf{H} = \mathrm{spec}(G)$, in the cases where there could be more than one choice, is determined by $\mathsf{H} = G \cap \mathsf{G}$ (with G given by scalar multiples of elements of G), or as $\mathsf{H} = \{g \in G : \det(g) = 1\}$.
- The scalar a is obtained from $|Z(G)|$ or $|G|$ using the formulas in Table 4.

In other words, each finite irreducible subgroup G of $GL(\mathbb{C}^2)$ is uniquely determined by

$$|G|, \quad \frac{G}{Z(G)}, \quad \mathrm{spec}(G).$$

Example 3.3 *The (primitive) reflection groups with collineation groups A_4 and S_4 are*

$$\begin{aligned}
G_5 &= G(3,\mathsf{T},\mathsf{T}) = M_3(\mathsf{T}), & G_7 &= G(6,\mathsf{T},\mathsf{T}) = M_6(\mathsf{T}),\\
G_4 &= G(1,\mathsf{T},\mathsf{D}_2) = M_1(H_0^{A_4}), & G_6 &= G(2,\mathsf{T},\mathsf{D}_2) = M_2(H_0^{A_4}),\\
G_9 &= G(4,\mathsf{O},\mathsf{O}) = M_4(\mathsf{O}), & G_{11} &= G(12,\mathsf{O},\mathsf{O}) = M_{12}(\mathsf{O}),\\
G_{13} &= G(2,\mathsf{O},\mathsf{O}) = M_2(\mathsf{O}), & G_{15} &= G(6,\mathsf{O},\mathsf{O}) = M_6(\mathsf{O}),\\
G_{12} &= G(1,\mathsf{O},\mathsf{T}) = M_1(H_0^{S_4}), & G_{14} &= G(3,\mathsf{O},\mathsf{T}) = M_3(H_0^{S_4}),\\
G_8 &= G(2,\mathsf{O},\mathsf{T}) = M_1(H_1^{S_4}), & G_{10} &= G(6,\mathsf{O},\mathsf{T}) = M_3(H_1^{S_4}),
\end{aligned}$$

where the reflection groups in grey are minimal subgroups of $GL(\mathbb{C}^2)$.

We now seek to identify the imprimitive reflection groups $G(m,p,2)$. We observed in (2.29) that these have order $2m^2/p$ and collineation group D_n, where

$$n = \frac{m}{\gcd(p,2)} \quad\Longrightarrow\quad \mathsf{G} = \begin{cases} \mathsf{D}_m, & p \text{ odd};\\ \mathsf{D}_{m/2}, & p \text{ even}. \end{cases} \tag{3.60}$$

If m (and hence p) is odd, then $G(m,p,2)$ has order $2nb = 2mb$, with $b = m/p$ odd, so (see Table 4)

$$G(m,p,2) = G(\frac{m}{p},\mathsf{D}_m,\mathsf{C}_m), \qquad p \text{ odd}, \quad \tfrac{m}{p} \text{ odd},$$

otherwise, $G(m,p,2)$ has order

$$|G(m,p,2)| = 4na = 4\frac{m}{\gcd(p,2)}a, \quad a = \frac{m}{p}\frac{\gcd(p,2)}{2} \quad\Longrightarrow\quad G(m,p,2) = G(a,\mathsf{G},\mathsf{H}),$$

where H is to be determined. In this regard, by (2.13), we have

$$G(m,p,2) = \langle \begin{pmatrix} \zeta_m & 0\\ 0 & \zeta_m \end{pmatrix}, \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{m}{p}} & 0\\ 0 & 1 \end{pmatrix} \rangle, \qquad \mathsf{C}_m = \mathsf{C}_{\gcd(p,2)n} \subset G(m,p,2).$$

We recall $D_n = \langle A, B\rangle$ as given in (3.54). For $\frac{m}{p}$ even, we have

$$B = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta_{\frac{m}{p}} & 0\\ 0 & 1 \end{pmatrix}^{\frac{m}{2p}} \in G(m,p,2),$$

so that H is not cyclic in this case, and, since $\mathsf{C}_{2m} \not\subset G(m,p,2)$, we have

$$G(m,p,2) = G(\frac{m}{2p},\mathsf{D}_m,\mathsf{D}_{m/2}), \qquad p \text{ odd}, \quad \tfrac{m}{p} \text{ even},$$

$$G(m,p,2) = G(\frac{m}{p},\mathsf{D}_{m/2},\mathsf{D}_{m/2}), \qquad p \text{ even}, \quad \tfrac{m}{p} \text{ even}.$$

Finally, for $\frac{m}{p}$ odd (and p even), we cannot have $B \in G(m,p,2)$, and so we have

$$G(m,p,2) = G(\frac{m}{p},\mathsf{D}_{m/2},\mathsf{C}_m), \qquad p \text{ even}, \quad \tfrac{m}{p} \text{ odd}.$$

Example 3.4 *The (irreducible) imprimitive reflection subgroups of $GL(\mathbb{C}^2)$ are*

$$G(m,p,2) = \begin{cases} G(\frac{m}{p}, \mathsf{D}_m, \mathsf{C}_m), & p \text{ odd}, \ \frac{m}{p} \text{ odd}; \\ G(\frac{m}{2p}, \mathsf{D}_m, \mathsf{D}_{m/2}), & p \text{ odd}, \ \frac{m}{p} \text{ even}; \\ G(\frac{m}{p}, \mathsf{D}_{m/2}, \mathsf{D}_{m/2}), & p \text{ even}, \ \frac{m}{p} \text{ even}; \\ G(\frac{m}{p}, \mathsf{D}_{m/2}, \mathsf{C}_m), & p \text{ even}, \ \frac{m}{p} \text{ odd}. \end{cases}$$

This agrees with the theorem on page 394 of [Coh76], except for the group for p even and $\frac{m}{p}$ even, which is given as $G(\frac{m}{p}, \mathsf{D}_m, \mathsf{D}_m)$ (which has the wrong order). The case $(m,p) = (2,2)$ must be excluded, since $G(2,2,2)$ is not irreducible.

We now give a generalisation of Lemma 2.2, which will allow for the classification of all rank two quaternionic reflection groups with imprimitive complexifications.

Lemma 3.2 *All the finite irreducible subgroups of $U(\mathbb{C}^2)$ are closed under conjugation by the quaternionic matrix s of (2.8), except for all groups G in the two families*

(i) $G(a, \mathsf{T}, \mathsf{D}_2) = M_c(H_r^{A_4})$, $a \geq 1$.

(ii) $G(b, \mathsf{D}_n, \mathsf{C}_n) = M_b(D_n)$, *$n$ odd, b odd.*

For these groups, $\hat{G} = \langle G, G^s \rangle = \langle G, \det(G) \rangle$ (the closure under conjugation by s) is

$$\hat{G}(a, \mathsf{T}, \mathsf{D}_2) = G(3a, \mathsf{T}, \mathsf{T}), \qquad \hat{G}(b, \mathsf{D}_n, \mathsf{C}_n) = G(b, \mathsf{D}_n, \mathsf{C}_{2n}).$$

Proof: As in Lemma 2.2, we verify whether or not $\det(G) \subset G$, to determine whether G is closed under conjugation by s or not (respectively). This can easily be done by checking whether $\det(g_1), \ldots, \det(g_m) \in Z(G)$ for any generators $g_1, \ldots, g_m$ of G.

The condition $\det(g_j) \in Z(G)$ is trivially satisfied for a generator with $\det(g_j) = 1$ or a generator which is a scalar matrix. The relevant calculations for the groups listed in Table 3 (with their centres and generators) are

$$\begin{aligned} M_c(H_r^{A_4}): &\quad \det(\zeta_{3^{r+1}} \hat{Z}) = \zeta_{3^{r+1}}^2 \notin \langle \zeta_{3^r \cdot 2c} \rangle, \\ M_b(H_r^{S_4}): &\quad \det(\zeta_{2^{r+2}} \hat{R}) = \zeta_{2^{r+2}}^2 = \zeta_{2^{r+1}} \in \langle \zeta_{2^r \cdot 2b} \rangle, \\ M_b(H_r^{D_n}): &\quad \det(\zeta_{2^{r+2}} B) = \zeta_{2^{r+2}}^2 = \zeta_{2^{r+1}} \in \langle \zeta_{2^r \cdot 2b} \rangle, \\ M_b(D_n): &\quad \det(-A) = 1 \in \langle \zeta_b \rangle, \ \det(iB) = -1 \notin \langle \zeta_b \rangle, \\ M_b(H_r^{D_n}), M_b(K_r^{D_n}): &\quad \det(\zeta_{2^{r+2}} A), \det(\zeta_{2^{r+2}} B) = \zeta_{2^{r+1}} \in \langle \zeta_{2^r \cdot 2b} \rangle, \end{aligned}$$

For the two families not closed under conjugation by s, we calculate $\hat{G} = \langle G, \det(G) \rangle$

$$\begin{aligned} \hat{G}(a, \mathsf{T}, \mathsf{D}_2) &= \hat{M}_c(H_r^{A_4}) = \langle \hat{R}^2, \zeta_{3^{r+1}} \hat{Z}, \zeta_{3^r \cdot 2c}, \zeta_{3^{r+1}} \rangle = \langle \hat{R}^2, \hat{Z}, \zeta_{3^{r+1} \cdot 2c} \rangle \\ &= M_{3^{r+1} \cdot c}(\mathsf{T}) = G(3a, \mathsf{T}, \mathsf{T}), \end{aligned}$$

$$\begin{aligned} \hat{G}(b, \mathsf{D}_n, \mathsf{C}_n) &= \hat{M}_b(D_n) = \langle -A, iB, \zeta_b, -1 \rangle = \langle -A, iB, \zeta_{2b} \rangle \\ &= M_{2b}(D_n) = G(b, \mathsf{D}_n, \mathsf{C}_{2n}). \end{aligned}$$

□

Remark 3.4 *During our calculations, we observed that the groups given in Table 3 are all closed under the transpose, equivalently (since they are unitary) closed under entry-wise conjugation, except those in the family $G(a, \mathsf{T}, \mathsf{D}_2)$ (which is not closed under conjugation by s). The mechanics for this appears to be related to the fact*

$$J \det(g) g^{-1} J^{-1} = g^T, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (s = jJ),$$

so that if G is closed under conjugation by s and $J \in G$ then G is closed under the transpose, and that our groups are chosen so that they satisfy natural inclusions.

As an example, the maximal complex reflection group G_7 appears $G_7 = G(6, \mathsf{T}, \mathsf{T})$, which is invariant under conjugation by s and under the transpose. This contains the normal reflection subgroups $G_4 = G(1, \mathsf{T}, \mathsf{D}_2)$ and $G_6 = G(2, \mathsf{T}, \mathsf{D}_2)$, which are not invariant under s or the transpose. The groups by applying conjugation by s or the transpose to these is the second (normal) copy of G_4 and G_6 in G_7 (as depicted in Figure 1).

By our previous discussion (see Lemma 2.3), all of the rank two finite irreducible subgroups of $GL(\mathbb{H}^2)$ with imprimitive complexifications have the form $\langle G, s\rangle = \mathcal{H}_1(G)$, with G a finite subgroup of $U(\mathbb{C}^2)$ which is closed under conjugation by s (i.e., $\hat{G} = G$). In light of Lemma 3.2, we define quaternionic groups

$$G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H}) := \langle G(a, \mathsf{G}, \mathsf{H}), s\rangle, \qquad a \geq 1, \quad \mathsf{G} = \mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I}, \quad \mathsf{G}/\mathsf{H} = 1, C_2, \tag{3.61}$$

where the condition $\mathsf{G}/\mathsf{H} = 1, C_2$ excludes the families (i),(ii), which have $\mathsf{G}/\mathsf{H} = C_3, C_4$. This has order

$$|G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H})| = 2|G(a, \mathsf{G}, \mathsf{H})| = 2a|\mathsf{G}|.$$

and quotients

$$\frac{G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H})}{\langle \zeta_{2a}, s\rangle} = \mathsf{G}, \qquad \frac{G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H})}{\langle \zeta_{2a}\rangle} = C_2 \times \frac{\mathsf{G}}{\langle -1\rangle}, \qquad \frac{G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H})}{\langle s\rangle} = G(a, \mathsf{G}, \mathsf{H}).$$

We can now generalise the classification of Theorems 2.1 and 2.2.

