Multivariate Lagrange interpolation and polynomials of one quaternionic variable

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Abstract

This paper considers the extension of classical Lagrange interpolation in one real or complex variable to "polynomials of one quaternionic variable". To do this we develop some aspects of the theory of such polynomials. We then give a number of related multivariate polynomial interpolation schemes for $\mathbb{R}^4$ and $\mathbb{C}^2$ with good geometric properties, and some aspects of least interpolation and of Kergin interpolation.

Key Words: quaternions, quaternionic polynomials, Lagrange interpolation, Lagrange polynomials, Newton form, Kergin interpolation, least interpolation, quaternionic equiangular lines.

AMS (MOS) Subject Classifications: primary 12E15, 41A05, 41A10, 41A63, secondary 12E10, 16D10.

1 Introduction

The **quaternions** $\mathbb{H}$ are a celebrated extension of the field of complex numbers to a noncommutative associative algebra over the real numbers (a skew-field) with elements

$$q = q_1 + q_2 i + q_3 j + q_4 k = (q_1 + q_2 i) + (q_3 + q_4 i)j \in \mathbb{H}, \qquad q_j \in \mathbb{R}.$$

and (see [Rod14] for the basic theory of $\mathbb{H}$, which is assumed)

$$i^2 = j^2 = k^2 - 1, \quad ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j.$$

The Lagrange interpolant Lf to a function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $\mathbb{R}$ or $\mathbb{C}$ is the unique polynomial of degree n matching the values of f at these points, which can be given explicitly as

$$Lf(x) := \sum_j \ell_j(x) f(x_j), \qquad \ell_j(x) := \prod_{j \neq k} \frac{x - x_k}{x_j - x_k}. \tag{1.1}$$

Formally, the above formula makes sense for points in $\mathbb{H}$, giving an interpolant. However, due to the noncommutativity of the quaternions, the Lagrange polynomials depend on the order in which the product is evaluated. This is the first indication that the quaternionic polynomials of degree n (as a right $\mathbb{H}$-module) might have a dimension greater than $n+1$ (depending on how they are defined). To resolve this impasse one could

- Look for an interpolant from a fixed $(n+1)$-dimensional subspace of quaternionic polynomials of degree n.
- Seek a "best" choice of Lagrange polynomials ℓ_j for given interpolation points, which would implicitly define an $(n+1)$-dimensional subspace of interpolants that is related to the geometry of the points.

The first approach has been considered by [Bol15], which we discuss in the next section. Our approach is the second. The essential features of each are (respectively):

- Interpolation is not possible for all configurations of points. The condition for unique interpolation and the interpolation space are not translation invariant. The Lagrange polynomials ℓ_j may have zeros which are not interpolation points. It is possible to develop a Newton form and notion of divided difference.
- Interpolation is possible for all configurations of points, and the interpolation space depends continuously on the points. The interpolant is translation invariant. The Lagrange polynomials ℓ_j can be chosen to be zero only at the interpolation points, and a Newton form can be developed.

We want polynomials and functions of a quaternionic variable to be an "$\mathbb{H}$-vector space". To do this, we view such spaces a right $\mathbb{H}$-module (and co-opt the language of linear algebra). Linear maps (which are technically $\mathbb{H}$-homomorphisms) then act on the left, with the usual algebra of matrices then extending in the obvious way (cf. [Rod14]). The Lagrange interpolant, as defined above, is an $\mathbb{H}$-linear map, since

$$L(f\alpha + g\beta)(x) = \sum_j \ell_j(x)(f\alpha + g\beta)(x_j) = \sum_j \ell_j(x)(f(x_j)\alpha + g(x_j)\beta) = Lf(x)\alpha + Lg(x)\beta.$$

2 Lagrange interpolation from $\mathbb{H}[z]$

Bolotnikov [Bol15], [Bol20] uses the formal polynomials $\mathbb{H}[z]$ in z

$$f(z) = z^n f_n + \cdots + z f_1 + f_0, \qquad f_0, \ldots, f_n \in \mathbb{H},$$

on which a left and right evaluation at $a \in \mathbb{H}$ are defined by

$$f^{e_l}(a) := \sum_j a^j f_j, \qquad f^{e_r}(a) := \sum_j f_j a^j.$$

For $\mathbb{H}[z]$ as right vector space, left evaluation is linear but right evaluation is not.

