Refinements of the Peano kernel theorem

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Abstract. It is shown that by starting with a general form of the Peano kernel theorem which makes no reference to the interchange of linear functionals and integrals, the most general results can be obtained in an elementary manner. In particular, we classify how the Peano kernels become increasingly smooth and satisfy boundary (or equivalently moment) conditions as the linear functionals they represent become continuous on wider classes of functions. These results are then used to give new representations of the continuous duals of $C^r[a,b]$ and $W_p^r[a,b]$, $1 \le p < \infty$.

1. Introduction

Recall the formula for Taylor interpolation at a from Π_k (polynomials of degree $\le k$) together with integral remainder. For a sufficiently smooth function f defined on $[a,b]$

$$f = \sum_{j=0}^{k} D^j f(a) \frac{(\cdot - a)^j}{j!} + \int_a^b D^{k+1} f(t) \frac{(\cdot - t)_+^k}{k!}\, dt, \tag{1.1}$$

where $(\cdot)_+^k$ is the truncated power function. Suppose that λ is a linear functional which vanishes on Π_k. Then applying λ to (1.1) gives

$$\lambda(f) = \lambda \left(\int_a^b D^{k+1} f(t) \frac{(\cdot - t)_+^k}{k!}\, dt \right). \tag{1.2}$$

In these terms (following Peano's paper [P13] on quadrature errors) the *Peano kernel theorem* is usually stated (and proved) as follows.

Theorem 1.3 (Taylor error form). *Suppose that λ is a linear functional which vanishes on Π_k. Then under "certain conditions" on λ it is possible to interchange λ with the integral $\int_a^b$ in (1.2) to obtain the representation*

$$\lambda(f) = \int_a^b D^{k+1} f(t) K(t)\, dt, \tag{1.4}$$

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for "suitable" f, where the Peano kernel K is given by

$$K(t) := \frac{1}{k!}\lambda\left((\cdot - t)_+^k\right), \tag{1.5}$$

and sometimes (1.5) *has to be interpreted in an "appropriate" way.*

Representations of this type can also be found in the earlier work of Birkhoff [B06]. See Sard [S63:p. 25], Davis [D75:p. 70], or Brass and Förster [BF98] for typical examples.

In this paper we take an alternative approach based on a factorisation theorem of functional analysis. This is used in the form given by Theorem 2.2, which can be paraphrased as follows:

> *If the linear functional λ vanishes on Π_k, and is continuous on a certain space X (i.e., satisfies "certain conditions"), then*
>
> $$\lambda(f) = Q(D^{k+1}f), \qquad \forall f \in X \quad (\textit{the "suitable" } f),$$
>
> *and the (continuous) linear functional Q can be represented by integration against some kernel K, or more generally a measure (the "appropriate" interpretation).*

The above measure can be constructed, and so this gives (1.4) and its variations thereof. A key part of this result is that the correspondence between λ and Q is 1–1. This allows us to characterise the space of Peano kernels corresponding to functionals from a number of continuous dual spaces X^*, each of which can be interpreted as a subspace of $(C^{k+1}[a,b])^*$. The rest of the paper is set out as follows.

In Section 2, the general Peano kernel theorem (Theorem 2.2) is presented, together with some technical lemmas which are needed in the applications of it.

In Section 3, these results are used to give a complete characterisation of the spaces of Peano kernels corresponding to the nested subspaces of $(C^{k+1}[a,b])^*$

$$(C[a,b])^* \subset (C^1[a,b])^* \subset \cdots \subset (C^k[a,b])^*$$

(see Theorem 3.3), and

$$(L_p[a,b])^* \subset (W_p^1[a,b])^* \subset \cdots \subset (W_p^k[a,b])^* \subset (W_p^{k+1}[a,b])^*, \qquad 1 \le p < \infty$$

(see Theorem 3.19). In addition to becoming increasingly smooth (which is to be expected), these spaces also satisfy certain boundary conditions. These boundary conditions are related to certain moment and orthogonality conditions which are illustrated with some familiar examples including B–splines.

In Section 4, the Peano kernel classification of Section 3 is used to give new representations of the continuous dual spaces of $C^r[a,b]$ and $W_p^r[a,b]$, $1 \le p < \infty$.

