A variational characterisation of projective spherical designs over the quaternions

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November 17, 2020

Abstract

We give an inequality on the packing of vectors/lines in quaternionic Hilbert space $\mathbb{H}^d$, which generalises those of Sidelnikov and Welch for unit vectors in $\mathbb{R}^d$ and $\mathbb{C}^d$. This has a parameter t, and depends only on the vectors up to projective unitary equivalence. The sequences of vectors in $\mathbb{F}^d=\mathbb{R}^d,\mathbb{C}^d,\mathbb{H}^d$ that give equality, which we call spherical (t,t)-designs, are seen to satisfy a cubature rule on the unit sphere in $\mathbb{F}^d$ for a suitable polynomial space $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$. Using this, we show that the projective spherical t-designs on the Delsarte spaces $\mathbb{F}P^{d-1}$ coincide with the spherical (t,t)-designs of unit vectors in $\mathbb{F}^d$. We then explore a number of examples in quaternionic space. The unitarily invariant polynomial space $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$ and the inner product that we define on it so the reproducing kernel has a simple form are of independent interest.

Key Words: Sidelnikov inequality, Welch bound/inequality, Delsarte space, projective spherical t-designs, spherical (t,t)-designs, harmonic polynomials, reproducing kernels, apolar inner product, Bombieri inner product, finite tight frames, quaternionic equiangular lines, projective unitary equivalence over the quaternions.

AMS (MOS) Subject Classifications: primary 05B25, 05B30, 15B33, secondary 42C15, 51M20, 65D30,

1 Introduction

Two unit vectors v_1 and v_2 on the unit sphere in $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$ (or the lines they represent) are spaced far apart if $|\langle v_1, v_2\rangle|^2$ is *small*. The maximal possible separation $\langle v_1, v_2\rangle = 0$ occurs for orthogonal vectors (lines). For a sequence (v_j) of vectors in $\mathbb{F}^d$, and an integer $t = 1, 2, \ldots$, the following inequality holds

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j, v_k\rangle|^{2t} \geq c_t(\mathbb{F}^d)\Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t}\Big)^2, \tag{1.1}$$

where

$$c_t(\mathbb{R}^d) = \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+1)\cdots(d+2(t-1))}, \qquad c_t(\mathbb{C}^d) = \frac{1}{\binom{d+t-1}{t}}. \tag{1.2}$$

Sequences of vectors which give equality above can be thought of as being an optimal packing (of well separated lines), e.g., an orthonormal basis gives equality for $t = 1$. For unit vectors in $\mathbb{R}^d$ this inequality is due to Sidelnikov [Sid74], and for unit vectors in $\mathbb{C}^d$ it is due to Welch [Wel74]. It can be shown that vectors giving equality in (1.1) give a type of cubature rule [Wal17], which we call a spherical (t, t)-design. From this, it follows that for any t and d there is equality in (1.1) for some sufficiently large n. There is considerable interest in finding the smallest possible n (for a given t and d) [HW21], and even the case of t not integral has been considered [CGG$^+$20], [GP19]. Sequences giving this equality have long been used in information theory [MM93], [DF07].

The primary aim of this paper is to establish the analogue of (1.1) for quaternionic Hilbert space $\mathbb{H}^d$, and a corresponding theory of spherical (t, t)-designs (cubature rules). One consequence of this, is that we show the projective spherical t-designs on the Delsarte spaces $\mathbb{F}P^{d-1}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ studied by Hoggar [Hog82], [Hog84] are precisely the spherical (t, t)-designs of unit vectors in $\mathbb{F}^d$. This gives a simple characterisation of projective spherical t-designs which was previously unknown, and formally makes sense for the octonionic space $\mathbb{O}^d$ also.

Our original intent was to extend the unified proof of (1.1) given in [Wal18] to quaternionic Hilbert space $\mathbb{H}^d$, to prove an inequality which was observed numerically (including the constant). Since the quaternions $\mathbb{H}$ are not commutative, this is nontrivial. The special case $t = 1$ corresponds to tight frames, and was treated in [Wal20], where much of the needed theory of quaternionic Hilbert space was developed. Tensor products are central to the argument of [Wal20]. There is the "commutativity relation"

$$\mathrm{Re}(ab) = \mathrm{Re}(ba), \qquad \forall a, b \in \mathbb{H}. \tag{1.3}$$

which we hoped to apply (as in the case $t = 1$) together with a theory of tensor products in quaternionic Hilbert space (see [MW20]) to establish such an inequality. Ultimately, this was unsuccessful, with our "faux proof" failing to clearly identify the polynomials for the cubature rule. Instead, we have adapted an argument of [KP17] (for the complex case), which is both elegant (it uses Cauchy-Schwarz) and insightful (equality naturally identifies the constant $c_t(\mathbb{H}^d)$ and the space $\mathrm{Hom}_{\mathbb{H}^d}(t, t)$ of polynomials for the cubature rule). We first present this argument for all three cases (Theorem 4.1), and then show the existence of the inner product that it is predicated on (Theorems 8.1 and 8.2).

We now give a brief summary of quaternionic Hilbert space and polynomials on it, referring back to [Wal20] as appropriate (also see [GMP13]).

2 Quaternionic Hilbert space

We assume basic familiarity with the **quaternions** $\mathbb{H}$ which are an extension of the complex numbers $x+iy$ to a noncommutative associative algebra over the real numbers (skew field) consisting of elements:

$$q = q_1 + q_2 i + q_3 j + q_4 k = (q_1 + q_2 i) + (q_3 + q_4 i)j \in \mathbb{H}, \qquad q_j \in \mathbb{R},$$

with the (noncommutative) multiplication given by

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j.$$

The **conjugate** and **norm** of a quaternion $q = q_1 + q_2 i + q_3 j + q_4 k \in \mathbb{H}$ are given by

$$\overline{q} := q_1 - q_2 i - q_3 j - q_3 k, \qquad |q|^2 = q\overline{q} = \overline{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

Since the multiplication is not commutative, we must distinguish between left and right vector spaces (modules) over $\mathbb{H}$. In [Wal20], we considered $\mathbb{H}^d$ to be a right $\mathbb{H}$-vector space (module), so that the usual rules of matrix multiplication extend.

We define a generalisation $\langle \cdot, \cdot \rangle : \mathbb{H}^d \times \mathbb{H}^d \to \mathbb{H}$ of the Euclidean inner product by

$$\langle v, w \rangle := \sum_j \overline{v_j} w_j. \tag{2.4}$$

Above we have used j as an index, rather than as a quaternion unit, which is common practice, where no confusion can arise. This **inner product** on a right $\mathbb{H}$-vector space satisfies the defining conditions:

1. Conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
2. Linearity in the second variable: $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
   $\langle v, w\alpha \rangle = \langle v, w \rangle \alpha$,
   which gives $\langle v\alpha, w \rangle = \overline{\alpha} \langle v, w \rangle$.
3. Positive definiteness: $\langle v, v \rangle > 0$, $v \neq 0$.

Here linearity is in the second variable (a change of convention by the author, to benefit from more natural formulas). Moreover, the **Euclidean** inner product (2.4) satisfies

$$\langle \alpha v, w \rangle = \langle v, \overline{\alpha} w \rangle, \qquad \forall \alpha \in \mathbb{H}.$$

Much of the theory of the Euclidean inner product extends, including the notions of Hermitian and unitary matrices, Cauchy-Schwarz, and Gram-Schmidt orthogonalisation.

We now consider multivariate quaternionic polynomials $\mathbb{H}^d \to \mathbb{H}$. The **quaternionic monomials** of degree r are the polynomials of the form

$$q = (q_1, \ldots, q_d) \mapsto \alpha_0 q_{j_1} \alpha_1 q_{j_2} \alpha_2 \cdots q_{j_{r-1}} \alpha_{r-1} q_{j_r} \alpha_r, \qquad \alpha_j \in \mathbb{H},\ j_1, \ldots, j_r \in \{1, \ldots, d\}.$$

Their $\mathbb{H}$-span (as a right $\mathbb{H}$-vector space) is $\mathrm{Hom}_r(\mathbb{H})$ the **homogeneous polynomials** of degree r, and $\mathrm{Pol}_n(\mathbb{H})$ the **polynomials** of degree n is the $\mathbb{H}$-span of the homogeneous polynomials of degrees $\leq n$.

It is clear from the definitions, that the quaternionic polynomials are a graded ring, i.e., the product of homogeneous polynomials of degrees j and k is a homogeneous polynomial of degree $j+k$. To understand the dimensions of these spaces, we write each coordinate q_a of $q = (q_1, \ldots, q_d) \in \mathbb{H}^d$ as

$$q_a = t_a + ix_a + jy_a + kz_a, \qquad t_a, x_a, y_a, z_a \in \mathbb{R},$$

and observe (see [Sud79]) that

$$\begin{aligned} t_a &= \frac{1}{4}(q_a - iq_a i - jq_a j - kq_a k), & x_a &= \frac{1}{4i}(q_a - iq_a i + jq_a j + kq_a k), \\ y_a &= \frac{1}{4j}(q_a + iq_a i - jq_a j + kq_a k), & z_a &= \frac{1}{4k}(q_a + iq_a i + jq_a j - kq_a k). \end{aligned} \tag{2.5}$$

Hence t_a, x_a, y_a, z_a are homogeneous monomials (in q_a), as are $\overline{q_a}$ and $|q_a|^2 = q_a \overline{q_a}$.

Every monomial of degree r can be written as a homogeneous polynomial of degree r in the $4d$ (real) variables t_a, x_a, y_a, z_a, $1 \leq a \leq d$, with quaternionic coefficients. The monomials in these $4d$ real variables are linearly independent over $\mathbb{H}$ by the usual argument (of taking Taylor coefficients), and so we have

$$\dim_{\mathbb{H}}(\mathrm{Hom}_r(\mathbb{H}^d)) = \dim_{\mathbb{R}}(\mathrm{Hom}_r(\mathbb{R}^{4d})) = \binom{r+4d-1}{4d-1}, \tag{2.6}$$

$$\dim_{\mathbb{H}}(\mathrm{Pol}_n(\mathbb{H}^d)) = \dim_{\mathbb{R}}(\mathrm{Pol}_n(\mathbb{R}^{4d})) = \binom{n+4d}{4d}. \tag{2.7}$$

It is, at times, convenient and insightful to treat the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ simultaneously, with

$$m = m_{\mathbb{F}} := \dim_{\mathbb{R}}(\mathbb{F}) = \begin{cases} 1, & \mathbb{F} = \mathbb{R}; \\ 2, & \mathbb{F} = \mathbb{C}; \\ 4, & \mathbb{F} = \mathbb{H}. \end{cases} \tag{2.8}$$

throughout this paper, e.g.,

$$\dim_{\mathbb{F}}(\mathrm{Hom}_r(\mathbb{F}^d)) = \dim_{\mathbb{R}}(\mathrm{Hom}_r(\mathbb{R}^{md})) = \binom{r+md-1}{md-1}.$$

We define a subspace of $\mathrm{Hom}_{2t}(\mathbb{F}^d)$ by

$$\mathrm{Hom}_{\mathbb{F}^d}(t,t) := \mathrm{span}\{|\langle v, \cdot\rangle|^{2t} : v \in \mathbb{F}^d\}, \qquad t = 1, 2, \ldots. \tag{2.9}$$

Since $|\langle v, \cdot\rangle|^{2t}$ maps $\mathbb{F}^d$ to $\mathbb{R}$, we may take the span over $\mathbb{R}$ or $\mathbb{F}$, with the dimension unchanged. For $U : \mathbb{F}^d \to \mathbb{F}^d$ unitary,

$$|\langle v, U\cdot\rangle|^{2t} = |\langle U^* v, \cdot\rangle|^{2t},$$

so that $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ is a unitarily invariant subspace. See Section 8 for further detail.

3 Integration on the real, complex and quaternionic spheres

Though it is not immediately apparent from the inequality (1.1) itself, those vectors giving equality provide discrete approximations to surface area measure on the real and complex spheres (this is clear from Sidelnikov [Sid74], but not Welch [Wel74]).