The (left) Lagrange interpolation of [Bol15] is from the $(n+1)$-dimensional subspace of polynomials of degree n in $\mathbb{H}[z]$ to left evaluation at $n+1$ points in $\mathbb{H}$. Let us consider an example, to see the nature of this interpolation.

Example 2.1 *There is a unique linear interpolant $p(z) = p_0 + z p_1$ to a function f at any distinct points $a, b \in \mathbb{H}$ given by the Lagrange polynomial formula*

$$\begin{aligned} p(z) &= (z-b)(a-b)^{-1} f(a) + (z-a)(b-a)^{-1} f(b) \\ &= \bigl(b(b-a)^{-1} f(a) a(a-b)^{-1} f(b)\bigr) + z\left((a-b)^{-1} f(a) - a(b-a)^{-1} f(b)\right). \end{aligned}$$

Now we consider interpolation at the three points $i, j, c \in \mathbb{H}$. Up to a scalar, the quadratic Lagrange polynomial from $\mathbb{H}[z]$ which is zero at i and j is

$$p(z) = 1 + z^2,$$

while those given by the Lagrange polynomial formula are

$$p_1(x) = (x-i)(x-j), \qquad p_2(x) = (x-j)(x-i).$$

We note that p is zero at all quaternions q with $q^2 = -1$, equivalently, $\mathrm{Re}(q) = 0$, $|q| = 1$, e.g., $q = k$, whereas p_1 and p_2 are zero precisely at $q = i, j$, and they are not the same polynomial since $p_1(1) \neq p_2(1)$. The theory of [Bol15] is based on the Euclidean algorithm for the associative multiplication on $\mathbb{H}[z]$ given by

$$\Bigl(\sum_j z^j a_j\Bigr) * \Bigl(\sum_k z^k b_k\Bigr) := \sum_{j,k} z^{j+k} a_j b_k.$$

As an example, the "root" $z = i$ of $p(z)$ gives a "linear factor" as follows

$$(z^2+1) - ((z-i)*z + (z-i)*i) = 0 \quad \Longrightarrow \quad z^2 + 1 = (z-i)*(z+i).$$

Note that $f(z) := (z-i)*(z-j) = z^2 - z(i+j) + k$ has $z = i$ as a left root, i.e., $f^{e_l}(i) = 0$, but not $z = j$ (which is a right root).

Two quaternions q_1 and q_2 are said to be **similar** (the term *equivalent* is used in [Bol15]) if $q_2 = a q_1 a^{-1}$ for some nonzero $a \in \mathbb{H}$. This is equivalent to $\mathrm{Re}(q_1) = \mathrm{Re}(q_2)$ and $|q_1| = |q_2|$. Hence i, j, k are similar and 1 and either of j, k are not.

The left point evaluations $\delta_a : \mathbb{H}[z] \to \mathbb{H} : f \mapsto f^{e_l}(a)$, $a \in \mathbb{H}$, are $\mathbb{H}$-linear functionals on the right vector space $\mathbb{H}[z]$. These span a left vector space, with linear dependencies given by following:

Lemma 2.1 *([Bol15] Lemma 3.1) For $f \in \mathbb{H}[z]$ and $a, b, c \in H$ distinct and similar*

$$f^{e_l}(c) = (c-b)(a-b)^{-1} f^{e_l}(a) + (c-a)(b-a)^{-1} f^{e_l}(b), \tag{2.2}$$

i.e.,

$$\delta_c = (c-b)(a-b)^{-1}\delta_a + (c-a)(b-a)^{-1}\delta_b. \tag{2.3}$$

To see this in play, we consider left quadratic Lagrange interpolation from $\mathbb{H}[z]$.

