Tight frames over the quaternions and equiangular lines

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Abstract

We show that much of the theory of finite tight frames can be generalised to vector spaces over the quaternions. This includes the variational characterisation, group frames, and the characterisations of projective and unitary equivalence. We are particularly interested in sets of equiangular lines (equi-isoclinic subspaces) and the groups associated with them, and how to move them between the spaces $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{H}^d$. We present and discuss the analogue of Zauner's conjecture for equiangular lines in $\mathbb{H}^d$.

Key Words: finite tight frames, quaternionic equiangular lines, equi-isoclinic subspaces, equichordal subspaces, projective unitary equivalence over the quaternions, group frames, quaternionic reflection groups, double cover of A_6.

AMS (MOS) Subject Classifications: primary 05B30, 15B33, 42C15, 51M20, secondary 15B57, 51M15, 65D30, 94A12.

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1 Introduction

Tight frames are a notion of redundant orthonormal basis which is of both theoretical and practical interest [Wal18]. Their recent development has been driven by connections with algebraic combinatorics and applications to quantum physics, signal analysis and engineering. In all of these settings, tight frames for which the vectors/lines are "well spread out" are desired, with equiangular tight frames being of the most interest.

We consider tight frames over the quaternions, motivated by equiangular tight frames in $\mathbb{R}^d$ and $\mathbb{C}^d$. Given enough care, much of the theory generalises to quaternionic Hilbert space $\mathbb{H}^d$, including the variational characterisation, group frames and G-matrices, and the characterisation of projective and unitary equivalence. We consider in detail how to move between tight frames (and associated linear operators) in $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{H}^d$.

The maximum possible number of equiangular lines in $\mathbb{R}^d$ is $\frac{1}{2}d(+1)$, and in $\mathbb{C}^d$ it is d^2. The bound for real equiangular lines is rarely met, but for complex lines the bound is conjectured to hold in all cases: Zauner's conjecture on the existence of Weyl-Heisenberg SICs [Zau10], [ACFW18]. For $\mathbb{H}^d$ the bound is $2d^2 - d$, for a maximum of six equiangular lines in $\mathbb{H}^2$. We give an elementary construction of five equiangular lines in $\mathbb{H}^2$, and investigate the maximal configuration of six equiangular lines in $\mathbb{H}^2$ recently obtained independently by [KF08] and [ET20]. Based on this single data point (and the beauty of the quaternions):

We conjecture the existence of $2d^2 - d$ *equiangular lines in* $\mathbb{H}^d$,

or, at least, alert the reader to the possibility of the existence of ten to fifteen equiangular lines in $\mathbb{H}^3$, and give some hints about this might play out.

We now give the basic theory of inner product spaces over the quaternions (which are not commutative), to a point where we are able to define and discuss tight frames over $\mathbb{H}$. We then develop the theory of tight frames over $\mathbb{H}$, introducing further properties of quaternionic spaces, as required.

1.1 Inner products over the quaternions

The reader is assumed to be familiar with the **quaternions** $\mathbb{H}$ which are an extension of the complex numbers $x + iy$ to a noncommutative associative algebra over the real numbers (skew field) consisting of elements:

$$q = q_1 + q_2 i + q_3 j + q_4 k = (q_1 + q_2 i) + (q_3 + q_4 i)j \in \mathbb{H}, \qquad q_j \in \mathbb{R},$$

with the (noncommutative) multiplication given by Hamilton's famous formula that $i^2 = j^2 = k^2 = ijk = -1$, which implies

$$ij = k, \quad jk = i, \quad ki = j, \qquad ji = -k, \quad kj = -i, \quad ik = -j.$$

Since the multiplication is not commutative, we must distinguish between left and right vector spaces (modules) over $\mathbb{H}$. Since we wish to appropriate much of matrix theory, we take our vector spaces to be right $\mathbb{H}$-vector spaces. Thus $\mathbb{H}$-linear maps have the form

$$L(v_1\alpha_1 + \cdots + v_n\alpha_n) = L(v_1)\alpha_1 + \cdots + L(v_n)\alpha_n,$$

and are represented by matrices, with the usual rules for multiplication, i.e.,

$$(AB)_{jk} = \sum_\ell a_{j\ell} b_{\ell k},$$

where order of multiplication in $a_{j\ell}b_{\ell k}$ cannot be reversed. For those who may have noticed, I apologise for using j and k above as indices for matrix entries, and elsewhere as quaternian units (as is often done with the complex unit i).

The **conjugate** and **norm** of a quaternion $q = q_1 + q_2 i + q_3 j + q_4 k \in \mathbb{H}$

$$\overline{q} := q_1 - q_2 i - q_3 j - q_3 k, \qquad |q| := \sqrt{q\overline{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2},$$

generalise the conjugate and modulus of a complex number $x + iy$, and allow the inner product (and associated norm) to be extended to $\mathbb{H}$ as follows. We note that

$$\overline{ab} = \overline{b}\,\overline{a}, \qquad a, b \in \mathbb{H}.$$

Definition 1.1 *Let $\mathcal{V}$ be a finite-dimensional (right) vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then an $\mathbb{F}$-valued map $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ is called an* **inner product** *if it satisfies*

1. *Conjugate symmetry:* $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
2. *Linearity in the first variable:* $\langle v\alpha, w \rangle = \langle v, w \rangle \alpha$,
   $\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$,
3. *Positive definiteness:* $\langle v, v \rangle > 0$, $v \neq 0$.

for all vectors $v, w, u \in \mathcal{V}$ and scalars $\alpha \in \mathbb{F}$.

We will say that $\mathcal{V}$ is a real, complex or quaternionic inner product space (respectively). The theory of inner product spaces evolves as in the real and complex cases, though it is not well known, e.g., the Cauchy-Schwarz inequality

$$|\langle v, w \rangle| \leq \|v\| \|w\|, \qquad \|v\| := \sqrt{\langle v, v \rangle},$$

holds (with equality if and only if v and w are linearly dependent), though it is not mentioned in the monograph [Rod14]. I think this is in part due to the fact that real and complex inner products are often also defined on $\mathbb{H}$-vector spaces. A good treatment is given in [Coh80] ("unitary inner products") and [GMP13] ("Hermitean quaternionic scalar products", which includes Cauchy-Schwarz). The prototype of such an inner product is the **Euclidean** (or **standard**) inner product

$$\langle v, w \rangle := \sum_j \overline{w_j} v_j, \qquad v, w \in \mathbb{H}^d.$$

Throughout, we will use the notation $\langle v, w \rangle$ for the Euclidean inner product, sometimes writing $\langle v, w \rangle_{\mathbb{F}}$ to emphasize when all the entries of vectors v and w are in $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The Euclidean inner product on the entries of a matrix is the **Frobenius inner product**

$$\langle A, B \rangle_F := \mathrm{trace}(B^* A), \qquad \|A\|_F^2 = \sum_j \sum_k |a_{jk}|^2.$$

In light of the noncommutativity of the quaternians, we note that scalars come outside an inner product (as we have defined it) as follows

$$\langle \alpha v, \beta w\rangle = \overline{\beta}\langle v, w\rangle \alpha.$$

The notion of orthogonality, and the Gram-Schmidt process extends in the obvious fashion. The Riesz representation also extends to inner products over $\mathbb{H}$, and so the **adjoint** of a linear map $T : \mathcal{V} \to \mathcal{W}$ between finite-dimensional inner product spaces can be defined as the unique linear map $T^* : \mathcal{W} \to \mathcal{V}$ satisfying

$$\langle T^* w, v\rangle = \langle w, Tv\rangle, \qquad \forall v \in \mathcal{V},\ w \in \mathcal{W}.$$

If T and T^* are represented as matrices $[T]$ and $[T^*]$ with respect to orthonormal bases (v_j) and (w_k), so that $v = \sum_j v_j\langle v, v_j\rangle$, $\forall v \in \mathcal{V}$, and $w = \sum_k w_k\langle w, w_k\rangle$, $\forall w \in \mathcal{W}$, then

$$[T]_{jk} = \langle T v_k, w_j\rangle = \langle v_k, T^* w_j\rangle = \overline{\langle T^* w_j, v_k\rangle} = \overline{[T^*]_{kj}},$$

and hence the matrix $[T^*]$ is the conjugate transpose of the matrix $[T]$. For this reason, it is often assumed that the inner product is the standard inner product on $\mathbb{H}^d$, and all calculations are done with matrices, with A^* defined to be the conjugate transpose (or **Hermitian transpose**) of the matrix A, as is the case in [Rod14]. The adjoint (or Hermitian transpose) satisfies some (but not all) of the usual properties, including

$$(AB)^* = B^* A^*, \qquad (A+B)^* = A^* + B^*, \qquad (A^*)^* = A,$$
$$(A^*)^{-1} = (A^{-1})^* \quad \text{(for } A \text{ invertible).}$$

It can be shown that if $AB = I$ (for matrices over $\mathbb{H}$) then $BA = I$, and so a right inverse exists for A if and only if a left inverse exists, and these inverses are equal (and denoted by A^{-1}).

One subtle point, which is not obvious from the matrix formulation, is that scalar multiplication by $\beta \in \mathbb{H} \setminus \mathbb{R}$, i.e., $R_\beta : \mathcal{V} \to \mathcal{V} : v \mapsto v\beta$ is not an $\mathbb{H}$-linear map, since

$$R_\beta(v\alpha) = (v\alpha)\beta = v(\alpha\beta) \neq v(\beta\alpha) = (v\beta)\alpha = (R_\beta v)\alpha \qquad \text{(in general).}$$

Left multiplication of $\mathbb{H}^d$ by β defines an $\mathbb{H}$-linear map $L_\beta : \mathbb{H}^d \to \mathbb{H}^d : v \mapsto \beta v$, but this is dependent on a choice of basis: it is the linear map which maps $e_j \mapsto e_j\beta$, i.e., the linear map whose matrix representation with respect to the standard basis (e_j) is βI (see the discussion of [GMP13] §3.1). On the other hand, multiplication of a fixed vector $v \in \mathcal{V}$ by a scalar $[v] : \mathbb{H} \to \mathcal{V} : \alpha \mapsto v\alpha$ is an $\mathbb{H}$-linear map:

$$[v](\beta_1\alpha_1 + \beta_2\alpha_2) = v(\beta_1\alpha_1 + \beta_2\alpha_2) = (v\beta_1)\alpha_1 + (v\beta_2)\alpha_2 = ([v]\beta_1)\alpha_1 + ([v]\beta_2)\alpha_2.$$

Its adjoint $[v]^* : \mathcal{V} \to \mathbb{H}$ is given by $[v]^* = \langle \cdot, v\rangle$, since

$$\langle [v]^* w, \alpha\rangle = \langle w, [v]\alpha\rangle = \langle w, v\alpha\rangle = \overline{\alpha}\langle w, v\rangle = \overline{\alpha}\langle w, v\rangle\langle\langle w, v\rangle, \alpha\rangle.$$

The map $[v]$ is sometimes abbreviated simply as v, especially when $v \in \mathbb{H}^d$ is thought of as a column vector, i.e., as an element of $\mathbb{H}^{d\times 1}$. More generally, a **synthesis map**

$$V = [v_1, \ldots, v_n] : \mathbb{H}^n \to \mathcal{V} : a \mapsto v_1 a_1 + \cdots + v_n a_n,$$

for a sequence of vectors $v_1, \ldots, v_n \in \mathcal{V}$, has adjoint the **analysis map**

$$V^* : \mathcal{V} \to \mathbb{H}^n : v \mapsto (\langle v, v_j\rangle)_{j=1}^n.$$

2 Tight frames

A **frame** for a Hilbert space $\mathcal{H}$ is a sequence of vectors (v_j) satisfying the (somewhat cryptic, but easier to verify in the infinite-dimensional setting) condition

$$A\|v\|^2 \le \sum_j |\langle v, v_j\rangle|^2 \le B\|v\|^2, \qquad \forall v \in \mathcal{H}, \tag{2.1}$$

where $A, B > 0$ are constants, with the case $A = B$ giving a "tight frame". From this, a "frame expansion" follows, which for tight frames takes the particularly simple form

$$v = \frac{1}{A}\sum_j v_j\langle v, v_j\rangle \qquad \forall v \in \mathcal{H}.$$

A prominent early example of the use of such "generalised orthonormal bases" is in the theory of wavelets. Recently, frames have been considered for quaternionic Hilbert space, see, e.g., [KTS17] and [VSSS20]. These papers deal primarily with the the frame operator and the construction of dual frames. Here we consider tight frames (where the dual frame is the frame itself) with a particular emphasis on the classification and construction of such frames. This is related to earlier work of Hoggar [Hog77], [Hog82] and others, which implicitly considers tight frames over quaternionic (and even octonionic) Hilbert spaces.

2.1 Tight frames defined and unitary equivalence

We will say that a sequence of vectors with synthesis map $V = [v_1, \ldots, v_n]$ is a **tight frame** for a (finite-dimensional) quaternionic Hilbert space $\mathcal{H}$ if it satisfies (2.1), where $A = B$. It is said to be **normalised** if $A = 1$, which can be achieved by multiplying the vectors by a suitable positive scalar. The **frame operator** (for a sequence of vectors) is $S = VV^*$ and the **Gramian** (matrix) is $G = V^*V$.

The monograph [Wal18] is a good reference for those parts of the theory of finite tight frames which we now develop. First we consider equivalent conditions for being a tight frame. For this, we need the polarisation identity for quaternionic Hilbert space. Since this is not well known (it is not in [Rod14]), we provide it with proof.

Lemma 2.1 *(Polarisation identity) For an inner product space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have*

$$\langle v, w\rangle = \frac{1}{4}\sum_{r=0}^{m}\Big(\|v + wi_r\|^2 - \|v - wi_r\|^2\Big)i_r,$$

where $m = \dim_{\mathbb{R}}(\mathbb{F})$, $(i_0, i_1, i_2, i_3) = (1, i, j, k)$, and $\langle\cdot,\cdot\rangle$ is linear in the first variable.