We now provide the basic theory of integration on the sphere, and calculate the constant $c_t(\mathbb{H}^d)$, which is an average over the quaternionic sphere. Let

$$\mathbb{S} = \mathbb{S}(\mathbb{F}^d) := \{x \in \mathbb{F}^d : \|x\| = 1\} = \{x \in \mathbb{R}^{md} : \|x\| = 1\}$$

be the unit sphere in $\mathbb{F}^d$, and σ be the surface area measure on $\mathbb{S}$, normalised so that $\sigma(\mathbb{S}) = 1$. We note that surface area measure invariant under unitary maps on $\mathbb{F}^d$, i.e., for U unitary

$$\int_{\mathbb{S}(\mathbb{F}^d)} f(Ux)\, d\sigma(x) = \int_{\mathbb{S}(\mathbb{F}^d)} f(x)\, d\sigma(x), \qquad \forall f.$$

This follows from the result for $\mathbb{R}^{md}$ and the fact that the unitary maps on $\mathbb{F}^d$ correspond to a subgroup of the unitary maps on $\mathbb{R}^{md}$. Moreover, for any pair of unit vectors $x, y \in \mathbb{F}^d$ there is a unitary map U with $y = Ux$. To prove this, take $x = e_1$ and use Gram-Schmidt and the fact that that unitary matrices have orthonormal columns (which extend to $\mathbb{H}^d$). From these observations, it follows that there is a constant $c_t(\mathbb{F}^d)$ with

$$\int_{\mathbb{S}(\mathbb{F}^d)} |\langle x, y\rangle|^{2t}\, d\sigma(x) = \|y\|^{2t} c_t(\mathbb{F}^d), \qquad \forall y \in \mathbb{F}^d. \tag{3.10}$$

We now calculate $c_t(\mathbb{F}^d)$ using the well known integrals for the monomials

$$\int_{\mathbb{S}(\mathbb{R}^d)} x^{2\alpha}\, d\sigma(x) = \frac{(\frac{1}{2})_\alpha}{(\frac{d}{2})_{|\alpha|}}, \qquad \alpha \in \mathbb{Z}_+^d,$$

where $x^\alpha = \prod_j x_j^{\alpha_j}$ and $(x)_\alpha = \prod_j (x_j)_{\alpha_j}$, with $(a)_n = a(a+1)\cdots(a+n-1)$.

Lemma 3.1 *The constant of (3.10) for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is given by*

$$c_t(\mathbb{F}^d) = c_{t,m} = \frac{(\frac{m}{2})_t}{(\frac{md}{2})_t} = \prod_{j=0}^{t-1} \frac{m+2j}{md+2j}, \qquad m := \dim_{\mathbb{R}}(\mathbb{F}). \tag{3.11}$$

It satisfies $c_t(\mathbb{R}) = c_t(\mathbb{C}) = c_t(\mathbb{H}) = 1$ and

$$c_1(\mathbb{R}^d) = c_1(\mathbb{C}^d) = c_1(\mathbb{H}^d) = \frac{1}{d}, \qquad c_t(\mathbb{R}^d) > c_t(\mathbb{C}^d) > c_t(\mathbb{H}^d), \quad t > 1,\ d > 1.$$

Proof: Take $x = z$, $y = e_1$ in (3.10) and use the multinomial formula, to obtain

$$c_t(\mathbb{F}^d) = \int_{\mathbb{S}(\mathbb{F}^d)} (|z_1|^2)^t\, d\sigma(z) = \int_{\mathbb{S}(\mathbb{R}^{md})} (x_1^2 + \cdots + x_m^2)^t\, d\sigma(x)$$

$$= \int_{\mathbb{S}(\mathbb{R}^{md})} \sum_{\substack{|\alpha|=t \\ \alpha\in\mathbb{Z}_+^m}} \binom{t}{\alpha} x^{2\alpha}\, d\sigma(x) = \sum_{\substack{|\alpha|=t \\ \alpha\in\mathbb{Z}_+^m}} \binom{t}{\alpha} \frac{(\frac{1}{2})_\alpha}{(\frac{md}{2})_{|\alpha|}} = \frac{(\frac{m}{2})_t}{(\frac{md}{2})_t}.$$

Here the final simplification follows from the multivariate Rothe theorem.

The strict inequality $c_t(\mathbb{R}^d) > c_t(\mathbb{C}^d)$ is given in [Wal18] (Exercise 6.9). We adapt the method used there for the second inequality. Since

$$\frac{c_t(\mathbb{H}^d)}{c_t(\mathbb{C}^d)} = \frac{c_{t-1}(\mathbb{H}^d)}{c_{t-1}(\mathbb{C}^d)} \frac{\frac{4+2(t-1)}{4d+2(t-1)}}{\frac{2+2(t-1)}{2d+2(t-1)}} = \frac{c_{t-1}(\mathbb{H}^d)}{c_{t-1}(\mathbb{C}^d)} \Big(1 - \frac{(t-1)(d-1)}{(t+2d-1)t}\Big),$$

the strict inequality holds by induction on t. □

The value (3.11) coincides with formulas of (1.2) for $\mathbb{R}^d, \mathbb{C}^d$, and

$$c_t(\mathbb{H}^d) = \prod_{j=0}^{t-1} \frac{4+2j}{4d+2j} = \frac{2 \cdot 3 \cdots (t+1)}{2d(2d+1)\cdots(2d+t-1)} = \frac{t+1}{\binom{2d+t-1}{t}}.$$

4 The variational inequality

To prove our quaternionic version of the Sidelnikov–Welch inequality (1.1), we require the existence of an inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ on $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$ for which

$$K_w(z) := |\langle w, z \rangle|^{2t}$$

is the reproducing kernel, i.e.,

$$\langle K_w, f \rangle_{\mathbb{H}} = f(w), \qquad \forall f \in \mathrm{Hom}_{\mathbb{H}^d}(t,t), \quad \forall w \in \mathbb{H}^d.$$

Such an inner product does exist (see Theorems 8.1 and 8.2). For our purposes, it is not necessary to know it explicitly (it follows from the reproducing property), or even the dimension of $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$ (which is not obvious). We also take as given, the existence of such an inner product $\langle \cdot, \cdot \rangle_{\mathbb{F}}$ for $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$, for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ also (which is well known), i.e.,

$$\langle K_w, f \rangle_{\mathbb{F}} = f(w), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t), \quad \forall w \in \mathbb{F}^d. \tag{4.12}$$

We now prove a generalised form of (1.1) given by Sidelnikov [Sid74] for $\mathbb{F} = \mathbb{R}$ and Kotelina and Pevnyi [KP17] for $\mathbb{F} = \mathbb{C}$, by using the method of the latter.

Theorem 4.1 *Let μ be a Borel measure on $X \subset \mathbb{F}^d$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\int_X \|x\|^{2t}\, d\mu(x) < \infty$, and $t \in \mathbb{N}$. Then*

$$\int_X \int_X |\langle x, y \rangle|^{2t}\, d\mu(x)\, d\mu(y) \geq c_t(\mathbb{F}^d) \Big(\int_X \|z\|^{2t}\, d\mu(z) \Big)^2, \tag{4.13}$$

with equality if and only if

$$\frac{1}{\int_X \|x\|^{2t}\, d\mu(x)} \int_X |\langle w, z \rangle|^{2t}\, d\mu(w) = c_t(\mathbb{F}^d) \|z\|^{2t}, \qquad \forall z \in \mathbb{F}^d, \tag{4.14}$$

which is equivalent to the cubature rule

$$\int_{\mathbb{S}} f\, d\sigma = \frac{1}{\int_X \|x\|^{2t}\, d\mu(x)} \int_X f(w)\, d\mu(w), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t). \tag{4.15}$$

There is equality in (4.13) for $X = \mathbb{S}$, $\mu = \sigma$ and certain finitely supported measures.

Proof: We define polynomials f and ω_t in $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ by

$$f(z) := \int_X K_w(z)\, d\mu(w), \qquad \omega_t(z) := \|z\|^{2t}.$$

The integral defining $f(z)$ converges, since Cauchy-Schwarz gives

$$\int_X |K_w(z)|\, d\mu(w) \le \int_X (\|w\|\|z\|)^{2t}\, d\mu(w) = \|z\|^{2t} \int_X \|w\|^{2t}\, d\mu(w) < \infty.$$

These are in $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$, since $K_w \in \mathrm{Hom}_{\mathbb{F}^d}(t,t)$ and by (3.10), respectively.

For the (apolar) inner product (4.12), we have

$$\begin{aligned}
\langle f, \omega_t\rangle_{\mathbb{F}} &= \langle \int_X K_w\, d\mu(w), \|\cdot\|^{2t}\rangle_{\mathbb{F}} = \int_X \langle K_w, \|\cdot\|^{2t}\rangle_{\mathbb{F}}\, d\mu(w) = \int_X \|w\|^{2t}\, d\mu(w), \\
\langle f, f\rangle_{\mathbb{F}} &= \langle \int_X K_x\, d\mu(x), \int_X K_y\, d\mu(y)\rangle_{\mathbb{F}} = \int_X \int_X \langle K_x, K_y\rangle_{\mathbb{F}}\, d\mu(x)\, d\mu(y) \\
&= \int_X \int_X |\langle x, y\rangle|^{2t}\, d\mu(x)\, d\mu(y), \\
\langle \omega_t, \omega_t\rangle_{\mathbb{F}} &= \langle \frac{1}{c_t(\mathbb{F}^d)} \int_{\mathbb{S}} K_x\, d\sigma(x), \|\cdot\|^{2t}\rangle_{\mathbb{F}} = \frac{1}{c_t(\mathbb{F}^d)} \int_{\mathbb{S}} \langle K_x, \|\cdot\|^{2t}\rangle_{\mathbb{F}}\, d\sigma(x) \\
&= \frac{1}{c_t(\mathbb{F}^d)} \int_{\mathbb{S}} \|x\|^{2t}\, d\sigma(x) = \frac{1}{c_t(\mathbb{F}^d)}.
\end{aligned}$$

The integral formula for ω_t used in the last equation above is (3.10).

Thus the inequality (4.13) is given by the Cauchy-Schwarz inequality (which also holds for quaternionic Hilbert space) in the form

$$\langle f, f\rangle_{\mathbb{F}} \ge \frac{1}{\langle \omega_t, \omega_t\rangle_{\mathbb{F}}} (\langle f, \omega_t\rangle_{\mathbb{F}})^2,$$

with equality if and only if f and ω_t are scalar multiples of each other, i.e.,

$$\int_X |\langle w, z\rangle|^{2t}\, d\mu(w) = C\|z\|^{2t}.$$

The scalar C above can be determined by integrating with respect to σ, and using (3.10)

$$\begin{aligned}
C &= \int_{\mathbb{S}} C\|z\|^{2t}\, d\sigma(z) = \int_{\mathbb{S}} \int_X |\langle w, z\rangle|^{2t}\, d\mu(w)\, d\sigma(z) = \int_X \int_{\mathbb{S}} |\langle w, z\rangle|^{2t}\, d\sigma(z)\, d\mu(w) \\
&= \int_X c_t(\mathbb{F}^d) \|w\|^{2t}\, d\mu(w) = c_t(\mathbb{F}^d) \int_X \|x\|^{2t}\, d\mu(x).
\end{aligned}$$

and so we obtain the condition (4.14) for equality.

By homogeneity, (4.14) holds if and only if it holds for $z \in \mathbb{S}$, i.e.,

$$\frac{1}{\int_X \|x\|^{2t}\, d\mu(x)} \int_X K_z(w)\, d\mu(w) = c_t(\mathbb{F}^d) = \int_{\mathbb{S}} K_z\, d\sigma, \qquad \forall z \in \mathbb{S}(\mathbb{F}^d),$$

which is (4.15) for $f = K_z = |\langle z, \cdot\rangle|^{2t}$. Since the integral is linear, and $\{K_z : z \in \mathbb{S}(\mathbb{F}^d)\}$ spans $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$, we obtain the equivalent condition (4.15).