Example 2.2 *We seek a quadratic polynomial $p(z) = p_0 + zp_1 + z^2 p_2$ which left Lagrange interpolates a function f at distinct points $a, b, c \in \mathbb{H}$, i.e.,*

$$\begin{aligned} p^{e_l}(a) &= p_0 + ap_1 + a^2 p_2 = f(a), \\ p^{e_l}(b) &= p_0 + bp_1 + b^2 p_2 = f(b), \\ p^{e_l}(c) &= p_0 + cp_1 + c^2 p_2 = f(c). \end{aligned}$$

Gauss elimination gives the row echelon form

$$\begin{aligned} p_0 + ap_1 + a^2 p_2 &= f(a), \\ p_1 + (b-a)^{-1}(b^2-a^2)p_2 &= (b-a)^{-1}(f(b)-f(a)), \\ \{(c-a)^{-1}(c^2-a^2) - (b-a)^{-1}(b^2-a^2)\}p_2 &= (c-a)^{-1}(f(c)-f(a)) - (b-a)^{-1}(f(b)-f(a)), \end{aligned}$$

and so there is a unique interpolant to every f if and only if

$$(c-a)^{-1}(c^2-a^2) - (b-a)^{-1}(b^2-a^2) \neq 0.$$

It is easily seen that equality above is equivalent to taking $f(z) = z^2$ in (2.2), i.e.,

$$c^2 = (c-b)(a-b)^{-1}a^2 + (c-a)(b-a)^{-1}b^2 \iff (c-a)^{-1}(c^2-a^2) = (b-a)^{-1}(b^2-a^2).$$

The Gauss elimination argument above shows that a necessary condition for left (or right) Lagrange interpolation by a polynomial in $\mathbb{H}[z]$ of degree n to any f to $n+1$ distinct points in $\mathbb{H}$ is that no three of the points are similar. This is in fact sufficient.

Theorem 2.1 *([Bol15] Theorem 3.3) Left (or right) Lagrange interpolation from the polynomials of degree n in $\mathbb{H}[z]$ to $n+1$ points in $\mathbb{H}$ is uniquely possible if and only if the no three of the points are similar, i.e., have the same modulus and real part.*

Corollary 2.1 *For a set $A \subset \mathbb{H}$ the linear functionals $\{\delta_a\}_{a\in A}$ given by*

$$\delta_a : \mathbb{H}[z] \to \mathbb{H} : f \mapsto f^{e_l}(a),$$

are $\mathbb{H}$-linearly independent if and only if no three of them are similar.

Proof: If three of the points $a, b, c \in A$ are similar, then we have the nontrivial linear dependency (2.3). Conversely, suppose that no three points are similar. Take a linear combination

$$\sum_{j=0}^{n} c_j \delta_{a_j} = 0, \qquad a_j \in A, \quad c_j \in \mathbb{H},$$

and apply both sides of this to the unique Lagrange interpolant to the function which is zero at all the points in $\{a_0, a_1, \dots, a_n\}$, except a_j, to conclude that $c_j = 0$. □

These results were developed from the notion of *P-independence* [Lam86], [LL88], i.e., the set of $n+1$ points A is (left) P-independent (polynomial independent) if the linear functionals $\{\delta_a\}_{a\in A}$ above are linearly independent, or, equivalently, there is a subspace of $\mathbb{H}[z]$ of dimension $n+1$ from which unique (left) Lagrange interpolation is possible. This has recently been explored in the multivariate setting [MPnK19], [Mar20].

Corollary 2.2 *There is a unique quadratic left Lagrange interpolant from $\mathbb{H}[z]$ to the distinct points $a, b, c \in \mathbb{H}$ if and only if the points are not all similar. The condition for the points to be similar can be expressed as*

$$(c-a)^{-1}(c^2-a^2) = (b-a)^{-1}(b^2-a^2). \tag{2.4}$$

A symmetric form of (2.4) can be obtained by evaluating the symmetrised form of the linear dependence (2.3) at $f(z) = z^2$.

The condition for unique left Lagrange interpolation is not translation invariant, except for some real translations or interpolation to less than three points. For example, quadratic interpolation at i, j, k is not possible, but it is possible at $2i, j+i, k+i$ (since $|2i| = 2 \neq \sqrt{2} = |j+i|$). In a similar vein, the polynomials of degree n in $\mathbb{H}[z]$ when viewed as functions $f : \mathbb{H} \to \mathbb{H} : q \mapsto f_0 + qf_1 + \cdots + q^n f_n$ are not translation invariant, e.g.,

$$(q+a)^2 = q^2 + qa + aq + a^2 = q^2 + qb + a^2, \qquad \forall q \in \mathbb{H},$$

for some $b \in \mathbb{H}$, if and only if a is real.