Proof: We first observe that for a quaternion $q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3$, $q_r \in \mathbb{R}$, a calculation gives

$$\overline{i_r}q + \overline{q}i_r = 2q_r, \qquad r = 0, 1, 2, 3, \tag{2.2}$$

and we write $(q)_r = q_r$. Expanding, using the properties of the inner product, gives

$$\|v \pm wi_r\|^2 = \langle v, v\rangle + \langle \pm wi_r, \pm wi_r\rangle + \langle v, \pm wi_r\rangle + \langle \pm wi_r, v\rangle$$

$$= \|v\|^2 + \|w\|^2 \pm \overline{i_r}\langle v, w\rangle \pm \langle w, v\rangle i_r,$$

so that

$$\|v + wi_r\|^2 - \|v + wi_r\|^2 = 2\big(\overline{i_r}\langle v, w\rangle + \overline{\langle v, w\rangle} i_r\big) = 4(\langle v, w\rangle)_r,$$

which gives the result. □

Proposition 2.1 *Let $V = [v_1, \ldots, v_n]$ be sequence of vectors in a d-dimensional (right) quaternionic Hilbert space $\mathcal{H}$, such as $\mathbb{H}^d$. Then the following are equivalent*

(i) *V is a normalised tight frame for $\mathcal{H}$, i.e.,*

$$\|v\|^2 = \sum_j |\langle v, v_j\rangle|^2, \qquad \forall v \in \mathcal{H}.$$

(ii) *The frame operator $S = VV^* = I$, i.e., we have the frame expansion*

$$v = \sum_j v_j \langle v, v_j\rangle, \qquad \forall v \in \mathcal{H}.$$

(iii) *The Plancherel identity*

$$\langle v, w\rangle = \sum_j \langle v_j, w\rangle\langle v, v_j\rangle, \qquad \forall v \in \mathcal{H}.$$

(iv) *The Gramian $P = V^*V$ is a rank d orthogonal projection, i.e., $P^2 = P$, $P^* = P$.*

Proof: The implications (ii)$\Longrightarrow$(iii)$\Longrightarrow$(i) follow by taking the inner product with w and then letting $w = v$, respectively. Suppose that (i) holds. By Lemma 2.1, we have

$$4(\langle v, w\rangle)_r = \|v + wi_r\|^2 - \|v - wi_r\|^2 = \sum_j \Big(|\langle v + wi_r, f_j\rangle|^2 - |\langle v - wi_r, f_j\rangle|^2\Big)$$

$$= \sum_j \Big(2\overline{i_r}\langle f_j, w\rangle\langle v, f_j\rangle + 2\langle f_j, v\rangle\langle w, f_j\rangle i_r\Big) = 4\sum_j (\langle f_j, w\rangle\langle v, f_j\rangle)_r.$$

Thus (by the Riesz representation)

$$\langle v, w\rangle = \sum_j \langle f_j, w\rangle\langle v, f_j\rangle = \langle \sum_j f_j\langle v, f_j\rangle, w\rangle \quad \Longrightarrow \quad v = \sum_j f_j\langle v, f_j\rangle,$$

which is (i).

We now show (iii) $\Longleftrightarrow$ (iv). We observe that by construction $P = (\langle v_k, v_j\rangle)_{j,k}$ is Hermitian. The condition $P^2 = P$ can be written entrywise as

$$\langle v_k, v_j\rangle = P_{jk} = \sum_\ell P_{j\ell}P_{\ell k} = \sum_\ell \langle v_\ell, v_j\rangle\langle v_k, v_\ell\rangle,$$

which is the Plancherel identity for $v = v_k$ and $w = v_j$. The implications then follow by extending the Plancherel identity (using linearity and symmetry of the inner product), and calculating $\mathrm{rank}(P) = \mathrm{trace}(P) = \mathrm{Re}(\mathrm{trace}(VV^*)) = d$, by (ii). □

A linear map U is unitary if it preserves angles, i.e., $\langle Uv, Uw\rangle = \langle v, w\rangle$, $\forall v, w$, or, equivalently $U^*U = I$. Unitary maps can be defined in the same way on quaternionic Hilbert spaces. If $V = [v_1, \ldots, v_n]$ is a tight frame for a quaternionic Hilbert space, then so is any unitary image $UV = [Uv_1, \ldots, Uv_n]$, and these frames have the same Gramian since $(UV)^*UV = V^*U^*UV = V^*V$, and we say that they are **unitarily equivalent**. Tight frames are studied up to unitary equivalence (which is an equivalence relation) and multiplication by a nonzero scalar.

For ease of presentation, we will now consider $\mathbb{H}^d$, rather than saying let $\mathcal{H}$ be a quaternionic Hilbert space of dimension d. We also write $\mathbb{F}^d$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The following characterisation extends the real and complex cases (see [Wal18] Theorem 2.1).

Proposition 2.2 *An $n \times n$ matrix P is the Gramian matrix of a normalised tight frame $V = [v_1, \ldots, v_n]$ for $\mathbb{H}^d$ if and only if it is an orthogonal projection matrix of rank d.*

Proof: We have already seen that a normalised tight frame is determined by its Gramian, which is an orthogonal projection of rank d (Proposition 2.1). It remains only to show that such a matrix P corresponds to a normalised tight frame. Let $v_j = Pe_j$. Then with the Euclidean norm on $\mathbb{H}^n$, we have that

$$\langle v_k, v_j\rangle = \langle Pe_k, Pe_k\rangle = \langle e_k, Pe_j\rangle = \overline{P_{kj}} = P_{jk},$$

so that (v_j) is such a tight frame (for its d-dimensional span). □

A finite sequence of unit vectors (v_j) (or the lines they represent) are said to be **equiangular** if

$$|\langle v_j, v_k\rangle|^2 = \lambda = c^2 = \cos\theta, \qquad \forall j \neq k. \tag{2.3}$$

The constants λ, c and θ all occur in the literature, and are called the (common) angle.

Example 2.1 *Four equiangular lines in $\mathbb{H}^2$ with $\lambda = \frac{1}{3}$ are given in [Hog77], namely*

$$w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\end{pmatrix}, \quad w_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1-\sqrt{2}i\\ j-\sqrt{2}k\end{pmatrix},$$
$$w_3 = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{2}+\sqrt{3}+i\\ \sqrt{2}j-\sqrt{3}j+k\end{pmatrix}, \quad w_4 = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{2}-\sqrt{3}+i\\ \sqrt{2}j+\sqrt{3}j+k\end{pmatrix}.$$

The Gramian of these vectors (which are a tight frame for $\mathbb{H}^2$) has only complex entries, and so they are unitarily equivalent to an equiangular tight frame for $\mathbb{C}^2$. They have the same Gramian as the Weyl-Heisenberg SIC $v_1 = v$, $v_2 = Sv$, $v_3 = \Omega v$, $v_4 = iS\Omega v$, where

$$v = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3+\sqrt{3}}\\ \frac{1}{\sqrt{2}}(1+i)\sqrt{3-\sqrt{3}}\end{pmatrix}, \quad S = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \ \Omega = \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Therefore, there is a unitary map U with $v_j = Uw_j$, which we calculate as

$$U = \begin{pmatrix}z_1 & -jz_1\\ z_2 & -kz_2\end{pmatrix}, \quad z_1 := \frac{\sqrt{3+\sqrt{3}}}{2\sqrt{3}} + \frac{\sqrt{3-\sqrt{3}}}{2\sqrt{3}}i, \quad z_2 := \frac{\sqrt{3+\sqrt{6}}}{2\sqrt{3}} - \frac{\sqrt{3-\sqrt{6}}}{2\sqrt{3}}i.$$

Though this first example of quaternionic equiangular lines are not "quaternionic", we will see that such lines do exist, and are very intriguing.

2.2 The variational characterisation of tight frames

We now seek to extend the variational characterisation for tight frames [BF03], [Wal03]. This is most easily proved from the spectral decomposition of the frame operator using the formula $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ (see [Wal18], Theorem 6.1). This trace formula no longer holds over the quaternions, even for 1×1 matrices. Instead, we will use the fact

$$\mathrm{Re}(\mathrm{trace}(AB)) = \mathrm{Re}(\mathrm{trace}(BA)), \tag{2.4}$$

which follows from the special case $\mathrm{Re}(ab) = \mathrm{Re}(ba)$, $\forall a, b \in \mathbb{H}$.

The general spectral theory of matrices over $\mathbb{H}$ is fraught (see [Rod14]), since

$$Av = v\lambda \quad \Longrightarrow \quad A(v\alpha) = (v\alpha)\alpha^{-1}\lambda\alpha,$$

so that if v is a (right) eigenvector for λ, then $v\alpha$ is an eigenvector for eigenvalue $\alpha^{-1}\lambda\alpha$. However, **Hermitian matrices** (those with $A^* = A$), have real eigenvalues and are unitarily diagonalisable, as in the complex case.

Lemma 2.2 *Let $V = [v_1, \ldots, v_n]$ be vectors in $\mathbb{F}^d$, with frame operator $S = VV^*$ and Gramian $G = V^*V$. Then $\mathrm{trace}(S^k) = \mathrm{trace}(G^k)$, $k = 1, 2, \ldots$. In particular,*

$$\mathrm{trace}(S) = \sum_j \|v_j\|^2, \qquad \mathrm{trace}(S^2) = \sum_j \sum_k |\langle v_j, v_k\rangle|^2. \tag{2.5}$$

Proof: The trace of an Hermitian matrix A is real, since $\overline{\langle Ax, x\rangle} = \langle x, Ax\rangle = \langle Ax, x\rangle$. Since S^k and G^k are Hermitian, they have real trace, and so by (2.4), we have

$$\begin{aligned}\mathrm{trace}(S^k) &= \mathrm{Re}(\mathrm{trace}(VV^*(VV^*)^{k-1})) = \mathrm{Re}(\mathrm{trace}(V^*(VV^*)^{k-1}V)) \\ &= \mathrm{Re}(\mathrm{trace}((V^*V)^k)) = \mathrm{trace}(G^k).\end{aligned}$$

The formulas for $\mathrm{trace}(G)$ and $\mathrm{trace}(G^2)$ given on the left hand side of (2.5) are easily calculated from $(G)_{jk} = \langle v_k, v_j\rangle$. □

Theorem 2.1 *Let $v_1, \ldots, v_n$ be vectors in $\mathbb{F}^d$, which are not all zero. Then*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^2 \ge \frac{1}{d}\Bigl(\sum_{j=1}^n \|v_j\|^2\Bigr)^2, \tag{2.6}$$

with equality if and only if $(v_j)_{j=1}^n$ is a tight frame for $\mathbb{F}^d$.

Proof: Let $V = [v_j]$. Since $S = VV^*$ is positive definite, it is unitarily diagonalisable $S = U\Lambda U^*$, $\Lambda = \mathrm{diag}(\lambda_j)$, with real eigenvalues $\lambda_1, \ldots, \lambda_d \ge 0$. From (2.4), we have

$$\mathrm{trace}(S^k) = \mathrm{Re}(\mathrm{trace}(U\Lambda^k U^*)) = \mathrm{Re}(\mathrm{trace}(\Lambda^k U^* U)) = \mathrm{Re}(\mathrm{trace}(\Lambda^k)) = \mathrm{trace}(\Lambda^k).$$

Thus, the Cauchy-Schwarz inequality gives

$$\mathrm{trace}(S)^2 = (\sum_j \lambda_j)^2 = \langle (1), (\lambda_j)\rangle^2 \le \|(1)\|^2 \|(\lambda_j)\|^2 = d \sum_j \lambda_j^2 = d\,\mathrm{trace}(S^2),$$

which, by (2.5), is (2.6), with equality if and only if $\lambda_j = A$, $\forall j$, $A > 0$, i.e.,

$$S = U(AI)U^* = AI \quad \Longleftrightarrow \quad (v_j) \text{ is a tight frame for } \mathbb{F}^d.$$

Note above, since one vector is nonzero, $S = \sum_j v_j v_j^* \ne 0$, and so $A \ne 0$. □

This variational characterisation of tight frames depends only on the Gramian, and hence the frame up to unitary equivalence. It is easy to verify, and plays a key role in Theorems 3.1 and 3.2. We now consider its implications for equiangular lines.

2.3 Bounds on equiangular lines

We recall that unit vectors (v_j) in $\mathbb{F}^d$ are equiangular if they satisfy (2.3), i.e.,

$$|\langle v_j, v_k\rangle|^2 = \lambda = c^2 = \cos\theta, \qquad \forall j \neq k.$$

Those of the most interest have the maximum separation of the corresponding lines, i.e., $\lambda = c^2$ *small*, or, equivalently, θ *large*. Examples that exist in every dimension d are orthonormal bases of $n = d$ vectors ($\lambda = 0$, $\theta = 90°$) and the $n = d+1$ vertices of a regular simplex ($\lambda = \frac{1}{d^2}$). As an example of Theorem 2.1, we have the following bound.

Example 2.2 *If all the n vectors (v_j) in $\mathbb{F}^d$ have unit norm, then (2.6) reduces to*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^2 \geq \frac{1}{d}\Big(\sum_{j=1}^n 1^2\Big)^2 = \frac{n^2}{d}.$$

Moreover, if the (v_j) are equiangular, then the left hand side is $(n^2-n)\lambda + n$, and the inequality rearranges to

$$\lambda \geq \frac{n-d}{d(n-1)}, \tag{2.7}$$

with equality (and maximum possible separation) when the vectors are a tight frame, and for $\lambda < \frac{1}{d}$ it rearranges to the **relative bound** *for equiangular lines*

$$n \leq \frac{1-\lambda}{\frac{1}{d}-\lambda}, \qquad \lambda < \frac{1}{d}.$$

The next bound (which is well known for $\mathbb{F} = \mathbb{R}, \mathbb{C}$) depends on the underlying field.

