Pseudometrics, Distances and Multivariate Polynomial Inequalities

Len Bos, Norm Levenberg and Shayne Waldron

Abstract. We discuss three natural pseudodistances and pseudometrics on a bounded domain in $I\!R^N$ based on polynomial inequalities.

Key words and phrases: pseudodistance, pseudometric, polynomial inequalities

AMS-MOS classification numbers: Primary 41A17; Secondary 41A63, 26D10

§0. Introduction.

In [3], for a compact set $K \subset \mathbb{R}^n$, we defined a Carathéodory type distance due to Dubiner [6] and a Finsler type distance based on Baran's generalization of the van der Corput - Schaake polynomial inequality [1], [2]. These distances are intimately connected to the distribution of optimal points for multivariate polynomial interpolation, as well as to the distribution of nodes for "good" quadrature rules (cf., the Introduction and the references of [3]).

Let $K = \overline{\Omega} \subset \mathbb{R}^N$ where Ω is a domain. We expand upon the definitions given in [3] in proving some general relationships among *three* natural pseudodistances as well as three natural pseudometrics on Ω.

The classical Markov inequality, or more precisely, the van der Corput - Schaake inequality, says that for $p : \mathbb{R} \to \mathbb{R}$ a real polynomial such that $||p||_I = \sup_{x\in[-1,1]} |p(x)| \leq 1$,

$$\left| \frac{p'(x)}{\sqrt{1-p^2(x)}} \right| \leq \deg(p) \frac{1}{\sqrt{1-x^2}},\ x \in (-1,1).$$

This is equivalent to

$$\frac{1}{\deg(p)} \left| \frac{d}{dx} \cos^{-1}(p(x)) \right| \leq \left| \frac{d}{dx} \cos^{-1}(x) \right|$$

which motivates the definition of the *Dubiner* pseudodistance (Definition 1.4).

Analogously, estimates on $\frac{1}{\deg p}\frac{|D_y p(x)|}{\sqrt{1-p(x)^2}}$ for polynomials $p:\mathbb{R}^N\to\mathbb{R}$ normalized with $||p||_K=\sup_{x\in K}|p(x)|\le 1$, where $D_y p(x)$ denotes the directional derivative of p at x in the direction y, give rise to the definition of the *Markov* pseudometric (Definition 1.5).

Next, we recall for a compact set $K\subset\mathbb{C}^N$, the function

$$V_K(z):=\sup\{\log|p(z)|^{1/\deg(p)}\,:\,p:\mathbb{C}^N\to\mathbb{C},\,\deg(p)\ge 1,\,||p||_K\le 1\}$$

is known as the Siciak-Zaharjuta extremal function. If $V_K(z)$ is finite, which it is for all $z\in\mathbb{C}^N$ when $K=\overline{\Omega}\subset\mathbb{R}^N$ where Ω is a domain, then for any polynomial p and any point z, from the definition of V_K we have the Bernstein-Walsh inequality

$$|p(z)|\le e^{\deg(p)V_K(z)}||p||_K.$$

The function V_K will be utilized, in particular, in defining and analyzing the *Baran* pseudometric and pseudodistance (Definition 1.6).

The organization of the paper is as follows: in section 1, we define the notions of pseudometric and pseudodistance on domains in $\mathbb{R}^N$. We follow closely the presentation in Jarnicki-Pflug [7], but we also recommend Dineen's monograph [5]. Then we define the Dubiner, Markov and Baran pseudodistances and pseudometrics for a bounded domain $\Omega\subset\mathbb{R}^N$ and recall the results of the relevant calculations from [3]. In section 2, we give relationships among these pseudodistances and pseudometrics for general Ω and we prove certain properties (monotonicity, invariance, etc.). Finally, in the last section, we show that all three pseudometrics coincide when $K=\overline{\Omega}$ is a symmetric convex body in $\mathbb{R}^N$ (Proposition 3.6). The corresponding pseudodistances are shown to coincide for symmetric convex bodies in $\mathbb{R}^2$ that satisfy a technical condition; we conjecture that this additional condition is not needed, and that indeed the result is true in $\mathbb{R}^N$. This is *not* the case, in general, for non-symmetric convex bodies as was shown in [3] via the example of the simplex in $\mathbb{R}^2$.

§1. Definition of the pseudodistances and pseudometrics.

We begin our discussion with the definitions of pseudodistances and pseudometrics; we refer the reader to section 4.3 of [7] for details and proofs of Propositions 1.1-1.6. A word of warning: in [7], the field of scalars is $\mathbb{C}$. Let $K=\overline{\Omega}\subset\mathbb{R}^N$ where Ω is a domain.

Definition 1.1. *We call $F : \Omega \times \mathbb{R}^N \to \mathbb{R}^+$ a pseudometric if*

a. *$F(x;\lambda)$ is uppersemicontinuous (usc) as a function of $(x,\lambda) \in \Omega \times \mathbb{R}^N$;*
b. *$F(x;\lambda)$ is positive definite in λ: $F(x;\lambda) \geq 0$ and $F(x;\lambda) = 0$ if and only if $\lambda = 0$;*
c. *$F(x;\lambda)$ is positively homogeneous in λ: $F(x;t\lambda) = |t|F(x;\lambda)$ for $t \in \mathbb{R}$.*

Remark 1.1. It follows from a. and c. that

d. $F(x;\lambda)$ is locally Lipschitz in λ: $F(x;\lambda) \leq c|\lambda|$ where $c = c(x)$ depends on x and is locally bounded above.

More precisely, we should call F an *usc* pseudometric; but we omit the adjective usc. All of our pseudometrics will, in addition, satisfy a bi-Lipschitz condition:

d'. $c_1|\lambda| \leq F(x;\lambda) \leq c_2|\lambda|$ where $c_i = c_i(x)$, $i = 1,2$ depend on x with c_1 locally bounded below and c_2 locally bounded above.

Definition 1.2. *We call $d : \Omega \times \Omega \to \mathbb{R}^+$ a pseudodistance if*

A. *$d(a,b) = d(b,a) \geq 0$;*
B. *$d(a,b) \leq d(a,c) + d(c,b)$;*
C. *d is locally dominated by the Euclidean distance: for all $c \in \Omega$ there exists $M > 0$, $r > 0$ with $d(a,b) \leq M|a-b|$ if $a,b \in \Omega$ with $\max\{|a-c|,|b-c|\} < r$.*

All of our pseudodistances will *locally dominate* the Euclidean distance; hence:

D. for all $c \in \Omega$ there exist $m, M > 0$, $r > 0$ with $m|a-b| \leq d(a,b) \leq M|a-b|$ if $a,b \in \Omega$ with $\max\{|a-c|,|b-c|\} < r$.