Since there is a unique one-dimensional subspace of polynomials of degree n in $\mathbb{H}[z]$ whose left evaluation at n points (with no three similar) is zero, a Newton form for (left) Lagrange interpolation can be developed.

3 Quaternionic polynomials

It is now time to understand the nature of the quaternionic polynomials, viewed as functions $\mathbb{H} \to \mathbb{H}$, obtained from the formula (1.1) for the Lagrange polynomials. These involve polynomials of the form

$$q \mapsto \alpha_0 q \alpha_1 q \alpha_2 \cdots q \alpha_{r-1} q \alpha_r, \qquad \alpha_j \in \mathbb{H},$$

which [Sud79] calls a **quaternionic monomial** of degree r. We define the $\mathbb{H}$-span of these monomials to be $\mathrm{Hom}_r(\mathbb{H})$ the **homogeneous polynomials** of degree r, and $\mathrm{Pol}_n(\mathbb{H})$ the **polynomials** of degree k to be the $\mathbb{H}$-span of the homogeneous polynomials of degrees $\leq n$. These definitions extend to multivariate polynomials $\mathbb{H}^d \to \mathbb{H}$, where each occurrence of q in the formula for a monomial is replaced by some coordinate q_j.

It is clear from the definitions, that the quaternionic polynomials are a graded ring, i.e., the product of homogeneous polynomials of degrees j and k is a homogeneous polynomial of degree $j+k$. To understand the dimensions of these spaces, we write $q \in \mathbb{H}$ as

$$q = t + ix + jy + kz, \qquad t, x, y, z \in \mathbb{R},$$

and observe (see [Sud79]) that

$$\begin{aligned} t &= \frac{1}{4}(q - iqi - jqj - kqk), \\ x &= \frac{1}{4i}(q - iqi + jqj + kqk), \\ y &= \frac{1}{4j}(q + iqi - jqj + kqk), \\ z &= \frac{1}{4k}(q + iqi + jqj - kqk). \end{aligned} \tag{3.5}$$

Hence t, x, y, z are homogeneous monomials (in q), as are $\overline{q}$ and $|q|^2 = q\overline{q}$, i.e.,

$$\overline{q} = t - ix - jy - kz = -\frac{1}{2}(q + iqi + jqj + kqk),$$

$$|q|^2 = q\overline{q} = -\frac{1}{2}(q^2 + (qi)^2 + (qj)^2 + (qk)^2).$$

Every monomial of degree r can be written as a homogeneous polynomial of degree r in the (real) variables t, x, y, z with quaternionic coefficients. The monomials in t, x, y, z are linearly independent over $\mathbb{H}$ by the usual argument (of taking Taylor coefficients), and so we have

$$\dim_{\mathbb{H}}(\mathrm{Hom}_r(\mathbb{H})) = \dim_{\mathbb{R}}(\mathrm{Hom}_r(\mathbb{R}^4)) = \binom{r+3}{3}, \tag{3.6}$$

$$\dim_{\mathbb{H}}(\mathrm{Pol}_n(\mathbb{H})) = \dim_{\mathbb{R}}(\mathrm{Pol}_k(\mathbb{R}^4)) = \binom{n+4}{4}. \tag{3.7}$$

In particular, the vector spaces of constant, linear and quadratic polynomials of a single quaternionic variable have dimensions $1, 5$ and 15, respectively. This contrasts sharply with the real and complex cases, where the dimensions are $1, 2$ and 3.

Example 3.1 *In view of (3.5), a basis for the linear polynomials is given by* $1, q, iq, jq, kq$. *There is a unique linear Lagrange interpolant to any five points* $x_0, \dots, x_5$*, which are affinely independent as points in* $\mathbb{R}^4$. *An explicit formula for the Lagrange interpolant* $p(q) = p_0 + qp_1 + iqp_2 + jqp_3 + kqp_4$ *can be obtained by solving the "linear system"*

$$p(x_j) = p_0 + x_j p_1 + ix_j p_2 + x_j p_3 + kx_j p_4 = f(x_j), \qquad 0 \le j \le 4,$$

for $p_0, \dots, p_4 \in \mathbb{H}$ *in the skew-field* $\mathbb{H}$. *The corresponding Lagrange polynomials are the barycentric coordinates for the interpolation points. If the points are taken to be* $0, 1, i, j, k$*, then the interpolant can be written as*

$$(1 - t - x - y - z)f(0) + tf(1) + xf(i) + yf(j) + zf(k).$$

From this a (multivariate) Bernstein interpolant could be developed, if desired.