2. The general Peano kernel theorem

The spaces X of "suitable" f (on which D^{k+1} must be defined) will be subspaces of the distributions $\mathcal{D}'(a,b)$. The corresponding continuous dual spaces X^* will be interpreted as subspaces of $(C^{k+1}[a,b])^*$ in the following way. For each X we consider, $C^{k+1}[a,b]$ is a dense subset and the embedding map $C^{k+1}[a,b] \hookrightarrow X$ is continuous. This implies that each $\lambda \in X^*$ is uniquely determined by its restriction to $C^{k+1}[a,b]$, which is an element of $(C^{k+1}[a,b])^*$, and we write

$$X^* \subset (C^{k+1}[a,b])^*, \tag{2.1}$$

etc, without further explanation. The general Peano kernel theorem is the following.

Theorem 2.2 (general form). *Suppose that $X \subset \mathcal{D}'(a,b)$ is a space containing Π_k. Then there is a 1–1 (linear) correspondence between the linear functionals $\lambda : X \to \mathbb{R}$ which vanish on Π_k and the linear functionals $\mathcal{Q} : Y \to \mathbb{R}$ ($Y := D^{k+1}X$) given by the representation*

$$\lambda(f) = \mathcal{Q}(D^{k+1}f), \qquad \forall f \in X. \tag{2.3}$$

Note that (2.3) *defines $\mathcal{Q}$. Further, if X and Y are given topologies for which*

$$D^{k+1} : X \to Y$$

is a continuous open map, then the $\lambda \in X^*$ (which vanish on Π_k) correspond to the $Q \in Y^*$. If X and Y are normed linear spaces, then $\lambda \mapsto \|\mathcal{Q}\|$ is an equivalent norm on those $\lambda \in X^*$ which vanish on Π_k.

As mentioned in the introduction, the 1–1 correspondence between the $\lambda \in X^*$ and the (Peano kernels) $Q \in Y^*$ is vital to our applications. This result is a special case of the following quotient theorem of functional analysis, the first part of which is a simple algebraic result (called the key lemma by some algebraists).

Theorem 2.4 (Quotient theorem). *Suppose that $\mathcal{U} : X \to Y$ is a linear map onto Y. Then there is a 1–1 (linear) correspondence between the linear maps $\mathcal{R} : X \to Z$ which vanish on the kernel of $\mathcal{U}$ and the linear maps $\mathcal{Q} : Y \to Z$ given by*

$$\mathcal{R} = \mathcal{Q} \circ \mathcal{U}.$$

Further, if X, Y, Z are topological vector spaces, and $\mathcal{U}$ is a continuous and open map (maps open sets to open sets), then under this correspondence the continuous maps $\mathcal{R}$ correspond to the continuous maps $\mathcal{Q}$. If X, Y, Z are normed linear spaces and $\mathcal{U}$ is a continuous open map, then

$$\mathcal{R} \mapsto \|\mathcal{Q}\| \tag{2.5}$$

is an equivalent norm on those continuous linear maps $\mathcal{R} : X \to Z$ which vanish on the kernel of $\mathcal{U}$.

Sard [S63:p. 311] gives a version of this quotient theorem for Banach spaces where $\mathcal{U}$ is continuous. By the open mapping theorem (a continuous map from one Banach space onto another is open) these assumptions imply that $\mathcal{U}$ is open. The topological space version of what is referred to there as *Sard's factorisation theorem* is given by Atteia in [At92:p. 98] (where the condition that $\mathcal{U}$ be an open map is built into the definition of homomorphism used there). Neither of these results mentions the equivalence of norms (2.5), for which we now provide a proof.

Proof (equivalence of norms): From $\mathcal{R} = \mathcal{Q} \circ \mathcal{U}$ we obtain

$$\|\mathcal{R}\| \leq \|\mathcal{Q}\| \|\mathcal{U}\|.$$

Since $\mathcal{U}$ is open, there exists an $r > 0$ for which

$$B_Y \subset \mathcal{U}(rB_X),$$

where B_X, B_Y are the unit balls in X, Y. Thus

$$\|\mathcal{Q}\| = \sup \|\mathcal{Q}B_Y\| \leq \sup \|\mathcal{Q}\mathcal{U}(rB_X)\| \leq \|\mathcal{R}\| \sup \|rB_X\| = \|\mathcal{R}\| r,$$

which is the reverse inequality. □

Theorem 2.2 is obtained from Theorem 2.4 by taking:

$$\mathcal{R} = \lambda : X \to \mathbb{R}, \qquad \mathcal{U} = D^{k+1} : X \to Y \quad \text{(which has kernel } \Pi_k).$$