Theorem 2.2 *Suppose $d > 1$. Let (v_j) be a sequence of n unit vectors in $\mathbb{F}^d$ giving a set of n equiangular lines, then the orthogonal projections*

$$P_j = v_j v_j^* : v \mapsto v_j\langle v, v_j\rangle, \qquad j = 1, \ldots, n,$$

are linearly independent over $\mathbb{R}$, and hence

$$n \leq \begin{cases} \frac{1}{2}d(d+1), & \mathbb{F} = \mathbb{R}; \\ d^2, & \mathbb{F} = \mathbb{C}; \\ 2d^2 - d, & \mathbb{F} = \mathbb{H}, \end{cases} \tag{2.8}$$

with equality if and only if (P_j) is a basis for the $\mathbb{R}$-vector space of Hermitian matrices. In these cases, the angle is

$$\lambda = \begin{cases} \frac{1}{d+2}, & \mathbb{F} = \mathbb{R}; \\ \frac{1}{d+1}, & \mathbb{F} = \mathbb{C}; \\ \frac{1}{d+\frac{1}{2}}, & \mathbb{F} = \mathbb{H}. \end{cases} \tag{2.9}$$

Proof: Since $d > 1$, the equiangularity constant λ is less than 1. Using (2.4), we calculate

$$\begin{aligned}\mathrm{Re}(\mathrm{trace}(P_jP_k)) &= \mathrm{Re}(\mathrm{trace}(v_jv_j^*v_kv_k^*)) \\ &= \mathrm{Re}(\mathrm{trace}(v_j^*v_kv_k^*v_j)) = |\langle v_j, v_k\rangle|^2 = \lambda, \qquad j \neq k.\end{aligned}$$

The $\mathbb{R}$-linear combination $\sum_j c_jP_j$ is Hermitian, and hence its Frobenius norm satisfies

$$\begin{aligned}\|\sum_j c_jP_j\|_F^2 &= \mathrm{Re}(\mathrm{trace}(\sum_j c_jP_j\sum_k c_kP_k)) = \sum_j\sum_k c_jc_k\,\mathrm{Re}(\mathrm{trace}(P_jP_k)) \\ &= \sum_j\sum_k c_jc_k\lambda + \sum_j c_jc_j(1-\lambda) = \lambda\Bigl(\sum_j c_j\Bigr)^2 + (1-\lambda)\sum_j c_j^2,\end{aligned}$$

which is zero only for the trivial linear combination.

The n projections $\{P_j\}$ belong to the real vector space of $d\times d$ Hermitian matrices which has dimension given by the right hand side of (2.8). For example, for $\mathbb{F} = \mathbb{H}$ the Hermitian matrices are determined by their real diagonal, and the entries above it which can be any quaternions, giving a dimension of $d + \frac{1}{2}(d^2 - d)\cdot 4 = 2d^2 - d$. □

This result for $\mathbb{H}^d$, the inequality (2.8), is given in [Hog76], without proof, and does not seem to be widely known.

We are now in position to discuss quaternionic equiangular lines. We first observe:

- Quaternionic equiangular lines do exist (for $\lambda < 1$, $d > 1$).

You will recall from Example 2.1 that Hoggar's example of four equiangular lines in $\mathbb{H}^2$ were in fact lines in $\mathbb{C}^2$ (most likely the very first occurrence of a SIC in the literature). For $d = 1$, any sequence of unit quaternions is an equiangular tight frame (with $\lambda = 1$), which is quaternionic if any ratio of the quaternions is not a complex number. Even though this is a trivial example, we will be able to use such frames to construct unit-norm tight frames in $\mathbb{C}^2$ and $\mathbb{R}^4$ (Example 3.5). We now give a simple example in $\mathbb{H}^2$.

Example 2.3 *(Five equiangular lines in $\mathbb{H}^2$). Fix $0 < t < 1$, and consider the four unit vectors*

$$v_r = \begin{pmatrix} t \\ \sqrt{1-t^2}\, i_r\end{pmatrix}, \qquad i_1 = 1, \quad i_2 = i, \quad i_3 = j, \quad i_4 = k.$$

These are equiangular, with

$$|\langle v_r, v_s\rangle|^2 = \lambda := t^4 + (1-t^2)^2, \qquad j \neq k,$$

where $\frac{1}{2} \leq \lambda < 1$. By Theorem 2.2, the maximal number of equiangular lines in $\mathbb{C}^2$ is four, with $\lambda = \frac{1}{3}$, so these lines are quaternionic. For the maximal separation $\lambda = \frac{1}{2}$, we may add a fifth equiangular line, to obtain five equiangular lines in $\mathbb{H}^2$ given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\j\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\k\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix} \ \Bigl(\mathit{or}\ \begin{pmatrix}0\\1\end{pmatrix}\Bigr).$$

These lines are not tight, since they do not give equality in (2.7), i.e.,

$$\lambda = \frac{1}{2} > \frac{3}{8} = \frac{5-2}{2(5-1)} = \frac{n-d}{d(n-1)}.$$

Another method to obtain tight equiangular lines is via the **complementary tight frame**. The construction is as follows. Let G be the Gramian of $n > d$ equiangular unit vectors in $\mathbb{F}^d$ at an angle $\lambda = \frac{n-d}{d(n-1)} \neq 0$, so that $P = \frac{d}{n}G$ is an orthogonal projection matrix (Proposition 2.1). The complementary orthogonal projection $Q = I - P$ gives an equiangular tight frame of n vectors for $\mathbb{F}^{n-d}$ with Gramian G_c given by

$$G_c = \frac{n}{n-d}I - \frac{d}{n-d}G,$$

and common angle $\lambda_c = \frac{d^2}{(n-d)^2}\lambda = \frac{d}{(n-d)(n-1)}$.

Let c_d be the right hand side of (2.8), which we can write as

$$c_d = d + \frac{1}{2}d(d-1)\cdot m, \qquad m := \dim_{\mathbb{R}}(\mathbb{F}).$$

Since the complementary tight frame also must satisfy the bound (2.8), for $n - d \neq 1$, we have that an equiangular tight frame of $n > d + 1$ unit vectors in $\mathbb{F}^d$ must satisfy

$$n \leq \min\{c_d, c_{n-d}\}. \tag{2.10}$$

This gives the following (see Theorem 2.18 of [KF08]).

Proposition 2.3 *An equiangular tight frame of n vectors for $\mathbb{F}^d$ satisfies*

$$d + \frac{1}{2} + \frac{\sqrt{\frac{8}{m}d + 1}}{2} \leq n \leq d + \frac{m}{2}d(d-1), \qquad m = \dim_{\mathbb{R}}(\mathbb{F}), \tag{2.11}$$

so that

$$n \geq d + 2 + j, \qquad \textit{for} \quad d > \frac{m}{2}j(j+1). \tag{2.12}$$

Proof: The condition $n \leq c_{n-d}$ in (2.10) can be written as

$$n^2 - (2d+1)n + d(d+1) - \frac{2}{m}d \geq 0.$$

By considering the roots of this quadratic polynomial in n, this is satisfied if and only if

$$n \leq d + \frac{1}{2} - \frac{\sqrt{\frac{8}{m}d + 1}}{2} < d, \quad \text{or} \quad n \geq d + \frac{1}{2} + \frac{\sqrt{\frac{8}{m}d + 1}}{2},$$

which gives the lower bound in (2.11). The upper bound is the condition $n \leq c_d$.

Rearranging the right hand inequality in

$$n \geq d + \frac{1}{2} + \frac{\sqrt{\frac{8}{m}d + 1}}{2} \geq d + 2 + j,$$

gives

$$d \geq \frac{m}{8}\big((2j+3)^2 - 1\big) = \frac{m}{2}(j+1)(j+2),$$

which gives (2.12). □

The lower bound in (2.11) is a decreasing function of m and the upper bound is an increasing function of m. This says that there is more room in $\mathbb{H}^d$ for tight equiangular lines than there is in $\mathbb{C}^d$, and in turn $\mathbb{R}^d$.

Example 2.4 *(Five tight equiangular lines in $\mathbb{H}^3$) By Proposition 2.3, there cannot be five tight equiangular lines in $\mathbb{R}^3$ or $\mathbb{C}^3$, but they could exist in $\mathbb{H}^3$. We now construct such lines as a complementary tight frame. The following five tight equiangular lines in $\mathbb{H}^2$ with $\lambda = \frac{3}{8}$ are given by [ET20]*

$$V = \begin{pmatrix} 1 & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{5}}{2\sqrt{2}} & -\frac{\sqrt{5}}{6\sqrt{2}} + \frac{\sqrt{5}}{3}i & -\frac{\sqrt{5}}{6\sqrt{2}} - \frac{\sqrt{5}}{6}i + \frac{\sqrt{5}}{2\sqrt{3}}j & -\frac{\sqrt{5}}{6\sqrt{2}} - \frac{\sqrt{5}}{6}i - \frac{\sqrt{5}}{2\sqrt{3}}j \end{pmatrix}.$$

The complementary tight frame therefore gives five equiangular lines in $\mathbb{H}^3$ at angle $\lambda = \frac{1}{6}$. A concrete presentation of these lines is

$$W = \begin{pmatrix} 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{5}}{\sqrt{6}} & -\frac{\sqrt{5}}{3\sqrt{6}} - \frac{\sqrt{5}}{3\sqrt{3}}i & -\frac{\sqrt{5}}{3\sqrt{6}} + \frac{\sqrt{5}}{6\sqrt{3}}i - \frac{\sqrt{5}}{6}j & -\frac{\sqrt{5}}{3\sqrt{6}} + \frac{\sqrt{5}}{6\sqrt{3}}i + \frac{\sqrt{5}}{6}j \\ 0 & 0 & \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{6} + \frac{\sqrt{5}}{2\sqrt{3}}k & -\frac{\sqrt{5}}{6} - \frac{\sqrt{5}}{2\sqrt{3}}k \end{pmatrix}.$$

This was obtained by the following general method. The condition $VV^* = AI$ for V to be a tight frame is that the conjugates of the rows of V are orthogonal and of equal length, i.e., V^* has orthogonal columns of equal length. By using Gram-Schmidt, add orthogonal columns of equal length to obtain $[V^*, W^*]$ a scalar multiple of a unitary matrix. Then W is a tight frame, which is the complement of V, since

$$\begin{pmatrix} V^* & W^* \end{pmatrix} \begin{pmatrix} V^* & W^* \end{pmatrix}^* = \begin{pmatrix} V^* & W^* \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} = V^*V + W^*W = AI.$$

Above we used the fact that the conjugates of the rows of the matrix V giving a tight frame are orthogonal. For frames over $\mathbb{C}$ this is equivalent to the rows being orthogonal. For frames over the quaternions, it is necessary to make this distinction. Indeed, there exist unitary matrices (orthogonal columns) whose rows are not orthogonal, e.g.,

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \qquad U^*U = UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (U^T)^*(U^T) = \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}.$$

Example 2.5 *(Six tight equiangular lines in $\mathbb{H}^4$) By Proposition 2.3, there cannot be six tight equiangular lines in $\mathbb{R}^4$ or $\mathbb{C}^4$, but they do exist in $\mathbb{H}^4$, by taking the complementary tight frame to the six tight equiangular lines in $\mathbb{H}^2$ of [ET20].*

We now consider tight equiangular lines in general, before giving a striking summary of the known results for two dimensions. For n tight equiangular lines in $\mathbb{H}^d$ (or $\mathbb{C}^d$, $\mathbb{R}^d$), the angle is

$$\lambda = \frac{n-d}{d(n-1)}, \qquad n > d,$$

with the following specific cases (in order of the number of vectors)

$$\lambda = 0 \quad \text{(orthonormal basis)}, \qquad \lambda = \frac{1}{d^2} \quad \text{(vertices of a simplex)},$$

and sets of lines giving the bounds of Theorem 2.2

$$\lambda = \frac{1}{d+2}, \qquad \lambda = \frac{1}{d+1} \quad \text{(SIC)}, \qquad \lambda = \frac{1}{d+\frac{1}{2}} \quad \text{(maximal set of lines in } \mathbb{H}^d).$$

The theory as is stands does not preclude the bounds above being reached by lines from a larger space, e.g., $n = \frac{1}{2}d(d+1)$ complex or even quaternionic lines in $\mathbb{H}^d$. This does not occur for two dimensions. Since

$$\frac{d\lambda}{dn} = \frac{d-1}{d(n-1)^2} > 0,$$

λ increases with the number of tight equiangular lines n (for d fixed), taking the possible values

$$\lambda = 0, \frac{1}{d^2}, \ldots, \frac{1}{d+2}, \ldots, \frac{1}{d+1}, \ldots, \frac{2}{2d+1}.$$

Equiangular lines are classified up to projective unitary equivalence (see Section 5). In two dimensions, the tight equiangular lines given by an orthonormal basis, the Mercedes-Benz frame and the SIC (two, three and four vectors, respectively) are well known, as is their uniqueness in $\mathbb{C}^2$. Putting these examples together with the five and six sets of equianglar lines of [KF08], [ET20] gives a complete characterisation of equiangular lines in $\mathbb{H}^2$.

Theorem 2.3 *There is a unique set of n tight equiangular lines in $\mathbb{H}^2$ for $n = 2, 3, 4, 5, 6$, with corresponding angles $\lambda = 0, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}$.*

2.4 Equi-isoclinic and equichordal subspaces

We now consider generalisations of equiangularity to r-subspaces (r-dimensional subspaces). Let P_j and P_k be the orthogonal projections onto r-subspaces V_j and V_k. Then

$$\|P_j - P_k\|_F^2 = \operatorname{trace}((P_j - P_k)^2) = 2r - \operatorname{trace}(P_jP_k + P_kP_j) \geq 0.$$

For $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$, we have $\operatorname{trace}(P_jP_k) = \operatorname{trace}(P_kP_j) \in \mathbb{R}$, and a collection of r-subspaces is a said to be **equichordal** (see [FJMW17]) if the corresponding orthogonal projections satisfy

$$\operatorname{trace}(P_jP_k) = \lambda r, \qquad j \neq k,$$

which reduces to the equiangularity condition (2.3) in the case of lines ($r = 1$).