As is evident from this example, Lagrange interpolation from $\mathrm{Pol}_n(\mathbb{H})$ is essentially interpolation from $\mathrm{Pol}_n(\mathbb{R}^4)$, with the additional feature that formulas can be developed using a single quaternionic variable q, or two complex variables v, w, where

$$q = v + jw, \qquad v = t + ix = \frac{1}{2}(q - iqi), \qquad y - iz = \frac{1}{2}(-jq + kqi). \tag{3.8}$$

We now consider interpolation from subspaces of $\mathrm{Pol}_n(\mathbb{H})$ that are a quaternionic analogue of the holomorphic functions. A function $f : \mathbb{H} \to \mathbb{H}$ is said to be **regular** if it is in the kernel of the **Cauchy-Feuter operator** $2\partial_\ell$, i.e.,

$$2\partial_\ell f := \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0,$$

and to be **harmonic** if

$$\Delta f := \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

If f is regular, then it is harmonic. The dimensions of $\mathrm{Reg}_n(\mathbb{H})$ and $\mathrm{Harm}_n(\mathbb{H})$, the regular and harmonic homogeneous polynomials of degree n, are

$$\dim_{\mathbb{H}}(\mathrm{Reg}_n(\mathbb{H})) = \frac{1}{2}(n+1)(n+2), \qquad \dim_{\mathbb{H}}(\mathrm{Harm}_n(\mathbb{H})) = (n+1)^2.$$

Example 3.2 *The harmonic polynomial $q = t + ix + jy + kz$ is not regular, since*

$$\frac{\partial q}{\partial t} + i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial f}{\partial z} = 1 + i(i) + j(j) + k(k) = -2 \neq 0.$$

A basis for (the right $\mathbb{H}$-vector space) $\mathrm{Reg}_1(\mathbb{H})$ is given by $t + ix, t + jy, t + kz$.

An explicit basis $\{P^n_{k\ell} - jP^n_{k-1,\ell}\}_{0\le k\le \ell\le 1}$, for $\mathrm{Reg}_n(\mathbb{H})$ is given in [Sud79], where

$$P^n_{k\ell}(v + jw) := \sum_r (-1)^r v^{[n-k-\ell+r]}\overline{v}^{[r]} w^{[k-r]}\overline{w}^{[\ell-r]}, \quad v, w \in \mathbb{C}, \qquad z^{[j]} := \begin{cases} \frac{z^j}{j!}, & j \ge 0; \\ 0, & j < 0. \end{cases}$$

Here

$$q = t + ix + jy = kz = (t + ix) + j(y - iz) = v + jw,$$

so for $n = 1$, $P^1_{00}(q) = v$, $P^1_{01}(q) = \overline{w}$, $P^1_{11}(q) = -\overline{v}$. But $Q^1_{01}(q) = \overline{w}$ is not regular. Interchanging k and ℓ in the formula (presumably a typo), gives $P^1_{01}(q) = w$, and the basis

$$v = t + ix, \qquad w = y - iz = (t + jy)(-j) + (t + kz)j,$$

$$-\overline{v} - jw = -(t - ix) - j(y - iz) = (t + ix) - (t + jy) - (t + kz).$$

The product of regular functions is not (in general) regular, since

$$2\partial_\ell\big((t + ix)(t + jy)\big) = 2ix, \qquad 2\partial_\ell\big((t + jy)(t + ix)\big) = 2jy.$$

However, multiplying the basis of Example 3.2 by i, j, k, to get $it - x, jt - y, kt - z$, we have

$$2\partial_\ell\bigl((it - x)(jt - y)\bigr) = 2kt, \qquad 2\partial_\ell\bigl((jt - y)(it - x)\bigr) = -2kt,$$

so that the average $\frac{1}{2}\{(it-x)(jt-y)+(jt-y)(it-x)\}$ is regular. In this way, [Sud79] gives a basis for $\mathrm{Reg}_n(\mathbb{H})$ consisting of symmetrised products of the linear regular polynomials $it - x, jt - y, kt - z$, where the factors occur n_1, n_2, n_3 times, $n_1 + n_2 + n_3 = n$.