Lemma 3.3 *Let $G \subset U(\mathbb{C}^2)$ be a primitive group. Then $\mathcal{H}_a(G)$ is an imprimitive quaternionic group in precisely the seven cases*

$$\mathcal{H}_a(\mathsf{G}) = G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{G}), \quad \mathsf{G} = \mathsf{T}, \mathsf{O}, \mathsf{I}, \ a = 1, 2, \qquad \mathcal{H}_1(H_0^{S_4}) = G_{\mathbb{H}}(1, \mathsf{O}, \mathsf{T}). \tag{3.62}$$

Proof: Since the quaternionic systems of imprimitivity for a group are shared by its subgroups (which could have more), we first consider the quaternionic systems of imprimitivity for the minimal primitive complex groups. In this regard, since

$$G_{12} = G(1, \mathsf{O}, \mathsf{T}) = M_1(H_0^{S_4}), \quad G_{13} = G(2, \mathsf{O}, \mathsf{O}) = M_2(\mathsf{O}), \quad G_{22} = G(2, \mathsf{I}, \mathsf{I}) = M_2(\mathsf{I}),$$

have quaternionic systems of imprimitivity, so do the minimal groups $H_0^{S_4}, \mathsf{O}, \mathsf{I}$. Each of the primitive minimal groups H (see Table 2) contains the monomial subgroup

$$\langle \hat{R}^2, \hat{Z}^{-1} \hat{R}^2 \hat{Z}\rangle = \langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\rangle \subset \mathsf{T}.$$

By the proof of Lemma 4.1 of [Wal26b] (a simple calculation), this implies that if H has a quaternionic system of imprimitivity, then it has the form (2.26), and by the calculation in the proof of Lemma 2.4, it is therefore given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -j \\ j & -1 \end{pmatrix}.$$

All that remains is to determine when this U conjugates generators for $G = M_a(H)$ to monomial matrices (it conjugates s to a monomial matrix). We have already seen that $\hat{R}, \hat{Z}, \hat{M}$ conjugate to monomial matrices, and that a scalar matrix ξ does if and only if ξ is a 4-th root of unity. Hence, by considering the generators given in Table 3, the conditions for $G = M_a(H)$ to have a quaternionic system of imprimitivity are

$$\begin{aligned} M_a(\mathsf{T}): &\quad a = 1, 2, \\ M_c(H_r^{A_4}): &\quad \text{not possible}, \\ M_a(\mathsf{O}): &\quad a = 1, 2, \\ M_b(H_r^{S_4}): &\quad r = 0,\ b = 1, \\ M_a(\mathsf{I}): &\quad a = 1, 2. \end{aligned}$$

Thus we obtain the seven imprimitive quaternionic groups of (3.62). □

Theorem 3.4 *(Classification) The finite irreducible subgroups of $GL(\mathbb{H}^2)$ which have imprimitive complexifications are precisely the groups*

$$G_{\mathbb{H}}(a, \mathsf{G}, \mathsf{H}) = \langle G(a, \mathsf{G}, \mathsf{H}), s \rangle, \qquad a \geq 1, \quad \mathsf{G} = \mathsf{D}_n, \mathsf{T}, \mathsf{O}, \mathsf{I}, \quad \frac{\mathsf{G}}{\mathsf{H}} = 1, C_2.$$

which are primitive, except for when $\mathsf{G} = \mathsf{D}_n$ and the seven exceptional cases of (3.62). Of these, the following are primitive (quaternionic) reflection groups

$$\begin{aligned} G_{\mathbb{H}}(3a, \mathsf{T}, \mathsf{T}), &\qquad a \geq 1, \\ G_{\mathbb{H}}(2a, \mathsf{O}, \mathsf{O}), &\qquad a \geq 2, \\ G_{\mathbb{H}}(a, \mathsf{O}, \mathsf{T}), &\qquad a \geq 2, \quad \gcd(a, 4) \neq 4, \\ G_{\mathbb{H}}(a, \mathsf{I}, \mathsf{I}), &\qquad a \geq 3, \quad 2 \mid a,\ 3 \mid a,\ \textit{or}\ \ 5 \mid a, \end{aligned} \tag{3.63}$$

and imprimitive reflection groups

$$G_{\mathbb{H}}(1, \mathsf{O}, \mathsf{T}), \qquad G_{\mathbb{H}}(2, \mathsf{O}, \mathsf{O}), \qquad G_{\mathbb{H}}(2, \mathsf{I}, \mathsf{I}), \tag{3.64}$$

$$\begin{aligned} G_{\mathbb{H}}(2a, \mathsf{D}_n, \mathsf{D}_n), &\qquad a \geq 1, \quad n\ \textit{even}, \\ G_{\mathbb{H}}(b, \mathsf{D}_n, \mathsf{C}_{2n}), &\qquad b\ \textit{odd}, \\ G_{\mathbb{H}}(2^r b, \mathsf{D}_n, \mathsf{D}_{n/2}), &\qquad b\ \textit{odd}, \quad n\ \textit{even}, \quad r := \nu_2(\tfrac{n}{2}). \end{aligned} \tag{3.65}$$

Proof: It remains to show that the groups of (3.61) are different (not conjugate) and to identify those which are the quaternionic reflection groups of Theorem 2.1.

By Lemma 2.3, each $G_{\mathbb{H}}(a,\mathsf{G},\mathsf{H}) = \mathcal{H}_a(G(a,\mathsf{G},\mathsf{H}))$ has a unique complex subgroup $G(a,\mathsf{G},\mathsf{H})$ of index 2 (the type A elements), and so by Corollary 3.1 these (normal) subgroups are distinct, and hence none of groups themselves are conjugate to each other.

We first consider the primitive quaternionic reflection groups of Theorem 2.1. For the A_4 groups,

$$\mathcal{T}_a(G_5) = \langle M_3(\mathsf{T}), \zeta_{6a}, s\rangle = \langle M_{3a}(\mathsf{T}), s\rangle = G_{\mathbb{H}}(3a,\mathsf{T},\mathsf{T}).$$

For the S_4 groups, it is convenient to split them into those that evolve from O, and the others (see Figure 2), i.e.. $\mathcal{H}_{2a}(\mathsf{O})$ and $\mathcal{H}_b(G_{12}), \mathcal{H}_b(G_8)$, for which

$$\begin{aligned}
\mathcal{H}_{2a}(\mathsf{O}) &= \langle \mathsf{O}, \zeta_{4a}, s\rangle = \langle M_{2a}(\mathsf{O}), s\rangle = G_{\mathbb{H}}(2a,\mathsf{O},\mathsf{O}),\\
\mathcal{H}_b(G_{12}) &= \langle H_0^{S_4}, \zeta_{2b}, s\rangle = \langle M_b(H_0^{S_4}), s\rangle = G_{\mathbb{H}}(b,\mathsf{O},\mathsf{T}),\\
\mathcal{H}_b(G_8) &= \langle H_1^{S_4}, \zeta_{4b}, s\rangle = \langle M_b(H_1^{S_4}), s\rangle = G_{\mathbb{H}}(2b,\mathsf{O},\mathsf{T}),
\end{aligned}$$

and so we obtain the groups $G_{\mathbb{H}}(a,\mathsf{O},\mathsf{T})$, where a is odd or a is an even multiple of an odd number, i.e., $\gcd(a,4) \neq 4$. For the next case, we have

$$\mathcal{I}_a(\mathsf{I}) = \langle \mathsf{I}, \zeta_{2a}, s\rangle = \langle M_a(\mathsf{I}), s\rangle = G_{\mathbb{H}}(a,\mathsf{I},\mathsf{I}),$$

with conditions for $M_a(\mathsf{I})$ to be a reflection group. Lastly, we use Lemma 3.3 to restrict the parameters so as to split these groups into the primitive and imprimitive groups.

We now consider the (imprimitive) D_n groups. It suffices to find those $G = G(m,p,2)$, with collineation group D_n and $\hat{G} = G$, which are minimal, and to then obtain the quaternionic reflection groups from them. With n given by (3.60), the relevant cases of Example 3.4 are

$$G(m,p,2) = \begin{cases} G(\frac{n}{2p}, \mathsf{D}_n, \mathsf{D}_{n/2}), & p \text{ odd},\ \frac{n}{p} \text{ even}; \quad (n \text{ even})\\ G(\frac{2n}{p}, \mathsf{D}_n, \mathsf{D}_n), & p \text{ even},\ \frac{2n}{p} \text{ even}; \quad (n \text{ even})\\ G(\frac{2n}{p}, \mathsf{D}_n, \mathsf{C}_{2n}), & p \text{ even},\ \frac{2n}{p} \text{ odd}. \end{cases}$$

(Here the case p and $\frac{m}{p}$ both odd is excluded, as this gives a group $G = G(b,\mathsf{D}_n,\mathsf{C}_n)$ with $\hat{G} \neq G$), For a given n, and fixed parity of p and $\frac{n}{p}$, we may minimise $\frac{n}{p}$ (equivalently maximise p) to the obtain minimal reflection groups from which all others are obtained by adding scalars. This minimisation gives

$$\begin{cases} G(2^r, \mathsf{D}_n, \mathsf{D}_{n/2}), & p = b,\ \frac{n}{p} = 2^{r+1}; \quad (\text{where } n = 2^{r+1}b,\ b \text{ odd})\\ G(2, \mathsf{D}_n, \mathsf{D}_n), & p = n,\ \frac{2n}{p} = 2; \quad (n \text{ even})\\ G(1, \mathsf{D}_n, \mathsf{C}_{2n}), & p = 2n,\ \frac{2n}{p} = 1. \end{cases}$$

Above 2^r is the 2-part of $\frac{n}{2}$, i.e., $r = \nu_2(\frac{n}{2})$ is the 2-adic valuation of $\frac{n}{2}$. From Table 4, we can identify these minimal complex reflection groups, and then write down the families

of quaternionic reflection groups that they give rise to

$$\begin{aligned} G(2^r, \mathsf{D}_n, \mathsf{D}_{n/2}) = M_1(K_r^{D_n}) \quad &\Longrightarrow \quad G_{\mathbb{H}}(2^r b, \mathsf{D}_n, \mathsf{D}_{n/2}) = \langle M_b(K_r^{D_n}), s\rangle, \\ G(2, \mathsf{D}_n, \mathsf{D}_n) = M_2(\mathsf{D}_n) \quad &\Longrightarrow \quad G_{\mathbb{H}}(2a, \mathsf{D}_n, \mathsf{D}_n) = \langle M_{2a}(\mathsf{D}_n), s\rangle, \\ G(1, \mathsf{D}_n, \mathsf{C}_{2n}) = M_1(H_0^{D_n}) \quad &\Longrightarrow \quad G_{\mathbb{H}}(b, \mathsf{D}_n, \mathsf{C}_{2n}) = \langle M_b(H_0^{D_n}), s\rangle, \end{aligned}$$

where the restriction on the parameters, i.e., b is odd, is so that no group is repeated (see Theorem 3.3). □

The above minimisation of reflection groups G with collineation group D_n and $\hat{G} = G$ (in the proof) gives two which are minimal groups (see Theorem 3.3), i.e.,

$$K_r^{D_n} = G(n, \frac{n}{2^{r+1}}, 2) \quad (n \text{ even}), \qquad H_0^{D_n} = G(2n, 2n, 2) \quad (n \text{ even}). \tag{3.66}$$

We now complete the minimisation (for reflection groups with $\hat{G} \neq G$) to obtain the minimal reflection groups for D_n. We recall from [Wal26a], that all reflection groups with collineation group D_n are (collineation preserving) normal reflection subgroups of the maximal reflection group $G(2n, 2, 2)$, which can be indexed by the divisors of $(2, 2, n)$.