If $d(a,b) > 0$ for $a \neq b$, we call d a *distance*; from D., all of our pseudodistances will be distances.

We summarize four operations with d, F:

1. *The operator $d \to d^i$:*

Given a pseudodistance d, let $\alpha : [0,1] \to \Omega$ denote a continuous curve. Define

$$L_d(\alpha) := \sup\{\sum_{j=1}^n d(\alpha(t_{j-1}),\alpha(t_j)) : 0 = t_0 < \cdots < t_n = 1\},$$

the $d-$length of α. Define $d^i : \Omega \times \Omega \to \mathbb{R}^+$ via

$$d^i(a,b) := \inf\{L_d(\alpha) : \alpha \text{ continuous curve in } \Omega \text{ joining } a,b\}.$$

We call d^i the *inner pseudodistance* associated to d.

Proposition 1.1. *d^i is a pseudodistance; $d \leq d^i$; and $L_{d^i} = L_d$.*

2. *The operator $F \to \int F$:*
   Given a pseudometric F and $\alpha : [0,1] \to \Omega$ a piecewise C^1 curve, define

$$L_F(\alpha) := \int_0^1 F(\alpha(t); \alpha'(t))dt,$$

the $F-$length of α. Define, for $a, b \in \Omega$,

$$(\int F)(a,b) := \inf\{L_F(\alpha) : \alpha \text{ piecewise } C^1 \text{ curve in } \Omega \text{ joining } a, b\}.$$

Proposition 1.2. *$\int F$ is a pseudodistance; and $L_{\int F} \leq L_F$ for each piecewise C^1 curve; hence $(\int F)^i = \int F$.*

3. *The operator $d \to Dd$:*
   Given a pseudodistance d, define, for $x \in \Omega$ and $y \in \mathbb{R}^N$,

$$Dd(x; y) := \limsup_{t \to 0^+,\ z \to x} \frac{d(z, z + ty)}{t}.$$

Proposition 1.3. *Dd is a pseudometric;*
(i) $Dd(x; y) := \limsup_{x_1, x_2 \to x,\ \frac{x_1 - x_2}{|x_1 - x_2|} \to y} \frac{d(x_1, x_2)}{|x_1 - x_2|},\ |y| = 1$;
(ii) $d \leq \int(Dd)$;
(iii) *for any pseudometric F, $D(\int F) \leq F$.*

4. *The operator $F \to \widehat{F}$:*
   Given $f : \mathbb{R}^N \to \mathbb{R}^+$ satisfying $f(tx) = |t|f(x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, and $f(x) \leq M|x|$, define

$$\Gamma(f) := \{y \in \mathbb{R}^N : |y \cdot z| = |\sum_{j=1}^N y_j z_j| \leq f(z),\ \text{for all } z \in \mathbb{R}^N\}$$

$$= \{y \in \mathbb{R}^N : y \cdot z \leq f(z),\ \text{for all } z \in \mathbb{R}^N\}$$
$$= \{y \in \mathbb{R}^N : y \cdot z \leq 1,\ \text{for all } z \in \mathbb{R}^N \text{ with } |f(z)| = 1\};$$

the first equality occurs since $f(-x) = f(x)$, the second from $f(tx) = |t|f(x)$. The (filled-in) indicatrix of such an absolutely homogeneous f is the set

$$E = \{x \in \mathbb{R}^N : f(x) \leq 1\};$$

and the polar of a set $E \subset \mathbb{R}^N$ is the set

$$E^* := \{y \in \mathbb{R}^N : y \cdot z \leq 1,\ \text{for all } z \in E\};$$

thus $\Gamma(f)$ is the polar of the "filled-in" indicatrix of f. Next, define

$$\widehat{f}(x) := \sup\{x \cdot y : y \in \Gamma(f)\};$$

this is the support function of $\Gamma(f)$. Note that $f \leq g$ implies $\hat{f} \leq \hat{g}$. Recalling that an absolutely homogeneous f defines a *seminorm* if $f(x+y) \leq f(x) + f(y)$, we have

a. $\widehat{f} \leq f$;
b. $\widehat{f}$ is always a seminorm and f is a seminorm iff $\widehat{f} = f$;
c. $\Gamma(\widehat{f}) = \Gamma(f)$;
d. $\{\widehat{f} \leq 1\}$ is the closed convex hull of $\{f \leq 1\}$.

(cf., Remark 4.3.4 in [7]). Now given a pseudometric F, define $\widehat{F}(x;y) := F(x;\widehat{\cdot})$ ("hat" operation in second variable).

Proposition 1.4. *$\widehat{F}$ is a pseudometric and $\int \widehat{F} = \int F$. Moreover, $D(\int F) \leq \widehat{F}$; for F satisfying d' (of Remark 1.1), we have equality if F is continuous in $(x;y)$.*

Proposition 1.5. *We have the following relations between the operations d^i, $\int$, Dd, $\widehat{F}$:*

(i) $D(\int F) \leq \widehat{F}$;
(ii) $\int(Dd) \geq d^i$;
(iii) $\int(\widehat{F}) = \int F$;
(iv) $\widehat{Dd} = Dd$.

For use in section 3, we define the notion of a C^1 *pseudodistance.* Below, $B(x,r)$ denotes the Euclidean ball of radius r centered at x.

Definition 1.3 (C^1 pseudodistance). *A pseudodistance d on Ω is a C^1 pseudodistance if for all $E \subset\subset \Omega$, and all $\epsilon > 0$, there exists $\eta > 0$ such that*

$$|d(x_1,x_2) - (Dd)(x;x_1 - x_2)| \leq \epsilon |x_1 - x_2|$$

for $x \in E$ and $x_1, x_2 \in B(x,\eta)$.

Proposition 1.6. *Let d be a C^1 pseudodistance. Then $d^i = \int(Dd)$ and d^i is a C^1 pseudodistance.*

We now define our natural pseudodistances and pseudometrics on a bounded domain $\Omega \subset \mathbb{R}^N$. The applications we have in mind and some of the fundamental notions we utilize deal with compact sets; thus we often consider one or more of the six items below as associated to $K = \overline{\Omega}$.

Definition 1.4 (Dubiner pseudodistance and pseudometric).

$$d_D^K(a,b) = d_D(a,b) := \sup_{||p||_K \leq 1,\ \deg p \geq 1} \frac{1}{\deg p} |\cos^{-1}(p(a)) - \cos^{-1}(p(b))|$$

is the Dubiner pseudodistance on K. Note that this is well-defined on $K \times K$ for any compact set K. For $x \in \Omega$ and $y \in \mathbb{R}^N$,

$$\delta_D^K(x;y) = \delta_D(x;y) := Dd_D(x;y) := \limsup_{t \to 0^+,\ z \to x} \frac{d_D(z, z+ty)}{t}$$

is the Dubiner pseudometric for K.