It is easy to verify that there is equality in (4.13) for $X = \mathbb{S}$, $\mu = \sigma$, by using (3.10), or to observe that (4.15) holds trivially. It follows from a result of [SZ84] that (4.15) holds for a finitely supported measure. □

5 Spherical (t,t)-designs

Let δ_v be the Dirac δ-measure concentrated at $v \in \mathbb{F}^d$. A finitely supported measure

$$\mu = \sum_{j=1}^n w_j \delta_{v_j}, \qquad v_j \in \mathbb{F}^d,\ v_j \neq 0, \quad w_j, > 0$$

will be called a **spherical** (t,t)**-design** for $\mathbb{F}^d$ (or $\mathbb{S}$) if it gives equality in (4.13), i.e., by (4.15),

$$\int_{\mathbb{S}} f\, d\sigma = C \sum_j w_j f(v_j), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t),$$

for a fixed constant C. Since $\mathrm{Hom}_{\mathbb{F}^d}(t,t) \subset \mathrm{Hom}_{2t}(\mathbb{F}^d)$, we have

$$C \sum_j w_j f(v_j) = C \sum_j f\big((w_j)^{\frac{1}{2t}} v_j\big) = C \sum_j w_j \|v_j\|^{2t} f\big(\frac{v_j}{\|v_j\|}\big), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t),$$

so that the measure $\mu = \sum_j w_j \delta_{v_j}$ giving a spherical (t,t)-design could be replaced by one where the weights w_j are 1, or the vectors v_j have unit length, i.e.,

$$\sum_j \delta_{(w_j)^{\frac{1}{2t}} v_j}, \qquad \sum_j \|v_j\|^{2t} \delta_{\frac{v_j}{\|v_j\|}}.$$

The particular choice taken (there are many others) makes not essential difference to the theory of spherical (t,t)-designs, and we consider all such (t,t)-designs as equivalent. There is some crossover with the theory of "Euclidean t-designs", which, in addition, seek to integrate polynomials of lower degree, and some of these measures (that we consider equivalent) may correspond to Euclidean designs (see [HW21]).

Sometimes, it is convenient for us to "normalise" by choosing the weights to be 1, or the vectors to be in $\mathbb{S}$. We now give the corresponding presentations of Theorem 4.1.

Corollary 5.1 *Fix $t \in \mathbb{N}$. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, not all zero. Then*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k \rangle|^{2t} \geq c_t(\mathbb{F}^d) \Big(\sum_{\ell=1}^n \|v_\ell\|^{2t} \Big)^2, \tag{5.16}$$

with equality when one of the following equivalent conditions hold

(a) *The generalised Bessel identity*

$$c_t(\mathbb{F}^d) \|x\|^{2t} = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n |\langle v_j, x \rangle|^{2t}, \qquad \forall x \in \mathbb{F}^d. \tag{5.17}$$

(b) *The cubature rule for* $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$

$$\int_{\mathbb{S}} f\, d\sigma = \frac{1}{\sum_{\ell=1}^n \|v_\ell\|^{2t}} \sum_{j=1}^n f(v_j), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t). \tag{5.18}$$

Proof: Take $\mu = \sum_j \delta_{v_j}$ in Theorem 4.1. □

In light of the above, we will say that $(v_j) \subset \mathbb{F}^d$ is a **spherical** (t,t)**-design** if

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2t} = c_t(\mathbb{F}^d)\Big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Big)^2. \tag{5.19}$$

Corollary 5.2 *Fix $t \in \mathbb{N}$. Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and (w_j) be nonnegative weights with $\sum_j w_j = 1$. Then*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle v_j, v_k\rangle|^{2t} \ge c_t(\mathbb{F}^d), \tag{5.20}$$

with equality when one of the following equivalent conditions hold

(a) *The generalised Bessel identity*

$$c_t(\mathbb{F}^d)\|x\|^{2t} = \sum_{j=1}^n w_j |\langle v_j, x\rangle|^{2t}, \qquad \forall x \in \mathbb{F}^d. \tag{5.21}$$

(b) *The cubature rule for* $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$

$$\int_{\mathbb{S}} f\, d\sigma = \sum_{j=1}^n w_j f(v_j), \qquad \forall f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t). \tag{5.22}$$

Proof: Take $\mu = \sum_j w_j \delta_{v_j}$ in Theorem 4.1. □

Since $|\langle v, \cdot\rangle|^{2r}\|\cdot\|^{2t-2r} \in \mathrm{Hom}_{\mathbb{F}^d}(t,t)$ (see Example 8.5), it follows from the cubature rule characterisation that a spherical (t,t)-design is a spherical (r,r)-design for $1 \le r \le t$. This takes a more natural form in the presentation with weights (w_j) and vectors on the sphere.

Proposition 5.1 *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then*

(a) *If $(v_j) \subset \mathbb{F}^d$ is a spherical (t,t)-design (for $\mathbb{F}^d$), then $(\|v_j\|^{t/r-1} v_j)$ is a spherical (r,r)-design, $1 \le r \le t$, i.e.,*

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2r} \|v_j\|^{2t-2r}\|v_k\|^{2t-2r} = c_r(\mathbb{F}^d)\Big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Big)^2. \tag{5.23}$$

(b) *If (w_j), $(v_j) \subset \mathbb{S}(\mathbb{F}^d)$ is a (weighted) spherical (t,t)-design, then (w_j), (v_j) is a spherical (r,r)-design, $1 \le r \le t$, i.e.,*

$$c_r(\mathbb{F}^d) := \int_{\mathbb{S}}\int_{\mathbb{S}} |\langle x, y\rangle|^{2r}\, d\sigma(x)\, d\sigma(y) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle v_j, v_k\rangle|^{2r}. \tag{5.24}$$

Proof: Let $f \in \mathrm{Hom}_{\mathbb{F}^d}(r,r)$, so that $\|\cdot\|^{2t-2r} f \in \mathrm{Hom}_{\mathbb{F}^d}(t,t)$, and (5.22) gives

$$\int_{\mathbb{S}} f\, d\sigma = \int_{\mathbb{S}} \|\cdot\|^{2t-2r} f\, d\sigma = \sum_{j=1}^n w_j \|v_j\|^{2t-2r} f(v_j) = \sum_{j=1}^n w_j f(v_j),$$

which, by Corollary 5.2, gives (b). Part (a) follows similarly from Corollary 5.1 (also see [Wal18] Proposition 6.2). □

6 Quaternionic spherical (t,t)-designs

In view of Theorem 7.1, examples of quaternionic spherical (t,t)-designs are given by the known projective t-designs. In particular, see the listing in [Hog82].

Example 6.1 *There are nine quaternionic spherical (t,t)-designs listed in [Hog82]. All, except Examples 27 and 30, have rational angles $\alpha = |\langle v_j, v_k\rangle|^2$. The Example 27 is a 315 vector $(5,5)$-design for $\mathbb{H}^3$, for which (7.31) holds as (see [Hog84], Table 3)*

$$1 + 10(0)^5 + 32(\tfrac{3-\sqrt{5}}{8})^5 + 160(\tfrac{1}{4})^5 + 80(\tfrac{1}{2})^5 + 32(\tfrac{3+\sqrt{5}}{8})^5 = \tfrac{15}{2} = n \cdot c_5(\mathbb{H}^3) = 315\tfrac{1}{42}.$$

For the octonions (Cayley numbers) $\mathbb{O}^d$, one can formally define the Euclidean inner product as for $\mathbb{R}, \mathbb{C}, \mathbb{H}$. It satisfies

$$\overline{\langle v, w\rangle} = \langle w, v\rangle, \qquad |\langle v, w\rangle|^2 = |\langle w, v\rangle|^2, \qquad \langle v, v\rangle > 0, \quad v \neq 0,$$

but is not linear in the second variable (only additive, in both). Therefore the variational inequality (5.16) and *octonionic* spherical (t,t)-designs can be defined formally. A notion of (projective) unitary equivalence of such designs is not obvious. It has not yet been established whether the octonionic version of the Welch-Sidelnikov inequality holds.

Example 6.2 *(MUBs) Consider the $n = 2m+2$ unit vectors (v_j) in $\mathbb{F}^d$ given by*

$$\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\} \cup \{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\a\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-a\end{pmatrix}\}_{a\in\{1,i,j,k\}},$$

The left-hand side of (5.19) is

$$(2m+2)\cdot 1^t + 2m(2m+2)\cdot\left(\frac{1}{2}\right)^t + (2m+2)\cdot 0^t = 2m+2+\frac{4}{2^t}m(m+1),$$

and the right-hand side is

$$c_t(\mathbb{F}^2)(2m+2)^2 = \frac{m(m+2)\cdots(m+2t-2)}{md(md+2)\cdots(md+2t-2)}(2m+2)^2.$$

These are equal for $t = 1,2,3$ (and all values for m), giving spherical $(3,3)$-designs. They are Examples 1,2,3 of [Hog82], with Example 4 giving the octonionic version. The ten vectors in the quaternionic case can be interpreted as a set of five mutually unbiased bases (or MUBs) in $\mathbb{H}^2$. These meet the bound $2d+1$ on the number of MUBs in $\mathbb{H}^d$ (see [CD08]). There is a general bound of $\frac{m}{2}d+1$ on the number of MUBs in $\mathbb{F}^d$, which is obtained by this example.

A sequence of unit vectors (v_j) in $\mathbb{F}^d$ (or the lines they give) is **equiangular** if

$$|\langle v_j, v_k\rangle|^2 = C, \qquad j \neq k,$$

for some constant C. The case $C = 0$ gives orthonormal vectors.

Example 6.3 *(SICs) It can be shown [Wal20], that the number of equiangular lines in* $\mathbb{F}^d$ *is less than or equal to* $d+\frac{m}{2}(d^2-d)$*, and such a (maximal) set of* $n=d+\frac{m}{2}(d^2-d)$ *equiangular lines is a tight frame, with equiangularity constant* $C=\frac{m}{md+2}$*. There is considerable interest in such maximal sets of equiangular lines, especially in the complex case, where they are known as SICs (see [ACFW18]). It follows that such a configuration is a spherical* $(2,2)$*-design by verifying (5.19) via the calculation*

$$n+(n^2-n)C^2 = n\big(1+(n-1)C^2\big) = n\Big(1+\Big(d+\frac{m}{2}(d^2-d)-1\Big)\Big(\frac{m}{md+2}\Big)^2\Big)$$
$$= n\frac{(md-m+2)(m+2)}{2(md+2)} = n\Big(d+\frac{m}{2}(d^2-d)\Big)\frac{m(m+2)}{md(md+2)} = n^2c_2(\mathbb{F}^d).$$

There are six equiangular lines in $\mathbb{H}^2$ *(see [KF08],[Wal20]), and Example 15 of [Hog82] gives a construction of* $n=2d$ *equiangular lines in* $\mathbb{H}^d$*.*

The variational characterisation (Corollary 5.1) of spherical (t,t)-designs allows for a numerical search for them (see [HW21] for the real and complex cases), by minimising the left-hand side of (5.16). Naive calculations readily identified many of the known quaternionic spherical (t,t)-designs above (which have a putatively optimal number of vectors). We also noticed some *near* designs, with rational angles (to machine precision).

Example 6.4 *A numerical search for* $(2,2)$*-designs with a fixed number of vectors/lines in* $\mathbb{H}^2$*, by minimising the left-hand side of (5.16), gave the six equiangular lines. A search with five vectors gave five of these six lines, with the variational inequality (5.16) being*

$$5(1)^2+20(\frac{3}{8})^2=\frac{125}{16}=7.812500>7.5=5^2c_2(\mathbb{H}^5),$$

and a search with seven vectors (of unit length), gave a near $(2,2)$*-design, with angles* $\frac{1}{4},\frac{1}{3},\frac{1}{2}$ *(to high precision), with the variational inequality being*

$$7(1)^2+24(\frac{1}{2})^2+12(\frac{1}{3})^2+6(\frac{1}{4})^2=\frac{353}{24}=14.708333\cdots>14.7=c_2(\mathbb{H}^2)(7)^2.$$

Example 6.5 *A numerical search for* $(4,4)$*-designs for* $\mathbb{H}^2$ *gave various* $(3,3)$*-designs, including one of* 12 *vectors and one of* 14 *vectors, with the corresponding variational inequalities*

$$12(1)^4+12(0)^4+60(\frac{2}{5})^4+60(\frac{3}{5})^4=\frac{2664}{125}=21.31200000>20.57142857\cdots=\frac{12^2}{7},$$

$$16(1)^4+80(1/5)^4+160(\frac{3}{5})^4=36.86400000>36.57142857\cdots=\frac{16^2}{7}.$$

Currently there is no method for determining whether or not quaternionic spherical (t,t)-designs are unitarily equivalent (as is there is in the real complex cases [CW16]), and so it is not yet possible to see whether these numerical designs (and near designs) are unique up to projective unitary equivalence.