Corollary 3.2 *(Minimal reflection groups) The minimal imprimitive complex reflection groups with collineation group D_n are the following normal reflection subgroups of the maximal reflection group $G(2n, 2, 2)$*

$$\begin{aligned} &n \text{ odd}: \quad G(n, n, 2) \quad \textit{(twice)}, \\ &n \text{ even}: \quad G(2n, 2n, 2) \quad G(n, \frac{n}{2^r}, 2), \quad r = \nu_2(n), \quad \textit{(twice)}. \end{aligned}$$

They give all reflection groups with collineation group D_n (see Figures 6, 7), as follows

$$n \text{ odd:} \qquad \langle \zeta_q \rangle G(n, n, 2) = \begin{cases} G(n, \frac{n}{q}, 2), & q \text{ odd}; \\ G(2n, 2\frac{2n}{q}, 2), & q \text{ even}, \end{cases} \qquad q \mid 2n$$

$$\begin{aligned} n \text{ even:} \qquad &\langle \zeta_{2q} \rangle G(2n, 2n, 2) = G(2n, 2\frac{n}{q}, 2), \qquad q \mid n, \\ &\langle \zeta_{2^r p} \rangle G(n, \frac{n}{2^r}, 2) = G(n, \frac{n}{2^r p}, 2), \qquad p \mid \frac{n}{2^r}. \end{aligned}$$

Proof: We firstly observe that the middle index in the first group in (3.66) is

$$\frac{n}{2^{r+1}}, \quad r = \nu_2(\frac{n}{2}) \quad \Longrightarrow \quad r + 1 = \nu_2(n),$$

and so we redefine the index r for a more natural formula.

We now consider $G(m, p, 2)$ for p and $\frac{m}{p}$ odd ($m = n$), which was not treated in the proof of Theorem 3.4, i.e.,

$$G = G(m, p, 2) = G(\frac{n}{p}, \mathsf{D}_n, \mathsf{C}_n), \qquad p \text{ odd}, \ \ \tfrac{n}{p} \text{ odd}; \quad (n \text{ odd}),$$

for which minimising $\frac{n}{p}$ gives

$$G(1, \mathsf{D}_n, \mathsf{C}_n) = D_n = G(n, n, 2), \qquad p = n, \ \ \tfrac{n}{p} = 1; \quad (n \text{ odd}). \tag{3.67}$$

Thus (3.66) and (3.67) give all of the minimal reflection groups for D_n. The lattice of reflection groups between these and the maximal group $G(2n, 2, 2)$, together with their indices and centres, is given in Figure 6 (n odd) and Figure 7 (n even). We note, in particular, that $G(n, n, 2)$ and $G(n, \frac{n}{2^r}, 2)$ appear twice, and those groups that evolve from them appear with multiplicity one or two, as shown in the Figures. □

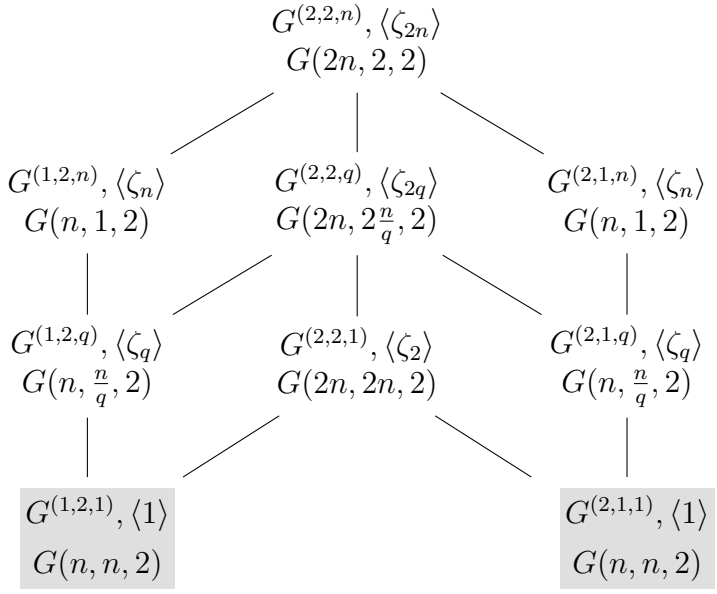


Figure 6: The $\tau(2n)$ reflection groups with collineation group D_n for n odd, as they evolve from the minimal reflection group $G(n, n, 2) = D_n$ (in grey), where $q \mid n$. They are given as they appear as collineation preserving normal reflection subgroups of the maximal reflection group $G(2n, 2, 2)$, with the reflection orbit index and centre given.

Example 3.5 *(Twins) For n even, there are two minimal groups, and so there can be two collineation preserving normal reflection subgroups of $G(2n, 2, 2)$ of the same order, namely when*

$$|G(n, \frac{n}{2^r p}, 2)| = |G(2n, \frac{2n}{q}, 2)| \quad \Longleftrightarrow \quad \frac{2n^2}{\frac{n}{2^r p}} = \frac{2(2n)^2}{\frac{2n}{q}} \quad \Longleftrightarrow \quad q = 2^{r-1}p.$$

Indeed, corresponding to each group $G(n, \frac{n}{2^r p}, 2)$, there is group $G(2n, \frac{2n}{2^{r-1}p}, 2)$ of the same order, and these groups are different (since spec(G) *differs), except in the case $n = 2$ where $G(2, 1, 2) \cong G(4, 4, 2)$.*

We now seek to identify the imprimitive quaternionic reflection groups $G(n, a, b, r)$ within the classification of Theorem 3.4. By (2.35) and (3.4), we have

$$G(n, a, b, r) = \langle G(\frac{2n}{a}, \frac{2n}{ar}, 2), \zeta_{\frac{2n}{b}}, s\rangle, \qquad [n, a, b, r] \in \Lambda_n,$$

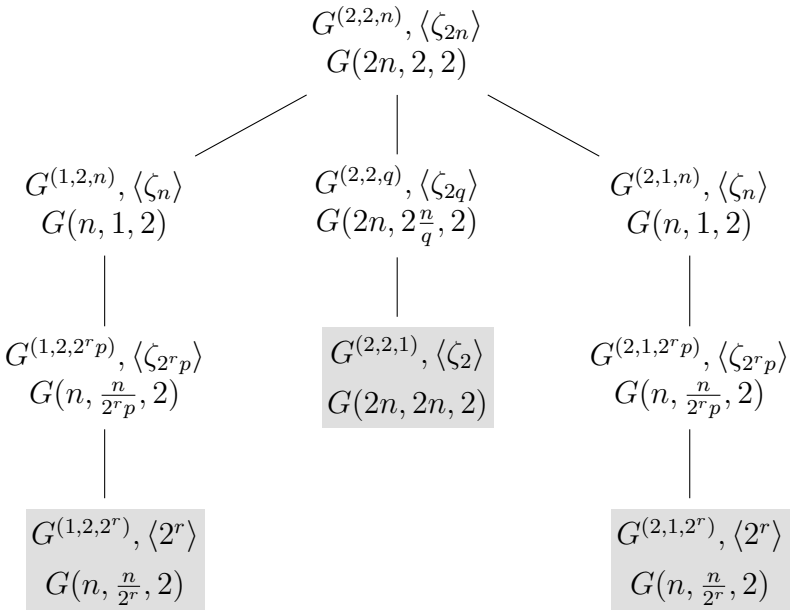


Figure 7: The $\tau(n) + \tau(\frac{n}{2^r})$ reflection groups with collineation group D_n for n even, as they evolve from the minimal reflection groups $G(n\frac{n}{2^r}, 2)$ and $G(2n, 2n)$ (in grey), where $\frac{n}{2^r}$ is odd ($\nu_2(n) = r$), $p \mid \frac{n}{2^r}$ and $q \mid n$. They are given as they appear as collineation preserving normal reflection subgroups of the maximal reflection group $G(2n, 2, 2)$, with the reflection orbit index and centre given. For $n = 2$, the minimal groups are equal: $G(2, 1, 2) \cong G(4, 4, 2)$.

where

$$G(\frac{2n}{a}, \frac{2n}{ar}, 2) = \begin{cases} G(r, \mathsf{D}_{2n/a}, \mathsf{C}_{2n/a}), & \frac{2n}{ar} \text{ odd}, \ \ r \text{ odd}; \\ G(\frac{r}{2}, \mathsf{D}_{2n/a}, \mathsf{D}_{n/a}), & \frac{2n}{ar} \text{ odd}, \ \ r \text{ even}; \\ G(r, \mathsf{D}_{n/a}, \mathsf{D}_{n/a}), & \frac{2n}{ar} \text{ even}, \ \ r \text{ even}; \\ G(r, \mathsf{D}_{n/a}, \mathsf{C}_{2n/a}), & \frac{2n}{ar} \text{ even}, \ \ r \text{ odd}. \end{cases} \tag{3.68}$$

The groups with $r = \frac{n}{ab}$ are called the *base groups*, and those with $r = \frac{2n}{ab}$ (ab odd), which can be obtained from them, by adding a scalar matrix are the *higher order groups*. Now

$$\begin{aligned} \text{base group:} \qquad & r = \frac{n}{ab} \implies \frac{2n}{ar} = 2b \text{ is even}, \\ \text{higher order group:} \qquad & r = \frac{2n}{ab} \implies \frac{2n}{ar} = b \text{ is odd}, \quad r = 2\frac{n}{ab} \text{ is even}, \end{aligned}$$

and so there are only three cases that appear in (3.68), i.e.,

$$G(\frac{2n}{a}, \frac{2n}{ar}, 2) = \begin{cases} G(\frac{r}{2}, \mathsf{D}_{2n/a}, \mathsf{D}_{n/a}), & 2n = abr, \ r \text{ even}; \\ G(r, \mathsf{D}_{n/a}, \mathsf{D}_{n/a}), & n = abr, \ r \text{ even}; \\ G(r, \mathsf{D}_{n/a}, \mathsf{C}_{2n/a}), & n = abr, \ r \text{ odd}. \end{cases}$$

which we can write as

$$G(\frac{2n}{a},\frac{2n}{ar},2)=\begin{cases}G(\frac{n}{ab},\mathsf{D}_{br},\mathsf{D}_{br/2}), & 2n=abr;\\ G(\frac{n}{ab},\mathsf{D}_{br},\mathsf{D}_{br}), & n=abr,\ r \text{ even};\\ G(\frac{n}{ab},\mathsf{D}_{br},\mathsf{C}_{2br}), & n=abr,\ r \text{ odd}.\end{cases} \tag{3.69}$$

Therefore

$$\langle G(\frac{2n}{a},\frac{2n}{ar},2),\zeta_{\frac{2n}{b}}\rangle = G(\frac{n}{b},\mathsf{D}_{br},\mathsf{H}),$$

with H as in (3.69), except when the scalar $-1\in SL(\mathbb{C}^2)$ is introduced, which can happen in the one case

$$r=\tfrac{n}{ab} \text{ is odd}, \qquad ra=\tfrac{n}{b} \text{ is even},$$

for which C_{2br} enlarges to $\langle -1\rangle\mathsf{C}_{2br}=\mathsf{D}_{br}$. Thus, we obtain

$$\langle G(\frac{2n}{a},\frac{2n}{ar},2),\zeta_{\frac{2n}{b}}\rangle=\begin{cases}G(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br/2}), & 2n=abr;\\ G(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br}), & n=abr,\ ar \text{ even};\\ G(\frac{n}{b},\mathsf{D}_{br},\mathsf{C}_{2br}), & n=abr,\ ar \text{ odd}.\end{cases} \tag{3.70}$$

We finish this section with a technical calculation of the somewhat complicated correspondence between the $G(n,a,b,r)$ and $G_{\mathbb{H}}(a,\mathsf{D}_n,\mathsf{H})$ quaternionic reflection groups.