Definition 1.5 (Markov pseudodistance and pseudometric).

$$\delta_M^K(x;y) = \delta_M(x;y) := \sup_{||p||_K \le 1,\ \deg p \ge 1} \frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1-p(x)^2}},$$

(for $x \in \Omega$ and $y \in \mathbb{R}^N$) defined for compacta K for which it is usc, is the Markov pseudometric for K and

$$d_M^K = d_M = \int \delta_M$$

is the Markov pseudodistance for K. From the results of [2], δ_M^K is continuous at $x \in \Omega = K^o$ if K is a centrally symmetric convex body (see Cor. 3.5).

Definition 1.6 (Baran pseudodistance and pseudometric).

$$\delta_B^K(x;y) = \delta_B(x;y) := \limsup_{t \to 0^+} \frac{V_K(x+ity)}{t},$$

(for $x \in \Omega$ and $y \in \mathbb{R}^N$) defined for compacta K for which it is usc, is the Baran pseudometric for K and

$$d_B^K = d_B = \int \delta_B$$

is the Baran pseudodistance for K. From the results of [4], δ_B^K is continuous for $x \in K^o$ if K is an arbitrary convex body. Moreover, in this case, the limit in the definition of δ_B^K exists.

Remark 1.2. When the set K is understood, we delete the superscript K for our pseudodistances and pseudometrics.

Remark 1.3. For the unit cube C in $\mathbb{R}^N$, one can explicitly compute

$$\delta_M^C(x;y) = \delta_B^C(x;y) = \max_{j=1,\ldots,N} \frac{|y_j|}{\sqrt{1-x_j^2}}$$

(see [3]). Since $K_1 \subset K_2$ clearly implies $\delta_M^{K_1}(x;y) \geq \delta_M^{K_2}(x;y)$ and $\delta_B^{K_1}(x;y) \geq \delta_B^{K_2}(x;y)$ for x in the interior of K_1, for any $K = \overline{\Omega}$ in $\mathbb{R}^N$ we see by taking a cube inside K and another containing K that δ_M^K and δ_B^K are pseudometrics satisfying the bi-Lipschitz property d'. Proposition 1.2 shows that d_M^K and d_B^K are pseudodistances; i.e., they satisfy A.-C. of Definition 1.2. The fact that δ_D^K is a pseudometric satisfying the bi-Lipschitz property d' will follow from Proposition 2.1 (equation (2.2)). Finally, the verification of property C. of Definition 1.2 for d_D^K – A. and B. are trivial – will follow from Proposition 2.1 (equation (2.1)). To verify the other half of property D. for d_D^K, take $r > 0$ so that the Euclidean ball $B(c, r) \subset \Omega$ and let p be the polynomial of degree one which is (normalized) linear projection to the line joining a and b. Proposition 2.1 (equation (2.1)) will imply the same property for d_M^K and d_B^K.

Remark 1.4. We see from Proposition 1.2 that each of the Markov and Baran pseudodistances are inner; i.e., $d_M^i = d_M$ and $d_B^i = d_B$.

As concrete examples, we summarize the following calculations in [3]:

(i) For $K = \overline{\Omega} = \{x = (x_1, ..., x_N) \in \mathbb{R}^N : |x|^2 = \sum_{j=1}^N x_j^2 \leq 1\}$ the closed unit ball, $d_D(a,b) = d_B(a,b) = \cos^{-1}(\tilde{a} \cdot \tilde{b})$ where $\tilde{a} = (a, \sqrt{1-|a|^2})$, $\tilde{b} = (b, \sqrt{1-|b|^2})$ are the liftings of a, b to the surrounding unit sphere $S^N \subset \mathbb{R}^{N+1}$. From Proposition 2.1 in the next section, we conclude that $d_D(a,b) = d_M(a,b) = d_B(a,b)$.

(ii) For $K = \overline{\Omega} = I^N = \{x = (x_1, ..., x_N) \in \mathbb{R}^N : \max_{j=1,...,N} |x_j| \leq 1\}$ the closed unit cube, $d_D(a,b) = d_B(a,b) = \max_{j=1,...,N} d_D^I(a_j, b_j) = \max_{j=1,...,N} |\cos^{-1} b_j - \cos^{-1} a_j|$. From Proposition 2.1 in the next section, we conclude that $d_D(a,b) = d_M(a,b) = d_B(a,b)$.

(iii) For $K = \overline{\Omega} = \{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_j \geq 0, \sum_{j=1}^N x_j \leq 1\}$ the standard simplex, $d_B(a,b) = 2[\cos^{-1}(\tilde{a} \cdot \tilde{b})]$. Here, $d_D(a,b) \not\equiv d_B(a,b)$.

§2. Pseudodistances and pseudometrics: general K.

In this section, we let $K = \overline{\Omega} \subset \mathbb{R}^N$ where Ω is a bounded domain such that δ_M and δ_B are usc, and we derive the following inequalities relating the Dubiner, Markov and Baran pseudodistances and pseudometrics.

Proposition 2.1. *For $K = \overline{\Omega} \subset \mathbb{R}^N$ we have*

$$d_D \leq d_D^i \leq d_M \leq d_B \tag{2.1}$$

and

$$\delta_D = \delta_M \leq \delta_B. \tag{2.2}$$

Proof. First note that for any polynomial p with $||p||_K \leq 1$, and any two points $a, b \in \Omega$, if $\alpha : [0,1] \to \Omega$ is a C^1 curve with $\alpha(0) = a$ and $\alpha(1) = b$, then

$$|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| = |\int_0^1 \frac{d}{dt}[\cos^{-1}(p(\alpha(t))]dt|$$

$$\leq \int_0^1 \frac{\left|(D_{\alpha'(t)}p(\alpha(t))\right|}{\sqrt{1-(p(\alpha(t)))^2}}dt.$$

Taking the supremum over all such polynomials and the infimum over all such C^1 curves shows that $d_D \leq d_M$. Moreover, we have

$$\int_0^1 \frac{\left|(D_{\alpha'(t)}p(\alpha(t))\right|}{\sqrt{1-(p(\alpha(t)))^2}}dt \leq \text{deg}p \int_0^1 \delta_B(\alpha(t);\alpha'(t))dt.$$