7 Projective spherical t-designs on Delsarte spaces

We now seek to make the connection between spherical (t,t)-designs (as we have defined them) and the projective spherical t-designs. For this purpose, it is convenient to work with weights (w_j) and vectors (v_j) in $\mathbb{S}$.

The condition of equality in (5.20) used to define a spherical (t,t)-design can be written

$$c_t(\mathbb{F}^d) := \int_{\mathbb{S}}\int_{\mathbb{S}} |\langle x,y\rangle|^{2t}\, d\sigma(x)\, d\sigma(t) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k |\langle v_j, v_k\rangle|^{2t}, \tag{7.25}$$

or, equivalently,

$$\int_{\mathbb{S}}\int_{\mathbb{S}} g(|\langle x,y\rangle|^2)\, d\sigma(x)\, d\sigma(y) = \sum_{j=1}^n \sum_{k=1}^n w_j w_k g(|\langle v_j, v_k\rangle|^2), \tag{7.26}$$

for $g=(\cdot)^t$, the univariate monomial of degree t. By (5.24) of Proposition 5.1, (7.26) also holds for the univariate monomials $g=(\cdot)^r$, $1\le r\le t-1$, and it holds trivially for the constant monomial $g=(\cdot)^0=1$. Thus

Lemma 7.1 *Let μ_m be the Borel (probability) measure defined on $[0,1]\subset\mathbb{R}$ by*

$$\int_0^1 g(s)\, d\mu_m(s) := \int_{\mathbb{S}}\int_{\mathbb{S}} g(|\langle x,y\rangle|^2)\, d\sigma(x)\, d\sigma(y), \tag{7.27}$$

so that (7.26) can be written as

$$\int_0^1 g\, d\mu_m = \sum_{j=1}^n \sum_{k=1}^n w_j w_k g(|\langle v_j, v_k\rangle|^2), \tag{7.28}$$

and let $Q_0^{(m)}, Q_1^{(m)}, \ldots$ the orthogonal polynomials for the measure μ_m. Then the condition for (w_j), $(v_j)\subset\mathbb{S}(\mathbb{F})$ to be a spherical (t,t)-design for $\mathbb{F}^d$ is equivalent to the following

(a) *The equation (7.28) holds for the monomial $g=(\cdot)^t$.*

(b) *The equation (7.28) holds for all $g\in\mathrm{Pol}_t(\mathbb{R})$.*

(c) *The equation (7.28) holds for $g=Q_1^{(m)},\ldots,Q_t^{(m)}$, i.e.,*

$$\sum_{j=1}^n \sum_{k=1}^n w_j w_k Q_\ell^{(m)}(|\langle v_j, v_k\rangle|^2) = 0, \qquad \ell=1,\ldots,t. \tag{7.29}$$

Proof: We have already observed the conditions (a) and (b). Since $Q_0^{(m)},\ldots,Q_t^{(m)}$ is a basis for $\mathrm{Pol}_t(\mathbb{R})$, and (7.28) holds trivially for the constant polynomial $Q_0^{(m)}=1$, we obtain the condition that (7.28) holds for $Q_1^{(m)},\ldots,Q_t^{(m)}$. The orthogonality condition gives

$$\int_0^1 Q_\ell^{(m)}\, d\mu_m = \int_0^1 Q_\ell^{(m)} Q_0^{(m)}\, d\mu_m = 0, \qquad \ell=1,2,\ldots,$$

and therefore we obtain (c). □

The (induced) measure of (7.27) is absolutely continuous with respect to Lebesgue measure, and is given by (see [Hog82], Theorem 2.11) $d\mu_m(z) = W(z)\,dz$, where

$$W(z) := \frac{\Gamma(\frac{md}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}(d-1))} z^{\frac{m}{2}-1}(1-z)^{\frac{m}{2}(d-1)-1}, \qquad m := \dim_{\mathbb{R}}(\mathbb{F}). \tag{7.30}$$

This can be checked, using the density of polynomials in $L_1(\mu_m)$, by the calculation

$$\begin{aligned}\int_0^1 z^r\, W(z)\,dz &= \frac{\Gamma(\frac{md}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}(d-1))}\int_0^1 z^{\frac{m}{2}+r-1}(1-z)^{\frac{m}{2}(d-1)-1}\,dz \\ &= \frac{\Gamma(\frac{md}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}(d-1))}\frac{\Gamma(\frac{m}{2}+r)\Gamma(\frac{m}{2}(d-1))}{\Gamma(\frac{md}{2}+r)} = \frac{(\frac{m}{2})_r}{(\frac{md}{2})_r} \\ &= c_r(\mathbb{F}^d) = \int_{\mathbb{S}}\int_{\mathbb{S}} (|\langle x,y\rangle|^2)^r\,d\sigma(x)\,d\sigma(y).\end{aligned}$$

It is evident from (7.30) that the orthogonal polynomials $Q_k^{(m)}$ of Lemma 7.1 are Jacobi polynomials (on $[0,1]$). Hence, we have

$$\begin{aligned}Q_k^{(m)}(x) &= P_k^{(\frac{m}{2}-1,\frac{m}{2}(d-1)-1)}(1-2x) = \frac{(\frac{m}{2})_k}{k!}{}_2F_1(-k,\tfrac{md}{2}-1+k,\tfrac{m}{2};x) \\ &= \frac{(\frac{m}{2})_k}{k!}\sum_{j=0}^k (-1)^j\binom{k}{j}\frac{(\frac{md}{2}-1+k)_j}{(\frac{m}{2})_j}x^j.\end{aligned}$$

The condition (7.29) does not depend on the particular normalisation of the $Q_k^{(m)}$. The norm can be calculated from the orthogonality relations for the Jacobi polynomials

$$\int_{-1}^1 (1-z)^\alpha(1+z)^\beta P_j^{(\alpha,\beta)}(z)P_k^{(\alpha,\beta)}(z)\,dz = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+1)k!}\delta_{jk}.$$

The substitution $z = 1-2x$, so that $1-z = 2x$, $1+z = 2(1-x)$ and $dz = -2dx$, gives

$$\int_0^1 P_j^{(\alpha,\beta)}(1-2x)P_k^{(\alpha,\beta)}(1-2x)\,\frac{\Gamma(\alpha+\beta+2)x^\alpha(1-x)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,dx = \frac{1}{2k+\alpha+\beta+1}\frac{(\alpha+1)_k(\beta+1)_k}{(\alpha+\beta+2)_{k-1}k!}\delta_{jk},$$

where $(x)_{-1} := 1/(x-1)$. Taking $\alpha = \frac{m}{2}-1$, $\beta = \frac{m}{2}(d-1)-1$, then gives

$$\int_0^1 Q_j^{(m)}Q_k^{(m)}\,d\mu_m = \frac{1}{2k+\frac{md}{2}-1}\frac{(\frac{m}{2})_k(\frac{m}{2}(d-1))_k}{(\frac{md}{2})_{k-1}}\frac{1}{k!}\delta_{jk}.$$

The condition (c) of Lemma 7.1 is essentially Hoggar's definition of a t-design in the projective space $\mathbb{F}P^{d-1}$ (a **projective t-design**) [Hog82],[Hog84],[Hog90] which is an example of a more general theory of t-designs on Delsarte spaces (which we discuss later). There only the case with constant weights $w_j = 1$ is considered, but the "weighted" version of projective t-designs extends in the obvious fashion, see [Lev98]. This connection is generally understood for $\mathbb{F} = \mathbb{R},\mathbb{C}$ (see [RS07], [Wal18] Theorem 6.7), and is a new result for $\mathbb{F} = \mathbb{H}$.

Theorem 7.1 *The spherical (t,t)-designs for $\mathbb{F}^d$ are precisely the (projective) t-designs on the Delsarte spaces $\mathbb{F}P^{d-1}$, for $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$.*

Proof: The Neumaier construction of t-designs in Delsarte spaces [Neu81], which Hoggar [Hog82],[Hog84] used to construct projective t-designs, involves the distance $d([x],[y]) = \sqrt{1-|\langle x,y\rangle|^2}$ between lines given by unit vectors $x,y\in\mathbb{F}^d$, which are reformulated in terms of the "angle" $|\langle x,y\rangle|^2=\cos^2\theta_{xy}$. It is enough to observe that the condition (1) in [Hog84] is condition (c) of Lemma 7.1, where the polynomials Q_k defined in (4) are multiples of our $Q_k^{(m)}$, since

$$Q_k^{(m)}(x) = \frac{(\frac{md}{2}-1+k)_k}{k!}\sum_{j=0}^k(-1)^j\binom{k}{j}\frac{(\frac{m}{2}+j)_{k-j}}{(\frac{md}{2}-1+k+j)_{k-j}}x^j,$$

and, with ${}_j(x):=x(x-1)\cdots(x-(j-1))=(x-j+1)_j$,

$$\begin{aligned}
Q_k(x) &:= \frac{(\frac{md}{2})_{2k}}{(\frac{m}{2})_k k!}\sum_{j=0}^k(-1)^j\binom{k}{j}\frac{{}_j(k+\frac{m}{2}-1)}{{}_j(2k+\frac{md}{2}-2)}x^{k-j}\\
&= \frac{(\frac{md}{2})_{2k}}{(\frac{m}{2})_k k!}\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}\frac{{}_{k-j}(k+\frac{m}{2}-1)}{{}_{k-j}(2k+\frac{md}{2}-2)}x^{j}\\
&= \frac{(\frac{md}{2})_{2k}}{(\frac{m}{2})_k k!}(-1)^k\sum_{j=0}^k(-1)^{j}\binom{k}{j}\frac{(\frac{m}{2}+j)_{k-j}}{(\frac{md}{2}+k+j-1)_{k-j}}x^{j}.
\end{aligned}$$

□

Hoggar [Hog84] considered **regular schemes** $\mathcal{B}$, i.e., finite sets of projective points (unit vectors in $\mathbb{F}^d$) with angles $A=\{\alpha_1,\ldots,\alpha_s\}\subset[0,1]$, for which the number d_{α_j} of points making an angle α_j with $x\in\mathcal{B}$ is independent of x, e.g., those given by an orbit.

Corollary 7.1 *Let $\mathcal{B}$ be a regular scheme of n points in $\mathbb{F}^d$. Then $\mathcal{B}$ is a projective t-design if and only if*

$$1+\alpha_1^r d_{\alpha_1}+\cdots+\alpha_s^r d_{\alpha_s} = n\frac{(\frac{m}{2})_r}{(\frac{md}{2})_r}, \tag{7.31}$$

for $r=t$.

Proof: Since $\mathcal{B}=(v_j)$ is a regular scheme, the condition (5.19) for being a spherical (t,t)-design (and hence a t-design) reduces to

$$\sum_j\sum_k|\langle v_j,v_k\rangle|^{2t} = n^{\text{`}}\sum_k|\langle v_j,v_1\rangle|^{2t} = n\bigl(1+\alpha_1^r d_{\alpha_1}+\cdots+\alpha_s^r d_{\alpha_s}\bigr) = c_t(\mathbb{F}^d)(n)^2,$$

which (after division by n) is (7.31). □

This illuminates and refines the Theorem 2.4 of [Hog84], which gives the condition for a regular scheme $\mathcal{B}$ to be a projective t-design is that (7.31) holds for $r = 1, \ldots, t$.