Proposition 3.1 *(Correspondence) The imprimitive quaternionic reflection groups $G(n,a,b,r)$ can be written in terms of those in (3.65) as follows*

$$G(n,a,b,r)=\begin{cases}G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br}), & n=abr,\ \frac{n}{b}\ \textit{even};\\ G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{C}_{2br}), & n=abr,\ \frac{n}{b}\ \textit{odd};\\ G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br/2}), & 2n=abr,\end{cases} \tag{3.71}$$

and conversely those of (3.65) as

$$\begin{aligned}G_{\mathbb{H}}(2a,\mathsf{D}_n,\mathsf{D}_n)&=G(\frac{2an}{r},\frac{2a}{r},\frac{n}{r},r), && r:=\gcd(2a,n),\ \textit{(n even)},\\ G_{\mathbb{H}}(b,\mathsf{D}_n,\mathsf{C}_{2n})&=G(\frac{bn}{r},\frac{b}{r},\frac{n}{r},r), && r:=\gcd(b,n),\\ G_{\mathbb{H}}(2^rb,\mathsf{D}_n,\mathsf{D}_{n/2})&=G(\frac{bn}{2q},\frac{b}{q},\frac{n}{2^{r+1}q},2^{r+1}q), && q:=\gcd(b,n),\ r:=\nu_2(\tfrac{n}{2}),\ \textit{(n even)},\end{aligned}$$

where $\frac{b}{q}$ and $\frac{n}{2^{r+1}q}$ should be swapped when $2^{r+1}b>n$, so as to obtain an index in in $\Lambda_{\frac{bn}{2q}}$.

Proof: Since

$$G(n,a,b,r)=\langle G(\frac{2n}{a},\frac{2n}{ar},2),\zeta_{\frac{2n}{b}},s\rangle, \qquad [n,a,b,r]\in\Lambda_n,$$

we obtain (3.71) from (3.70), where we write ar as $\frac{n}{b}$ and change the ordering of cases for aesthetic purposes.

For the converse formulas, we solve for the parameters in the three separate cases, using the fact that the fraction $\frac{a}{b}$ is reduced, since $\gcd(a,b)=1$. For the $(\mathsf{D}_n,\mathsf{D}_n)$ case, we write

$$G(2A,\mathcal{D}_N,\mathcal{D}_N)=G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br}),\qquad n=abr,\quad \tfrac{n}{b}\text{ even},\quad (N\text{ even}),$$

and equating coefficients gives

$$\begin{aligned}\frac{n}{b}=2A,\quad br=\frac{n}{a}=N\quad&\Longrightarrow\quad \frac{a}{b}=\frac{2A}{N}=\frac{\frac{2A}{\gcd(2A,N)}}{\frac{N}{\gcd(2A,N)}}\\&\Longrightarrow\quad a=\frac{2A}{\gcd(2A,N)},\quad b=\frac{N}{\gcd(2A,N)},\\&\Longrightarrow\quad r=\frac{N}{b}=\gcd(2A,N),\quad n=abr=\frac{2AN}{\gcd(2A,N)},\end{aligned}$$

which gives the first equation, i.e.,

$$G_{\mathbb{H}}(2a,\mathsf{D}_n,\mathsf{D}_n)=G(\frac{2an}{r},\frac{2a}{r},\frac{n}{r},r),\qquad r:=\gcd(2a,n),\quad (n\text{ even}),$$

where $\frac{2a}{r}\le\frac{n}{r}$. Similarly, for the $(\mathsf{D}_n,\mathsf{C}_{2n})$ case

$$G_{\mathbb{H}}(B,\mathcal{D}_N,\mathsf{C}_{2N})=G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{C}_{2br}),\qquad n=abr,\quad \tfrac{n}{b}\text{ odd},\quad (B\text{ odd}),$$

gives (replace $2A$ by B in the previous argument)

$$\frac{n}{b}=B,\quad br=\frac{n}{a}=N\quad\Longrightarrow\quad n=\frac{BN}{r},\quad a=\frac{B}{r},\quad b=\frac{N}{r},\quad r=\gcd(B,N),$$

and we obtain the second equation.

For the higher order groups, the calculations are slightly more involved. Let

$$G_{\mathbb{H}}(2^RB,\mathsf{D}_N,\mathsf{D}_{N/2})=G_{\mathbb{H}}(\frac{n}{b},\mathsf{D}_{br},\mathsf{D}_{br/2}),\qquad B\text{ odd},\quad N\text{ even},\quad R=\nu_2(N/2).$$

Since $(N/2)/2^R=N/2^{R+1}$ is odd, we have

$$\begin{aligned}\frac{n}{b}=2^RB,\quad br=\frac{2n}{a}=N\quad&\Longrightarrow\quad \frac{a}{b}=\frac{2^RB}{N/2}=\frac{B/q}{(N/2^{R+1})/q},\qquad q=\gcd(B,N),\\&\Longrightarrow\quad a=\frac{B}{q},\quad b=\frac{N}{2^{R+1}q},\quad n=\frac{BN}{2q},\quad r=2^{R+1}q,\end{aligned}$$

and we have

$$G_{\mathbb{H}}(2^rb,\mathsf{D}_n,\mathsf{D}_{n/2})=G(\frac{bn}{2q},\frac{b}{q},\frac{n}{2^{r+1}q},2^{r+1}q),\qquad q=\gcd(b,n),\quad r=\nu_2(\tfrac{n}{2}).$$

To ensure that the index is $\Lambda_{\frac{bn}{2q}}$, we must exchange $\frac{b}{q}$ and $\frac{n}{2^{r+1}q}$ when $\frac{b}{q}>\frac{n}{2^{r+1}q}$, to obtain the index $[\frac{bn}{2q},\frac{\min(2^{r+1}b,n)}{2^{r+1}q},\frac{\max(2^{r+1}b,n)}{2^{r+1}q},2^{r+1}q]\in\Lambda_{\frac{bn}{2q}}$. □

4 Reflection orbits and normal reflection subgroups

We now consider the reflection orbits of the rank two quaternionic reflection groups

$$\mathcal{H}_a(G) = \langle G, \zeta_m, s\rangle, \qquad m := |Z(G)|a, \quad \hat{G} = G,$$

where G is a finite irreducible unitary complex reflection group with $\hat{G} = G$. For any reflection group, the reflections with a given root form a (parabolic) subgroup, and the group acts on these subgroups by conjugation. We call the orbits the "reflection orbits" of the reflection group, and use the symbolic notation (see [Wal26a])

$$n_1H_1, \ldots, n_\ell H_\ell \qquad \text{(reflection type)}$$

to indicate that there are ℓ orbits of sizes $n_1, \ldots, n_\ell$ with the reflection subgroup for a given root being isomorphic to the abstract groups $H_1, \ldots, H_\ell$. In the case of complex reflection groups, and hence the quaternionic reflection groups that we consider (which have additional reflections of order 2), the groups $H_1, \ldots, H_\ell$ are all cyclic, and the notation

$$(k_1, \ldots, k_\ell) \qquad \text{(reflection type)}$$

is adopted, where k_j is the order of the cyclic group H_j. The normal reflection subgroups of a given reflection group are obtained by taking the reflections corresponding to taking subgroups C_{α_j} of the $H_j = C_{k_j}$, i.e., divisors $(\alpha_1, \ldots, \alpha_\ell)$ of $(k_1, \ldots, k_\ell)$ (see Lemma 2.1 of [Wal26a]). This labelling of the normal reflection subgroups is 1–1 for complex reflection groups (Theorem 2.1 of [Wal26a]), but may not be so for quaternionic reflection groups, as we will see in this section. We recall that the reflections of $\mathcal{H}_a(G)$ have two types

- The complex reflections in $\langle G, \zeta_m\rangle$. (type A)
- The m quaternionic reflections ξs, where $\xi \in \langle\zeta_m\rangle$. (type Bj)

We now consider the orbit of s in $\mathcal{H}_a(G)$. For any $g \in \langle G, \zeta_m\rangle$, by (2.9), we have

$$\begin{aligned} s^g &= gsg^{-1} = g(sg^{-1}s)s = g\det(g)g^{-1}s = \det(g)s, \\ s^{gs} &= gs(s)(gs)^{-1} = gsg^{-1} = s^g = \det(g)s, \end{aligned}$$

so that the orbit of s is

$$s^{\mathcal{H}_a(G)} = \det(\langle G, \zeta_m\rangle)s. \tag{4.72}$$

This orbit contains at least the half of the type Bj reflections given by $\langle\zeta_m^2\rangle s$, and the orbit of $\zeta_m s$ contains (at least) the other half. Thus, there is either one mC_2 reflection orbit of type Bj reflections or two $\frac{m}{2}C_2, \frac{m}{2}C_2$ orbits, as determined by calculating (4.72).

Lemma 4.1 *The number of reflection orbits of quaternionic reflections (type Bj) in the groups of Theorems 2.1 and 2.2 are as follows*

One orbit:	$\mathcal{O}_b(G_8)$ *and* $\mathcal{O}_1(G_{12})$,
	$G(n,a,b,r)$, $\frac{n}{b}$ *odd or* $\nu_2(r) = \nu_2(\frac{2n}{b}) \geq 2$,
Two orbits:	$\mathcal{T}_a(G_5)$, $\mathcal{O}_a(G_{12})$ *with* $a \neq 1$, *and* $\mathcal{I}_a(\mathsf{I})$,
	$G(n,a,b,r)$, $\frac{n}{b}$ *even and* $\nu_2(r) \neq \nu_2(\frac{2n}{b})$.

Proof: For each of the groups, we calculate (4.72), with there being one orbit if $\det(\langle G, \zeta_m\rangle) = \langle \zeta_m\rangle$ and otherwise two. We observe that the determinant of a reflection of order k is a primitive k–th root of unity. Since G is generated by reflections, and the determinant is multiplicative, we have $\det(G) = \langle \zeta_t\rangle$, where t is the least common multiple of the orders of the reflections in G, and hence

$$\det(\langle G, \zeta_m\rangle) = \det(G)\langle \zeta_m^2\rangle = \langle \zeta_t\rangle\langle \zeta_m^2\rangle.$$

For the primitive groups, we calculate

$$\begin{aligned}
\mathcal{T}_a(G_5): &\quad \langle \zeta_t\rangle\langle \zeta_m^2\rangle = \langle \zeta_3\rangle\langle \zeta_{6a}^2\rangle = \langle \zeta_{3a}\rangle \neq \langle \zeta_{6a}\rangle,\\
\mathcal{O}_a(G_{12}): &\quad \langle \zeta_t\rangle\langle \zeta_m^2\rangle = \langle \zeta_2\rangle\langle \zeta_{2a}^2\rangle = \langle \zeta_a\rangle \neq \langle \zeta_{2a}\rangle \iff a \neq 1,\\
\mathcal{O}_b(G_8): &\quad \langle \zeta_t\rangle\langle \zeta_m^2\rangle = \langle \zeta_4\rangle\langle \zeta_{4b}^2\rangle = \langle \zeta_4\rangle\langle \zeta_{2b}\rangle = \langle \zeta_{4b}\rangle, \quad \text{(since } b \text{ is odd)},\\
\mathcal{I}_a(\mathsf{I}): &\quad \langle \zeta_t\rangle\langle \zeta_m^2\rangle = \langle 1\rangle\langle \zeta_{2a}^2\rangle = \langle \zeta_a\rangle \neq \langle \zeta_{2a}\rangle.
\end{aligned}$$

For the imprimitive groups $G(n, a, b, r)$ generated by the reflections given in (2.35), we have $r \mid m = \frac{2n}{b}$, $t = r$ for r even, and $t = 2r$ for r odd. If r is even and r does not divide $\frac{n}{b}$ then $\nu_2(r) = \nu_2(\frac{2n}{b}) = k$. Accordingly, we calculate the cases

$$\begin{aligned}
r \text{ even},\ r \mid \tfrac{n}{b}: &\quad \langle \zeta_r\rangle\langle \zeta_m^2\rangle = (\langle -1\rangle\langle \zeta_r\rangle)\langle \zeta_{\frac{2n}{b}}^2\rangle = \langle -1\rangle\langle \zeta_{\frac{n}{b}}\rangle = \langle \zeta_{\frac{2n}{b}}\rangle \iff \tfrac{n}{b} \text{ is odd},\\
r \text{ even},\ r \nmid \tfrac{n}{b}: &\quad \langle \zeta_r\rangle\langle \zeta_m^2\rangle = \langle \zeta_{2^k}\rangle\langle \zeta_{\frac{r}{2^k}}\rangle\langle \zeta_{\frac{m}{2}}\rangle = \langle \zeta_{2^k}\rangle\langle \zeta_m\rangle = \langle \zeta_m\rangle,\\
r \text{ odd}: &\quad \langle \zeta_{2r}\rangle\langle \zeta_m^2\rangle = \langle \zeta_{2r}\rangle\langle \zeta_{\frac{n}{b}}\rangle = \langle \zeta_2\rangle\langle \zeta_{\frac{n}{b}}\rangle = \langle \zeta_{\frac{2n}{b}}\rangle \iff \tfrac{n}{b} \text{ is odd}.
\end{aligned}$$

The middle condition above for there to be a single orbit is that

$$\nu_2(r) = \nu_2(\tfrac{2n}{b}) \geq 1 \quad \iff \quad r \text{ is even and } \tfrac{n}{b} \text{ is odd, or } \nu_2(r) = \nu_2(\tfrac{2n}{b}) \geq 2,$$

and we can combine the cases as in the theorem, which completes the proof. □

The result for the $G(n, a, b, r)$ is consistent with that of (ii) in Lemma 5.3 of [Wal25], which was stated and proved at the level of reflection systems.