For $\mathbb{H}^d$, $\operatorname{trace}(P_jP_k)$ need not be real, nor equal to $\operatorname{trace}(P_kP_j)$, e.g., for

$$P = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}, \qquad PQ = \frac{1}{4}\begin{pmatrix} 1-k & -i-j \\ i+j & 1-k \end{pmatrix},$$

and trace$(PQ) = \frac{1}{2}(1-k) \neq \frac{1}{2}(1+k) = $ trace(QP). However, by (2.4), we do have

$$\operatorname{trace}(P_jP_k + P_kP_j) = \operatorname{Re}(\operatorname{trace}(P_jP_k + P_kP_j)) = 2\operatorname{Re}(P_jP_k),$$

and so we say that r-subspaces in $\mathbb{H}^d$ (or $\mathbb{R}^d, \mathbb{C}^d$) are **equichordal** if

$$\operatorname{Re}(\operatorname{trace}(P_jP_k)) = \lambda r, \quad j \neq k \quad \Longleftrightarrow \quad \|P_j - P_k\|_F^2 = 2(1-\lambda)r, \quad j \neq k. \tag{2.13}$$

Two r-subspaces V_j and V_k, $j \neq k$, are **isoclinic** with parameter $0 \leq \lambda \leq 1$ (see [LS73], [Hog77]) if the orthogonal projection P_{jk} onto $V_j + V_k$ satisfies

$$(1-\lambda)P_{jk} = (P_j - P_k)^2.$$

An equivalent condition to being isoclinic is

$$P_jP_kP_j = \lambda P_j, \qquad P_kP_jP_k = \lambda P_k, \qquad j \neq k, \tag{2.14}$$

which follows from the observation

$$(1-\lambda)P_j = (P_j - P_k)^2 P_j \quad \Longleftrightarrow \quad P_jP_kP_j = \lambda P_j.$$

Hoggar [Hog77] claims that just one of the conditions (2.14) is required (over $\mathbb{H}$), which follows by writing $P_j = V_jV_j^*$, $V_j^*V_j = I$, and the implications

$$P_jP_kP_j = \lambda P_j \quad \Longleftrightarrow \quad V_j^*V_kV_k^*V_j = \lambda I \quad \Longleftrightarrow \quad V_k^*V_jV_j^*V_k \quad \Longleftrightarrow \quad P_kP_jP_k = \lambda P_k.$$

Subspaces (V_j) are said to be **equi-isoclinic** with parameter $0 \leq \lambda \leq 1$ if (2.14) holds. Equi-isoclinic subspaces are equichordal, since

$$P_jP_kP_j = \lambda P_j \quad \Longrightarrow \quad \operatorname{Re}(\operatorname{trace}(P_jP_k)) = \operatorname{Re}(\operatorname{trace}(P_jP_kP_j)) = \operatorname{trace}(\lambda P_j) = \lambda r.$$

The orthogonal complement $(V_j^\perp)$ of equichordal subspaces is equichordal, since

$$\operatorname{Re}(\operatorname{trace}((I-P_j)(I-P_k))) = d - r - r + \operatorname{Re}(\operatorname{trace}(P_jP_k)) = d - 2r + \lambda r, \quad j \neq k.$$

However, the orthogonal complements $(V_j^\perp)$ of equi-isoclinic subspaces (V_j) are not in general equi-isoclinic, as the following example shows.

Example 2.6 *(Two isoclinic planes do not exist in $\mathbb{R}^3$). Consider the equi-isoclinic 1-dimensional subspaces given by*

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \sqrt{1-a^2-b^2} \\ a \\ b \end{pmatrix}.$$

The orthogonal projections $Q_j = I - v_jv_j^*$ onto the complementary subspaces satisfy

$$Q_1Q_2Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a^2 & -ab \\ 0 & -ab & 1-b^2 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, for $V_1^\perp$ and $V_2^\perp$ to be isoclinic, we must have that $a = b = 0$, i.e., $V_1 = V_2$. Thus there cannot be two (nonequal) isoclinic planes in $\mathbb{R}^3$, despite the fact that there can be up to six equi-isoclinic lines in $\mathbb{R}^3$.

3 From $\mathbb{R}$ to $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{H}$, and back

There is a natural inclusion $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ and hence of $\mathbb{R}^d \subset \mathbb{C}^d \subset \mathbb{H}^d$. Since tight frames are determined up to unitary equivalence by their Gramians:

- There is a unitary map of a tight frame to $\mathbb{R}^d$ if and only if its Gramian has real entries, and we say the tight frame is **real**.
- There is a unitary map of a tight frame to $\mathbb{C}^d$ if and only if its Gramian has complex entries, and we say the tight frame is **complex** if its Gramian has a nonreal entry.
- If the Gramian of a tight frame has a noncomplex entry, then we say that it is a **quaternionic** tight frame.

As an example, the four equiangular lines in $\mathbb{H}^2$ of Hoggar [Hog77] are lines in $\mathbb{C}^2$ (see Example 2.1). For tight frames up to projective unitary equivalence, i.e., thought of as lines, the corresponding analogue is more involved, see Section 5.

There is also a natural identification of a point $z = x+iy \in \mathbb{C}$ (in the complex plane) with a point $(x, y) \in \mathbb{R}^2$ (in the plane). We generalise this, by defining an invertible $\mathbb{R}$-linear map

$$[\cdot]_{\mathbb{R}} : \mathbb{C}^d \to \mathbb{R}^{2d} : v \mapsto \begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix}, \qquad \operatorname{Re} v = \frac{v+\overline{v}}{2}, \quad \operatorname{Im} v = \frac{v-\overline{v}}{2i}. \tag{3.15}$$

Based on a thorough analysis of this, we will then define an analogous map $\mathbb{H}^d \to \mathbb{C}^{2d}$. The first subtle point, is that $[\cdot]_{\mathbb{R}}$ maps k-dimensional complex-subspaces of $\mathbb{C}^d$ to real $(2k)$-dimensional subspaces of $\mathbb{R}^{2d}$. To see why this is, we first calculate the image of a complex scalar multiple $\alpha + i\beta$ of a vector $v = x + iy$

$$(\alpha + i\beta)v = (\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\alpha y + \beta x),$$

which gives

$$[(\alpha + i\beta)v]_{\mathbb{R}} = \alpha \begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix} + \beta \begin{pmatrix} -\operatorname{Im} v \\ \operatorname{Re} v \end{pmatrix} = \alpha[v]_{\mathbb{R}} + \beta[iv]_{\mathbb{R}}. \tag{3.16}$$

Thus the one-dimensional complex subspace spanned by $v \in \mathbb{C}^d$ is mapped to the real two-dimensional subspace

$$[\operatorname{span}_{\mathbb{C}}\{v\}]_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix}, \begin{pmatrix} -\operatorname{Im} v \\ \operatorname{Re} v \end{pmatrix}\} \qquad \text{(orthogonal vectors in } \mathbb{R}^{2d}\text{)}.$$

The general result then follows from the correspondence between linear dependencies

$$\sum_\ell (\alpha_\ell + i\beta_\ell)v_\ell = 0 \quad \iff \quad \sum_\ell \Big\{\alpha_\ell \begin{pmatrix} \operatorname{Re} v_\ell \\ \operatorname{Im} v_\ell \end{pmatrix} + \beta_\ell \begin{pmatrix} -\operatorname{Im} v_\ell \\ \operatorname{Re} v_\ell \end{pmatrix}\Big\} = 0.$$

We also calculate

$$\begin{aligned} \langle v, w\rangle &= \langle \operatorname{Re} v + i \operatorname{Im} v, \operatorname{Re} w + i \operatorname{Im} w\rangle \\ &= \langle \operatorname{Re} v, \operatorname{Re} w\rangle + \langle \operatorname{Im} v, \operatorname{Im} w\rangle + i(\langle \operatorname{Im} v, \operatorname{Re} w\rangle - \langle \operatorname{Re} v, \operatorname{Im} w\rangle), \end{aligned}$$

so that

$$\operatorname{Re}(\langle v,w\rangle_{\mathbb{C}}) = \langle [v]_{\mathbb{R}}, [w]_{\mathbb{R}}\rangle_{\mathbb{R}}, \quad \operatorname{Im}(\langle v,w\rangle_{\mathbb{C}}) = \langle [v]_{\mathbb{R}}, [iw]_{\mathbb{R}}\rangle_{\mathbb{R}}, \tag{3.17}$$

$$\langle [v]_{\mathbb{R}}, [iv]_{\mathbb{R}}\rangle_{\mathbb{R}} = 0. \tag{3.18}$$

Let $A : \mathbb{C}^n \to \mathbb{C}^m$ a $\mathbb{C}$-linear map be represented as an $\mathbb{R}$-linear map $[A]_{\mathbb{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ under this identification, i.e., $[A]_{\mathbb{R}} := [\cdot]_{\mathbb{R}} A [\cdot]_{\mathbb{R}}^{-1}$. Then

$$\begin{aligned} A(u+iv) &= (\operatorname{Re}(A) + i\operatorname{Im}(A))(u+iv) \\ &= \operatorname{Re}(A)u - \operatorname{Im}(A)v + i\operatorname{Im}(A)u + i\operatorname{Re}(A)v, \quad u,v \in \mathbb{R}^n, \end{aligned}$$

and $\operatorname{Re}(A^*) = \operatorname{Re}(A)^T$, $\operatorname{Im}(A^*) = -\operatorname{Im}(A)^T$, so that

$$[A]_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re}(A) & -\operatorname{Im}(A) \\ \operatorname{Im}(A) & \operatorname{Re}(A) \end{pmatrix}, \qquad \operatorname{rank}([A]_{\mathbb{R}}) = 2\operatorname{rank}(A),$$

$$[A^*]_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re}(A)^T & \operatorname{Im}(A)^T \\ -\operatorname{Im}(A)^T & \operatorname{Re}(A)^T \end{pmatrix} = [A]_{\mathbb{R}}^T.$$

The usual rules for matrix multiplication follow, e.g, $[A]_{\mathbb{R}}[B]_{\mathbb{R}} = [AB]_{\mathbb{R}}$. One must be careful if a vector $v \in \mathbb{C}^d$ is being thought of as a $d \times 1$ matrix, i.e., the linear map $[v] : \mathbb{C} \to \mathbb{C}^d : \alpha \mapsto \alpha v$, since $[v]_{\mathbb{R}} \in \mathbb{R}^{2d\times 1}$, $[[v]]_{\mathbb{R}} \in \mathbb{R}^{2d\times 2}$. In particular, the familiar formula $P = vv^*$ for the orthogonal projection onto a unit vector $v \in \mathbb{C}^d$, is $P = [v][v]^*$, which maps as follows

$$[P]_{\mathbb{R}} = [[v]]_{\mathbb{R}}[[v]^*]_{\mathbb{R}} = [[v]]_{\mathbb{R}}[[v]]_{\mathbb{R}}^T, \qquad [[v]]_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re} v & -\operatorname{Im} v \\ \operatorname{Im} v & \operatorname{Re} v \end{pmatrix}.$$

This is the orthogonal projection onto

$$[\operatorname{span}_{\mathbb{C}}\{v\}]_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{[v]_{\mathbb{R}}, [iv]_{\mathbb{R}}\}, \qquad [v]_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re} v \\ \operatorname{Im} v \end{pmatrix}, \quad [iv]_{\mathbb{R}} = \begin{pmatrix} -\operatorname{Im} v \\ \operatorname{Re} v \end{pmatrix}.$$

The identification $[\cdot]_{\mathbb{R}}$ preserves various properties of linear maps, see Theorem 3.3. In particular, orthogonal projections map to orthogonal projections, and hence:

- Equi-isoclinic subspaces of dimension r in $\mathbb{C}^d$ correspond to equi-isoclinic subspaces of dimension $2r$ in $\mathbb{R}^{2d}$, and similarly for equichordal subspaces.

We now consider the situation for tight frames, which is somewhat more involved, e.g., a basis for $\mathbb{C}^d$ does not correspond to a basis for $\mathbb{R}^{2d}$ (which has twice the dimension). Let $V = V_1 + iV_2$ be the synthesis map for a sequence of vectors $v_1, \ldots, v_n \in \mathbb{C}^d$, and $V_{\mathbb{R}}$ be the corresponding map for the sequence $[v_1]_{\mathbb{R}}, \ldots, [v_n]_{\mathbb{R}} \in \mathbb{R}^{2d}$, i.e.,

$$V_{\mathbb{R}} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{R}^{2d\times n}.$$

Then V gives a tight frame for $\mathbb{C}^d$ if and only if

$$VV^* = (V_1 + iV_2)(V_1^* - iV_2^*) = V_1V_1^* + V_2V_2^* + i(V_2V_1^* - V_1V_2^*) = AI,$$

where $dA := \sum_j \|v_j\|^2 = \operatorname{trace}(VV^*) = \operatorname{trace}(V_{\mathbb{R}}V_{\mathbb{R}}^T) =$, i.e.,

$$V_1V_1^T + V_2V_2^T = AI, \qquad V_2V_1^T - V_1V_2^T = 0,$$

and $V_{\mathbb{R}}$ gives a tight frame for $\mathbb{R}^{2d}$ if and only if

$$V_{\mathbb{R}}V_{\mathbb{R}}^* = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} V_1^T & V_2^T \end{pmatrix} = \begin{pmatrix} V_1V_1^T & V_1V_2^T \\ V_2V_1^T & V_2V_2^T \end{pmatrix} = \frac{1}{2}A \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

i.e.,

$$V_1V_1^T = V_2V_2^T = \frac{1}{2}AI, \qquad V_1V_2^T = V_2V_1^T = 0. \tag{3.19}$$

Thus all tight frames for $\mathbb{R}^{2d}$ map to tight frames for $\mathbb{C}^d$, and a tight frame for $\mathbb{C}^d$ gives a tight frame for $\mathbb{R}^{2d}$ if and only if (3.19) holds. This condition says that V_1 and V_2 are tight frames for $\mathbb{R}^d$ (with the same frame bound) which are orthogonal (see [Wal18] §3.5). We now show that (3.19) depends only on V up to unitary equivalence.

Let $U = U_1 + iU_2$ be unitary, then $UU^* = U_1U_1^T + U_2U_2^T + i(U_2U_1^T - U_1U_2^T) = I$, which is equivalent to

$$U_1U_1^T + U_2U_2^T = I, \qquad U_2U_1^T - U_1U_2^T = 0. \tag{3.20}$$

Suppose that V satisfies (3.19), then

$$UV = [Uv_1, \ldots, Uv_n] = (U_1 + iU_2)(V_1 + iV_2) = (U_1V_1 - U_2V_2) + i(U_2V_1 + U_1V_2),$$

$A = \sum_j \|v_j\|^2 = \sum_j \|Uv_j\|^2$, and using (3.20), we calculate

$$\begin{aligned}
\operatorname{Re}(UV)\operatorname{Re}(UV)^T &= (U_1V_1 - U_2V_2)(V_1^TU_1^T - V_2^TU_2^T) = \frac{1}{2}A(U_1U_1^T + U_2U_2^T) = \frac{1}{2}AI, \\
\operatorname{Im}(UV)\operatorname{Im}(UV)^T &= (U_2V_1 + U_1V_2)(V_1^TU_2^T + V_2^TU_1^T) = \frac{1}{2}A(U_2U_2^T + U_1U_1^T) = \frac{1}{2}AI, \\
\operatorname{Re}(UV)\operatorname{Im}(UV)^T &= (U_1V_1 - U_2V_2)(V_1^TU_2^T + V_2^TU_1^T) = \frac{1}{2}A(U_1U_2^T - U_2U_1^T) = 0,
\end{aligned}$$

so that UV satisfies (3.19).