We now consider the Delsarte space construction in more detail. If (X, d) is a metric space with finite diameter, and ω a finite measure on X, then it is a **Delsarte space** (with respect to ω) if there exist polynomials f_{jk} of degree $\le \min\{j,k\}$, for which

$$\int_X d(a,x)^{2j} d(b,x)^{2k}\, d\omega(x) = f_{jk}(d(a,b)^2), \qquad \forall j,k = 0,1,2,\ldots. \tag{7.32}$$

The metric on lines $[x] = \{\lambda x : \lambda \in \mathbb{F}, |\lambda| = 1,\, x \in \mathbb{F}^d,\, \|x\| = 1$ in $X = \mathbb{F}P^{d-1}$ is

$$d([x],[y]) = \sqrt{1 - |\langle x,y\rangle|^2}, \tag{7.33}$$

and the measure on X is given by

$$\int_X f([x])\, d\omega([x]) = \int_{\mathbb{S}} \tilde{f}(x)\, \sigma(x), \qquad \tilde{f}(x) := f([x]).$$

The condition (7.32) to be a Delsarte space is that

$$\int_X (1-|\langle a,x\rangle|^2)^j (1-|\langle b,x\rangle|^2)^k\, d\omega([x]) = \int_{\mathbb{S}} (1-|\langle a,x\rangle|^2)^j (1-|\langle b,x\rangle|^2)^k\, d\sigma(x) = f_{jk}(1-|\langle a,b\rangle|^2),$$

which is equivalent to

$$\int_{\mathbb{S}} |\langle a,x\rangle|^{2j} |\langle b,x\rangle|^{2k}\, d\sigma(x) = p_{jk}(|\langle a,b\rangle|^2), \qquad \forall a,b \in \mathbb{S}, \tag{7.34}$$

for some polynomials p_{jk} with degree $\le \min\{j,k\}$. This has been proved by [Neu81] ($\mathbb{F} = \mathbb{R}, \mathbb{C}$) and [Gan67] ($\mathbb{F} = \mathbb{H}$), so that $\mathbb{F}P^{d-1}$ is indeed a Delsarte space.

We can give a constructive proof of (7.34), as follows. As motivation, we note that Lemma 3.1 gives the special case $k = 0$ (a constant polynomial)

$$\int_{\mathbb{S}} |\langle a,x\rangle|^{2j} |\langle b,x\rangle|^{0}\, d\sigma(x) = \int_{\mathbb{S}} |\langle a,x\rangle|^{2j}\, d\sigma(x) = c_j(\mathbb{F}^d).$$

Assume, without loss of generality, that $k \le j$. By Gram-Schmidt, for $a, b \in \mathbb{S}$, we have

$$b = (b - a\langle a,b\rangle) + a\langle a,b\rangle, \qquad (b - a\langle a,b\rangle) \perp a, \qquad \|b - a\langle a,b\rangle\| = \sqrt{1 - |\langle a,b\rangle|^2}.$$

Thus, we may choose a unitary U with

$$U(a\langle a,b\rangle) = |\langle a,b\rangle| e_1, \qquad U(b - a\langle a,b\rangle) = \sqrt{1 - |\langle a,b\rangle|^2}\, e_2,$$

so that, by the unitary invariance of surface area measure, we have

$$\begin{aligned}
\int_{\mathbb{S}} &|\langle a,x\rangle|^{2j} |\langle b,x\rangle|^{2k}\, d\sigma(x) = \int_{\mathbb{S}} |\langle Ua,x\rangle|^{2j} |\langle Ub,x\rangle|^{2k}\, d\sigma(x) \\
&= \int_{\mathbb{S}} |\langle e_1,x\rangle|^{2j} |\langle e_1 |\langle a,b\rangle| + \sqrt{1-|\langle a,b\rangle|^2} e_2, x\rangle|^{2k}\, d\sigma(x) \\
&= \int_{\mathbb{S}} |x_1|^{2j} \Big| |\langle a,b\rangle| x_1 + \sqrt{1-|\langle a,b\rangle|^2} x_2 \Big|^{2k}\, d\sigma(x) \\
&= \int_{\mathbb{S}} |x_1|^{2j} \big(|\langle a,b\rangle|^2 |x_1|^2 + (1-|\langle a,b\rangle|^2)|x_2|^2 + 2|\langle a,b\rangle| \sqrt{1-|\langle a,b\rangle|^2}\, \mathrm{Re}(x_1 \overline{x_2}) \big)^k\, d\sigma(x).
\end{aligned}$$

It is easily verified that the integral of an odd power of $\mathrm{Re}(x_1\overline{x_2})$ is zero (in each of the cases $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$), and so the above integral gives a polynomial of degree k in $|\langle a,b\rangle|^2$.

As the calculation above suggests, for the Delsarte space $X=\mathbb{F}P^{d-1}$, it is more convenient to work with $|\langle x,y\rangle|^2$, rather than the metric d of (7.33). This view point is taken in the unified development of Levenshtein [Lev98], who gives bounds for a large class of "codes" $C\subset X$, with weights m. In addition to a metric space (X,d) with a finite measure ω and weights m, there is a **substitution** σ_s, i.e., continuous strictly monotone function $[0,\mathrm{diam}(X)]\to\mathbb{R}$. In this setup, a finite set C with weights m (which add to $|C|$) is a **weighted τ-design** (in X with respect to the substitution $\sigma_s(d)$) if

$$\int_X\int_X g(\sigma_s(d(x,y)))\,d\omega(x)\,d\omega(y)=\frac{1}{|C|^2}\sum_{x,y\in C} g(\sigma_s(d(x,y)))\,m(x)m(y), \qquad (7.35)$$

holds for all univariate polynomials $g:\mathbb{R}\to\mathbb{R}$ of degree $\le\tau$. Since this definition depends only on σ_s up to a linear change of variables, we may choose σ_s to have the **standard** form (to be a **standard substitution**)

$$\sigma_s(\mathrm{diam}(X))=-1\le\sigma_s(d)\le 1=\sigma_s(0)).$$

For $\mathbb{F}P^{d-1}$, [Lev98] takes the following variant of the metric (7.33) and the standard substitution

$$d([x],[y])=\sqrt{1-|\langle x,y\rangle|},\qquad \sigma_s(d)=2(1-d^2)^2-1, \qquad (7.36)$$

where

$$\sigma_s(d([x],[y]))=2|\langle x,y\rangle|^2-1=\cos(2\theta_{xy}),\quad |\langle x,y\rangle|=\cos(\theta_{xy}).$$

The general form of condition (c) of Lemma 7.1 for a weighted τ-design, as defined by (7.35), is given in Corollary 2.14 of [Lev98].

A general form of the variational inequality (Theorem 4.1, Corollary 5.1) is given in [Lev98] for real and complex valued functions, which includes the Welch and Sidelnikov inequalities, but not our quaternionic version. This "inequality on the mean" of a **FDNDF** (**finite dimensional nonnegative definite function** $F:X\times X\to\mathbb{F}$, where $\mathbb{F}=\mathbb{R},\mathbb{C}$, is as follows. A function F is said to be **Hermitian** if

$$\overline{F(x,y)}=F(y,x),\qquad \forall x,y\in X,$$

and moreover to be **nonnegative definite** if $F|_{C\times C}$ is positive semidefinite for all finite subsets $C\subset X$, i.e.,

$$\sum_{x,y\in C}\overline{v(x)}F(x,y)v(y)\ge 0,\qquad \forall v:X\to\mathbb{C}.$$

Such an F is finite dimensional if it can be written

$$F(x,y)=\sum_{j=1}^n\overline{g_j(x)}g_j(y),$$

for finitely many functions $g_j:X\to\mathbb{C}$. Important examples of FDNDF are $\langle x,y\rangle$ and $|\langle x,y\rangle|^2$ on $\mathbb{F}^d$, $\mathbb{F}=\mathbb{R},\mathbb{C}$. We note that $|\langle x,y\rangle|$ is not a FDNDF, and that products of

FDNDFs are FDNDF, so that $|\langle x,y\rangle|^{2t}$ is a FDNDF. A FDNDF F is said to satisfy the **inequality on the mean** if

$$\frac{1}{|C|^2}\sum_{x,y\in C} F(x,y) \ge \int_X\int_X F(x,y)\,d\omega(x)\,d\omega(y). \tag{7.37}$$

It is shown (Corollary 3.10 [Lev98]) that F satisfies the inequality on the mean if $\int_X F(x,y)\,d\omega(y)$ does not depend on $x\in X$. For

$$\omega=\sigma \text{ on } X=\mathbb{S}(\mathbb{F}), \quad \mathbb{F}=\mathbb{R},\mathbb{C}, \qquad F(x,y)=|\langle x,y\rangle|^{2t},$$

this condition follows from (3.10), with (7.37) becoming the (unweighted) version of the Welch and Sidelnikov inequalities, respectively. A theory of quaternion valued FDNDFs could be developed (cf. [TM14]), which would yield a corresponding inequality on the mean, giving Theorem 4.1 for $\mathbb{F}=\mathbb{H}$ (as a particular case). Instead, we present our original approach, which is more constructive.

8 Reproducing kernels and inner products on $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$

We now establish (Theorem 8.2) a key fact used to prove the variational inequality of Theorem 4.1, i.e., the existence of an inner product on

$$\mathrm{Hom}_{\mathbb{F}^d}(t,t) := \mathrm{span}\{|\langle v,\cdot\rangle|^{2t}\},$$

with the property that

$$\langle K_w, f\rangle_{\mathbb{F}} = f(w), \quad \forall f\in \mathrm{Hom}_{\mathbb{F}^d}(t,t), \qquad K_w(z):=|\langle z,w\rangle|^{2t},$$

or, in other words, there is an inner product for which $|\langle v,w\rangle|^{2t}$ is the reproducing kernel.

Reproducing kernels for real and complex Hilbert space are well studied, and the extension to quaternionic Hilbert space follows in the obvious way [TM14]. We say that an $\mathbb{F}$-Hilbert space $\mathcal{H}$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$) consisting of functions on a set X is a **reproducing kernel Hilbert space** if there is a "kernel" $K_w\in\mathcal{H}$, $w\in X$, for which

$$\langle K_w, f\rangle = f(w), \quad \forall f\in\mathcal{H}, \quad \forall w\in X. \tag{8.38}$$

Such a kernel $K_w(z)$ can exist if only if point evaluation is a continuous linear functional. We now present the basic structure theorem for (finite dimensional) reproducing kernel Hilbert spaces, in terms of tight frames. A finite set (f_j) in an $\mathbb{F}$-Hilbert space $\mathcal{H}$ is a **normalised tight frame** (see [Wal20], [Wal18]) if

$$f=\sum_j f_j\langle f_j,f\rangle, \qquad \forall f\in\mathcal{H}. \tag{8.39}$$

Proposition 8.1 *Let (K_w) be the reproducing kernel for a finite dimensional $\mathbb{F}$-Hilbert space, with normalised tight frame (f_j). Then its reproducing kernel is*

$$K_w(z)=\sum_j f_j(z)\overline{f_j(w)}.$$

Proof: Since all linear functionals on finite dimensional Hilbert spaces are continuous, in particular the point evaluations, the Hilbert space has a reproducing kernel. By the tight frame expansion (8.39) and the reproducing property (8.38), we have

$$K_w = \sum_j f_j \langle f_j, K_w \rangle = \sum_j f_j \overline{\langle K_w, f_j \rangle} = \sum_j f_j \overline{f_j(w)},$$

so that

$$K_w(z) = \langle K_z, K_w \rangle = \langle \sum_k f_k \overline{f_k(z)}, \sum_j f_j \overline{f_j(w)} \rangle = \sum_j \Big(\sum_k f_k(z) \langle f_k, f_j \rangle \Big) \overline{f_j(w)}$$
$$= \sum_j f_j(z) \overline{f_j(w)}.$$

□

The desired inner product on the spaces $\mathrm{Hom}_{\mathbb{R}^d}(t,t)$ and $\mathrm{Hom}_{\mathbb{C}^d}(t,t)$ is well known. We now give these, as a consequence of the multinomial theorem. Let $i_1, \ldots, i_4 \in \mathbb{H}$ be given by

$$i_1 := 1, \qquad i_2 := i, \qquad i_3 = j, \qquad i_4 := k. \tag{8.40}$$

For a polynomial $f(x)$, in the variables $x = (x_1, \ldots, x_d) \in \mathbb{F}^d$,

$$x_j = x_{j1} i_1 + x_{j2} i_2 + \cdots + x_{jm} i_m \in \mathbb{F}, \quad x_{j1}, \ldots, x_{jm} \in \mathbb{R}, \qquad 1 \le j \le d, \tag{8.41}$$

we the define the differential operator $f(D)$ by replacing x_{jk} by $\frac{\partial}{\partial x_{jk}}$, in the usual way. Also for $f(z) = z^\alpha \overline{x}^\beta$, $z \in \mathbb{C}^d$, $z_j = x_j + i y_j$, we define $f(\partial)$ to be the differential operator obtained by replacing z^α and $\overline{z}^\beta$ by ∂^α and $\overline{\partial}$, the multivariate Wirtinger operators given by

$$\partial_j = \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \qquad \overline{\partial}_j = \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