This last inequality is *Baran's inequality* (Theorem 1.14 of [1]) and actually holds with $\delta_B(x;y)$ replaced by

$$\tilde{\delta}_B(x;y) := \liminf_{t\to 0^+} \frac{V_K(x+ity)}{t}.$$

In particular, we get $\delta_M \leq \delta_B$ and hence, from the definitions of the Markov and Baran pseudodistances, that $d_M \leq d_B$. It follows that

$$d_D \leq d_M \leq d_B. \tag{2.3}$$

We also conclude that

$$Dd_D \leq Dd_M \leq Dd_B. \tag{2.4}$$

Next we show that

$$\delta_M \leq Dd_D. \tag{2.5}$$

For, by definition of d_D, for any polynomial p with $||p||_K \leq 1$,

$$\frac{1}{\text{deg}p}\frac{|\cos^{-1}(p(x+ty)) - \cos^{-1}(p(x))|}{t} \leq \frac{d_D(x+ty,x)}{t}.$$

Thus

$$\frac{1}{\text{deg}p}\frac{|D_y p(x)|}{\sqrt{1-(p(x))^2}} \leq \limsup_{t\to 0^+}\frac{d_D(x+ty,x)}{t} \leq (Dd_D)(x;y).$$

Hence $\delta_M \leq Dd_D$. Combining (2.4) and (2.5) we have

$$\delta_M \leq Dd_D \leq Dd_M \leq Dd_B. \tag{2.6}$$

Now from Proposition 1.5, $\widehat{Dd} = Dd$ (property (iv)), and $D(\int F) \le \widehat{F}$ (property (i)); thus, taking "hats" of (2.6),

$$\widehat{\delta_M} \le \widehat{Dd_D} = Dd_D \le \widehat{Dd_M} = Dd_M \le \widehat{\delta_M}.$$

Thus equality holds throughout and, in particular,

$$Dd_M = \widehat{\delta_M}.$$

But $\widehat{\delta_M} \le \delta_M \le Dd_D = \widehat{\delta_M}$ so that

$$\delta_D = Dd_D = Dd_M = \widehat{\delta_M} = \delta_M. \tag{2.7}$$

Together with (2.6) and the conclusion from Baran's inequality that $\delta_M \le \delta_B$, this completes the proof of (2.2). Finally, integrating (2.7) to get a relation among the pseudodistances, we have

$$d_D^i \le \int Dd_D = \int Dd_M = \int \widehat{\delta_M} = \int \delta_M = d_M$$

using (ii) from Proposition 1.5. Together with (2.3), this completes the proof of (2.1). ♣

Based on Remark 1.3 and property D. of Definition 1.2, we delete the "pseudo" in referring henceforth to the Dubiner, Baran and Markov *distances.* We make a few useful observations about the Dubiner distance and pseudometric.

Lemma 2.2. *For $a, b \in K$ and a positive integer k, we have*

$$d_D(a,b) = d_D^{(k)}(a,b) := \sup_{||p||_K \le 1,\ \deg p \le k} \frac{1}{\deg p} |\cos^{-1}(p(b)) - \cos^{-1}(p(a))|$$

for $k \ge \frac{\pi}{d_D^{(1)}(a,b)}$.

Proof. If $\deg p > k$, then $\frac{1}{\deg p}|\cos^{-1}(p(a)) - \cos^{-1}(p(b))| \le \frac{\pi}{\deg p} < \frac{\pi}{k} \le d_D^{(1)}(a,b)$. ♣

Lemma 2.3. *We have*

$$\delta_D(x;y) = \lim_{t \to 0+} \frac{d_D(x, x+ty)}{t},$$

i.e., the limit in the definition of the Dubiner pseudometric exists.

Proof. Recall that

$$\delta_D^K(x;y) = \delta_D(x;y) := Dd_D(x;y) := \limsup_{t \to 0^+,\ z \to x} \frac{d_D(z, z+ty)}{t}.$$

By definition of d_D, for any polynomial p with $||p||_K \le 1$,

$$\frac{1}{\mathrm{deg}p}\frac{|\cos^{-1}(p(x+ty))-\cos^{-1}(p(x))|}{t} \le \frac{d_D(x+ty,x)}{t}.$$

Thus

$$\frac{1}{\mathrm{deg}p}\frac{|D_y p(x)|}{\sqrt{1-(p(x))^2}} = \liminf_{t\to 0^+}\frac{1}{\mathrm{deg}p}\frac{|\cos^{-1}(p(x+ty))-\cos^{-1}(p(x))|}{t}$$

$$\le \liminf_{t\to 0^+}\frac{d_D(x+ty,x)}{t} \le \limsup_{t\to 0^+}\frac{d_D(x+ty,x)}{t} \le \delta_D(x;y).$$

By (2.2), $\delta_D(x;y)=\delta_M(x;y)$; moreover the above inequality for any polynomial p with $||p||_K \le 1$ implies that

$$\delta_M(x;y) \le \liminf_{t\to 0^+}\frac{d_D(x+ty,x)}{t};$$

combining these inequalities,

$$\delta_M(x;y) \le \liminf_{t\to 0^+}\frac{d_D(x+ty,x)}{t} \le \limsup_{t\to 0^+}\frac{d_D(x+ty,x)}{t} \le \delta_M(x;y)$$

so that the limit exists. ♣

Next we discuss invariance properties. We begin with the Dubiner distance.

Lemma 2.4. *For a polynomial map $P=(p_1,...,p_N):\mathbb{R}^N\to\mathbb{R}^N$ with $\mathrm{deg}P := \max(\mathrm{deg}p_1,...,\mathrm{deg}p_N)$ and $a,b\in K$,*

$$d_D^K(a,b) \ge \frac{1}{\mathrm{deg}P} d_D^{P(K)}(P(a),P(b)).$$

For an invertible linear map $T:\mathbb{R}^N\to\mathbb{R}^N$ and $a,b\in K$,

$$d_D^K(a,b) = d_D^{T(K)}(T(a),T(b)).$$

Proof. The inequality follows from the definition of d_D^K and $d_D^{P(K)}$. In particular, this inequality holds for an invertible linear map $T:\mathbb{R}^N\to\mathbb{R}^N$. The reverse inequality in this case follows by applying the above inequality with $K, P(K)$ and the map P replaced by the sets $T(K), T^{-1}(T(K))=K$ and the map T^{-1}. ♣

The Markov pseudometric is invariant under invertible linear maps.