Since the condition for a tight frame for $\mathbb{C}^d$ to be a tight frame for $\mathbb{R}^{2d}$ depends only on V up to unitary equivalence, it follows that this condition can be written in terms of the Gramian of V. The Gramians of V and $V_{\mathbb{R}}$ are

$$V^*V = (V_1^* - iV_2^*)(V_1 + iV_2) = V_1^TV_1 + V_2^TV_2 + i(V_1^TV_2 - V_2^TV_1),$$

$$V_{\mathbb{R}}^*V_{\mathbb{R}} = \begin{pmatrix} V_1^T & V_2^T \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = V_1^TV_1 + V_2^TV_2.$$

The variational characterisation for being a tight frame for $\mathbb{C}^d$ and for $\mathbb{R}^{2d}$ are

$$\|V^*V\|_F^2 = \frac{1}{d}(\operatorname{trace}(V^*V))^2, \qquad \|V_{\mathbb{R}}^*V_{\mathbb{R}}\|_F^2 = \frac{1}{2d}(\operatorname{trace}(V_{\mathbb{R}}^*V_{\mathbb{R}}))^2.$$

Since $\operatorname{trace}(V^*V) = \operatorname{trace}(V_1^TV_1 + V_2^TV_2) = \operatorname{trace}(V_{\mathbb{R}}^*V_{\mathbb{R}})$, a tight frame for $\mathbb{C}^d$ gives a tight frame for $\mathbb{R}^{2d}$ if and only if

$$2\|V_{\mathbb{R}}^*V_{\mathbb{R}}\|_F^2 - \|V^*V\|_F^2 = 0. \tag{3.21}$$

By writing this explicitly in terms of V^*V (cf [Wal20b]), we obtain the following.

Theorem 3.1 *Let* $[\cdot]_{\mathbb{R}} : \mathbb{C}^d \to \mathbb{R}^{2d}$ *be the correspondence (3.15) between* $\mathbb{C}^d$ *and* $\mathbb{R}^{2d}$*. Then*

1. *Tight frames for* $\mathbb{R}^{2d}$ *correspond to tight frames for* $\mathbb{C}^d$*.*
2. *A tight frame* $V = [v_1, \ldots, v_n]$ *for* $\mathbb{C}^d$ *corresponds to a tight frame for* $\mathbb{R}^{2d}$ *if and only if it satisfies*

$$\sum_j \sum_k \langle v_j, v_k \rangle^2 = 0, \tag{3.22}$$

which can also be written as

$$\sum_j \sum_k (\mathrm{Re}\langle v_j, v_k \rangle)^2 = \sum_j \sum_k (\mathrm{Im}\langle v_j, v_k \rangle)^2. \tag{3.23}$$

Proof: In light of our previous discussion, it remains only to show that (3.21) can be written as (3.22) and (3.23). Using (3.17), we have

$$\begin{aligned} \|V_{\mathbb{R}}^* V_{\mathbb{R}}\|_F^2 - \|V^* V\|_F^2 &= 2 \sum_j \sum_k \langle [v_j]_{\mathbb{R}}, [v_k]_{\mathbb{R}} \rangle^2 - \sum_j \sum_k |\langle v_j, v_k \rangle|^2 \\ &= 2 \sum_j \sum_k (\mathrm{Re}\langle v_j, v_k \rangle)^2 - \sum_j \sum_k |\langle v_j, v_k \rangle|^2 = 0. \end{aligned}$$

By taking $z = \langle v_j, v_k \rangle$ in

$$2(\mathrm{Re}(z))^2 - |z|^2 = 2\Bigl(\frac{z + \overline{z}}{2}\Bigr)^2 - z\overline{z} = \frac{1}{2}(z^2 + \overline{z}^2),$$

we see that this condition can be written as

$$\frac{1}{2} \sum_j \sum_k \bigl(\langle v_j, v_k \rangle^2 + \langle v_k, v_j \rangle^2\bigr) = \sum_j \sum_k \langle v_j, v_k \rangle^2 = 0.$$

which gives (3.22). By substituting in $|\langle v_j, v_k \rangle|^2 = (\mathrm{Re}\langle v_j, v_k \rangle)^2 + (\mathrm{Im}\langle v_j, v_k \rangle)^2$, we obtain (3.23). □

Example 3.1 *A tight frame* (z_j) *for* $\mathbb{C}$ *corresponds to a tight frame for* $\mathbb{R}^2$ *if and only if*

$$\sum_j \sum_k (z_j \overline{z_k})^2 = \Bigl(\sum_j z_j^2\Bigr)\Bigl(\sum_k \overline{z_k}^2\Bigr) = \Bigl|\sum_j z_j^2\Bigr|^2 = 0 \quad \Longleftrightarrow \quad \sum_j z_j^2 = 0.$$

The complex number $z_j^2 = (x_j + i y_j)^2$ *corresponding to a point* (x_j, y_j) *is sometimes called a diagram vector, and the condition that a frame for* $\mathbb{R}^2$ *is tight if and only if its diagram vectors sum to zero is well known.*

We now give a map $\mathbb{H}^d \to \mathbb{C}^{2d}$ that has similar properties to $[\cdot]_{\mathbb{R}} : \mathbb{C}^d \to \mathbb{R}^{2d}$. This is based on the following analogue of the polar decomposition for $\mathbb{C}$, the Cayley-Dickson construction, that every quaternion $q \in \mathbb{H}$ can be written uniquely

$$q = z + wj, \qquad z, w \in \mathbb{C}. \tag{3.24}$$

Moreover, we observe the "commutativity" relation

$$jz = \overline{z}j, \qquad \forall z \in \mathbb{C},$$

which implies

$$jA = \overline{A}j, \qquad \forall A \in \mathbb{C}^{m\times n}.$$

Let $\mathbb{H}^d$ be a right vector space, and define a $\mathbb{C}$-linear map

$$[\cdot]_{\mathbb{C}} : \mathbb{H}^d \to \mathbb{C}^{2d} : z + wj \mapsto \begin{pmatrix} z \\ \overline{w} \end{pmatrix}, \tag{3.25}$$

The conjugation $\overline{w}$ is necessary for $\mathbb{C}$-linearity: $(z+wj)\alpha = z\alpha + wj\alpha = z\alpha + w\overline{\alpha}j$ gives

$$[(z+wj)\alpha]_{\mathbb{C}} = \begin{pmatrix} z\alpha \\ \overline{w}\alpha \end{pmatrix} = \begin{pmatrix} z \\ \overline{w} \end{pmatrix}\alpha = [z+wj]_{\mathbb{C}}\alpha \qquad \forall \alpha \in \mathbb{C}.$$

Let Co_1 and Co_2 be the $\mathbb{C}$-linear maps $\mathbb{H}^d \to \mathbb{C}^d$ giving the "complex coordinates" of $q = z + wj$, i.e.,

$$\mathrm{Co}_1(z+wj) := z, \qquad \mathrm{Co}_2(z+wj) := \overline{w}.$$

We note in particular, that

$$|q|^2 = |\,\mathrm{Co}_1(q)|^2 + |\,\mathrm{Co}_2(q)|^2.$$

From

$$\begin{aligned} \langle v, w\rangle_{\mathbb{H}} &= \langle v_1 + v_2 j, w_1 + w_2 j\rangle \\ &= (\langle v_1, w_1\rangle + \langle \overline{v_2}, \overline{w_2}\rangle) + (\langle v_2, w_1\rangle - \langle w_2, v_1\rangle)j, \\ &= \langle [v]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}} - \langle [vj]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}} j \end{aligned}$$

we get the analogues of (3.17) and (3.18)

$$\mathrm{Co}_1(\langle v, w\rangle_{\mathbb{H}}) = \langle [v]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}}, \qquad \mathrm{Co}_2(\langle v, w\rangle_{\mathbb{H}}) = -\langle [vj]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}}. \tag{3.26}$$

$$\langle [v]_{\mathbb{C}}, [vj]_{\mathbb{C}}\rangle_{\mathbb{C}} = 0. \tag{3.27}$$

$$\mathrm{Re}(\langle v, w\rangle_{\mathbb{H}}) = \mathrm{Re}(\mathrm{Co}_1(\langle v, w\rangle_{\mathbb{H}})) = \mathrm{Re}(\langle [v]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}}) = \langle [[v]_{\mathbb{C}}]_{\mathbb{R}}, [[w]_{\mathbb{C}}]_{\mathbb{R}}\rangle_{\mathbb{R}}. \tag{3.28}$$

The analogue of (3.29) for $v = z + wj$ is

$$[v(\alpha+\beta j)]_{\mathbb{C}} = [v\alpha + v\beta j]_{\mathbb{C}} = [v\alpha + vj\overline{\beta}]_{\mathbb{C}} = [v]_{\mathbb{C}}\alpha + [vj]_{\mathbb{C}}\overline{\beta}, \qquad \alpha, \beta \in \mathbb{C}, \tag{3.29}$$

where

$$[v]_{\mathbb{C}} = \begin{pmatrix} z \\ \overline{w} \end{pmatrix}, \quad [vj]_{\mathbb{C}} = \begin{pmatrix} -w \\ \overline{z} \end{pmatrix}, \qquad \langle [v]_{\mathbb{C}}, [vj]_{\mathbb{C}}\rangle_{\mathbb{C}} = 0.$$

Thus $[\cdot]_{\mathbb{C}}$ maps k-dimensional $\mathbb{H}$-subspace of $\mathbb{H}^d$ to $(2k)$-dimensional $\mathbb{C}$-subspaces of $\mathbb{C}^{2d}$.

Let $L : \mathbb{H}^n \to \mathbb{H}^m$ an $\mathbb{H}$-linear map be represented as a $\mathbb{C}$-linear map $[L]_{\mathbb{C}} : \mathbb{C}^{2n} \to \mathbb{C}^{2m}$ under this identification, i.e., $[L]_{\mathbb{C}} := [\cdot]_{\mathbb{C}} L [\cdot]_{\mathbb{C}}^{-1}$. In view of (3.24), its standard matrix $[L]_{\mathbb{H}} \in \mathbb{H}^{m\times n}$ has a unique decomposition

$$[L]_{\mathbb{H}} = A + Bj, \qquad A, B \in \mathbb{C}^{m\times n}.$$

We have

$$L(z+wj) = (A+Bj)(z+wj) = Az + Awj + Bjz + Bjwj$$
$$= Az + Awj + B\overline{z}j - B\overline{w} = Az - B\overline{w} + (\overline{\overline{B}z + \overline{A}\overline{w}})j,$$

and

$$[L^*]_{\mathbb{H}} = (A+Bj)^* = A^* + (-j)B^* = A^* - \overline{B^*}j = A^* - B^T j,$$

so that

$$[L]_{\mathbb{C}} = \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix}, \qquad \operatorname{rank}([L]_{\mathbb{C}}) = 2\operatorname{rank}([L]_{\mathbb{H}}),$$
$$[L^*]_{\mathbb{C}} = \begin{pmatrix} A^* & B^T \\ -B^* & A^T \end{pmatrix} = [L]_{\mathbb{C}}^*.$$

The other observations for the previous case also hold (see Theorem 3.3), in particular

- Equi-isoclinic subspaces of dimension r in $\mathbb{H}^d$ correspond to equi-isoclinic subspaces of dimension $2r$ in $\mathbb{C}^{2d}$, and similarly for equichordal subspaces.

We now seek the analogue of Theorem 3.1, this time starting with the development in terms of the Gramian. The variational characterisation for $V = [v_1, \ldots, v_n]$ being a tight frame for $\mathbb{H}^d$ and for $V_{\mathbb{C}} := \bigl[[v_1]_{\mathbb{C}}, \ldots, [v_n]_{\mathbb{C}}\bigr]$ being a tight frame for $\mathbb{C}^{2d}$ are

$$\|V^*V\|_F^2 = \frac{1}{d}(\operatorname{trace}(V^*V))^2, \qquad \|V_{\mathbb{C}}^*V_{\mathbb{C}}\|_F^2 = \frac{1}{2d}(\operatorname{trace}(V_{\mathbb{C}}^*V_{\mathbb{C}}))^2.$$

Since $\operatorname{trace}(V^*V) = \operatorname{trace}(V_{\mathbb{C}}^*V_{\mathbb{C}})$, a tight frame for $\mathbb{H}^d$ gives a tight frame for $\mathbb{C}^{2d}$ if and only if

$$2\|V_{\mathbb{C}}^*V_{\mathbb{C}}\|_F^2 - \|V^*V\|_F^2 = 0. \tag{3.30}$$

Writing this explicitly in terms of the Gramian V^*V gives the following.

Lemma 3.1 *Let $V = [v_1, \ldots, v_n] = V_1 + V_2 j \in \mathbb{H}^{d\times n}$. Then the following are equivalent*

(i) $V_{\mathbb{C}} = \bigl[[v_1]_{\mathbb{C}}, \ldots, [v_n]_{\mathbb{C}}\bigr] = \begin{pmatrix} V_1 \\ \overline{V_2} \end{pmatrix} \in \mathbb{C}^{2d\times n}$ *is a tight frame for* $\mathbb{C}^{2d}$.

(ii)

$$V_1V_1^* = V_2V_2^* = \frac{1}{2}AI, \qquad V_1V_2^T = V_2V_1^T = 0, \qquad A := \frac{1}{d}\sum_j \|v_j\|^2.$$

(iii)

$$(V_1^*V_1 + V_2^T\overline{V_2})^2 = \frac{1}{2}A\,(V_1^*V_1 + V_2^T\overline{V_2}), \qquad A := \frac{1}{d}\sum_j \|v_j\|^2.$$

(iv)

$$\|\operatorname{Co}_1(V^*V)\|_F^2 = \sum_j\sum_k |\operatorname{Co}_1(\langle v_j, v_k\rangle)|^2 = \frac{1}{2d}\Bigl(\sum_j \|v_j\|^2\Bigr)^2.$$

Proof: In terms of the frame operator, the condition (i) is

$$\begin{pmatrix} V_1 \\ \overline{V_2} \end{pmatrix} \begin{pmatrix} V_1 \\ \overline{V_2} \end{pmatrix}^* = \begin{pmatrix} V_1 \\ \overline{V_2} \end{pmatrix} \begin{pmatrix} V_1^* & V_2^T \end{pmatrix}^* = \begin{pmatrix} V_1 V_1^* & V_1 V_2^T \\ \overline{V_2} V_1^* & \overline{V_2} V_2^T \end{pmatrix} = \frac{1}{2} A \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where $dA = \sum_j \|v_j\|^2$, which is clearly equivalent to (ii).