Example 8.1 *The space* $\mathrm{Hom}_{\mathbb{R}^d}(t,t)$ *is* $\mathrm{Hom}_{\mathbb{R}^d}(2t)$*, the homogeneous polynomials of degree* $2t$*. Each polynomial can be written in terms of the monomial basis*

$$f(x) = \sum_{|\alpha|=2t} f_\alpha x^\alpha, \qquad f_\alpha \in \mathbb{R}.$$

By the multinomial theorem,

$$K_w(z) = (\langle w, z \rangle)^{2t} = \Big(\sum_j w_j z_j \Big)^{2t} = \sum_{|\alpha|=2t} \binom{2t}{\alpha} w^\alpha z^\alpha,$$

so that

$$\langle K_w, (\cdot)^\beta \rangle_{\mathbb{R}} = \langle \sum_{|\alpha|=2t} \binom{2t}{\alpha} w^\alpha (\cdot)^\alpha, (\cdot)^\beta \rangle_{\mathbb{R}} = \sum_{|\alpha|=2t} \binom{2t}{\alpha} \langle (\cdot)^\alpha, (\cdot)^\beta \rangle_{\mathbb{R}} w^\alpha = w^\beta, \quad \forall w, \ \forall \beta$$

if and only if $\binom{2t}{\alpha} \langle (\cdot)^\alpha, (\cdot)^\beta \rangle_{\mathbb{R}} = \delta_{\alpha\beta}$*, which gives the inner product*

$$\langle f, g \rangle_{\mathbb{R}} = \frac{1}{(2t)!} \sum_{|\alpha|=2t} \alpha! f_\alpha g_\alpha = \frac{1}{(2t)!} \sum_{|\alpha|=2t} \frac{D^\alpha f(0) D^\alpha g(0)}{\alpha!} = \frac{1}{(2t)!} f(D) g. \tag{8.42}$$

The inner product (8.42) is variously known as the *Bombieri* inner product [Zei94] or the *apolar* inner product/pairing [Veg00]. In this (unitarily invariant) inner product, the monomials are orthogonal, and so it is not a scalar multiple of the one given by integration on $\mathbb{S}$ (for which x_1^2 and x_2^2 are not orthogonal).

Example 8.2 *The space* $\mathrm{Hom}_{\mathbb{C}^d}(t,t)$ *has a basis given by the monomials*

$$m_{\alpha,\beta} : z \mapsto z^\alpha \overline{z}^\beta, \qquad |\alpha| = |\beta| = t,$$

so that each $f \in \mathrm{Hom}_{\mathbb{C}^d}(t,t)$ *can be written uniquely*

$$f = \sum_{|\alpha|=|\beta|=t} f_{\alpha,\beta}\, m_{\alpha,\beta}, \qquad f_{\alpha,\beta} \in \mathbb{C}.$$

The multinomial theorem gives

$$K_w(z) = (\langle w, z\rangle)^t (\langle z, w\rangle)^t = \Big(\sum_j \overline{w_j} z_j\Big)^t \Big(\sum_k w_k \overline{z_k}\Big)^t = \sum_{|\alpha|=t} \binom{t}{\alpha} \overline{w}^\alpha z^\alpha \sum_{|\beta|=t} \binom{t}{\beta} w^\beta \overline{z}^\beta,$$

so that

$$\begin{aligned} \langle K_w, m_{a,b}\rangle_{\mathbb{C}} &= \langle \sum_{|\alpha|=|\beta|=t} \binom{t}{\alpha}\binom{t}{\beta} \overline{w}^\alpha w^\beta m_{\alpha,\beta}, m_{a,b}\rangle_{\mathbb{C}} \\ &= \sum_{|\alpha|=|\beta|=t} \binom{t}{\alpha}\binom{t}{\beta} w^\alpha \overline{w}^\beta \langle m_{\alpha,\beta}, m_{a,b}\rangle_{\mathbb{C}} = w^a \overline{w}^b, \qquad \forall w,\ \forall a, b, \end{aligned}$$

if and only if $\binom{t}{\alpha}\binom{t}{\beta}\langle m_{\alpha,\beta}, m_{a,b}\rangle_{\mathbb{C}} = \delta_{(\alpha,\beta),(a,b)}$*, which gives*

$$\langle f, g\rangle_{\mathbb{C}} = \frac{1}{t!^2} \sum_{|\alpha|=|\beta|=t} \alpha!\beta!\, \overline{f_{\alpha,\beta}} g_{\alpha,\beta} = \frac{1}{t!^2} \tilde{f}(\partial) g, \tag{8.43}$$

where $\tilde{f}(z) := \overline{f(\overline{z})}$*, i.e,* $\tilde{f} = \sum_{|\alpha|=|\beta|=t} \overline{f_{\alpha,\beta}}\, m_{\alpha,\beta}$.

The inner product (8.43) can be found in [KP17] and [Wal17]. Initially, it seemed to be impossible to find the quaternion analogue of (8.42) and (8.43), without first finding a basis for $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$ and an analogue of the Wirtinger calculus. With hindsight, we observe that

$$f(z) = z_j \quad \Longrightarrow \quad \overline{f}(z) := \overline{f(z)} = \overline{z_j} \quad \Longrightarrow \quad f(D) = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} = 2f(\partial_j),$$

$$f(z) = \overline{z_j} \quad \Longrightarrow \quad \overline{f}(z) := \overline{f(z)} = z_j \quad \Longrightarrow \quad f(D) = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j} = 2f(\overline{\partial}_j),$$

so that

$$\langle f, g\rangle_{\mathbb{C}} = \frac{1}{t!^2} \tilde{f}(\partial) g = \frac{1}{t!^2 2^{2t}} \overline{f}(D) g. \tag{8.44}$$

In Theorem 8.2, we give the quaternionic analogue of the (apolar) inner products (8.42) and (8.44).

To understand $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$, we first consider the simple example $t = 1$, $d = 2$.

Example 8.3 $\mathrm{Hom}_{\mathbb{H}^2}(1,1)$ *has the following basis of six real-valued polynomials*

$$\begin{aligned}
|q_1|^2 &= t_1^2+x_1^2+y_1^2+z_1^2, \\
|q_2|^2 &= t_2^2+x_2^2+y_2^2+z_2^2, \\
\mathrm{Re}(q_1\overline{q_2}) &= t_1t_2+x_1x_2+y_1y_2+z_1z_2, \\
\mathrm{Re}(q_1\overline{q_2}i) &= t_1x_2-x_1t_2+y_1z_2-z_1y_2, \\
\mathrm{Re}(q_1\overline{q_2}j) &= t_1y_2-x_1z_2-y_1t_2+z_1x_2, \\
\mathrm{Re}(q_1\overline{q_2}k) &= t_1z_2+x_1y_2-y_1x_2-z_1t_2, \qquad (8.45)
\end{aligned}$$

where $q_a=t_a+x_ai+y_aj+z_ak$. *To see this, we expand* $|\langle v,q\rangle|^2=\langle v,q\rangle\langle q,v\rangle$ *as*

$$|\langle v,q\rangle|^2=(\overline{v_1}q_1+\overline{v_2}q_2)(\overline{q_1}v_1+\overline{q_2}v_2)=|q_1|^2|v_1|^2+|q_2|^2|v_2|^2+\overline{v_1}q_1\overline{q_2}v_2+\overline{v_2}q_2\overline{q_1}v_1,$$

where

$$\overline{v_1}q_1\overline{q_2}v_2+\overline{v_2}q_2\overline{q_1}v_1=2\,\mathrm{Re}(\overline{v_1}q_1\overline{q_2}v_2)=2\,\mathrm{Re}(q_1\overline{q_2}v_2\overline{v_1}),$$

by (1.3). Write $v_2\overline{v_1}=\alpha=\alpha_1+\alpha_2i+\alpha_3j+\alpha_4k$, $\alpha_j\in\mathbb{R}$, *and expand, to obtain*

$$\begin{aligned}
|\langle v,q\rangle|^2 &= |q_1|^2|v_1|^2+|q_2|^2|v_2|^2+2\,\mathrm{Re}(q_1\overline{q_2}(\alpha_1+\alpha_2i+\alpha_3j+\alpha_4k)) \\
&= |q_1|^2|v_1|^2+|q_2|^2|v_2|^2+2\,\mathrm{Re}(q_1\overline{q_2})\alpha_1+2\,\mathrm{Re}(q_1\overline{q_2}i)\alpha_2+2\,\mathrm{Re}(q_1\overline{q_2}j)\alpha_3+2\,\mathrm{Re}(q_1\overline{q_2}k)\alpha_4,
\end{aligned}$$

where $\alpha_r=\mathrm{Re}(\overline{\alpha}i_r)=\mathrm{Re}(v_1\overline{v_2}i_r)$ *and* $(i_1,i_2,i_3,i_4)=(1,i,j,k)$. *Thus the six linearly independent polynomials in (8.45) span* $\mathrm{Hom}_{\mathbb{H}^2}(1,1)$, *and hence are a basis.*

The above calculation generalises, to give the following.

Lemma 8.1 *The spanning polynomials* $|\langle v,\cdot\rangle|^2$, $v\in\mathbb{H}^d$, *for* $\mathrm{Hom}_{\mathbb{H}^d}(1,1)$ *can be written*

$$|\langle v,q\rangle|^2=\sum_{j=1}^d|q_j|^2|v_j|^2+\sum_{1\le j<k\le d}\sum_{r=1}^4 2\,\mathrm{Re}(q_j\overline{q_k}i_r)\,\mathrm{Re}(v_j\overline{v_k}i_r), \qquad (8.46)$$

where $(i_1,i_2,i_3,i_4)=(1,i,j,k)$ *and the* $d+4\binom{d}{2}$ *polynomials*

$$\begin{aligned}
p_j &: q\mapsto|q_j|^2, && 1\le j\le d, \\
p_{jkr} &: q\mapsto\mathrm{Re}(q_j\overline{q_k}i_r), && 1\le j<k\le d,\ 1\le r\le 4, \qquad (8.47)
\end{aligned}$$

are a basis for $\mathrm{Hom}_{\mathbb{H}^d}(1,1)$.

Proof: The expansion (8.46) follows as in Example 8.3. Moreover, the polynomials in (8.47) are linearly independent. This is easily seen from the formulas for them given in (8.45), e.g., the coordinate functionals are given explicitly by

$$\begin{aligned}
f &\mapsto \frac{1}{2}\partial_{t_j}^2 f(0), && 1\le j\le d, \\
f &\mapsto \partial_{t_j}\partial_{(q_k)_r}f(0), && 1\le j<k\le d,\ 1\le r\le 4, \qquad (8.48)
\end{aligned}$$

where $q_a=t_a+x_ai+y_aj+z_ak$. □

The expansion (8.46) can also be written in the symmetric (but redundant) form

$$|\langle v,q\rangle|^2=\sum_{j=1}^d\sum_{k=1}^d\sum_{r=1}^4 \operatorname{Re}(q_j\overline{q_k}i_r)\ \operatorname{Re}(v_j\overline{v_k}i_r). \tag{8.49}$$

Let

$$P=(p_j)_{1\le j\le d}\cup(\sqrt{2}p_{jkr})_{1\le j<k\le d,1\le r\le 4},\qquad Q=(p_{jkr})_{1\le j,k\le d,1\le r\le 4}, \tag{8.50}$$

be the basis and spanning set for $\operatorname{Hom}_{\mathbb{H}^d}(1,1)$ given by the polynomials defined in (8.47). Since the coordinates of P and Q are real-valued functions, we can take monomials in P and Q in the usual way, i.e.,

$$P^\alpha=\Bigl(\prod_{1\le j\le d}p_j^{\alpha_j}\Bigr)\Bigl(\prod_{\substack{1\le j<k\le d\\ 1\le r\le 4}}(\sqrt{2}p_{jkr})^{\alpha_{jkr}}\Bigr),\qquad Q^\beta=\prod_{\substack{1\le j,k\le d\\ 1\le r\le 4}}p_{jkr}^{\beta_{jkr}},$$

where α and β are multi-indices defined on the index sets of P and Q.