We now consider the orbits of the complex reflections $g \in G$ in $\mathcal{H}_a(G)$. Since conjugation by ζ_m fixes g, its $\mathcal{H}_a(G)$-orbit does not depend on the value of a. Moreover, the $\mathcal{H}_a(G)$-orbit of g contains its G-orbit, and can only be larger when s conjugates g into another G-orbit. We therefore consider when (not often) this can happen.

Lemma 4.2 *The reflection orbits of all the irreducible unitary complex reflection groups $G \subset U(\mathbb{C}^2)$ are closed under conjugation by s, except for the groups*

G_5 *and* G_7,	*(the $4C_3, 4C_3$ orbits combine to a $8C_3$ orbit),*
$G(m, p, 2)$, $\gcd(m, 4) = \gcd(p, 4) = 2$,	*(the $\frac{m}{2}C_2, \frac{m}{2}C_2$ orbits combine to a mC_2 orbit).*

Proof: We consider when s conjugates elements of one $n_jC_{k_j}$ reflection orbit into another. Clearly, this could only happen when there are two orbits with reflections for a given root of the same order, i.e., the indices k_j are the same.

For $G_5 = \langle Z, Z^R\rangle$ and the larger group G_7, there are two $4C_3$ reflection orbits given by

$$O_Z = \{(Z^a)^g\}_{0\le a\le 2, g\in G_5}, \qquad O_{Z^R} = \{((Z^R)^a)^g\}_{0\le a\le 2, g\in G_5}.$$

Since

$$((Z^a)^g)^s = \det((gZ^ag^{-1})^*)gZ^ag^{-1} = g(\det(Z^*)Z)^ag^{-1} = g(Z^s)^ag^{-1},$$

it follows that if s conjugates Z to an element of O_{Z^R} then it conjugates O_Z to O_{Z^R}, and this is indeed the case, since

$$Z^s = \det(Z^*)Z = g(Z^R)^{-1}g^{-1}, \qquad g = (Z^2Z^R)^2 \in G_5.$$

The only other primitive groups where there are reflection orbits with the same orders of reflections are the $12C_2, 6C_2$ reflection orbits of G_{13}. However, these come from $12C_2, 6C_4$ reflection orbits of the larger group G_{11}, which are fixed by s, and so can be disregarded.

For the imprimitive reflection groups $G(m, p, 2)$, the reflection orbits were calculated in Lemma 8.1 of [Wal26a] (there can be one, two or three). The relevant cases are for p (and hence m) even, where there are $\frac{m}{2}C_2, \frac{m}{2}C_2$ reflection orbits, given by

$$O_1 = \Big\{\begin{pmatrix} 0 & \omega^{2j} \\ \overline{\omega}^{2j} & 0\end{pmatrix}\Big\}_{1\le j\le \frac{m}{2}}, \qquad O_2 = \Big\{\begin{pmatrix} 0 & \omega^{2j-1} \\ \overline{\omega}^{2j-1} & 0\end{pmatrix}\Big\}_{1\le j\le \frac{m}{2}}, \qquad \omega = \zeta_m.$$

and when $\frac{m}{p} = 2$ there is a third $2C_2$ reflection orbit

$$O_3 = \Big\{\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\Big\}.$$

Since all these reflections have determinant -1, O_3 is closed under conjugation by s, and

$$(O_1)^s = \Big\{-\begin{pmatrix} 0 & \omega^{2j} \\ \overline{\omega}^{2j} & 0\end{pmatrix}\Big\}_{1\le j\le \frac{m}{2}}, \qquad O_2 = \Big\{\begin{pmatrix} 0 & \omega^{2j-1} \\ \overline{\omega}^{2j-1} & 0\end{pmatrix}\Big\}_{1\le j\le \frac{m}{2}},$$

so that s conjugates O_1 to O_2 to give a single mC_2 orbit if and only if $\frac{m}{2}$ is odd. □

For clarity, we modify the labelling of the normal reflection subgroups of $\mathcal{H}_a(G)$ to

$$(\alpha_1, \dots, \alpha_\ell : \beta_1, \dots, \beta_{\ell'}), \quad \ell' \in \{1, 2\} \qquad \text{(reflection type)},$$

where the ":" separates the type A (complex) and type Bj (quaternionic) reflections. For $a > 1$, the addition of the scalars $\langle\zeta_m\rangle$ to $\mathcal{H}_a(G) = \langle G, \zeta_m, s\rangle$ may introduce additional complex reflections (previously hidden), and so it is convenient to denote the largest complex reflection group contained in $\langle G, \zeta_m\rangle$ by

$$G_a := \text{the reflection group generated by the reflections in } \langle G, \zeta_m\rangle \subset \mathcal{H}_a(G).$$

Clearly, $G_1 = G$, $G \subset G_a$, and G_a is contained in the maximal reflection group for G.

If $\mathcal{H}_a(G)$ has a single orbit of quaternionic reflections, then its normal subgroups are relatively easy to describe.

Example 4.1 *(One Bj orbit) Suppose that $\mathcal{H}_a(G) = H^{(k_1,\dots,k_\ell:2)}$ has a single orbit $\langle\zeta_m\rangle s$ of type Bj (quaternionic) reflections, i.e., $\ell' = 1$. Then its normal reflection subgroups are*

$$G^{(\alpha_1,\dots,\alpha_\ell)} := H^{(\alpha_1,\dots,\alpha_\ell:1)} \lhd G_a, \qquad H^{(\alpha_1,\dots,\alpha_\ell:2)} = \langle G^{(\alpha_1,\dots,\alpha_\ell)}, \zeta_m, s\rangle,$$

where $(\alpha_1,\dots,\alpha_\ell)$ (entrywise) divides $(k_1,\dots,k_\ell)$. Each $G^{(\alpha_1,\dots,\alpha_\ell)}$ is a different normal reflection subgroup of $G_a = H^{(k_1,\dots,k_\ell:1)}$ (uniquely determined by its reflection orbits), and it remains to determine whether any groups $H^{(\alpha_1,\dots,\alpha_\ell:2)} = \langle G^{(\alpha_1,\dots,\alpha_\ell)}, \zeta_m, s\rangle$ could have multiple labels.

For G primitive, Lemma 4.1 provides just two possibilities:

$$\mathcal{O}_b(G_8) = \mathcal{H}_b(G_8), \qquad \mathcal{O}_1(G_{12}) = \mathcal{H}_1(G_{12}),$$

where (in the first case)

$$G_b = \begin{cases} G_8, & \gcd(b,3) = 1; \\ G_{10}, & b \mid 3. \end{cases}$$

The reflection types of G_8, G_{10} and G_{12} are $6C_4$, $6C_4, 8C_3$ and $12C_2$, giving the following $H^{(k_1,\dots,k_\ell:2)}$

$$H^{(4:2)}, \quad H^{(4,3:2)}, \quad H^{(2:2)},$$

respectively. Since there are just a few divisors of $(k_1,\dots,k_\ell:2)$, we can check whether there are any multiple labels of the type $(\alpha_1,\dots,\alpha_\ell:1)$, and there is one case:

$$\mathcal{H}_b(G_8) = H^{(4,3,:2)} = H^{(4,1,:2)}, \qquad 3 \mid b.$$

The lattices of the normal reflection subgroups of $\mathcal{O}_b(G_8)$ and $\mathcal{O}_1(G_{12})$ are given in Figures 8 and 9. From these, we can conclude that if $\mathcal{H}_a(a)$ has one orbit of quaternionic reflections and is primitive, then its only possible normal complex reflection subgroups are G_8, $G(4,2,2)$ and G_{12}.

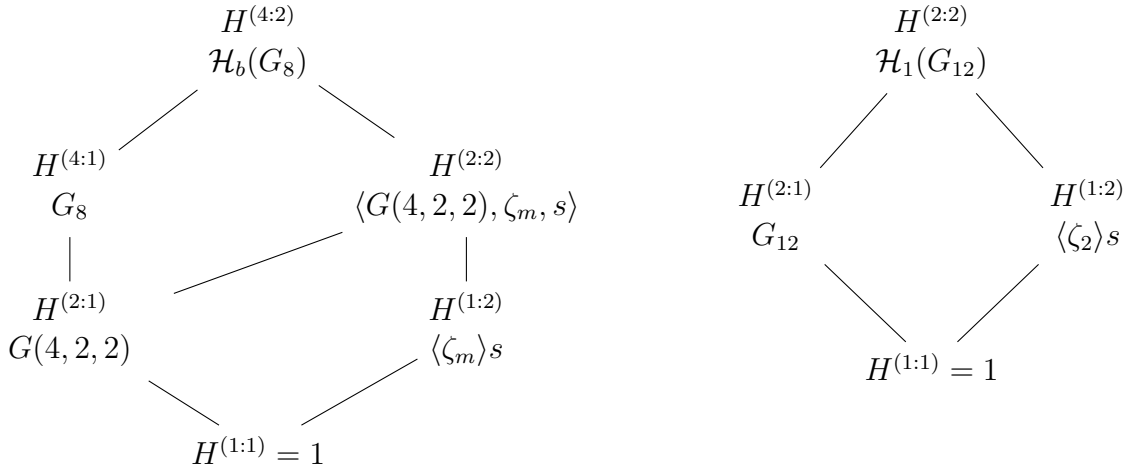


Figure 8: The lattice of the normal reflection subgroups of $\mathcal{O}_b(G_8) = \mathcal{H}_b(G_8) = H^{(4:2)}$, b odd (note the lattice size is independent of b), and $\mathcal{O}_1(G_{12}) = \mathcal{H}_1(G_{12})$.