In terms of the Gramian $V_{\mathbb{C}}^* V_{\mathbb{C}} = V_1^* V_1 + V_2^T \overline{V_2}$ being (a multiple of) an orthogonal projection matrix (Proposition 2.1), the condition (i) is (iii).

In terms of the variational characterisation (Theorem 2.1), the condition (i) is

$$\sum_j \sum_k |\langle [v_j]_{\mathbb{C}}, [v_k]_{\mathbb{C}} \rangle|^2 = \frac{1}{2d} \sum_j \Bigl(\|[v_j]_{\mathbb{C}}\|^2 \Bigr)^2,$$

which can be written as (iv), since $\langle [v]_{\mathbb{C}}, [w]_{\mathbb{C}} \rangle_{\mathbb{C}} = \mathrm{Co}_1(\langle v, w \rangle_{\mathbb{H}})$ and $\|[v]_{\mathbb{C}}\| = \|v\|_{\mathbb{H}}$. □

We observe that condition the (iv) depends only on V up to unitary equivalence, and so the others do also.

Theorem 3.2 *Let $[\cdot]_{\mathbb{C}} : \mathbb{H}^d \to \mathbb{C}^{2d}$ be the correspondence (3.25) between $\mathbb{H}^d$ and $\mathbb{C}^{2d}$. Then*

1. *Tight frames for $\mathbb{C}^{2d}$ correspond to tight frames for $\mathbb{H}^d$.*
2. *A tight frame $V = [v_1, \ldots, v_n]$ for $\mathbb{H}^d$ corresponds to a tight frame for $\mathbb{C}^{2d}$ if and only if it satisfies*

$$\sum_j \sum_k |\,\mathrm{Co}_1(\langle v_j, v_k \rangle)|^2 = \sum_j \sum_k |\,\mathrm{Co}_2(\langle v_j, v_k \rangle)|^2. \tag{3.31}$$

Proof: The sequence $V = V_1 + V_2 j$ is tight frame for $\mathbb{H}^d$ if and only if

$$VV^* = (V_1 + V_2 j)(V_1^* - V_2^T j) = (V_1 V_1^* + V_2 V_2^*) + (V_2 V_1^T - V_1 V_2^T) j = AI,$$

which is clearly satisfied if V corresponds to a tight frame for $\mathbb{C}^{2d}$ (by Proposition 3.1).

The variational characterisation for being a tight frame for $\mathbb{H}^d$ and for $\mathbb{C}^{2d}$ are

$$\|V^*V\|_F^2 = \frac{1}{d} \Bigl(\sum_j \|v_j\|^2 \Bigr)^2, \qquad \|\,\mathrm{Co}_1(V^*V)\|_F^2 = \frac{1}{2d} \Bigl(\sum_j \|v_j\|^2 \Bigr)^2.$$

Hence, if V gives a tight frame for $\mathbb{H}^d$, then it gives a tight frame for $\mathbb{C}^{2d}$ if and only if

$$2\|\,\mathrm{Co}_1(V^*V)\|_F^2 - \|V^*V\|_F^2 = 0.$$

Since $|\langle v_j, v_k \rangle|^2 = |\,\mathrm{Co}_1(\langle v_j, v_k \rangle)|^2 + |\,\mathrm{Co}_2(\langle v_j, v_k \rangle)|^2$, this is (3.31). □

The conditions (3.23) and (3.31) can be written insightfully as

$$\|\operatorname{Re}(V^*V)\|_F = \|\operatorname{Im}(V^*V)\|_F, \qquad \|\operatorname{Co}_1(V^*V)\|_F = \|\operatorname{Co}_2(V^*V)\|_F.$$

Example 3.2 *Let $V = [1, i, j, k]$, which is a tight frame for $\mathbb{H}$. The Gramian is*

$$V^*V = \begin{pmatrix} 1 & i & j & k \\ -i & 1 & -k & j \\ -j & k & 1 & -i \\ -k & -j & i & 1 \end{pmatrix} = \begin{pmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & i & j \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix} j,$$

so this gives a tight frame for $\mathbb{C}^2$, i.e., $W = [e_1, ie_1, e_2, ie_2]$, with Gramian

$$W^*W = \begin{pmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

so that this in turn gives a tight frame for $\mathbb{R}^4$, i.e., $[e_1, e_3, e_2, e_4]$.

Example 3.3 *Consider the Gramian of the SIC of four vectors in $\mathbb{C}^2$ (Example 2.1). The Frobenius norm of the diagonal entries, which are all real, is 4, and for the off diagonal entries it is $12\frac{1}{3} = 4$. Thus the SIC corresponds to a tight frame for $\mathbb{R}^4$ if and only if its vectors can be scaled so that the off diagonal entries of the Gramian are pure imaginary. This can in fact be done, e.g., take $V = [v, iSv, i\Omega v, -S\Omega v]$, to obtain*

$$\begin{pmatrix} \operatorname{Re}(V) \\ \operatorname{Im}(V) \end{pmatrix} = \begin{pmatrix} a & -b & 0 & b \\ b & 0 & b & -a \\ 0 & b & a & b \\ b & a & -b & 0 \end{pmatrix}, \qquad a = \frac{\sqrt{3+\sqrt{3}}}{\sqrt{6}}, \ b = \frac{\sqrt{3-\sqrt{3}}}{2\sqrt{3}}.$$

This is an orthonormal basis, by Proposition 2.1, or directly by using (3.17). Hence there is a norm-preserving (invertible) $\mathbb{R}$-linear map $\mathbb{C}^2 \to \mathbb{R}^4$ which maps the SIC to an orthonormal basis.

We now summarise some basic results about $[\cdot]_{\mathbb{F}}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and the associated linear maps, in a unified form. We first observe that in the literature, there is some variation in the definitions, in particular, the ordering of $[v]_{\mathbb{R}}$ can be either of

$$[v]_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re}(v) \\ \operatorname{Im}(v) \end{pmatrix}, \qquad \begin{pmatrix} \operatorname{Re}(v_1) \\ \operatorname{Im}(v_1) \\ \vdots \\ \operatorname{Re}(v_d) \\ \operatorname{Im}(v_d) \end{pmatrix},$$

and similarly for $[v]_{\mathbb{C}}$. In the latter case (cf [Hog77], [Rod14] for $[v]_{\mathbb{C}}$), the matrix representation $[A]_{\mathbb{R}}$ is then obtained by replacing the entry a_{jk} of the matrix A by the matrix

$$\begin{pmatrix} \operatorname{Re}(a_{jk}) & -\operatorname{Im}(a_{jk}) \\ \operatorname{Im}(a_{jk}) & \operatorname{Re}(a_{jk}) \end{pmatrix}.$$

Our choice of the former was governed by the simpler formulas (cf [Coh80]). Indeed, with $L = A + iB, A + Bj$ (respectively), we have the explicit formulas

$$[L]_{\mathbb{F}} = \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix}, \qquad [L^*]_{\mathbb{F}} = \begin{pmatrix} A^* & B^T \\ -B^* & A^T \end{pmatrix} = [L]_{\mathbb{F}}^*, \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}. \tag{3.32}$$

Theorem 3.3 *The $\mathbb{F}$-linear maps $[\cdot]_{\mathbb{F}}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ given by (3.15) and (3.25) have the following properties*

(a) *They map r-dimensional subspaces to $(2r)$-dimensional subspaces.*

(b) *They preserve the Euclidean norm of a vector.*

(c) *They map orthogonal vectors to orthogonal vectors.*

(d) *They map tight frames satisfying (3.23) and (3.31), respectively, to tight frames.*

(e) *They map equi-isoclinic r-subspaces to equi-isoclinic $(2r)$-subspaces.*

(f) *They map equichordal r-subspaces to equichordal $(2r)$-subspaces.*

Moreover, the associated $\mathbb{F}$-linear maps $L \mapsto [L]_{\mathbb{F}}$ to matrices over $\mathbb{F}$ satisfy

(i) $[AB]_{\mathbb{F}} = [A]_{\mathbb{F}}[B]_{\mathbb{F}}$, $[\lambda A]_{\mathbb{F}} = \lambda [A]_{\mathbb{F}}$, $\lambda \in \mathbb{R}$, *and* $[A^*]_{\mathbb{F}} = [A]_{\mathbb{F}}^*$.

(ii) *They map rank r linear maps to rank $2r$ linear maps.*

(iii) *They map invertible linear maps to invertible linear maps, with $[A^{-1}]_{\mathbb{F}} = [A]_{\mathbb{F}}^{-1}$.*

(iv) *They map self adjoint operators to self adjoint operators.*

(v) *They map unitary operators to unitary operators.*

(vi) *They map orthogonal projections to orthogonal projections, and in particular the identity to the identity.*

Proof: For the first part, (a) has already been observed, (b) and (c) follow directly from (3.17) and (3.26), (d) follows from Theorems 3.1 and 3.2, and (e) and (f) follow from the definitions (2.14) and (2.13), and the facts (i), (ii), (vi).

Now the second part. The first part of (i) follows from the definition, and the second part was a calculation that we did in each case. For (ii), we have $\ker([L]_{\mathbb{F}}) = [\ker(L)]_{\mathbb{F}}$, and the result follows from (a), with (iii) being a special case. If A is invertible, then (i) gives $I = [I]_{\mathbb{F}} = [AA^{-1}]_{\mathbb{F}} = [A]_{\mathbb{F}}[A^{-1}]_{\mathbb{F}}$, which gives the formula for the inverse. The properties (iv), (v) and (vi) are straightforward calculations using (3.32). □

Example 3.4 *From the observation*

$$j(A_1 + A_2 j) = (\overline{A_1} + \overline{A_2} j) j, \qquad A_1, A_2 \in \mathbb{C}^{m\times n},$$

it follows that the image of the $m \times n$ matrices over $\mathbb{H}$ is

$$[\mathbb{H}^{m\times n}]_{\mathbb{C}} = \{A \in \mathbb{C}^{2m\times 2n} : J_m A = \overline{A} J_n\}, \qquad J_\ell := [jI_\ell]_{\mathbb{C}} = \begin{pmatrix} 0 & -I_\ell \\ I_\ell & 0 \end{pmatrix}.$$

Example 3.5 *If G is a group of $d \times d$ matrices over $\mathbb{C}$ or $\mathbb{H}$, then it follows from Theorem 3.3 that $[G]_{\mathbb{F}} = \{[g]_{\mathbb{F}} : g \in G\}$ is an isomorphic group of $(2d) \times (2d)$ matrices. As an example, the quaternions $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ are generated by i and j, and so the groups of unitary matrices $[Q]_{\mathbb{C}}$ and $[[Q]_{\mathbb{C}}]_{\mathbb{R}}$ are generated by*

$$[i]_{\mathbb{C}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad [j]_{\mathbb{C}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$[[i]_{\mathbb{C}}]_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad [[j]_{\mathbb{C}}]_{\mathbb{R}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. These representations of Q are well known.

Example 3.6 *If $V = [v_1, \ldots, v_n] \in \mathbb{H}^{d\times n}$ gives a tight frame of n vectors for $\mathbb{H}^d$, i.e., $VV^* = AI$, then*

$$[V]_{\mathbb{C}} = \big[[v_1]_{\mathbb{C}}, \ldots, [v_n]_{\mathbb{C}}, [v_1 j]_{\mathbb{C}}, \ldots, [v_n j]_{\mathbb{C}}\big]$$

gives a tight frame of $2n$ vectors for $\mathbb{C}^{2d}$.

The equiangular lines in $\mathbb{H}^2$ of [ET20] were obtained by considering equi-isoclinic planes in $\mathbb{C}^4$. We now explain the mechanism.

Example 3.7 *Associated with a unit vector $v_a \in \mathbb{H}^d$, we have*

$$V_a := [[v_a]_{\mathbb{C}}, [v_a j]_{\mathbb{C}}] \in \mathbb{C}^{2d\times 2},$$

with orthonormal columns which span a plane in $\mathbb{C}^{2d}$. The entries of the "block Gramian" for $V = [V_1, \ldots, V_n]$ are $V_a^* V_b$ (with $V_a^* V_a = I$). These satisfy

$$(V_a^* V_b)^*(V_a^* V_b) = \begin{pmatrix} |\langle v_a, v_b\rangle_{\mathbb{H}}|^2 & 0 \\ 0 & |\langle v_a, v_b\rangle_{\mathbb{H}}|^2 \end{pmatrix}, \tag{3.33}$$

so that

$$|\langle v_a, v_b\rangle|^2 = \lambda \quad\Longleftrightarrow\quad (V_a^* V_b)^*(V_a^* V_b) = \lambda I.$$

Thus (v_a) gives a set of equiangular lines in $\mathbb{H}^d$ if and only if the the off diagonal entries of the block Gramian $[V_1, \ldots, V_n]^*[V_1, \ldots, V_n]$ are unitary matrices, up to a fixed scalar. An $n \times n$ block matrix with this structural form (2×2 blocks, positive semi-definite of rank $2d$), which corresponds to equi-isoclinic planes in $\mathbb{C}^{2d}$, can then be mapped back (under $[\cdot]_{\mathbb{C}}^{-1}$) to the Gramian of n equiangular lines in $\mathbb{H}^d$, see Theorem 13, [ET20].