Theorem 8.1 *Let P and Q be given by (8.50). There is a unique inner product on $\operatorname{Hom}_{\mathbb{H}^d}(t,t)$ for which $(\binom{t}{\alpha}^{1/2}P^\alpha)_{|\alpha|=t}$ and $(\binom{t}{\beta}^{1/2}Q^\beta)_{|\beta|=t}$ are normalised tight frames. The reproducing kernel for this inner product is $K_w(z)=|\langle z,w\rangle|^{2t}$.*

Proof: In [Wal11] a notion of *canonical coordinates* for a finite spanning sequence for a real or complex vector space was developed, from which it follows (Theorem 4.3 of [Wal18]) that there is a unique (canonical) inner product for which it is a normalised tight frame. We now appeal to the quaternionic version of this result, which holds. Briefly, for a given spanning sequence $(f_j)_{j=1}^n$ and f, the set of coefficients $c=(c_j)$, for which $f=\sum_j c_jf_j$, is an affine subspace of $\mathbb{F}^n$, and hence it has a unique element $c=c(f)\in\mathbb{F}^n$ which minimises $\sum_j|c_j|^2$. The functional $f\mapsto c(f)$ is linear, and the canonical inner product between f and g is defined to be $\langle c(f),c(g)\rangle$.

We consider P (the argument for Q being the same). We may write (8.46) as

$$|\langle w,z\rangle|^2=\sum_j P_j(z)\overline{P_j(w)},\qquad P=(P_j),$$

and so the multinomial theorem gives

$$|\langle w,z\rangle|^{2t}=\Bigl(\sum_j P_j(z)\overline{P_j(w)}\Bigr)^t=\sum_{|\alpha|=t}\binom{t}{\alpha}P^\alpha(z)\overline{P^\alpha(w)}.$$

This motivates our choice. Let $\langle\cdot,\cdot\rangle_P$ be the inner product for which $(\binom{t}{\alpha}^{1/2}P^\alpha)_{|\alpha|=t}$ is a normalised tight frame for its span $\mathcal{H}$. By Proposition 8.1 and the above, the reproducing kernel for $\mathcal{H}$ is

$$K_w(z)=\sum_{|\alpha|=t}\Bigl(\binom{t}{\alpha}^{1/2}\Bigr)^2P^\alpha(z)\overline{P^\alpha(w)}=|\langle w,z\rangle|^{2t},$$

with $\mathcal{H}=\operatorname{span}\{K_w:w\in\mathbb{H}^d\}=\operatorname{Hom}_{\mathbb{H}^d}(t,t)$. □

The polynomials $\{P^\alpha\}_{|\alpha|=t}$ are not a basis for $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$, in general.

Example 8.4 *The space* $\mathrm{Hom}_{\mathbb{H}^2}(2,2)$ *has dimension* 20*, and there are* 21 *polynomials in* $\{P^\alpha\}_{|\alpha|=2}$*. Therefore there is one linear dependency, which is*

$$\mathrm{Re}(q_1\overline{q_2})^2+\mathrm{Re}(q_1\overline{q_2}i)^2+\mathrm{Re}(q_1\overline{q_2}j)^2+\mathrm{Re}(q_1\overline{q_2}k)^2=|q_1|^2|q_2|^2.$$

From this, it follows that $\{P^\alpha\}_{|\alpha|=t}$ *is not a basis for* $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$*, for* $t\ge 2$ *(and* $d>1$*).*

Theorem 8.1 is sufficient to prove Theorem 4.1. We now give a more constructive version (Theorem 8.2). With the notation of (8.40), a simple calculation gives

$$\sum_{r=1}^m i_r q i_r = \begin{cases} q, & m=1;\\ 0, & m=2;\\ -2\overline{q}, & m=4,\end{cases}\qquad \sum_{r=1}^m i_r q\overline{i_r} = \begin{cases} q, & m=1;\\ 2q, & m=2;\\ 4\,\mathrm{Re}(q), & m=4,\end{cases}\qquad q\in\mathbb{F}, \tag{8.51}$$

where $m=\dim_{\mathbb{R}}(\mathbb{F})$. Let $\Delta=\sum_j\sum_r \frac{\partial^2}{\partial x_{jr}^2}$ be the Laplacian on functions $\mathbb{F}^d\to\mathbb{F}$.

Proposition 8.2 *For* $w,v\in\mathbb{F}^d$*, we have*

$$\langle w,D\rangle\langle\cdot,v\rangle=m\langle w,v\rangle,\qquad \langle w,D\rangle\langle v,\cdot\rangle=(2-m)\langle w,v\rangle, \tag{8.52}$$

$$\langle D,w\rangle\langle\cdot,v\rangle=(2-m)\langle w,v\rangle,\qquad \langle D,w\rangle\langle v,\cdot\rangle=\begin{cases} m\langle v,w\rangle, & m=1,2;\\ 4\,\mathrm{Re}\langle v,w\rangle, & m=4.\end{cases} \tag{8.53}$$

Proof: We use (8.51). Expanding gives

$$\langle v,x\rangle=\sum_j\overline{v_j}x_j=\sum_{j,r}\overline{v_j}x_{jr}i_r,\qquad \langle x,v\rangle=\sum_k\overline{x_k}v_k=\sum_{k,s}x_{ks}\overline{i_s}v_k,$$

so that (with x the variable)

$$\langle w,D\rangle\langle x,v\rangle=\sum_{j,r}\overline{w_j}i_r\frac{\partial}{\partial x_{jr}}\sum_{k,s}\overline{i_s}v_kx_{ks}=\sum_{j,r}\overline{w_j}i_r\overline{i_r}v_j=m\langle w,v\rangle.$$

Similarly, taking the sum over r for $m=4$, we have

$$\langle w,D\rangle\langle v,x\rangle=\sum_{j,r}\overline{w_j}i_r\frac{\partial}{\partial x_{jr}}\sum_{k,s}\overline{v_k}i_sx_{ks}=\sum_{j,r}\overline{w_j}i_r\overline{v_j}i_r=-2\sum_j\overline{w_j}v_j=-2\langle w,v\rangle,$$

with the cases $m=1,2$ following by similar calculations. This gives (8.52).

The remaining equations follow from

$$\langle D,w\rangle\langle x,v\rangle=\sum_{j,r}\overline{i_r}w_j\frac{\partial}{\partial x_{jr}}\sum_{k,s}\overline{i_s}v_kx_{ks}=\sum_{j,r}\overline{i_r}w_j\overline{i_r}v_j=\sum_j\Big(\sum_r i_rw_ji_r\Big)v_j,$$

$$\langle D,w\rangle\langle v,x\rangle=\sum_{j,r}\overline{i_r}w_j\frac{\partial}{\partial x_{jr}}\sum_{k,s}\overline{v_k}i_sx_{ks}=\sum_{j,r}\overline{i_r}w_j\overline{v_j}i_r=\sum_j\Big(\sum_r i_rw_j\overline{v_j}\overline{i_r}\Big),$$

and (8.51). □

In view of Proposition 8.2 the differential action of a plane wave on a plane wave is somewhat involved in the quaternionic case. Nevertheless, we have the following.

Lemma 8.2 *Let $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$. We have the differentiation formula for plane waves*

$$|\langle w,D\rangle|^2\left(|\langle v,\cdot\rangle|^{2t}\right)=2t(2t+m-2)|\langle v,w\rangle|^2|\langle v,\cdot\rangle|^{2t-2},\qquad v,w\in\mathbb{F}^d, \tag{8.54}$$

where $m=\dim_{\mathbb{R}}(\mathbb{F})$, and, in particular,

$$\Delta\left(|\langle v,\cdot\rangle|^{2t}\right)=2t(2t+m-2)\|v\|^2\langle v,\cdot\rangle|^{2t-2},\qquad v\in\mathbb{F}^d, \tag{8.55}$$

$$|\langle w,D\rangle|^2\left(\|\cdot\|^2\right)=2m\|w\|^2,\qquad w\in\mathbb{F}^d. \tag{8.56}$$

Proof: Since $|\langle w,D\rangle|^2=\langle w,D\rangle\langle D,w\rangle$ and $|\langle v,\cdot\rangle|^2$ is real-valued, the chain and product rules give

$$\begin{aligned}|\langle w,D\rangle|^2|\langle v,\cdot\rangle|^{2t}&=\langle w,D\rangle\left(t|\langle v,\cdot\rangle|^{2(t-1)}\langle D,w\rangle|\langle v,\cdot\rangle|^2\right)\\&=t(t-1)|\langle v,\cdot\rangle|^{2(t-2)}(\langle w,D\rangle|\langle v,\cdot\rangle|^2)(\langle D,w\rangle|\langle v,\cdot\rangle|^2)\\&\quad+t|\langle v,\cdot\rangle|^{2(t-1)}\langle w,D\rangle\langle D,w\rangle|\langle v,\cdot\rangle|^2.\end{aligned} \tag{8.57}$$

A calculation (to follow) gives

$$\langle w,D\rangle|\langle v,\cdot\rangle|^2=2\langle w,v\rangle\langle v,\cdot\rangle,\qquad\langle D,w\rangle|\langle v,\cdot\rangle|^2=2\langle\cdot,v\rangle\langle v,w\rangle, \tag{8.58}$$

$$\langle w,D\rangle\langle D,w\rangle|\langle v,\cdot\rangle|^2=|\langle w,D\rangle|^2|\langle v,\cdot\rangle|^2=2m|\langle v,w\rangle|^2, \tag{8.59}$$

and so (8.57) simplifies to (8.54).

Since $\Delta=\|D\|^2=|\langle e_1,D\rangle|^2+\cdots+|\langle e_d,D\rangle|^2$, we obtain (8.55) from (8.54), i.e.,

$$\begin{aligned}\Delta\left(|\langle v,\cdot\rangle|^{2t}\right)&=\sum_j|\langle e_j,D\rangle|^2|\langle v,\cdot\rangle|^{2t}=2t(2t+m-2)\sum_j|\langle v,e_j\rangle|^2|\langle v,\cdot\rangle|^{2t-2}\\&=2t(2t+m-2)\sum_j|v_j|^2|\langle v,\cdot\rangle|^{2t-2}=2t(2t+m-2)\|v\|^2|\langle v,\cdot\rangle|^{2t-2}.\end{aligned}$$

Since $\|\cdot\|^2=|\langle e_1,\cdot\rangle|^2+\cdots|\langle e_d,\cdot\rangle|^2+$, we obtain (8.56) from (8.54), i.e.,

$$|\langle w,D\rangle|^2\left(\|\cdot\|^2\right)=\sum_j|\langle w,D\rangle|^2|\langle e_j,\cdot\rangle|^2=\sum_j 2m|\langle e_j,w\rangle|^2=2m\sum_j|w_j|^2=2m\|w\|^2.$$

To prove (8.58) and (8.59), we need to to use the expansion (8.41), which gives

$$\langle v,x\rangle=\sum_{j=1}^d\overline{v_j}x_j=\sum_{j=1}^d\sum_{r=1}^m v_{jr}\overline{i_r}\sum_{s=1}^m x_{js}i_s=\sum_{j,r,s}\overline{i_r}i_s v_{jr}x_{js}.$$

Here the variables $v_{jr},x_{js}\in\mathbb{R}$, and so commute with any factor. Thus

$$\begin{aligned}\langle w,D\rangle|\langle v,x\rangle|^2&=\sum_{j,r,s}\overline{i_r}i_s w_{jr}\frac{\partial}{\partial x_{js}}\sum_{j_1,r_1,s_1}\overline{i_{r_1}}i_{s_1}v_{j_1r_1}x_{j_1s_1}\sum_{j_2,r_2,s_2}\overline{i_{r_2}}i_{s_2}x_{j_2r_2}v_{j_2s_2}\\&=\sum_{j,r,s}\sum_{\substack{j_1,r_1,s_1\\j_2,r_2,s_2}}\overline{i_r}i_s\overline{i_{r_1}}i_{s_1}\overline{i_{r_2}}i_{s_2}w_{jr}v_{j_1r_1}v_{j_2s_2}\frac{\partial}{\partial x_{js}}(x_{j_1s_1}x_{j_2r_2}).\end{aligned}$$