Though the (Feuter) regular polynomials are a proper subspace of the quaternionic polynomials (which share some aspects of holomorphic polynomials, but not that power series, since q is not regular), we do not readily see a practical way to interpolate from them. There has recently been considerable interest in Cullen-regular functions [GS07], which do have power series expansions (around 0 or a real centre), and so correspond to the $(n+1)$-dimensional subspace of formal polynomials $\mathbb{H}[z]$ in $\mathrm{Pol}_n(\mathbb{H})$, which we have already discussed.

4 Multivariate Lagrange interpolation

We now consider interpolation methods derived from (1.1). As already discussed, this is essentially multivariate polynomial interpolation to functions $\mathbb{R}^4 \to \mathbb{H}$.

Example 4.1 *The order in which the factors of $\ell_j(x)$ are calculated is important. For $x_0 = i$, $x_1 = j$, $x_2 = k$, we might take "$\ell_0(x)$" to be*

$$p(x) := (x - j)(x - k)(i - j)^{-1}(i - k)^{-1}, \qquad p(i) = \frac{1}{2}(-1 - i - j - k).$$

This is not 1 at x_0, and so care must be taken with the order of multiplication in (1.1). Natural choices for the multiplication order are to evaluate each quotient first (with right scalar multiplication), or to take the product of the numerators, and then right multiply this by the inverse of its value at x_j, in concrete terms for ℓ_0 for three points either of

$$(x - x_1)(x_0 - x_1)^{-1}(x - x_2)(x_0 - x_2)^{-1}, \qquad (x - x_1)(x - x_2)\bigl((x - x_1)(x - x_2))\bigr)^{-1}.$$

Either of the choices for computing $\ell_j(x)$ suggested above give

- A unique linear Lagrange interpolation operator to any points $x_0, \ldots, x_n \in \mathbb{H}$ from the $(n+1)$-dimensional subspace $\mathrm{span}\{\ell_j\}$ of $\mathrm{Pol}_n(\mathbb{H})$, where the Lagrange polynomials have precisely n zeros.
- The operator depends continuously on the interpolation points.
- It could be "symmetrised" to obtain an operator which doesn't depend on the ordering of the points (though the Lagrange polynomials might now have additional zeros).

In this vein, we now define a generic polynomial of degree n which is zero at n points. For points $x_1,\ldots,x_n \in \mathbb{H}$, let

$$p_{\{x_1,\ldots,x_n\}}(x) := \frac{1}{n!}\sum_{\sigma\in S_n}(x-x_{\sigma 1})(x-x_{\sigma 2})\cdots(x-x_{\sigma n}),$$

where S_n is the symmetric group. This polynomial of degree n is zero at the points $x_1,\ldots,x_n$, and (by construction) does not depend on their ordering.

We now present two possible choices for the Lagrange polynomials:

$$\ell_j(x) := p(x)p(x_j)^{-1}, \qquad p(x) = p_{\{x_0,\ldots,x_n\}\setminus\{x_j\}}(x), \tag{4.9}$$

provided that $p(x_j) \neq 0$, and

$$\ell_j(x) := \frac{1}{n!}\sum_{\substack{\sigma\in S_{n+1}\\ \sigma j=j}}\prod_{k\neq j}(x-x_{\sigma k})(x_j-x_{\sigma k})^{-1}, \tag{4.10}$$

where the factors above are multiplied in the order $k = 0,1,\ldots,n$ (or any fixed order). Both are independent of the point ordering, and depend continuously on the points. Let L_Θ be the corresponding Lagrange interpolation operator

$$L_\Theta f(x) := \sum_j \ell_j(x)f(x_j),$$

for the points $\Theta = \{x_0,\ldots,x_n\} \subset \mathbb{H}$, which does not depend on their ordering. We have

- The interpolation operator L_Θ depends continuously on the points Θ.
- The interpolation space $\Pi_\Theta := \operatorname{ran}(L_\Theta)$ depends continuously on the points Θ.
- The interpolation operator is translation invariant, i.e.,

$$L_{\Theta+a}f(x) = L_\Theta\big(f(\cdot+a)\big)(x-a).$$

We compare this Lagrange interpolation with the two most prominent multivariate generalisations of univariate Lagrange interpolation, where the interpolation points are not in some predetermined geometric configuration.