The equation (3.33) follows by a direct calculation, e.g., using (3.26), we have

$$\begin{aligned}(V_b^*V_aV_a^*V_b)_{11} &= [v_b]_{\mathbb{C}}^*[v_a]_{\mathbb{C}}[v_a]_{\mathbb{C}}^*[v_b]_{\mathbb{C}} + [v_b]_{\mathbb{C}}^*[v_aj]_{\mathbb{C}}[v_aj]_{\mathbb{C}}^*[v_b]_{\mathbb{C}} \\ &= \langle [v_a]_{\mathbb{C}}, [v_b]_{\mathbb{C}}\rangle_{\mathbb{C}}\langle [v_b]_{\mathbb{C}}, [v_a]_{\mathbb{C}}\rangle_{\mathbb{C}} + \langle [v_aj]_{\mathbb{C}}, [v_b]_{\mathbb{C}}\rangle_{\mathbb{C}}\langle [v_b]_{\mathbb{C}}, [v_aj]_{\mathbb{C}}\rangle_{\mathbb{C}} \\ &= |\operatorname{Co}_1(\langle v_a, v_b\rangle_{\mathbb{H}})|^2 + |\operatorname{Co}_2(\langle v_a, v_b\rangle_{\mathbb{H}})|^2 = |\langle v_a, v_b\rangle_{\mathbb{H}}|^2,\end{aligned}$$

$$\begin{aligned}(V_b^*V_aV_a^*V_b)_{12} &= [v_b]_{\mathbb{C}}^*[v_a]_{\mathbb{C}}[v_a]_{\mathbb{C}}^*[v_bj]_{\mathbb{C}} + [v_b]_{\mathbb{C}}^*[v_aj]_{\mathbb{C}}[v_aj]_{\mathbb{C}}^*[v_bj]_{\mathbb{C}} \\ &= \langle [v_a]_{\mathbb{C}}, [v_b]_{\mathbb{C}}\rangle_{\mathbb{C}}\langle [v_bj]_{\mathbb{C}}, [v_a]_{\mathbb{C}}\rangle_{\mathbb{C}} + \langle [v_aj]_{\mathbb{C}}, [v_b]_{\mathbb{C}}\rangle_{\mathbb{C}}\langle [v_bj]_{\mathbb{C}}, [v_aj]_{\mathbb{C}}\rangle_{\mathbb{C}} \\ &= \operatorname{Co}_1(\langle v_a, v_b\rangle_{\mathbb{H}})(-\operatorname{Co}_2(\langle v_b, v_a\rangle_{\mathbb{H}})) - \operatorname{Co}_2(\langle v_a, v_b\rangle_{\mathbb{H}})\operatorname{Co}_1(\langle v_bj, v_aj\rangle_{\mathbb{H}}) \\ &= \operatorname{Co}_1(\langle v_a, v_b\rangle_{\mathbb{H}})\operatorname{Co}_2(\langle v_a, v_b\rangle_{\mathbb{H}}) - \operatorname{Co}_2(\langle v_a, v_b\rangle_{\mathbb{H}})\operatorname{Co}_1(\langle v_a, v_b\rangle_{\mathbb{H}}) = 0,\end{aligned}$$

where in the second to last equality we used $\operatorname{Co}_2(\overline{q}) = -\operatorname{Co}_2(q)$, $q \in \mathbb{H}$.

Here is a construction of equiangular lines going in the opposite direction.

Example 3.8 *We consider the construction of* 64 *equiangular lines in* $\mathbb{C}^8$ *by [Hog98]. These were obtained by finding* 64 *unit vectors in* $\mathbb{H}^4$ *with angles* $\frac{1}{9}, \frac{1}{3}$ *(as vertices of a quaternionic polytope). These were then mapped by* $[\cdot]_{\mathbb{C}}$ *to* 64 *equiangular vectors in* $\mathbb{C}^8$*. We note that for* $v, w \in \mathbb{H}^d$*,* $\alpha \in \mathbb{H}$*, (3.26) gives*

$$\begin{aligned}\langle [v\alpha]_{\mathbb{C}}, [w]_{\mathbb{C}}\rangle_{\mathbb{C}} &= \operatorname{Co}_1(\langle v\alpha, w\rangle_{\mathbb{H}}) = \operatorname{Co}_1(\langle v, w\rangle_{\mathbb{H}}\alpha) \\ &= \operatorname{Co}_1(\alpha)\operatorname{Co}_1(\langle v, w\rangle_{\mathbb{H}}) - \operatorname{Co}_2(\alpha)\overline{\operatorname{Co}_2(\langle v, w\rangle_{\mathbb{H}})},\end{aligned}$$

so that multiplying vectors in $\mathbb{H}^d$ *by noncomplex unit scalars in* $\mathbb{H}$ *can change the angle between their images in* $\mathbb{C}^{2d}$*.*

4 Group frames and G-matrices

Many tight frames of interest are the orbit one or more vectors under the unitary action of a group, e.g., the Weyl-Heisenberg SICs. There is a well developed theory of such group frames based in the theory of group representations (over $\mathbb{R}$ and $\mathbb{C}$) [VW05], [VW16], [Wal18]. We now give an indication of how this theory extends to representations over $\mathbb{H}$ (see [SS95]).

A **representation** of a finite abstract group G on $\mathbb{H}^d$ is group homomorphism $\rho : G \to GL(\mathbb{H}^d)$ from G to the invertible $d \times d$ matrices over $\mathbb{H}$, with equivalence defined in the usual way. We will consider only **unitary representations**, i.e.. those where the matrices $\rho(g)$ are unitary. For these, we will write the unitary action as $gv := \rho(g)v$, and we note that $g^*v = g^{-1}v$. A frame (sequence of vectors) of the form $(gv)_{g\in G}$ is said to be a **group frame** (or G**-frame**). The frame operator of a group frame $(gv)_{g\in G}$ commutes with the frame operator, i.e.,

$$S(hv) = \sum_{g\in G} gv\langle hv, gv\rangle = h\sum_{g\in G} h^{-1}gv\langle v, h^{-1}gv\rangle = hS(v), \quad g, h \in G,\ v \in \mathbb{H}^d. \quad (4.34)$$

The Gramian of a group matrix has entries of the form

$$\langle hv, gv\rangle = \langle g^{-1}hv, v\rangle.$$

A matrix $A = [a_{gh}]_{g,h\in G} \in \mathbb{H}^{G\times G}$ is a G-**matrix** (or **group matrix**) if there exists a function $\nu : G \to \mathbb{H}$ such that

$$a_{gh} = \nu(g^{-1}h), \qquad \forall g, h \in G.$$

The Gramian of a G-frame is a G-matrix, and conversely if the Gramian of a frame $(v_g)_{g\in G}$ with vectors indexed by G is a G-matrix, then it is a G-frame (adapt the proof of [Wal18] Theorem 10.3). An action (representation) of G on $\mathbb{H}^d$ is **irreducible** if the only G-invariant subspaces of $\mathbb{H}^d$ are 0 and $\mathbb{H}^d$, i.e., $\mathrm{span}_{\mathbb{H}}\{gv\}_{g\in G} = \mathbb{H}^d$, for all $v \neq 0$.

The theory of G-frames for real and complex actions begins with irreducible actions, where it takes its simplest form. This extends without issue.

Proposition 4.1 *Suppose that a unitary action of a group G on $\mathbb{H}^d$ is irreducible. Then $(gv)_{g\in G}$ is a tight G-frame for $\mathbb{H}^d$ for any $v \neq 0$, i.e.,*

$$x = \frac{d}{|G|}\frac{1}{\|v\|^2}\sum_{g\in G} gv\langle x, gv\rangle, \qquad \forall x \in \mathbb{H}^d.$$

Proof: Fix $v \neq 0$, and let S be the frame operator of $(gv)_{g\in G}$. Since S is nonzero and positive semidefinite, it has an eigenvalue $\lambda > 0$, with corresponding eigenvector w. By (4.34), S commutes with the action of $g \in G$, we have

$$S(gw) = g(Sw) = g(w\lambda) = (gw)\lambda,$$

so that gw is an eigenvector for λ. But $(gw)_{g\in G}$ spans $\mathbb{H}^d$, so that $S = \lambda I$, i.e., $(gv)_{g\in G}$ is a tight frame. Since S is Hermitian, taking the trace gives

$$\mathrm{trace}(S) = \mathrm{Re}(\mathrm{trace}(S)) = \sum_g \|gv\|^2 = |G|\,\|v\|^2 = \mathrm{trace}(\lambda I) = d\lambda,$$

which gives the value of λ. □

The general theory [VW16], [Wal18], which allows for multiple orbits, involves the decomposition of the vector space into irreducible G-invariant subspaces.

Example 4.1 *Each finite subgroup of $\mathbb{H}^*$ corresponds to a (faithful) irreducible action on $\mathbb{H}^1$. These subgroups were classified by Stringham [Str81]. They are the infinite families of cyclic groups (generated by the n-th roots of unity) and binary dihedral groups, together with the binary tetrahedral, octahedral and icosahedral groups.*

Example 4.2 *The group generated by the matrices*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix},$$

has an irreducible unitary action on $\mathbb{H}^2$. It consists of all 128 invertible matrices with two zero entries and two entries in Q. It contains the scalar matrices from Q and its center is $\pm I$. Thus each orbit can be viewed as 16 lines in $\mathbb{H}^2$ (as a left vector space). This is an example of (quaternionic) reflection group, i.e., a finite group generated by reflections (linear maps which act as the identity on hyperplane). The finite irreducible quaternionic reflection groups have been classified (up to conjugacy) by Cohen [Coh80].

It is expected that the highly symmetric tight frames of [BW13] corresponding to complex reflection groups could be extended to the quaternionic reflection groups. In this regard we note the regular quaternionic polytopes have been classified by [Cuy95].

For G abelian, there are a finite number of tight G-frames (called harmonic frames) that can be obtained by "taking rows of the character table" (see [VW05], [CW11]). We now give an example to show how this can be extended to the quaternionic setting.

Example 4.3 *(Quaternionic harmonic frames). The irreducible representations over $\mathbb{C}$ for an abelian group G are all one-dimensional (this characterises abelian groups), and these "rows" of the character table are orthogonal, so by taking a set of rows of the character table one obtains a tight G-frame. Consider the quaternion group $G = Q$. This has four 1-dimensional and one 2-dimensional irreducible representations over $\mathbb{C}$. The 2-dimensional absolutely irreducible representation splits into four 1-dimensional representations over Q, corresponding to the outer automorphisms of the quaternions. In this way, one obtains a character table*

$q \in Q$	1	-1	i	$-i$	j	$-j$	k	$-k$
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	-1	-1	1	1	-1	-1
χ_4	1	1	-1	-1	-1	-1	1	1
χ_5	1	-1	i	$-i$	j	$-j$	k	$-k$
χ_6	1	-1	j	$-j$	i	$-i$	$-k$	k
χ_7	1	-1	$-i$	i	k	$-k$	j	$-j$
χ_8	1	-1	k	$-k$	$-i$	i	$-j$	j

where the rows are orthogonal (cf [SS95]). Taking rows gives a G-frame. The columns of the character table are also orthogonal, so taking columns also gives a tight frame, but these are not G-frames, in general (as follows for abelian groups by Pontryagin duality). As an example, the frame obtained by taking the characters χ_1 and χ_5 (rows 1 and 5 of the character table) gives a (unit-norm) tight Q-frame for $\mathbb{H}^2$, with the inner products $\{1 \pm 1, 1 \pm i, 1 \pm j, 1 \pm k\}$ occurring exactly once in every row (column) of the Gramian. This frame has two angles: each vector is orthogonal to one other, and makes a fixed angle with all the others. By comparison, taking columns 1 and 3 gives a unit-norm tight frame for $\mathbb{H}^2$, which is not a G-frame for any G, since its Gramian is not a G-matrix.

5 Projective unitary equivalence

Finally, we consider the equivalence of vectors thought of as lines in $\mathbb{H}^d$. Here the noncommutativity of scalar multiplication considerably complicates the theory.

We say that two sequences of vectors (v_j) and (w_j) in $\mathbb{H}^d$ are **projectively unitarily equivalent** if there a unitary U and unit norm scalars α_j with

$$w_j = (Uv_j)\alpha_j, \qquad \forall j.$$

Clearly, projective unitary equivalence is an equivalence relation. Moreover, one can define a **projective unitary symmetry group** of $(v_j)_{j\in J}$ to be all the permutations $\sigma : J \to J$ for which (v_j) and $(v_{\sigma j})$ are projectively unitarily equivalent (cf [CW18]).

To make a workable theory, one now needs a way to recognise projective unitary equivalence. In terms of the Gramians $V = [v_j]$ and $W = [w_j]$, the formal definition says that

$$W^*W = C^*V^*U^*UVC = C^*V^*VC, \qquad C := \operatorname{diag}(\alpha_j),$$

i.e.,

$$\langle w_j, w_k\rangle = \overline{\alpha_k}\langle v_j, v_k\rangle\alpha_j. \tag{5.35}$$

This leads to a "linear system" $C(W^*W) = (V^*V)C$ in the scalars α_j. However, due to the noncommutativity of the quaternions, this can not be solved by Gauss elimination, unless one first converts it to a linear system over $\mathbb{R}$ (in the coordinates of the α_j). What is usually done in the real and complex cases is to consider a collection of invariants: the m-products, which completely characterise projective unitary equivalence [CW16]. We now look at the analogue of these (also see [KMW19] for subspaces of $\mathbb{C}^d$).

For a sequence of vectors (v_j) in $\mathbb{H}^d$ the **m-products** are

$$\Delta(v_{j_1}, v_{j_2}, \ldots, v_{j_m}) := \langle v_{j_2}, v_{j_1}\rangle\langle v_{j_3}, v_{j_2}\rangle\langle v_{j_4}, v_{j_3}\rangle\cdots\langle v_{j_1}, v_{j_m}\rangle \in \mathbb{H}.$$

The 1-products and 2-products are clearly projective unitary invariants, since

$$\Delta(v_j) = \|v_j\|^2, \qquad \Delta(v_j, v_k) = |\langle v_j, v_k\rangle|^2.$$

From these, we can define the **frame graph** of (v_j) to be the graph with vertices $\{v_j\}$ and an edge between v_j and v_k $(j \neq k)$ if and only if $\langle v_j, v_k\rangle \neq 0$.