Lemma 2.5. *For an invertible linear map $T : \mathbb{R}^N \to \mathbb{R}^N$, $\delta_M^K(x;y) = \delta_M^{T(K)}(T(x);T(y))$.*

Proof. First of all, clearly δ_M^K is usc if and only if $\delta_M^{T(K)}$ is usc (the same is true for δ_B; this will be used in Corollary 2.7). From the definition,

$$\delta_M^K(x;y) = \sup_{||Q||_K \leq 1,\ \deg Q \geq 1} \frac{1}{\deg Q} \frac{|D_y Q(x)|}{\sqrt{1 - Q(x)^2}}.$$

Now if $Q(x) = (p \circ T)(x)$, and we call $x' = T(x)$, then

$$D_y Q(x) = \nabla_x Q(x) \cdot y = T^t(\nabla_{x'} p(x')) \cdot y = \nabla_{x'} p(x') \cdot T(y) = D_{T(y)} p(T(x)).$$

Note that if $||p||_{T(K)} \leq 1$, then $||Q||_K \leq 1$. We obtain

$$\delta_M^{T(K)}(T(x);T(y)) = \sup_{||p||_{T(K)} \leq 1,\ \deg p \geq 1} \frac{1}{\deg p} \frac{|D_{T(y)} p(T(x))|}{\sqrt{1 - [p(T(x))]^2}}$$

$$\leq \sup_{||Q||_K \leq 1,\ \deg Q \geq 1} \frac{1}{\deg Q} \frac{|D_y Q(x)|}{\sqrt{1 - Q(x)^2}} = \delta_M^K(x;y).$$

Applying the above argument with T^{-1}, we obtain

$$\delta_M^K(x;y) = \delta_M^{T^{-1}(T(K))}((T^{-1} \circ T)(x);(T^{-1} \circ T)y) \leq \delta_M^{T(K)}(T(x);T(y))$$

and equality holds. ♣

Finally we turn to the Baran distance and pseudometric. We recall a result of Klimek [8, Thm. 5.3.1]: if $P = (p_1, ..., p_N) : \mathbb{C}^N \to \mathbb{C}^N$ is a proper polynomial mapping of degree d, then $V_K(P(z)) = dV_{P^{-1}(K)}(z)$.

Lemma 2.6. *Suppose $P = (p_1, ..., p_N) : \mathbb{R}^N \to \mathbb{R}^N$ is a polynomial mapping satisfying the Klimek condition: $d :=\deg p_1 = ... = \deg p_N$ and $\hat{P}^{-1}(0) = \{0\}$ where $\hat{P}$ is the homogeneous part of P of degree d. Let $A, C \subset \mathbb{R}^N$ with $C = P(A)$ and suppose that if $x \in A$ with $\det JP(x) = 0$, then $P(x) \in \partial C$. Then*

$$\delta_B^A(x;y) = \frac{1}{d} \delta_B^C(P(x); JP(x) \cdot y)$$

and hence

$$d_B^A(a,b) = \frac{1}{d} d_B^C(P(a), P(b)).$$

Proof. Using Klimek's result, we have

$$\delta_B^A(x;y) = \limsup_{t \to 0^+} \frac{V_A(x + ity)}{t} = \frac{1}{d} \limsup_{t \to 0^+} \frac{V_C(P(x + ity))}{t}$$

$$= \frac{1}{d} \limsup_{t\to 0^+} \frac{V_C(P(x) + JP(x) \cdot ity + 0(t^2))}{t} = \frac{1}{d}\delta_B^C(P(x); JP(x) \cdot y).$$

Here, the last equality follows from the considerations of Remark 1.3.

Hence, letting γ vary over curves in the interior A^o of A joining two points a and b, and letting $\tilde{\gamma}$ vary over compositions $P \circ \gamma$,

$$d_B^A(a,b) = \inf_\gamma \int_0^1 \delta_B^A(\gamma(t); \gamma'(t))dt$$

$$= \frac{1}{d} \inf_\gamma \int_0^1 \delta_B^C(P(\gamma(t)); JP(\gamma(t)) \cdot \gamma'(t))dt$$

$$= \frac{1}{d} \inf_{\tilde{\gamma}} \int_0^1 \delta_B^C(\tilde{\gamma}(t); \tilde{\gamma}'(t))dt$$

$$= \frac{1}{d} \inf_\Gamma \int_0^1 \delta_B^C(\Gamma(t); \Gamma'(t))dt = \frac{1}{d} d_B^C(P(a), P(b)).$$

Here Γ varies over all curves joining $P(a), P(b)$ and the first equality in the last line follows from our hypothesis that $\det JP(x) \neq 0$ if $P(x) \in C^o$. ♣

Corollary 2.7. *For an invertible linear map $T : \mathbb{R}^N \to \mathbb{R}^N$,*

$$\delta^K(x; y) = \delta^{T(K)}(T(x); T(y)) \tag{2.8}$$

for each of the pseudometrics $\delta = \delta_D, \delta_M$, or δ_B; and

$$d^K(a,b) = d^{T(K)}(T(a), T(b)) \tag{2.9}$$

for each of the distances $d = d_D, d_M$, or d_B.

Remark 2.1 For $K_1 \subset K_2$, $d^{K_2} \leq d^{K_1}$ on $K_1^o \times K_1^o$ for each of the distances d_D, d_M, d_B.

§3. K convex and centrally symmetric.

At the end of section 1 we noted that the three distances coincide on balls and cubes. In this section, we study the connection between our three pseudometrics and distances for $K \subset \mathbb{R}^N$ a *centrally symmetric convex body*; i.e., K is compact and convex with $\Omega = K^o \neq \emptyset$ and $K = -K$. Let $|||x|||_K := \inf\{\lambda > 0 : x \in \lambda K\}$. Then K is the closed unit ball in this norm:

$$K = \{x \in \mathbb{R}^N : |||x|||_K \leq 1\}.$$

Motivated by some results due to Milev and Révész [10] in their investigation of the "inscribed ellipse" method of Sarantopoulos [12] (see also Kroó and Révész [9]) for investigating Markov inequalities in convex bodies, we obtain a geometric interpretation of the Markov pseudometric in Lemma 3.2. This will be used to verify equality of the three pseudometrics in Proposition 3.6. To begin, given $x \in K$ and $y \in \mathbb{R}^N$, let

$$E_b(x,y) := \{r(t) = x\cos t + yb\sin t : 0 \le t \le 2\pi\}. \tag{3.1}$$

This is a centrally symmetric ellipse containing the points $\pm x, \pm yb$. The point of the "inscribed ellipse" method is to scale b to fit inside K.

Lemma 3.1. *For* $b \le \frac{\sqrt{1-|||x|||_K^2}}{|||y|||_K}$, $E_b(x,y) \subset K$.