Some examples and preliminary results

Taking $X = C^{k+1}[a,b]$ in Theorem 2.2 (cf Sard [S63:p. 314]) gives a 1–1 correspondence between $\lambda \in (C^{k+1}[a,b])^*$ and functions $w \in \mathrm{NBV}[a,b]$ (normalised bounded variation on $[a,b]$), i.e., Riemann–Stieltjes measures, via

$$\lambda(f) = \int_a^b D^{k+1} f(t)\, dw(t), \qquad \forall f \in C^{k+1}[a,b]. \tag{2.6}$$

The map $\lambda \mapsto \mathrm{Var}(w)$ (the total variation of w) gives an equivalent norm on

$$\{\lambda \in (C^{k+1}[a,b])^* : \lambda(\Pi_k) = 0\}.$$

This result is constructive, since a λ of the form (2.6) can be applied to $(\cdot - t)_+^{k+1}$ whenever t does not belong to

$$J_{\lambda,k+1} := \text{the countable set of (jump) discontinuities of } w, \tag{2.7}$$

and in this way a function $w \in \mathrm{BV}[a,b]$ satisfying (2.6) can be obtained via the calculation

$$w(t) := w(b) - \frac{1}{(k+1)!} \lambda\left((\cdot - t)_+^{k+1}\right), \qquad \forall t \notin J_{\lambda,k+1}, \tag{2.8}$$

with the choice

$$w(b) := \frac{1}{(k+1)!} \lambda\left((\cdot)^{k+1} + g\right), \qquad \forall g \in \Pi_k \tag{2.9}$$

giving the (right continuous) w normalised to have $w(a) = 0$. Formally, λ is not defined on $(\cdot - t)_+^{k+1} \notin C^{k+1}[a,b]$ (for $t \neq a, b$), and the calculation (2.8) should be done by considering an appropriate sequence of approximations to $(\cdot - t)_+^{k+1}$ (see [S63:p. 139] for details). In practice this is not necessary.

A typical example of linear functional λ which requires the *mass representation* (2.6), and not simply a kernel K, is

$$f \mapsto D^{k+1}f(\xi), \quad \xi \in [a,b].$$

However, if $\lambda \in (C^k[a,b])^* \subset (C^{k+1}[a,b])^*$, then (2.8) can be 'differentiated' to obtain the representation (1.4), valid for $f \in C^{k+1}[a,b]$, where

$$K(t) := \frac{1}{k!}\lambda\left((\cdot - t)_+^k\right) = Dw(t).$$

The corresponding space of Peano kernels K is not all of $\mathrm{BV}[a,b]$ (see Theorem 3.3 for the general result), but rather a subspace of $L_1[a,b]$ (as the (1.4) implies it must be). This space, which we denote by $\mathrm{PK}[a,b]$, is defined to be the space

$$\mathcal{PK}[a,b] := \{w \in \mathrm{BV}[a,b] : w \text{ is right continuous on } (a,b),\ w(a) = w(b) = 0\} \qquad (2.10)$$

viewed as a subspace of $L_1[a,b]$. We now verify that $\mathcal{PK}[a,b]$ does define a subspace of $L_1[a,b]$, indeed it is a subspace of $L_\infty[a,b]$. For $w \in \mathcal{PK}[a,b]$, $\|w\|_{L_\infty[a,b]} \le \mathrm{Var}(w)$, and so w can be identified with an element $[w] \in L_\infty[a,b]$. This association is 1–1, since if $w \ne v$ at some point $a < \xi < b$, then because w and v have only countably many discontinuities, all of the first kind (removable or jump discontinuities), it follows that

$$\|[w] - [v]\|_{L_\infty[a,b]} \ge \lim_{x \to \xi^+} |w(x) - v(x)| = |w(\xi) - v(\xi)| > 0.$$

The spaces used in our Peano kernel classifications occur as antiderivatives of the subspaces $\mathrm{PK}[a,b], L_p[a,b] \subset L_1[a,b]$. To obtain them we need the following technical lemma.