Further, since

$$\begin{aligned}
&\Delta\big((Uv_{j_1})\alpha_{j_1}, (Uv_{j_2})\alpha_{j2}, \ldots, (Uv_{j_m})\alpha_{j_m}\big)\\
&\quad= \langle (Uv_{j_2})\alpha_{j_2}, (Uv_{j_1})\alpha_{j_1}\rangle\langle (Uv_{j_3})\alpha_{j_3}, (Uv_{j_2})\alpha_{j_2}\rangle\cdots\langle (Uv_{j_1})\alpha_{j_1}, (Uv_{j_m})\alpha_{j_m}\rangle\\
&\quad= \overline{\alpha_{j_1}}\langle v_{j_2}, v_{j_1}\rangle\alpha_{j_2}\overline{\alpha_{j_2}}\langle v_{j_3}, v_{j_2}\rangle\alpha_{j_3}\cdots\overline{\alpha_{j_m}}\langle v_{j_1}, v_{j_m}\rangle\alpha_{j_1}\\
&\quad= \overline{\alpha_{j_1}}\Delta(v_{j_1}, v_{j_2}, \ldots, v_{j_m})\alpha_{j_1},
\end{aligned}$$

the m-products are projective unitary invariants of (v_j) up to congruence, and real frames are characterised by having real m-products. A quaternion q is determined up to congruence by its real part $\operatorname{Re}(q)$ and its norm $|q|$, and so we can define (reduced) m-products as a pair of real numbers

$$\Delta_r(v_{j_1}, v_{j_2}, \ldots, v_{j_m}) := (\operatorname{Re}(q), |q|), \qquad q = \Delta(v_{j_1}, v_{j_2}, \ldots, v_{j_m}).$$

These are projective unitary invariants. For the complex case, the m-products are projective unitary invariants, which depend only the cycle $(j_1, \ldots, j_m)$, and small set of m-products corresponding to a basis for the cycle space of the frame graph of (v_j) provide a set of invariants which characterise projective unitary equivalence (see [CW16]). We can not yet make a similar claim in the quaternionic case, though we do imagine that the m-products do characterise projective unitary equivalence.

The dependence of m-products only the the associated m-cycle in the frame graph does follow, by the calculation

$$a\Delta(v_{j_1},v_{j_2},\ldots,v_{j_m})a^{-1}=\Delta(v_{j_2},v_{j_3},\ldots,v_{j_m},v_{j_1}),\qquad a=\frac{\langle v_{j_1},v_{j_2}\rangle}{|\langle v_{j_1},v_{j_2}\rangle|},$$

and so, in addition to the 1-products and 2-products, we need only consider the m-products for $m\ge 3$ which correspond to m-cycles in the frame graph, i.e., are nonzero. To check that the m-products for two sequences are equal (up to conjugation), it suffices to consider only the m-products corresponding to a cycle basis for the cycle space of the (common) frame graph:

Lemma 5.1 *(Cycle decomposition) For $1\le k\le m$, $n\ge 1$, we have*

$$\begin{aligned}&\Delta(v_k,v_{k+1},\ldots,v_m,v_1,\ldots,v_{k-1})\Delta(v_k,\ldots,v_1,w_1,\ldots,w_n)\\&\quad=|\langle v_1,v_2\rangle|^2|\langle v_2,v_3\rangle|^2\cdots|\langle v_{k-1}v_k\rangle|^2\Delta(v_k,v_{k+1},\ldots,v_m,v_1,w_1,w_2,\ldots,w_n).\end{aligned}$$

Proof: Expanding the left hand side gives

$$\begin{aligned}&\langle v_{k+1},v_k\rangle\langle v_{k+2},v_{k+1}\rangle\cdots\langle v_m,v_{m-1}\rangle\langle v_1,v_m\rangle\langle v_2,v_1\rangle\cdots\langle v_{k-1},v_{k-2}\rangle\langle v_k,v_{k-1}\rangle\\&\quad\times\langle v_{k-1},v_k\rangle\langle v_{k-2},v_{k-1}\rangle\cdots\langle v_1,v_2\rangle\langle w_1,v_1\rangle\langle w_2,w_1\rangle\cdots\langle w_n,w_{n-1}\rangle\langle v_k,w_n\rangle,\end{aligned}$$

which simplifies to the right hand side, since $\langle v_{j+1},v_j\rangle\langle v_j,v_{j+1}\rangle=|\langle v_j,v_{j+1}\rangle|^2\in\mathbb{R}$ commutes with any quaternion. □

This gives the following condition for projective unitary equivalence.

Theorem 5.1 *A necessary condition for sequences (v_j) and (w_j) of n vectors in $\mathbb{H}^d$ to be projectively unitarily equivalent is that the m-products corresponding to a cycle basis for the frame graph are are equal (up to conjugation).*

In the complex setting, this says that the m-products are equal, and the converse is proved by explicitly constructing scalars α_j which satisfy (5.35). The difficulties in extending this converse to the quaternionic setting include the fact that for $w_j=(Uv_j)\alpha_j$,

$$\Delta(w_{j_1},w_{j_2},\ldots,w_{j_m})=\overline{\alpha_{j_1}}\Delta(v_{j_1},v_{j_2},\ldots,v_{j_m})\alpha_{j_1},\tag{5.36}$$

which puts further constraints on the α_j (for $m\ge 3$ and the m-product nonzero). Indeed, in the complex setting one can assume that any α_j is 1, simply by replacing U by the unitary matrix $\alpha_j U$. Nevertheless, those parts of the theory that we do have allow us to investigate such things as the symmetries of lines, as our final example shows.

Example 5.1 *Consider the six tight equiangular lines in $\mathbb{H}^2$ at angle $\lambda=\frac{2}{5}$ of [ET20]*

$$v_1=\begin{pmatrix}1\\0\end{pmatrix},\ v_2=\begin{pmatrix}\frac{\sqrt2}{\sqrt5}\\\frac{\sqrt3}{\sqrt5}\end{pmatrix},\ v_3=\begin{pmatrix}\frac{\sqrt2}{\sqrt5}\\-\frac{\sqrt3}{4\sqrt5}+\frac34 i\end{pmatrix},\ v_4=\begin{pmatrix}\frac{\sqrt2}{\sqrt5}\\-\frac{\sqrt3}{4\sqrt5}-\frac14 i+\frac{1}{\sqrt2}j\end{pmatrix},$$

$$v_5 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} - \frac{1}{4}i + \frac{1}{2\sqrt{2}}j + \frac{\sqrt{3}}{2\sqrt{2}}k \end{pmatrix}, \; v_6 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} - \frac{1}{4}i + \frac{1}{2\sqrt{2}}j + \frac{\sqrt{3}}{2\sqrt{2}}k \end{pmatrix},$$

which are said to have "symmetry group" A_6. The reduced m-products $\Delta_r(v_{j_1}, \ldots, v_{j_m})$ of distinct vectors for $m = 1, 2, 3, 4, 6$ are all equal, taking the values

$$(1,1), \quad (\frac{2}{5}, \lambda^2), \quad (\frac{1}{10}, \lambda^3), \quad (-\frac{1}{50}, \lambda^4), \quad (-\frac{11}{250}, \lambda^6)$$

respectively, which puts no restriction on the possible projective symmetry group of the lines. However, the reduced 5-products (of distinct vectors) take two values

$$(-\frac{25 \pm 9\sqrt{5}}{500}, \lambda^5),$$

and the permutations of the vectors which preserve these 5-products is indeed A_6. With the present theory, this does not yet establish that A_6 is the projective symmetry group.

We now seek a corresponding projective unitary symmetry for each $\sigma \in A_6$, i.e., a unitary matrix U_σ and corresponding scalars α_j (also depending on σ) for which

$$w_j = v_{\sigma j} = (U_\sigma v_j)\alpha_j, \qquad \forall j.$$

Once the unit scalars α_j corresponding to a basis $[v_j]_{j \in J}$ of vectors from (v_j) are known, the matrix U_α is uniquely determined by

$$U_\sigma [v_j \alpha_j]_{j \in J} = [v_{\sigma j}]_{j \in J} \quad \Longrightarrow \quad U_\sigma = [v_{\sigma j}]_{j \in J} [v_j \alpha_j]_{j \in J}^{-1},$$

and it can then be checked whether or not the U_σ is unitary and permutes the other lines. By (5.36), for j, k, ℓ distinct, the unit scalar α_j satisfies

$$\alpha_j \Delta(v_{\sigma j}, v_{\sigma k}, v_{\sigma \ell}) = \Delta(v_j, v_k, v_\ell)\alpha_j,$$

which gives a homogeneous linear system of four equations for the four real coordinates of α_j. In the cases considered, this had a unique solution of unit norm up to a choice of sign, which was made in order to obtain a unitary matrix U_σ. For the generators

$$a = (12)(34) \quad \textit{(order 2)}, \qquad b = (1235)(46) \quad \textit{(order 4)}$$

for A_6, we obtained

$$\alpha_1 = \alpha_2 = -\frac{\sqrt{2}}{\sqrt{3}}i + \frac{1}{\sqrt{3}}k, \qquad U_a = \begin{pmatrix} \frac{2}{\sqrt{15}}i - \frac{\sqrt{2}}{\sqrt{15}}j & \frac{\sqrt{2}}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j \\ \frac{\sqrt{2}}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j & -\frac{2}{\sqrt{15}}i + \frac{\sqrt{2}}{\sqrt{15}}j \end{pmatrix}, \quad U_a^2 = -I,$$

and

$$\alpha_1 = \frac{1}{2\sqrt{2}} + \frac{\sqrt{5}}{2\sqrt{6}}i - \frac{3-\sqrt{5}}{4\sqrt{3}}j - \frac{\sqrt{5}+1}{4}k, \quad \alpha_2 = \frac{\sqrt{5}}{4} + \frac{1}{4\sqrt{3}}i - \frac{3\sqrt{5}+1}{4\sqrt{6}}j - \frac{\sqrt{5}-1}{4\sqrt{2}},$$

$$U_b = \begin{pmatrix} \frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{3}}i + \frac{3-\sqrt{5}}{2\sqrt{30}}j + \frac{\sqrt{5}+1}{2\sqrt{10}}k & \frac{\sqrt{3}}{2\sqrt{10}} - \frac{1}{2\sqrt{2}}i + \frac{3+\sqrt{5}}{4\sqrt{5}}j - \frac{\sqrt{3}}{5+\sqrt{5}}k \\ \frac{\sqrt{3}}{2\sqrt{10}} + \frac{1}{2\sqrt{2}}i + \frac{3-\sqrt{5}}{4\sqrt{5}}j + \frac{\sqrt{3}}{5-\sqrt{5}}k & -\frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{3}}i - \frac{3\sqrt{5}+5}{10\sqrt{6}}j + \frac{\sqrt{5}-1}{2\sqrt{10}}k \end{pmatrix}, \quad U_b^4 = -I.$$

These unitary matrices U_a and U_b do give the projective unitary symmetries supposed. Moreover, they generate the double cover $2 \cdot A_6$ of A_6, and so we have verified that A_6 is indeed the projective symmetry group of the six equiangular lines in $\mathbb{H}^2$. We note that our method did not require prior knowledge of what the symmetry group was.

The action group of the faithful representation of $2 \cdot A_6$ obtained in Example 5.1 contains 40 reflections (of order 3), and it is an irreducible reflection group which appears on the list of [Coh80]. Moreover, the vectors giving the lines are eigenvectors of nontrivial elements of the group, and so the six equiangular lines in $\mathbb{H}^2$ can be constructed directly from the reflection group as a group frame (or even from the abstract group $2 \cdot A_6$).

The sets of five and six equiangular lines in $\mathbb{H}^2$ were first calculated in [KF08] using the Hopf map. Though this technique does not immediately generalise to other dimensions, like that of [ET20], we recount the essential details, as it sheds further light on the geometry of these lines. The **Hopf map** ψ maps a point $\vec{a} = (a_1, \ldots, a_5)$ on the unit sphere in $\mathbb{R}^5$ to a line in the projective space $\mathbb{HP}^1$, i.e., a the unit vector $v \in \mathbb{H}^2$ in the line with $v_2 \geq 0$, and is given by $\psi(0,0,0,0,1) := (1,0)$ and

$$\psi(\vec{a}) := \begin{pmatrix} \frac{a}{\sqrt{2(1-a_5)}} \\ \frac{\sqrt{1-a_5}}{\sqrt{2}} \end{pmatrix}, \qquad a := a_1 + a_2 i + a_3 j + a_4 k, \quad a_5 \neq 1.$$

A calculation shows that

$$|\langle \psi(\vec{a}), \psi(\vec{b}) \rangle_{\mathbb{H}}|^2 = \frac{1 + \langle \vec{a}, \vec{b} \rangle_{\mathbb{R}}}{2}, \qquad \forall \vec{a}, \vec{b}, \tag{5.37}$$

so the $n \geq 3$ unit vectors (v_j), $v_j = \psi(\vec{v_j}) \in \mathbb{H}^2$, give tight equiangular lines if and only if

$$|\langle v_j, v_k \rangle|^2 = \frac{1 + \langle \vec{v_j}, \vec{v_k} \rangle}{2} = \frac{n-2}{2(n-1)} \quad \iff \quad \langle \vec{v_j}, \vec{v_k} \rangle = -\frac{1}{n-1}.$$

This latter condition says that the vectors $(\vec{v_j})$ are the vertices of a regular n-vertex simplex embedded in the unit sphere in $\mathbb{R}^5$, which can be done for $n = 3, 4, 5, 6$, with the corresponding image (v_j) giving n tight equiangular lines in $\mathbb{H}^2$. Moreover, for $n = 3$ we get real lines by choosing the simplex in $\{x : x = (x_1, 0, 0, 0, x_5)\}$, and complex lines for $n = 4$ by choosing the simplex in $\{x : x = (x_1, x_2, 0, 0, x_5)\}$.

5.1 Concluding remarks

We have shown how much of the theory of tight frames extends to quaternionic Hilbert space, with the characterisation of projective unitary equivalence of frames being the aspect that most depends intrinsically on the commutativity of the complex numbers. The notions of canonical coordinates and the canonical Gramian [Wal18] also extend to $\mathbb{H}$-vector spaces. In particular, there is a unique $\mathbb{H}$-inner product for which a finite spanning set for an $\mathbb{H}$-vector space becomes a normalised tight frame.

Our focus has been on group frames and equiangular lines. The maximal set of six equiangular lines in $\mathbb{H}^2$ comes as the orbit of a quaternionic reflection group, just as the SIC of four equiangular lines in $\mathbb{C}^2$ is the orbit of a complex reflection group. However, the known SICs in $\mathbb{C}^d$ (with one exception) are orbits of the Weyl-Heisenberg group, which is not a reflection group for $d \geq 3$. The key to constructing quaternionic equiangular lines will be knowing "the right group". This group might come from numerical constructions, using the techniques of this last section, or from the theory group representations over $\mathbb{H}$ (which is in its infancy). The construction of sets of tight quaternionic lines may also offer insight into Zauner's conjecture. Another direction of similar interest, would be that of optimal packings in quaternionic projective space $\mathbb{HP}^k$.

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