Proof. We have

$$|||r(t)|||_K \le |||x|||_K|\cos t| + |||y|||_K b|\sin t|$$

$$\le |||x|||_K|\cos t| + \sqrt{1-|||x|||_K^2}|\sin t| \le 1 \cdot \sqrt{|||x|||_K^2 + 1 - |||x|||_K^2} = 1.$$

♣

Now let

$$b^*(x,y) := \sup\{b : E_b(x,y) \subset K\}. \tag{3.2}$$

By definition and Lemma 3.1,

$$b^*(x,y) \ge \frac{\sqrt{1-|||x|||_K^2}}{|||y|||_K}.$$

Lemma 3.2. *Let $x \in K^o$, $y \in \mathbb{R}^N$. For p a polynomial with $||p||_K \le 1$ and $|p(x)| \ne 1$,*

$$\frac{1}{\deg p}\frac{|D_y p(x)|}{\sqrt{1-(p(x))^2}} \le \frac{1}{b^*(x,y)}. \tag{3.3}$$

Moreover,

$$\delta_M(x;y) = \frac{1}{b^*(x,y)}. \tag{3.4}$$

Proof. Fix $x \in K^o$, $y \in \mathbb{R}^N$, and $b < b^*(x,y)$. For p a polynomial with $||p||_K \le 1$ and $|p(x)| \ne 1$, let $T(t) := p(r(t))$ where $r(t)$ is as in (3.1). Then $T(t)$ is a trigonometric polynomial with $\deg T = \deg p$ and $||T||_{[0,2\pi]} \le ||p||_K$ since $E_b(x,y) \subset K$. By Szegö's inequality for trigonometric polynomials,

$$\frac{|T'(t)|}{\sqrt{1-T(t)^2}} \le \deg T,$$

so that, in particular,

$$\frac{1}{\deg T}\frac{|T'(0)|}{\sqrt{1-T(0)^2}} \le 1.$$

But $T(0) = p(r(0)) = p(x)$, $T'(0) = \nabla p(r(0)) \cdot r'(0) = \nabla p(x) \cdot by = bD_y p(x)$, thus

$$\frac{1}{\deg p}\frac{|D_y p(x)|}{\sqrt{1-(p(x))^2}} \le 1/b$$

which gives (3.3).

To show $\delta_M(x;y) \ge \frac{1}{b^*(x,y)}$, by definition of $b^*(x,y)$, there exists $u \in \partial K \cap E_{b^*}(x,y)$; by symmetry, $-u \in \partial K \cap E_{b^*}(x,y)$ as well. Let H and $-H$ be support hyperplanes to ∂K at $u, -u$ and let n be a unit normal vector for H (oriented "out" of K). Define the half-space

$$H_u := \{z \in \mathbb{R}^N : n \cdot z \le n \cdot u\}.$$

Then $K \subset H_u \cap -(H_u)$ and hence

$$E_{b^*}(x,y) \subset K \subset H_u \cap -(H_u).$$

Let $p(z) := \frac{n \cdot z}{n \cdot u}$. By construction, $||p||_K \le 1$ and p maps $E_{b^*}(x,y)$ onto $[-1,1]$. Hence, with $r(t) = x\cos t + yb^*(x,y)\sin t$, we can write

$$p(r(t)) = A\cos t + B\sin t$$

for some A, B with $A^2+B^2 = 1$. For if $A^2+B^2 > 1$, $p(E_{b^*}(x,y)) \not\subset [-1,1]$; if $A^2+B^2 < 1$, $p(E_{b^*}(x,y)) \subset [-1,1]$ but $p(E_{b^*}(x,y)) \ne [-1,1]$. Using the facts that $\deg p = 1$; $r(0) = x$; and $r'(0) = yb^*(x,y)$, it follows that

$$\delta_M(x;y) \ge \frac{|D_y p(x)|}{\sqrt{1-(p(x))^2}} = \frac{1}{b^*(x,y)}\Big(\frac{|\frac{d}{dt}(p(r(t)))|}{\sqrt{1-(p(r(t)))^2}}|\Big)_{t=0}$$

$$= \frac{1}{b^*(x,y)}\Big(\frac{|\frac{d}{dt}(A\cos t + B\sin t)|}{\sqrt{1-(A\cos t+B\sin t))^2}}|\Big)_{t=0}$$

$$= \frac{1}{b^*(x,y)}\frac{|B|}{\sqrt{1-A^2}} = \frac{1}{b^*(x,y)}\frac{|B|}{|B|} = \frac{1}{b^*(x,y)}$$

provided $A \ne 1$. But $A = p(x) \ne 1$ since $x \in K^o$. ♣

In [4], it was shown that the equality

$$\delta_B(x;y) = \frac{1}{b^*(x,y)}$$

holds for general convex bodies in $\mathbb{R}^n$, and, moreover, the function δ_B is continuous.

Corollary 3.3. *Let K be centrally symmetric and convex. Then $\delta_M = \delta_M^{(1)}$ where*

$$\delta_M^{(1)}(x;y) := \sup_{||p||_K \leq 1,\ \deg p=1} \frac{|D_y p(x)|}{\sqrt{1-p(x)^2}}.$$

Proof. This follows since the proof of Lemma 3.2 shows that the supremum in the definition of $\delta_M(x;y)$ is attained for linear polynomials. ♣

We next show that the Dubiner distance d_D is a C^1 pseudodistance (recall Definition 1.3 and equation (2.2)).

Proposition 3.4. *Let K be centrally symmetric and convex. For all $E \subset\subset K^o$ and all $\epsilon > 0$, there exists $\eta > 0$ with*

$$|d_D(a,b) - \delta_M(x;b-a)| \leq \epsilon |b-a|$$

for all $a,b \in B(x,\eta)$ and $x \in E$.