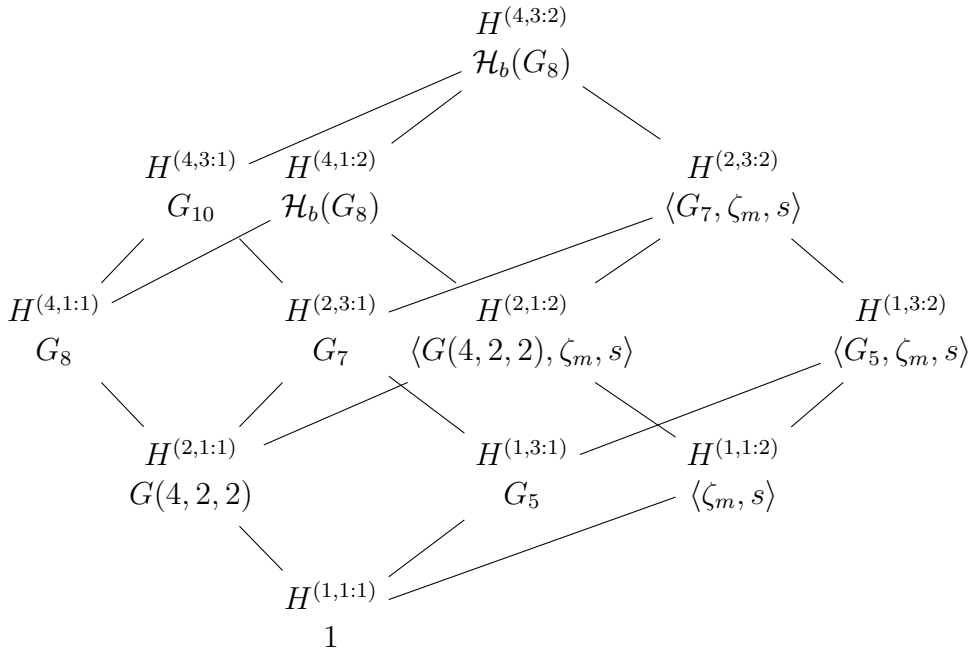


Figure 9: The lattice of the normal reflection subgroups of $\mathcal{O}_b(G_8) = \mathcal{H}_b(G_8) = H^{(4,2:2)}$, $3 \mid b$, b odd (note the lattice size is independent of b).

Example 4.2 *(Two Bj orbits) Suppose that $\mathcal{H}_a(G) = H^{(k_1,\ldots,k_\ell:2,2)}$ has two orbits $\langle\zeta_m^2\rangle s$ and $\langle\zeta_m^2\rangle\zeta_m s$ of quaternionic reflections, i.e., $\ell' = 2$. Then its normal reflection subgroups are*

$$G^{(\alpha_1,\ldots,\alpha_\ell)} := H^{(\alpha_1,\ldots,\alpha_\ell:1,1)} \lhd G, \qquad H^{(\alpha_1,\ldots,\alpha_\ell:2,2)} = \langle G^{(\alpha_1,\ldots,\alpha_\ell)}, \zeta_m, s\rangle,$$

and the pairs of isomorphic subgroups

$$H^{(\alpha_1,\ldots,\alpha_\ell:2,1)} = \langle G^{(\alpha_1,\ldots,\alpha_\ell)}, \zeta_{m/2}, s\rangle, \qquad H^{(\alpha_1,\ldots,\alpha_\ell:1,2)} = \langle G^{(\alpha_1,\ldots,\alpha_\ell)}, \zeta_{m/2}, \zeta_m s\rangle.$$

The above structural form of the pairs follows from the calculations

$$(\zeta_m^2 s)s^{-1} = \zeta_m^2 = \zeta_{m/2}, \qquad (\zeta_m^2\zeta_m s)(\zeta_m s)^{-1} = \zeta_m^2 = \zeta_{m/2}.$$

However, this does not preclude the possibility that ζ_m belongs to one, and hence both of the pairs, which would imply the multiple labelling

$$H^{(\alpha_1,\ldots,\alpha_\ell:2,2)} = H^{(\alpha_1,\ldots,\alpha_\ell:2,1)} = H^{(\alpha_1,\ldots,\alpha_\ell:1,2)}.$$

This can happen, e.g., for $\mathcal{H}_a(G_5) = H^{(3:2,2)}$ with a odd, the centre of G_5 has order 6 and the complex orbit of $\mathcal{H}_a(G_5)$ generates G_5, so that $m = 6a$ and $\zeta_m = \zeta_2\zeta_m^2 \in H^{(3:2,1)}$, giving

$$\mathcal{H}_a(G_5) = H^{(3:2,2)} = H^{(3:2,1)} = H^{(3:1,2)}.$$

On the other hand, if ξ_m does not belong to one (hence both) of the pairs then they are different groups containing either s or $\zeta_m s$, but not both, with the particular case

$$H^{(1,\ldots,1:2,2)} = \langle\zeta_m\rangle s, \qquad H^{(1,\ldots,1:1,2)} = \langle\zeta_{\frac{m}{2}}\rangle s, \qquad H^{(1,\ldots,1:2,1)} = \langle\zeta_{\frac{m}{2}}\rangle\zeta_m s.$$

We now give the reflection types for a general quaternionic reflection group from Theorem 2.1. Here we use the notation $n_jC_{k_j} = [g]$ to indicate that g is a reflection of order k_j on the $n_jC_{k_j}$ reflection orbit.

Example 4.3 *($\mathcal{T}_a(G_5)$ orbits) The reflection orbits for $\mathcal{T}_a(G_5)$ are*

$$8C_3 = [Z] \text{ (a combined orbit)}, \qquad 6C_2 = [R^2] \text{ (a even)}, \qquad 3aC_2 = [s], 3aC_2 = [\zeta_{6a}s],$$

giving $8 \cdot 2 + 6(\gcd(a,2)-1) + 6a$ reflections on $3 + (\gcd(a,2)-1)$ orbits. Here the reflection type of G_5 is $4C_3 = [Z], 4C_3 = [Z^R]$, and the orbits combine by Lemma 4.2.

Example 4.4 *($\mathcal{O}_a(G_{12})$ orbits) The reflection orbits for $\mathcal{O}_a(G_{12})$ are*

$$12C_2 = [F], \qquad 6C_2 = [R^2],\ 2 \mid a,\ 4 \nmid a \quad \text{or} \quad 6C_4 = [R],\ 4 \mid a, \qquad 8C_3 = [Z],\ 3 \mid a,$$

and the quaternionic reflections

$$aC_2 = [s], aC_2 = [\zeta_{2a}s] \text{ (for a even)} \quad \text{or} \quad 2aC_2 = [s] \text{ (for a odd)},$$

for a total number of reflections

$$12 + 6(\gcd(4,a)-1) + 8(\gcd(3,a)-1) + 2a.$$

The number of reflection orbits is

$$1 + (\gcd(2,a)-1) + \frac{1}{2}(\gcd(3,a)-1) + \gcd(2,a),$$

and there can be $2,3,4,5$ orbits, e.g.,

$$\mathcal{O}_1(G_{12}) = H^{(2:2)}, \quad \mathcal{O}_2(G_{12}) = H^{(2,2:2,2)}, \quad \mathcal{O}_3(G_{12}) = H^{(2,3:2)}, \quad \mathcal{O}_{12}(G_{12}) = H^{(2,4,3:2,2)}.$$

Example 4.5 *($\mathcal{O}_b(G_8)$ orbits) The reflection orbit for $\mathcal{O}_b(G_8)$ (b odd or even) are*

$$6C_4 = [R], \qquad 8C_3 = [Z],\ 3 \mid b, \qquad 12C_2 = [F],\ 2 \mid b,$$

$$4bC_2 = [s] \text{ (b odd)}, \quad 2bC_2 = [s], 2bC_2 = [\zeta_{4b}s] \text{ (b even)},$$

for a total number of reflections

$$6 \cdot 3 + 8(\gcd(3,b)-1) + 12(\gcd(2,b)-1) + 4b,$$

on $1 + \frac{1}{2}(\gcd(3,b)-1) + (\gcd(2,b)-1) + \gcd(2,b)$ orbits.

Example 4.6 *($\mathcal{I}_a(\mathsf{I})$ orbits) The reflection orbits of $\mathcal{H}_a(\mathcal{I})$ for $a = 2,3,4,5,6,8,9,\ldots$ (a is divisible by 2, 3 or 5) are*

$$30C_2 = [R^2],\ 2 \mid a, \quad 20C_3 = [F],\ 3 \mid a, \quad 12C_5 = [M],\ 5 \mid a, \quad aC_2 = [s], aC_2 = [\zeta_{2a}s]$$

so the number of reflections is

$$30(\gcd(2,a)-1) + 20(\gcd(3,a)-1) + 12(\gcd(5,a)-1) + 2a,$$

on $3,4,5$ orbits as given by

$$(\gcd(2,a)-1) + \frac{1}{2}(\gcd(3,a)-1) + \frac{1}{4}(\gcd(5,a)-1) + \gcd(2,a).$$

Curiously, many of these groups can have the same number of reflections, e.g.,

$$\begin{aligned} &46 \text{ reflections:} \quad \mathcal{H}_a(G_{12}), \quad a = 6, 8, 9, 14, 17, \\ &70 \text{ reflections:} \quad \mathcal{H}_a(G_{12}), \quad a = 12, 18, 20, 21, 26, 29, \\ &94 \text{ reflections:} \quad \mathcal{H}_a(G_{12}), \quad a = 24, 30, 32, 33, 38, 41, \end{aligned}$$

which leads to the following:

Example 4.7 *(Sixes) The reflection group $\mathcal{H}_a(G_{12})$ has $70 + 24k$ reflections for the six values*

$$a \in \{12, 18, 20, 21, 26, 29\} + 12k, \qquad k = 0, 1, 2, \ldots.$$

Table 5: The reflection types of groups $\mathcal{H}_a(G)$ giving the primitive reflection groups of Theorem 2.1, as detailed in Examples 4.3, 4.4, 4.5, 4.6.

group	reflection orbits	reflection type
$\mathcal{T}_a(G_5)$, a odd	$8C_3 : 3aC_2, 3aC_2$	$(3:2,2)$
$\mathcal{T}_a(G_5)$, a even	$8C_3, 6C_2 : 3aC_2, 3aC_2$	$(3,2:2,2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 1$	$12C_2 : 2aC_2$	$(2:2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 2$	$12C_2, 6C_2 : aC_2, aC_2$	$(2,2:2,2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 3$	$12C_2, 8C_3 : 2aC_2$	$(2,3:2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 4$	$12C_2, 6C_4 : aC_2, aC_2$	$(2,4:2,2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 6$	$12C_2, 6C_2, 8C_3 : aC_2, aC_2$	$(2,2,3:2,2)$
$\mathcal{O}_a(G_{12})$, $\gcd(a,12) = 12$	$12C_2, 6C_4, 8C_3 : aC_2, aC_2$	$(2,4,3:2,2)$
$\mathcal{O}_b(G_8)$, $\gcd(b,6) = 1$	$6C_4 : 4bC_2$	$(4:2)$
$\mathcal{O}_b(G_8)$, $\gcd(b,6) = 2$	$6C_4, 12C_2 : 2bC_2, 2bC_2$	$(4,2:2,2)$
$\mathcal{O}_b(G_8)$, $\gcd(b,6) = 3$	$6C_4, 8C_3 : 4bC_2$	$(4,3:2)$
$\mathcal{O}_b(G_8)$, $\gcd(b,6) = 6$	$6C_4, 12C_2, 8C_3 : 2bC_2, 2bC_2$	$(4,2,3:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 2$	$30C_2 : aC_2, aC_2$	$(2:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 3$	$20C_3 : aC_2, aC_2$	$(3:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 5$	$12C_5 : aC_2, aC_2$	$(5:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 6$	$30C_2, 20C_3 : aC_2, aC_2$	$(2,3:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 10$	$30C_2, 12C_5 : aC_2, aC_2$	$(2,5:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 15$	$20C_3, 12C_5 : aC_2, aC_2$	$(3,5:2,2)$
$\mathcal{I}_a(\mathsf{I})$, $\gcd(a,30) = 30$	$30C_2, 20C_3, 12C_5 : aC_2, aC_2$	$(2,3,5:2,2)$

We now consider the imprimitive quaternionic reflection groups $G(n,a,b,r)$, which is more involved as there can be 2, 3, 4, or 5 reflection orbits. To get an overview, see Table 6, which gives the numbers of labels and of normal reflection subgroups.

It appears that

- The subgroups of $G(n,a,b,r)$ can have multiple labels, except for when $n = ab$.