By the product rule, the derivative in the expression above is $\delta_{js,j_1s_1}x_{j_2r_2}+x_{j_1s_1}\delta_{js,j_2r_2}$. Using (8.51), for $m=4$, we calculate the $(j,s)=(j_2,r_2)$ terms to be

$$\sum_{j,r,s}\sum_{\substack{j_1,r_1,s_1\\ s_2}}\overline{i_r}i_s\overline{i_{r_1}}i_{s_1}\overline{i_s}i_{s_2}w_{jr}v_{j_1r_1}v_{js_2}x_{j_1s_1}=\sum_{j,s,j_1}\overline{w_j}i_s\overline{v_{j_1}}x_{j_1}\overline{i_s}v_j=\sum_j\overline{w_j}\Big(\sum_s i_s\langle v,x\rangle\overline{i_s}\Big)v_j$$
$$=\sum_j\overline{w_j}(4\operatorname{Re}\langle v,x\rangle)v_j=4\langle w,v\rangle\operatorname{Re}(\langle x,v\rangle),$$

and those for $(j,s)=(j_1,s_1)$ to be

$$\sum_{j,r,s}\sum_{\substack{r_1\\ j_2,r_2,s_2}}\overline{i_r}i_s\overline{i_{r_1}}i_s\overline{i_{r_2}}i_{s_2}w_{jr}v_{jr_1}v_{j_2s_2}x_{j_2r_2}=\sum_{j,s,j_2}\overline{w_j}i_s\overline{v_j}i_s\overline{x_{j_2}}v_{j_2}=\sum_j\overline{w_j}\Big(\sum_s i_s\overline{v_j}i_s\Big)\langle x,v\rangle$$
$$=\sum_j\overline{w_j}(-2v_j)\langle x,v\rangle=-2\langle w,v\rangle\langle x,v\rangle.$$

Hence

$$\begin{aligned}\langle w,D\rangle|\langle v,x\rangle|^2&=4\langle w,v\rangle\operatorname{Re}\langle x,v\rangle-2\langle w,v\rangle\langle x,v\rangle=2\langle w,v\rangle(2\operatorname{Re}\langle x,v\rangle-\langle x,v\rangle)\\&=2\langle w,v\rangle\overline{\langle x,v\rangle}=2\langle w,v\rangle\langle v,x\rangle,\end{aligned}$$

and, similarly,

$$\begin{aligned}\langle D,w\rangle|\langle v,x\rangle|^2&=\sum_{j,r,s}\sum_{\substack{j_1,r_1,s_1\\ j_2,r_2,s_2}}\overline{i_r}i_s\overline{i_{r_1}}i_{s_1}\overline{i_{r_2}}i_{s_2}w_{js}v_{j_1r_1}v_{j_2s_2}(\delta_{jr,j_1s_1}x_{j_2r_2}+\delta_{jr,j_2r_2}x_{j_1s_1})\\&=4\operatorname{Re}(\langle\overline{w},\overline{v}\rangle)\langle x,v\rangle-2\langle x,v\rangle\langle w,v\rangle=2\langle x,v\rangle(2\operatorname{Re}\langle w,v\rangle-\langle w,v\rangle)\\&=2\langle x,v\rangle\overline{\langle w,v\rangle}=2\langle x,v\rangle\langle v,w\rangle.\end{aligned}$$

The calculations for $m=1,2$ are simpler, and give the same formulas. Finally, by the first equation in (8.52), we have

$$\begin{aligned}|\langle D,w\rangle|^2|\langle v,\cdot\rangle|^2&=\langle w,D\rangle(\langle D,w\rangle|\langle v,\cdot\rangle|^2)=\langle w,D\rangle(2\langle\cdot,v\rangle\langle v,w\rangle)\\&=2m\langle w,v\rangle\langle v,w\rangle=2m|\langle v,w\rangle|^2.\end{aligned}\tag{8.60}$$

□

We note the subtlety in the calculations above, e.g., the product rule holds if one factor is real-valued, but not if both are $\mathbb{H}$-valued, and the differential operator $\langle w,D\rangle$ does not commute with quaternion scalars.

Example 8.5 *It follows from (8.54), that $|\langle w,D\rangle|^2$ maps $\operatorname{Hom}_{\mathbb{F}^d}(t+1,t+1)$ to $\operatorname{Hom}_{\mathbb{F}^d}(t,t)$. Using $\langle D,w\rangle\|\cdot\|^2=2\langle\cdot,w\rangle$ and $\langle w,D\rangle\|\cdot\|^2=2\langle w,\cdot\rangle$, we have*

$$\begin{aligned}|\langle w,D\rangle|^2\left(\|\cdot\|^{2t}\right)&=\langle w,D\rangle\langle D,w\rangle\left(\|\cdot\|^{2t}\right)=\langle w,D\rangle\left(t\|\cdot\|^{2(t-1)}(2\langle\cdot,w\rangle)\right)\\&=2t\|\cdot\|^{2(t-1)}m\langle w,w\rangle+t(t-1)\|\cdot\|^{2(t-2)}(2\langle w,\cdot\rangle)(2\langle\cdot,w\rangle)\\&=2mt\|w\|^2\|\cdot\|^{2(t-1)}+4t(t-1)|\langle w,\cdot\rangle|^2\|\cdot\|^{2(t-2)},\qquad t\ge1,\end{aligned}$$

so that $|\langle w,\cdot\rangle|^2\|\cdot\|^{2(t-1)}\in\operatorname{Hom}_{\mathbb{F}^d}(t,t)$. Continuing in this way, one obtains that $|\langle v,\cdot\rangle|^{2r}\|\cdot\|^{2t-2r}\in\operatorname{Hom}_{\mathbb{F}^d}(t,t)$, $0\le r\le t$ (see [MW20] for details).

Theorem 8.2 *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is a unique inner product on $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ for which $|\langle z, w\rangle|^{2t}$ is the reproducing kernel. It is given by*

$$\langle f, g\rangle_{\mathbb{F}} := \frac{1}{b_{t,m}} \overline{f}(D) g, \qquad b_{t,m} := \prod_{j=1}^{t} 2j(2j+m-2). \tag{8.61}$$

Proof: We first show that $\langle\!\langle f, g\rangle\!\rangle := \overline{f}(D)g$ defines an inner product on $\mathrm{Pol}_{2t}(\mathbb{F}^d)$, and hence on $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$. Using the notation of (8.41), we may write $f \in \mathrm{Pol}_{2t}(\mathbb{F}^d)$ as

$$f = \sum_{|\alpha|=2t} f_\alpha m_\alpha, \qquad m_\alpha(x) := (x_{jr})^{\alpha}_{1\le j\le d, 1\le r\le m}, \quad f_\alpha \in \mathbb{F}.$$

Then

$$\langle\!\langle f, g\rangle\!\rangle = \sum_{|\alpha|=2t} \overline{f_\alpha} m_\alpha(D) \sum_{|\beta|=2t} g_\beta m_\beta = \sum_{|\alpha|=2t} \alpha! \overline{f_\alpha} g_\alpha, \tag{8.62}$$

which is clearly the (weighted) Euclidean inner product (and hence is an inner product).

By t applications of (8.54) of Lemma 8.2, we have

$$\begin{aligned}|\langle w, D\rangle|^{2t} |\langle v, \cdot\rangle|^{2t} &= |\langle w, D\rangle|^{2(t-2)} \left(|\langle w, D\rangle|^2 |\langle v, \cdot\rangle|^{2t}\right) \\ &= 2t(2t+m-2)|\langle v, w\rangle|^2 \left(|\langle w, D\rangle|^{2(t-2)} |\langle v, \cdot\rangle|^{2t-2}\right) \\ &= \cdots = b_{t,m} |\langle v, w\rangle|^{2t},\end{aligned}$$

i.e., with $K_w(z) = |\langle z, w\rangle|^{2t}$ and $f = |\langle v, \cdot\rangle|^{2t}$,

$$\langle K_w, f\rangle_{\mathbb{F}} = |\langle v, w\rangle|^{2t} = f(w), \qquad \forall w \in \mathbb{F}^d,$$

so that $|\langle w, z\rangle|^{2t}$ is the reproducing kernel for $\langle\cdot,\cdot\rangle_{\mathbb{F}} = (1/b_{t,m})\langle\!\langle \cdot,\cdot\rangle\!\rangle$. □

We will refer to (8.61) as the **apolar** inner product, since it coincides with the apolar (Bombieri) inner product in the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$, i.e.,

$$\langle f, g\rangle_{\mathbb{R}} = \frac{1}{(2t)!} \overline{f}(D) g, \quad \langle f, g\rangle_{\mathbb{C}} = \frac{1}{2^{2t} t!^2} \overline{f}(D) g, \quad \langle f, g\rangle_{\mathbb{H}} = \frac{1}{2^{2t} t!(t+1)!} \overline{f}(D) g,$$

and see (8.42) and (8.44). Again, we observe this is not the inner product given by integration on the (quaternionic) sphere.

Example 8.6 *We have*

$$\langle |q_j|^{2t}, |q_j|^{2t}\rangle_{\mathbb{H}} = \langle K_{e_j}, K_{e_j}\rangle_{\mathbb{H}} = |\langle e_j, e_j\rangle|^{2t} = 1, \qquad 1 \le j \le d,$$

so the polynomials $p_j : q \mapsto q \mapsto |q_j|^{2t}$ in the normalised tight frames of Theorem 8.1 have unit norm, and hence (see [Wal18] Exercise 2.4) span orthogonal one-dimensional subspaces of $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$. By evaluating $\langle |x_1|^{2t}, |x_j|^{2t}\rangle_{\mathbb{F}}$ from (8.62), one gets the identity

$$\sum_{\substack{|\alpha|=t \\ \alpha \in \mathbb{Z}_+^m}} (2\alpha)! \binom{t}{\alpha}^2 = b_{t,m} = \prod_{j=1}^{t} 2j(2j+m-2).$$

9 Conclusion

We have proved the quaternionic analogue of the Welch-Sidlenikov inequality on the spacing of vectors/lines on unit sphere (Theorem 4.1), and shown that equality in it corresponds to a cubature rule for the unitarily invariant polynomial space $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$, which we call a spherical (t,t)-design. Consequences of this unified development for the real, complex and quaternionic cases include:

- A proof that projective spherical t-designs are precisely the spherical (t,t)-designs. Since (t,t)-designs are cubature rules for the path-connected topological space $\mathbb{S}$, it then follows from [SZ84] that these exist for any given t.

- The variational characterisation gives a simple condition for being a projective spherical t-design (Corollary 7.1).

- The variational characterisation allows for numerical constructions of spherical (t,t)-designs (Examples 6.4 and 6.5).

- The polynomial space $\mathrm{Hom}_{\mathbb{F}^d}(t,t)$ plays a key role. In the real and complex cases it is well understood with the bases given in Examples 8.1 and 8.2 implying that

$$\dim(\mathrm{Hom}_{\mathbb{R}^d}((t,t)) = \binom{d+2t-1}{2t}, \qquad \dim(\mathrm{Hom}_{\mathbb{C}^d}((t,t)) = \binom{d+t-1}{t}^2.$$

  We did not present a basis for $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$, instead using a normalised tight frame (Theorem 8.1). It can be shown (see [MW20]) that

$$\dim(\mathrm{Hom}_{\mathbb{H}^d}(t,t)) = \frac{1}{t+2d-1}\binom{t+2d-1}{t}\binom{t+2d-1}{t+1}.$$

Directions for further investigation include results for the octonionic sphere, such as the possibility of a Welch-Sidlenikov inequality, and cubature rules for unitarily invariant subspaces of $\mathrm{Hom}_{\mathbb{H}^d}(t,t)$.

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