Kergin interpolation [MM80] interpolates function values at $n+1$ points in $\mathbb{R}^d$ by a polynomial of degree n, with other "mean-value" interpolation conditions (see [Wal97]) that depend continuously on the points also matched. Here the interpolation space $\mathrm{Pol}_n(\mathbb{R}^d)$ is fixed, and hence depends continuously on the points. Kergin interpolation has also been extended to $\mathbb{C}^d$ (see [Fil97]). Using the identifications $\mathbb{H} \approx \mathbb{R}^4$, $\mathbb{H} \approx \mathbb{C}^2$ one can defined a Kergin interpolation to functions $\mathbb{H} \to \mathbb{H}$ from the whole space $\mathrm{Pol}_n(\mathbb{H})$ (with additional interpolation conditions). The explicit formulas for Kergin interpolation involve derivatives of f (the operator is defined for C^n-functions), and are not as easily computed as ours.

Least interpolation [dBR92] is a very general method, which seeks an interpolation space Π_Θ of dimension $n+1$ to the $n+1$ points Θ, which has polynomials of lowest (least) degree. It has many nice properties that include the continuity properties listed above, but there is no explicit formula. It could be applied to functions $\mathbb{H} \to \mathbb{H}$ in the same way that Kergin interpolation can be. It might even be possible to develop a least interpolation for "polynomials in $\mathbb{H}^d$", once such a theory is developed. The least interpolation has the advantage that polynomials of low degree are used. In particular, the Lagrange polynomials are a partition of unity, i.e.,

$$\sum_j \ell_j = 1.$$

For the Lagrange polynomials that we have proposed, this may not be the case. Indeed, the set of all possible "nice" Lagrange polynomials ℓ_j for x_j, i.e, those polynomials $p = p_{\{x_0,\ldots,x_n\}}$ of degree n, with $p(x_k) = \delta_{jk}$ and p not depending on the order of the points, form an affine subspace of $\mathrm{Pol}_n(\mathbb{H})$, from which we have suggested two choices. It may be that requiring, in addition, the partition of unity property, gives a unique choice, but we have not pursued this.

Since our Lagrange interpolation is effectively interpolation to functions $\mathbb{R}^4 \to \mathbb{H}$, we can define an interpolation operator to functions $\mathbb{R}^4 \to \mathbb{R}$ in the natural way, i.e.,

$$Lf := \mathrm{Re}(L_\Theta f) = \sum_j \mathrm{Re}(\ell_j) f(x_j), \qquad \text{where } f : \mathbb{H} \to \mathbb{R},$$

and Θ is the points viewed as a subset of $\mathbb{H}$. Heuristically, the real Lagrange polynomials $\hat{\ell}_j = \mathrm{Re}(\ell_j)$ are more likely to be "nice", e.g., form a partition of unity, have no extra zeros, or coincide for both choices, since there is the "commutativity relation"

$$\mathrm{Re}(ab) = \mathrm{Re}(ba), \qquad a, b \in \mathbb{H}.$$

In a similar way, one could define a Lagrange interpolation operator to functions $\mathbb{C}^2 \to \mathbb{C}$, by using the Cayley-Dickson construction (3.8).

It is also possible to develop Lagrange interpolants through a Newton form (either for functions $\mathbb{H} \to \mathbb{H}$ or $\mathbb{R}^4 \to \mathbb{R}$), and an associated theory of divided differences. For example, if $L_{n-1}f$ is a Lagrange interpolant at $x_1, \ldots, x_{n-1} \in \mathbb{H}$, then

$$L_n f(x) := L_{n-1} f(x) + p_n(x)[x_0, \ldots, x_n]f, \qquad \text{where } p \in \mathrm{Pol}_n(\mathbb{H}), \quad p(x_j) = \delta_{jn},$$

gives a Lagrange interpolant $L_n f$ to f at the points $x_0, \ldots, x_n$, and a "divided difference" $[x_0, \ldots, x_n]f \in \mathbb{H}$. Choices for p could include $p_{\{x_0,\ldots,x_{n-1}\}}$ or ℓ_n (for $x_0, \ldots, x_n$). The map $f \mapsto [x_0, \ldots, x_n]f \in \mathbb{H}$ is an $\mathbb{H}$-linear functional. We have not invesigated its divided difference type properties any further.