Proof. Fix a positive integer n and a polynomial p of degree at most n with $||p||_K \leq 1$.
Claim: For all ϵ there exists $\eta > 0$ (depending on n, E but not p) with

$$\left|\frac{1}{\deg p}|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| - \frac{1}{\deg p}\frac{|D_{b-a}(p(x))|}{\sqrt{1-p(x)^2}}\right| \leq \epsilon|b-a|$$

for all $a,b \in B(x,\eta)$ and $x \in E$.
Proof of Claim: Let $f(x) = \cos^{-1}(p(x))$. It suffices to show that

$$\left|\frac{|f(b)-f(a)|}{|b-a|} - |D_{\frac{b-a}{|b-a|}} f(x)|\right| \leq \epsilon$$

for all $a,b \in B(x,\eta)$ and $x \in E$. To verify this, let $g(t) := f(a+t(b-a))$. Then

$$|g(1)-g(0)| = |f(b)-f(a)| = \left|\int_0^1 g'(t)dt\right|$$

$$= \left|\int_0^1 D_{\frac{b-a}{|b-a|}} f(a+t(b-a))dt\right||b-a|$$

so that

$$\left|\frac{|f(b)-f(a)|}{|b-a|} - |D_{\frac{b-a}{|b-a|}} f(x)|\right| \leq \int_0^1 |D_{\frac{b-a}{|b-a|}} f(a+t(b-a)) - D_{\frac{b-a}{|b-a|}} f(x)|dt$$

$$\leq \sup_{z\in[a,b]} |D_{\frac{b-a}{|b-a|}} f(z) - D_{\frac{b-a}{|b-a|}} f(x)| \leq \sup_{z \in B(x,\eta)} |D_{\frac{b-a}{|b-a|}} f(z) - D_{\frac{b-a}{|b-a|}} f(x)|.$$

This last quantity is less than ϵ if $\eta = \eta(n, E)$ is sufficiently small by compactness of the family $\{p : \text{deg}p \leq n, \ ||p||_K \leq 1\}$. This proves the claim.

From Corollary 3.3, $\delta_M = \delta_M^{(1)}$, thus for $x \in E$ and $a, b \in B(x, \eta)$ we can take p with $\text{deg}p = 1$ and $||p||_K \leq 1$ such that $\frac{|D_{b-a}(p(x))|}{\sqrt{1-p(x)^2}} > \delta_M(x; b-a) - \epsilon|b-a|$. Applying the Claim (with $n = 1$),

$$\delta_M(x; b-a) < 2\epsilon|b-a| + |\cos^{-1}(p(b)) - \cos^{-1}(p(a))|$$

$$\leq 2\epsilon|b-a| + d_D(a, b).$$

On the other hand, from Lemma 2.2, for $a, b \in B(x, \eta)$ and $x \in E$ we can find an n with $d_D(a, b) = d_D^{(n)}(a, b)$. Choose p with $\text{deg}p \leq n$ and $||p||_K \leq 1$ such that $\frac{1}{\text{deg}p}|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| > d_D(a, b) - \epsilon|b-a|$. Applying the Claim,

$$d_D(a, b) < 2\epsilon|b-a| + \frac{1}{\text{deg}p}\frac{|D_{b-a}(p(x))|}{\sqrt{1-p(x)^2}} \leq 2\epsilon|b-a| + \delta_M(x; b-a).$$

♣

Corollary 3.5. *Let K be centrally symmetric and convex. Then $\delta_M = Dd_D$ is continuous and d_D is C^1.*

Proof. We have $\delta_M = Dd_D$ by (2.2) for general K. Thus δ_M is usc. By Corollary 3.3, δ_M is the supremum of a family of continuous functions and hence is lowersemicontinuous (lsc). The fact that d_D is a C^1 pseudodistance now follows from Proposition 3.4 and Definition 1.3. ♣

From Baran's work [1], for K centrally symmetric and convex,

$$\delta_B(x; y) = \sup\{\frac{|y \cdot w|}{\sqrt{1-(x \cdot w)^2}} : w \in K^*\} \tag{3.5}$$

where recall

$$K^* := \{x \in \mathbb{R}^N : x \cdot y \leq 1 \text{ for all } y \in K\}$$

is the polar of K. Note also that $|||x|||_K = \sup\{x \cdot w : w \in K^*\}$.

Proposition 3.6. *Let K be centrally symmetric and convex. Then*

$$b^*(x; y) = \inf\{\frac{\sqrt{1-(x \cdot w)^2}}{|y \cdot w|} : w \in K^*\}. \tag{3.6}$$

Hence

$$\delta_D = \delta_M = \delta_B. \tag{3.7}$$

Moreover,

$$d_D^i = d_M = d_B. \tag{3.8}$$

Proof. From the definition of $b^*(x;y)$ in (3.2) and K^* we can write

$$b^*(x;y) = \sup\{b : \sup_{w\in K^*,\ t\in[0,2\pi]} |x\cos t\cdot w + yb\sin t\cdot w| = 1\}$$

$$= \sup\{b : \sup_{w\in K^*} [(w\cdot x)^2 + b^2(w\cdot y)^2] = 1\}.$$

To see that this last supremum equals $\inf\{\frac{\sqrt{1-(x\cdot w)^2}}{|y\cdot w|} : w \in K^*\}$, take any b with $\sup_{w\in K^*}[(w\cdot x)^2 + b^2(w\cdot y)^2] = 1$; then, for any $w \in K^*$, $(w\cdot x)^2 + b^2(w\cdot y)^2 \le 1$ so that $b \le \frac{\sqrt{1-(x\cdot w)^2}}{|y\cdot w|}$ which shows that b^* is less than or equal to the right-hand-side of (3.6). Next, we observe that the infimum in the right-hand-side of (3.6) is attained. Let $b_0 = \min\{\frac{\sqrt{1-(x\cdot w)^2}}{|y\cdot w|} : w \in K^*\}$. Then $b_0 \le \frac{\sqrt{1-(x\cdot w)^2}}{|y\cdot w|}$ for all $w \in K^*$ with equality at some point(s); hence $(w\cdot x)^2 + b_0^2(w\cdot y)^2 \le 1$ for all $w \in K^*$ with equality at some point(s); i.e., $b_0 \le b^*$ and equality holds.

Equation (3.7) follows from (2.2), (3.4), (3.5) and (3.6). Using this, Proposition 1.6 gives

$$d_D^i = \int \delta_D = \int \delta_M = \int \delta_B$$

which is (3.8). ♣

As a concrete example, for K the closed unit ball in $\mathbb{R}^N$, given $x \in K^o$ and $y \in \mathbb{R}^N$, let

$$\tilde{w} := y(1-|x|^2) + (y\cdot x)x.$$

Then $w := \tilde{w}/|\tilde{w}|$ maximizes $\frac{|y\cdot w|}{\sqrt{1-(x\cdot w)^2}}$ and this maximal value is

$$\Big(\frac{(1-|x|^2)|y|^2 + (x\cdot y)^2}{1-|x|^2}\Big)^{1/2}\delta_B(x;y).$$

We conjecture that

$$d_D(a,b) = d_M(a,b) = d_B(a,b) \tag{3.9}$$

for K centrally symmetric and convex. We present some evidence supporting the validity of the conjecture.