Table 6: The size of the lattice of labels, i.e., the divisors of $(k_1, \ldots, k_m)$, and the size of the lattice of normal reflection subgroups of $G(n,a,b,r) = H^{(k_1,\ldots,k_m)}$ for various indices $[n,a,b,r] \in \Lambda_n$. These differ if and only if there are subgroups with multiple labels.

group	/ lattice sizes		group	/ lattice sizes		group	/ lattice sizes		group	/ lattice sizes	
$[2,1,2,1]$	8	8	$[3,1,1,6]$	16	14	$[4,1,2,2]$	32	23	$[5,1,1,10]$	16	14
$[2,1,1,4]$	12	11	$[3,1,3,1]$	4	4	$[4,1,1,8]$	16	14	$[5,1,5,1]$	4	4
$[2,1,1,2]$	32	27	$[3,1,1,3]$	8	7	$[4,1,4,1]$	8	8	$[5,1,5,2]$	8	8
			$[3,1,3,2]$	8	8	$[4,1,1,4]$	48	34	$[5,1,1,5]$	8	7
$[6,1,2,3]$	16	13	$[7,1,7,2]$	8	8	$[8,1,2,4]$	48	30	$[9,1,1,9]$	12	10
$[6,1,1,6]$	64	41	$[7,1,1,7]$	8	7	$[8,1,4,2]$	32	23	$[9,1,9,2]$	8	8
$[6,1,3,2]$	32	27	$[7,1,1,14]$	16	14	$[8,1,1,16]$	20	17	$[9,1,3,3]$	8	7
$[6,2,3,1]$	8	8	$[7,1,7,1]$	4	4	$[8,1,8,1]$	8	8	$[9,1,1,18]$	24	20
$[6,1,3,4]$	12	11				$[8,1,1,8]$	64	41	$[9,1,3,6]$	16	14
$[6,1,6,1]$	8	8							$[9,1,9,1]$	4	4
$[6,1,1,12]$	24	20									

To understand the mechanism behind this we consider the abelianisation, i.e., the surjective homomorphism

$$G \mapsto \frac{G}{[G,G]},$$

where $G_{\mathrm{ab}} = \frac{G}{[G,G]}$ is the largest quotient of G by a normal subgroup which is abelian (also called the abelianisation). This maps each reflection orbit $n_j C_{k_j}$ to a cyclic group of order dividing k_j. For G a complex reflection group, the abelianisation maps onto $C_{k_1} \times \cdots \times C_{k_\ell}$ (Lemma 7.1, [Wal26a]), thereby showing that the normal reflection subgroups of $G = G^{(k_1,\ldots,k_\ell)}$ are uniquely labelled by the divisors of $(k_1, \ldots, k_\ell)$. The connection of the abelianisation with braid groups is explored in [BMR98], [Bec11].

Theorem 4.1 *The abelianisation of $G = G(n,a,b,r)$, $[n,a,b,r] \in \Lambda_n$, has order*

$$|G(n,a,b,r)_{\mathrm{ab}}| = \frac{16nr}{\operatorname{lcm}(2, \frac{n}{a}, r)\operatorname{lcm}(2, \frac{n}{b}, r)} \in \{4, 8, 16\}, \tag{4.73}$$

and is given by

$$G(n,a,b,r)_{\mathrm{ab}} = \begin{cases} C_2 \times C_2, & \frac{n}{a}, \frac{n}{b}, r \text{ are odd;} \\ C_2 \times C_2 \times C_2 \times C_2, & \frac{n}{a}, \frac{n}{b}, r \text{ are even, } r \neq \frac{2n}{ab}; \\ C_2 \times C_2 \times C_2, & \text{otherwise.} \end{cases} \tag{4.74}$$

Therefore, the number of factors of C_2 above is a lower bound on the number of reflections of even order from different reflection orbits that are required to generate $G(n,a,b,r)$.

Proof: By direct computation, the commutators for the five generating reflections for G given in (2.35) are seen to generate the following subgroup of diagonal matrices

$$\langle \zeta_2, \zeta_{\frac{n}{b}}, \zeta_r, \begin{pmatrix} \zeta_2 & 0 \\ 0 & \frac{}{\zeta_2} \end{pmatrix}, \begin{pmatrix} \zeta_{\frac{n}{a}} & 0 \\ 0 & \frac{}{\zeta_{\frac{n}{a}}} \end{pmatrix}, \begin{pmatrix} \zeta_r & 0 \\ 0 & \frac{}{\zeta_r} \end{pmatrix} \rangle \subset [G, G], \tag{4.75}$$

which is normal in G, and hence is its commutator subgroup $[G, G]$. For example

$$[\begin{pmatrix} 0 & \omega^a \\ \frac{}{\omega^a} & 0 \end{pmatrix}, \begin{pmatrix} \zeta_r & 0 \\ 0 & 1 \end{pmatrix}] = \begin{pmatrix} \zeta_r & 0 \\ 0 & \frac{}{\zeta_r} \end{pmatrix}, \qquad [\begin{pmatrix} 0 & \omega^b j \\ -\omega^b j & 0 \end{pmatrix}, \begin{pmatrix} \zeta_r & 0 \\ 0 & 1 \end{pmatrix}] = \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r \end{pmatrix}.$$

The first three scalar matrices in (4.75) generate a cyclic group of order $\operatorname{lcm}(2, \frac{n}{b}, r)$, and, similarly, the last three matrices (all of determinant 1) generate a cyclic group of order $\operatorname{lcm}(2, \frac{n}{a}, r)$. The intersection of these two cyclic groups is the scalar matrices of determinant 1, i.e., the group of order two generated by $\zeta_2 = -1$. Thus, we have

$$|G(n, a, b, r)_{\mathrm{ab}}| = \frac{|G|}{|[G, G]|} = \frac{8nr}{\frac{1}{2}\operatorname{lcm}(2, \frac{n}{a}, r)\operatorname{lcm}(2, \frac{n}{b}, r)} = \frac{16nr}{\operatorname{lcm}(2, \frac{n}{a}, r)\operatorname{lcm}(2, \frac{n}{b}, r)}.$$

The minimum and maximum values of $|G_{\mathrm{ab}}|$ are easily seen to be taken at $\frac{n}{a}, \frac{n}{b}, r$ all odd and all even (respectively). Indeed, in these cases we have

$$4 = \frac{16nr}{2\frac{n}{a} \cdot 2\frac{n}{b}} \text{ divides } |G_{\mathrm{ab}}|, \qquad |G_{\mathrm{ab}}| \text{ divides } \frac{16nr}{\frac{n}{a} \cdot \frac{n}{b}} = 16,$$

and so one obtains to orders of (4.73) and (4.74).

The abelianisation maps a reflection and its orbit to an element dividing the order of the reflection. Therefore the images of the reflections of order 2 have order 1 or 2. The reflection of order r must have an image which is a power of 2, and so if r is odd the image has order 1, and if r is even, then the square of the reflection is

$$\begin{pmatrix} \zeta_r^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \zeta_r & 0 \\ 0 & \zeta_r \end{pmatrix} \begin{pmatrix} \zeta_r & 0 \\ 0 & \frac{}{\zeta_r} \end{pmatrix} \in [G, G],$$

so its image also has order 1 or 2. Since the images of the generating reflection generates the abelianisation, we therefore conclude that it is a product cyclic groups of order 2.

The abelianisation has minimal generating sets of size $2, 4, 3$, for the cases of (4.74). Therefore, a generating set of reflections for G, grouped by reflection orbits, must map to at least $2, 4, 3$ nontrivial elements of the abelianisation (respectively). □

Similar calculations show that

$$\mathcal{H}_a(G_5)_{\mathrm{ab}} = \begin{cases} C_2 \times C_3, & a \text{ odd}; \\ C_2 \times C_2 \times C_3, & a \text{ even}, \end{cases},$$

$$\mathcal{H}_a(G_8)_{\mathrm{ab}} = \mathcal{H}_a(G_{12})_{\mathrm{ab}} = \begin{cases} C_2 \times C_2, & a \text{ odd}; \\ C_2 \times C_2 \times C_2, & a \text{ even}. \end{cases}$$

The reflection orbits of

$$G(n, a, b, r) = \mathcal{H}_a(G(\frac{2n}{a}, \frac{2n}{ar}, 2))$$

can be calculated by

- Taking the (type A) reflections of $G(\frac{2n}{a}, \frac{2n}{ar}, 2)$ given in Lemma 8.1 of [Wal26a], and checking if any they combine under the action of s (Lemma 4.2).
- Taking the one or two type Bj orbits as given by Lemma 4.1 .

In this way, we obtain the following.

Table 7: The reflection types of the imprimitive reflection groups $G(n, a, b, r)$.

orbits	condition	reflection type	example
five orbits	$n = abr,\ r$ even (n even)	$(2, 2, r : 2, 2)$	$[2, 1, 1, 2]$
four orbits	$n = abr$ even, $r \neq 1$ odd, $\frac{n}{b}$ odd	$(2, 2, r : 2)$	$[6, 1, 2, 3]$
	$n = abr$ even, $r \neq 1$ odd, $\frac{n}{b}$ even	$(2, r : 2, 2)$	$[18, 2, 3, 3]$
three orbits	$n = abr$ even, $r = 1$, $\frac{n}{b}$ odd	$(2, 2 : 2)$	$[2, 1, 2, 1]$
	$n = abr$ even, $r = 1$, $\frac{n}{b}$ even	$(2 : 2, 2)$	$[6, 2, 3, 1]$
	$n = abr$ odd, $r \neq 1$ (r odd)	$(2, r : 2)$	$[3, 1, 1, 3]$
	$2n = abr$ (higher order group)	$(2, r : 2)$	$[2, 1, 1, 4]$
two orbits	$n = abr$ odd, $r = 1$	$(2 : 2)$	$[3, 1, 3, 1]$

Theorem 4.2 *The number of reflection orbits of the imprimitive quaternionic reflection groups*

$$G = G(n, a, b, r) = \mathcal{H}_a(G(\frac{2n}{a}, \frac{2n}{ar}, 2)), \qquad [n, a, b, r] \in \Lambda_n,$$

can be two, three, four or five, as given by the reflection types in Table 7.

Proof: We consider the case of five orbits.
For $G(\frac{2n}{a}, \frac{2n}{ar}, 2)$ to have three orbits, we must have (see Lemma 8.1 of [Wal26a])

$$\tfrac{2n}{a} \neq \tfrac{2n}{ar}, \quad \tfrac{2n}{ar} \text{ is even} \iff r \neq 1, \quad \tfrac{2n}{ar} \text{ is even.} \tag{4.76}$$

For these orbits to stay separate under the action of s, by Lemma 4.2, we must have

$$\gcd(\tfrac{2n}{a}, 4) \neq 2 \quad \text{or} \quad \gcd(\tfrac{2n}{ar}, 4) \neq 2. \tag{4.77}$$

Since $\gcd(\frac{2n}{a}, 4) \geq \gcd(\frac{2n}{ar}, 4) \geq 2$ (because $\frac{2n}{ar}$ is even), this condition becomes

$$\gcd(\tfrac{2n}{a}, 4) = 4 \implies \tfrac{n}{a} \text{ is even.}$$

By Lemma 4.1, the condition for there to be two Bj orbits is

$$\tfrac{n}{b} \text{ is even}, \quad \nu_2(r) \neq \nu_2(\tfrac{2n}{b}) \iff \tfrac{n}{b} \text{ is even}, \quad n = abr. \tag{4.78}$$

The equivalence above follows from the fact that if $2n = abr$ (so that ab is odd), then

$$\nu_2(\tfrac{2n}{b}) = \nu_2(ar) = \nu_2(a) + \nu_2(r) = \nu_2(r).$$

Since $\gcd(a,b)=1$, the conditions $\frac{n}{a}$ and $\frac{n}{b}$ above, imply that $r=\frac{n}{ab}$ is even. Hence for $G(n,a,b,r)$ to have five reflection orbits, we must have $n=abr$ and r is even. Conversely, if $n=abr$ and r is even, then it is easily checked that (4.76), (4.77), (4.78) hold, i.e.,

$$r\neq 1,\quad \tfrac{2n}{ar}=2b \text{ is even},\qquad \gcd(\tfrac{2n}{a},4)=\gcd(2br,4)=4\neq 2,\qquad \tfrac{n}{b}=ar \text{ is even},$$

so that there are five reflection orbits.