5 Concluding remarks

This work came about when investigating tight frames and spherical designs for $\mathbb{H}^d$ (see [Wal18], [Wal20]). It soon became apparent that the theory of quaternionic polynomials of one and several variables is involved, and not widely known. There is considerable work in at least two different directions. One is a formal (algebraic) approach dating back to [Ore33], [Lam86], where point evaluation and multiplication of polynomials are appropriately defined, and the other views the polynomials as functions, with the usual point evaluation and (pointwise) multiplication.

Here we have shown that

- The space of quaternionic polynomials required for the classical formula (1.1) for Lagrange interpolation to make sense has a high dimension (3.7).
- Lagrange interpolation methods with some geometric properties, e.g., translation invariance, are essentially multivariate polynomial interpolation methods.
- Many basic questions about quaternionic polynomials of one (and several) variables remain, e.g., the existence of Lagrange polynomials with no extra zeros which form a partition of unity.

Functions of quaternionic variables have long been used in physics, and geometric design (cf. [FGMS19]). We hope this paper gives some insight into polynomials of one or more quaternionic variables, and their use in interpolation and cubature (spherical designs).

References

[Bol15] Vladimir Bolotnikov. Polynomial interpolation over quaternions. *J. Math. Anal. Appl.*, 421(1):567–590, 2015.

[Bol20] Vladimir Bolotnikov. Lagrange interpolation over division rings. *Comm. Algebra*, 48(9):4065–4084, 2020.

[dBR92] Carl de Boor and Amos Ron. The least solution for the polynomial interpolation problem. *Math. Z.*, 210(3):347–378, 1992.

[FGMS19] Rida T. Farouki, Graziano Gentili, Hwan Pyo Moon, and Caterina Stoppato. Minkowski products of unit quaternion sets. *Adv. Comput. Math.*, 45(3):1607–1629, 2019.

[Fil97] Lars Filipsson. Complex mean-value interpolation and approximation of holomorphic functions. *J. Approx. Theory*, 91(2):244–278, 1997.

[GS07] Graziano Gentili and Daniele C. Struppa. A new theory of regular functions of a quaternionic variable. *Adv. Math.*, 216(1):279–301, 2007.

[Lam86] T. Y. Lam. A general theory of Vandermonde matrices. *Exposition. Math.*, 4(3):193–215, 1986.

[LL88] T. Y. Lam and A. Leroy. Vandermonde and Wronskian matrices over division rings. *J. Algebra*, 119(2):308–336, 1988.

[Mar20] Umberto Martínez-Peñas. Theory and applications of linearized multivariate skew polynomials. *arXiv e-prints*, page arXiv:2001.01273, January 2020.

[MM80] Charles A. Micchelli and Pierre Milman. A formula for Kergin interpolation in $\mathbf{R}^k$. *J. Approx. Theory*, 29(4):294–296, 1980.

[MPnK19] Umberto Martínez-Peñas and Frank R. Kschischang. Evaluation and interpolation over multivariate skew polynomial rings. *J. Algebra*, 525:111–139, 2019.

[Ore33] Oystein Ore. Theory of non-commutative polynomials. *Ann. of Math. (2)*, 34(3):480–508, 1933.

[Rod14] Leiba Rodman. *Topics in quaternion linear algebra.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2014.

[Sud79] A. Sudbery. Quaternionic analysis. *Math. Proc. Cambridge Philos. Soc.*, 85(2):199–224, 1979.

[Wal97] Shayne Waldron. Integral error formulæ for the scale of mean value interpolations which includes Kergin and Hakopian interpolation. *Numer. Math.*, 77(1):105–122, 1997.

[Wal18] Shayne F. D. Waldron. *An introduction to finite tight frames.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2018.

[Wal20] Shayne Waldron. Tight frames over the quaternions and equiangular lines. preprint, 1 2020.