Proposition 3.7. *For a centrally symmetric $E \subset \mathbb{R}^N$ bounded by an ellipsoid, $d_D^{(1)} = d_D = d_M = d_B$.*

Proof. We have equality of d_D, $d_D^{(1)}$ and d_B for $K = \overline{B}$, the unit ball, by explicit calculation in [3]. Thus by inequality (2.1),

$$d_D^{(1)} = d_D = d_M = d_B \tag{3.10}$$

for $K = \overline{B}$. Since $E = T(K)$ for some invertible linear mapping T, equality holds in (3.10) for E from (2.9) and the observation that $d_D^{(1)}$ for K and E coincide. ♣

We now specialize to centrally symmetric convex bodies in $\mathbb{R}^2$.

Theorem 3.8. *Let $K \subset \mathbb{R}^2$ be centrally symmetric and convex. For two points $a, b \in K$ with the property that there exists a centrally symmetric region $E = E(a, b) \subset K$ bounded by an ellipse with a, b lying on the same 'side' of the ellipse ∂E (with 'sides' separated by an axis joining supporting hyperplanes), we have $d_D^{(1)}(a, b) = d_D(a, b) = d_M(a, b) = d_B(a, b)$.*

Proof. The idea is similar to that utilized in Lemma 3.2. We expand E to construct a centrally symmetric region $\tilde{E} \subset K$ bounded by an ellipse with $a, b \in \partial\tilde{E}$ with the property that there exists $u \in \partial K \cap \partial\tilde{E}$, and hence $-u \in \partial K \cap \partial\tilde{E}$. (cf., Theorem 5.3 and its proof in [6]). Then, letting $H, -H$ be support hyperplanes to ∂K at $u, -u$ and calling n the unit normal vector for H (oriented "out" of K), the half-space $H_u := \{z : n \cdot z \leq n \cdot u\}$ satisfies $K \subset H_u \cap -(H_u)$ and hence

$$\tilde{E} \subset K \subset H_u \cap -(H_u).$$

Let $p(z) := \frac{n \cdot z}{n \cdot u}$. By construction, $||p||_K \leq 1$ and p maps $\tilde{E}$ and K onto $[-1, 1]$. Thus

$$d_D(a, b) = d_D^K(a, b) \geq |\cos^{-1}(p(b)) - \cos^{-1}(p(a))|$$

$$= d_D^{\tilde{E}}(a, b) = d_M^{\tilde{E}}(a, b) = d_B^{\tilde{E}}(a, b),$$

the last line coming from Proposition 3.7. But from Remark 2.1, (recall that $d_D^{\tilde{E}}$ is well-definded on $\partial\tilde{E}$) we have $d^K(a, b) \leq d^{\tilde{E}}(a, b)$ on $\tilde{E} \times \tilde{E}$ for each of our three distances and the result follows. ♣

From the proof of Theorem 3.8, we see that equality holds in (3.9) at points $(a, b) \in E_{b^*}(x, y) \times E_{b^*}(x, y)$ for each centrally symmetric ellipse $E_{b^*}(x, y)$ contained in K with $b^*(x; y)$ as in (3.6). Now recall from Remark 1.4 that the Markov and Baran distances are always inner; i.e., $d_M = d_M^i$ and $d_B = d_B^i$. Suppose we knew that the Dubiner distance d_D on a centrally symmetric convex body was an inner distance. Then from (3.8) of Proposition 3.6 we conclude that (3.9) holds. In this vein, we mention the following definition. For a subset X of a vector space equipped with a distance d, the pair (X, d) is called *metrically convex* if given $a, b \in X$, there exists $c \in X$ with $d(a, c) + d(c, b) = d(a, b)$. It is known [11] that *if (X, d) is metrically convex and complete, then through each pair of points a, b in X there is a shortest curve; i.e. there exists $\alpha : [0, 1] \to X$ a continuous curve joining a and b with $L_d(\alpha) = d^i(a, b)$, and, indeed, $d = d^i$.* Thus we make the following observation.

Corollary 3.9. *Let $K \subset \mathbb{R}^2$ be centrally symmetric and convex with the additional property that for any two points $a, b \in K$, there exists a centrally symmetric region $E = E(a, b) \subset K$ bounded by an ellipse with $a, b \in \partial E$. Then $d_D = d_M = d_B$.*

Proof. The property that d_D locally dominates the Euclidean distance in the interior of K (see Remark 1.3) extends to K, implying completeness of (K, d_D). Therefore it suffices to show that $d_D = d_D^i$ which will follow if d_D is metrically convex. But this follows by the hypothesized property, since we can take c to be any point on the (shorter) arc of the ellipse $\partial\widetilde{E}$ joining a and b which was constructed in the proof of Theorem 3.8. ♣

The geometric property hypothesized in Corollary 3.9 does not hold for every centrally symmetric convex body $K \subset \mathbb{R}^2$. For example, take K to be the square $[-a, a] \times [-a, a]$ (it can be shown, however, that a square is, indeed, metrically convex). By rounding off the edges of the square, we can even construct such a K which is strictly convex with smooth boundary.

References

1. Baran, M., Bernstein type theorems for compact sets in $\mathbb{R}^n$, J. Approx. Theory **69**, (1992), 156–166.
2. Baran, M., Bernstein type theorems for compact sets in $\mathbb{R}^n$ revisited, J. Approx. Theory **79**, (1994), 190–198.
3. Bos, L., Levenberg, N., and Waldron, S., Metrics associated to multivariate polynomial inequalities, *Advances in Constructive Approximation*, M. Neamtu and E. B. Saff eds., Nashboro Press, Nashville, 2004, 133–147.
4. Burns, D., Levenberg, N., Ma'u, S. and Révész, Sz., Monge-Ampère Measures for Convex Bodies and Bernstein-Markov Type Inequalities, submitted.
5. Dineen, S., *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
6. Dubiner, M., The theory of multidimensional polynomial approximation, J. Anal. Math. **67**, (1995), 39–116.
7. Jarnicki, M. and Pflug, P., *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, Berlin, 1993.
8. Klimek, M., *Pluripotential Theory*, Clarendon Press, Oxford, 1991.
9. Kroó, A. and Révész, Sz., *On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies*, J. Approx. Theory **99** (1999), 134–152.
10. Milev, L. and S. Révész, Bernstein's inequality for multivariate polynomials on the standard simplex, J. Ineq. Appl. 2005:2 (2005), 145–163.
11. Rinow, W., *Die Innere Geometrie der Metrischen Räume*, Springer-Verlag, 1961.
12. Sarantoupolos, *Bounds on the derivatives of polynomials on Banach spaces*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 307–312.

Len Bos
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4
lpbos@math.ucalgary.ca

Norm Levenberg
Department of Mathematics
Indiana University
Bloomington, IN 47405

USA
nlevenbe@indiana.edu

Shayne Waldron
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand
waldron@math.auckland.ac.nz