The other cases follow from similar elementary calculations. □

Example 4.8 *($n=ab$) From Table 6, we see that the labelling for the normal reflection subgroups of $G(n,a,b,r)$ can be unique, or there can be multiple labels. By considering the cases n even (8 subgroups) and n odd (4 subgroups), it can be shown that*

$$G(n,a,b,1)=\mathcal{H}_1(G(\frac{2n}{a},\frac{2n}{a},2))=\langle\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&\zeta_{\frac{2n}{a}}\\\zeta_{\frac{2n}{a}}^{-1}&0\end{pmatrix},\zeta_{\frac{2n}{b}},s\rangle$$

has unique labelling, as do the higher order groups $G(n,a,b,2)$ for $n=ab$ odd. Our calculations suggest these are the only cases of unique labelling (for all subgroups).

All of the reflections in the groups of Example 4.8 have order 2 (and there can be one or two Bj reflection orbits). This is not sufficient for unique labelling.

Example 4.9 *(Five orbits) The imprimitive quaternionic reflection group*

$$K=G(2,1,1,2)=\mathcal{G}(\{1,i,j,k\},\{\})=\langle G(4,2,2),\zeta_4,s\rangle$$

has five reflection orbits, and is notable for many reasons:

- *It is the unique quaternionic reflection group with the maximal number of five systems of imprimitivity [Wal26b].*
- *It has abelianisation $C_2\times C_2\times C_2\times C_2$ – the largest possible group (Theorem 4.1).*
- *It is intimately involved in the construction of the primitive quaternionic reflection groups of type P and the five MUBs (mutually unbiased bases) in $\mathbb{H}^2$ [BW26].*

It is generated by any four of its five reflection orbits, thereby giving six labels for K, and this accounts for all the multiple labelling, i.e., K has $2^5-5=27$ normal reflection subgroups. The other reflection groups with five reflection orbits (with type $(2,2,r:2,2)$ and $2^4\cdot r$ labels) have similar multiple labelling, e.g., we have the following numbers of labels and normal reflection subgroups (see Table 6)

$$G(4,1,2,2):\ 32,23,\quad G(4,1,1,4):\ 48,34,\quad G(6,1,1,6):\ 64,41,\quad G(6,1,3,2):\ 32,27.$$

Another mechanism for there to be multiple labelling is the existence of a reflection of order $r\neq 2$ (which is generated by the reflections of order 2).

Example 4.10 *The indices below give groups $G(n,a,b,r)$, $r \neq 2$, which are generated by the reflections of order 2, i.e., these reflections generate the reflections of order $r \neq 2$*

$$[3,1,1,3], \quad [4,1,1,4], \quad [5,1,1,5], \quad [6,1,2,3], \quad [6,1,1,6], \quad [7,1,1,7], \quad [8,1,2,4], \quad \ldots.$$

We consider the corresponding multiple labelling for the first

$$G(3,1,1,3) = \mathcal{H}_1(G(6,2,2)) = \langle G(6,2,2), \zeta_6, s \rangle,$$

which is a near miss for unique labelling. The complex reflection group $G(6,2,2)$ has reflection type $3C_2, 3C_2, 2C_3$, with the $3C_2, 3C_2$ combining to a single orbit in $G(3,1,1,3)$, which therefore has reflection type $6C_2, 2C_3 : 6C_2$ Direct calculation shows that

$$H^{(2,3:2)} = G(3,1,1,3) = H^{(2,1:2)},$$

i.e., $G(3,1,1,3)$ has two labels, and it is generated by its reflections of order 2. This behaviour does not happen for complex reflection groups.

It appears that the only case when $G = G(n,a,b,r)$ has a reflection of order $r \neq 2$ which is required in a generating set of reflections for G is

$$[n,a,b,r] = [n,1,b,\frac{2n}{b}], \quad b \neq n \quad \text{(higher order group, } b \text{ odd)},$$

and these groups do have subgroups with multiple labels.

We have not looked exhaustively at the lattice of normal reflection subgroups of the quaternionic reflection groups $\mathcal{H}_a(G)$, but this could, in principle, be done using the techniques of this paper – in particular to find a simple condition which ensures unique labels (or not). The mechanisms which are likely to lead to multiple labels include:

- Many reflection orbits, in particular two type Bj orbits.
- Reflections of order greater than 2.
- The number of factors of the abelianisation being less than the number of reflection orbits.

We conclude with a general summary of our current understanding.

Theorem 4.3 *(Normal reflection subgroups) Let $\mathcal{H}_a(G) = H^{(k_1,\ldots,k_m:\ell_1,\ldots,\ell_{m'})}$ be one of the quaternionic reflection groups of Theorems 2.1 and 2.2. Then*

$$(\alpha_1,\ldots,\alpha_m : \beta_1,\ldots,\beta_{m'}) \mapsto H^{(\alpha_1,\ldots,\alpha_m:\beta_1,\ldots,\beta_{m'})}$$

is a lattice homomorphism from the divisors of $(k_1,\ldots,k_m : \ell_1,\ldots,\ell_{m'})$ onto the normal reflection subgroups of $\mathcal{H}_a(G)$. This can be one-to-one (unique labelling) or not (multiple labelling), as detailed in the Examples 4.1, 4.2, 4.8, 4.9, 4.10.

5 Appendix

We constructed and calculated with the groups in this paper in the magma computer algebra system. We give some sample code (which may be cut and pasted), to show how the quaternionic algebras were used (there are some subtle technical issues). Code for the groups $G(n, a, b, r)$ and the indices Λ_n can be found in [Wal25].

```
// Given a matrix group H over a cyclotomic field F
// extend the field to K and define M_a(H) as group over K
// K is chosen to contain i, so as to avoid issues when j is added
Ma := function(a,H);
  F:=BaseRing(H);
  K:=CyclotomicField(LCM(CyclotomicOrder(F)*a,4));
  G:=ChangeRing(H,K);
  zeta:=ScalarMatrix(K,2,RootOfUnity(Order(Centre(H))*a));
  return MatrixGroup<2,K|Generators(G) join {zeta}>;
end function;

// Given a matrix group H over a cyclotomic field F
// extend the field to K and define G=M_a(H) as group over K
// extend K to a quaternionic algebra Q
// return the matrix group over Q generated by G and s
// _,Gp:=Ha(1,H) returns Gp the group H over Q
Ha := function(a,H);
  G:=Ma(a,H);
  F:=BaseRing(G);
  Q<i,j,k>:=QuaternionAlgebra<F|-1,-1>;
  s:=Matrix(Q,2,2,[0,j,-j,0]);
  Ggens:={Parent(s)![(z+ComplexConjugate(z))/2+(z-ComplexConjugate(z))
          /(2*RootOfUnity(4))*i:z in Eltseq(B)]: B in Generators(G)};
  return MatrixGroup<2,Q|Ggens join {s}>, MatrixGroup<2,Q|Ggens>;
end function;

// Check whether a matrix with entries in a quaternionic algebra
// over a cyclotomic field F is in the field F
// This can be used to see if a matrix is of type A or type Bj
IsComplex := function(g);
  quatparts:=&join[{ElementToSequence(z)[a]:a in [3..4]}:z in Eltseq(g)];
  return quatparts eq {0};
end function;
```

```
// The group H_r^A4, for r=0,1,2,...
// The choice r=-1 gives the binary tetrahedral group T
HA4 := function(r);
  F:=CyclotomicField(4*3^(r+1));
  i:=F!RootOfUnity(4); R:=Matrix(F,2,2,[1,0,0,i^3]);
  hZ:=1/2*Matrix(F,2,2,[-1+i,-1+i,1+i,-1-i]);
  return MatrixGroup<2,F|i*R^2,RootOfUnity(3^(r+1))*hZ>;
end function;

// The minimal group H_r^(S_4), for r=0,1,2,...
// The choice r=-1 gives the binary octahedral group O
HS4 := function(r);
  F:=CyclotomicField(LCM(8,2^(r+2)*3));
  i:=F!RootOfUnity(4); hR:=RootOfUnity(8)*Matrix(F,2,2,[1,0,0,i^3]);
  hZ:=1/2*Matrix(F,2,2,[-1+i,-1+i,1+i,-1-i]);
  return MatrixGroup<2,F|RootOfUnity(2^(r+2))*hR,hZ>;
end function;

// The minimal group I
bI := function();
  F:=CyclotomicField(120);
  i:=F!RootOfUnity(4); z5:=F!RootOfUnity(5); tau:=(z5+z5^-1+1);
  R:=Matrix(F,2,2,[1,0,0,i^3]);
  hM:=1/2*Matrix(F,2,2,[tau^-1-tau*i,-1,1,tau^-1+tau*i]);
  return MatrixGroup<2,F|i*R^2,hM>;
end function;

// The minimal groups H_r^(D_n) and K_r^(D_n), for r=0,1,2,...
// The choice r=-1 gives the binary dihedral group of order 4n
// The third argument gives the dihedral group D_n of order 2n (for n odd)
HKDn := function(n,r);
  F:=CyclotomicField(LCM(4*n,2^(r+2)));
  z:=F!RootOfUnity(2*n); i:=F!RootOfUnity(4);
  A:=Matrix(F,2,2,[z,0,0,z^-1]);
  B:=Matrix(F,2,2,[0,i,i,0]);  // B:=Matrix(F,2,2,[0,1,-1,0]);
  H:=MatrixGroup<2,F|A,RootOfUnity(2^(r+2))*B>;
  K:=MatrixGroup<2,F|RootOfUnity(2^(r+2))*A,RootOfUnity(2^(r+2))*B>;
  Dn:=MatrixGroup<2,F|-A,i*B>;
  return H,K,Dn;
end function;


// Calculates the primitive subgroups G(a,H,G) of GL(2,C)
// Call up as PrimG(a:G,H) assumes two or three arguments with (a:G) implying H=G
// with H,G taken from {T,D2,O,I}, the order being unimportant
```

```
PrimG := function(a : T:=false, D2:=false, O:=false, I:=false);
  if T and D2 then r,c:=Valuation(a,3); return Ma(c,HA4(r)); end if;
  if O and T  then r,b:=Valuation(a,2); return Ma(b,HS4(r)); end if;
  if T then return Ma(a,HA4(-1)); end if;
  if O then return Ma(a,HS4(-1)); end if;
  if I then return Ma(a,bI()); end if;
  return "There is no group with these parameters";
end function;

// Calculates the imprimitive irreducible subgroups G(a,D_n,H) of GL(2,C)
// Call up as ImPrimG(a,n:H) with H taken from {Dn,C2n,Cn,Dn2}
ImPrimG := function(a,n : Dn:=false, C2n:=false, Cn:=false, Dn2:=false);
  error if n lt 2, "n must be >= 2";
  error if not Type(a) eq RngIntElt or a lt 1, "a must be a positive integer";
  if Dn then return Ma(a,HKDn(n,-1)); end if;
  if C2n then r,b:=Valuation(a,2); return Ma(b,HKDn(n,r)); end if;
  if Dn2 and IsEven(n) then r,b:=Valuation(a,2); _,K:=HKDn(n,r);
                            return Ma(b,K); end if;
  if Cn then
    if IsOdd(n*a) then _,_,DnOdd:=HKDn(n,0); return Ma(a,DnOdd);
                  else error "There are no groups G(a,Dn,Cn) for na even";
    end if;
  end if;
  return "There is no group with these parameters";
end function;

// Calculate the orbits of the reflections in a group G
// This returns an orbit of an element which generates H_a
CalculateOrbits := function(G)
  d:=Ncols(Random(G)); STF:=BaseRing(G); I:=IdentityMatrix(STF,d);
  refs:={};
  for g in G do
    if Rank(g-I) eq 1 then Include(~refs,g); end if;
  end for;
  remainrefs:=refs;
  ords:=[]; orbs:=[];
  while #remainrefs gt 0 do
    // find a reflection h of maximal order
    maxord:=Maximum({Order(g):g in remainrefs});
    h:=Random({g : g in remainrefs | Order(g) eq maxord });
    horbit:={g*h*g^-1:g in G}; Include(~orbs,horbit);
    remainrefs:=remainrefs diff {g^j:g in horbit,j in [1..maxord-1]};
  end while;
  return orbs;
end function;
```

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