Lemma 2.11. *If $Y := Y[a,b]$ is a subspace of $L_1[a,b]$, then*

$$X = Y^j := Y^j[a,b] := \{f \in \mathcal{D}'(a,b) : D^j f \in Y\}, \qquad j = 0,1,2,\dots \qquad (2.12)$$

is a subspace of the Sobolev space $W_1^j[a,b]$, and

$$D^j : Y^j \to Y$$

maps onto Y. In particular, $Y^0 = Y$, while for $j = 1,2,3,\dots$,

$$Y^j = \{f \in C^{j-1}[a,b] : D^{j-1}f \text{ is absolutely continuous},\ D^j f \in Y\}. \qquad (2.13)$$

Proof: For subspaces $Z \subset Y \subset L_1[a,b]$, the definition (2.12) implies that $Z^j \subset Y^j$. Thus, to prove the rest of the result it is sufficient to show that (2.12) and (2.13) are equivalent for the particular choice $Y = L_1[a,b]$, where either of (2.12) or (2.13) is taken as the definition of $W_1^j[a,b] = Y^j$. We quickly sketch the proof of this well known result. For $f \in L_1[a,b]$, the function F defined by

$$F(x) := \int_a^x f(t)\,dt$$

is absolutely continuous with $DF = f$, and the only distributions with (distributional) derivative f are F plus a constant. Since the polynomials are absolutely continuous on $[a,b]$, each of the $j-1$ additional antiderivatives is absolutely continuous, which shows equivalence of both definitions of $W_1^j[a,b]$. $\square$

Notice from the proof, that if $w \in Y^{j-1}$, then

$$K(t) := -\int_a^b w(\xi)\, d\xi \tag{2.14}$$

defines a function $K \in Y^j$ with $DK = -w$ (in $L_1[a,b]$). This will be a key fact in several of our inductive arguments, which rely on integration by parts in the following form. If the Riemann–Stieltjes integral $\int_a^b f\, dw$ exists, then so does $\int_a^b w\, df$, and

$$\int_a^b f\, dw = [f(b)w(b) - f(a)w(a)] - \int_a^b w\, df.$$

The next lemma shows that the boundary conditions which occur in our classification can be interpreted as certain moment conditions on the Peano kernel K. Let $\langle \cdot, \cdot \rangle$ be the inner product

$$\langle f, g \rangle := \int_a^b f(t) g(t)\, dt.$$

Lemma 2.15. *For kernels $K \in W_1^j[a,b]$, $j = 1, 2, 3, \ldots$,*
(a) *The boundary conditions*

$$K(a) = DK(a) = \cdots = D^{j-1}K(a) = 0, \qquad K(b) = DK(b) = \cdots = D^{j-1}K(b) = 0\,.$$

imply the following:
(b) *The moment conditions*

$$\int_a^b DK(t)\, dt = \int_a^b tD^2K(t)\, dt = \int_a^b t^2 D^3 K(t)\, dt = \cdots = \int_a^b t^{j-1} D^j K(t)\, dt = 0.$$

(c) *The orthogonality conditions*

$$\langle D^r K, \Pi_{r-1} \rangle = 0, \qquad r = 1, 2, \ldots, j. \tag{2.16}$$

For $j \geq 2$, the conditions (a) *and* (c) *are equivalent.*

Proof: The proof is by induction, using integration by parts in the form

$$\int_a^b D^j K(t) g(t)\, dt$$
$$= D^{j-1}K(b)g(b) - D^{j-1}K(a)g(a) - \int_a^b D^{j-1}K(t) Dg(t)\, dt, \qquad \forall g \in \Pi_{j-1}.$$

□

3. A classification of the Peano kernels

In this section we use Theorem 2.2 to obtain a classification of the Peano kernels for linear functionals from the subspaces of $(C^{k+1}[a,b])^*$ given by

$$(C^{k-j}[a,b])^*, \qquad j=0,1,\ldots,k, \tag{3.1}$$

and

$$(W_p^{k+1-j}[a,b])^*, \qquad 1\le p<\infty,\ j=0,1,\ldots,k+1. \tag{3.2}$$

These satisfy the strict inclusions (by Sobolev's embedding theorem)

$$(W_p^{k-j}[a,b])^* \underset{\neq}{\subset} (C^{k-j}[a,b])^* \underset{\neq}{\subset} (W_p^{k+1-j}[a,b])^*, \qquad j=0,1,\ldots k,$$

and (by Hölder's inequality)

$$(W_{p_1}^{k+1-j}[a,b])^* \underset{\neq}{\subset} (W_{p_2}^{k+1-j}[a,b])^*, \qquad 1\le p_1<p_2<\infty,\ j=0,1,\ldots,k+1.$$

It is shown that as the space of linear functionals becomes more restrictive (the functionals are continuous on wider classes of functions) the corresponding Peano kernels become smoother and satisfy certain boundary (moment) conditions.

First we consider (3.1). The resulting Peano kernels are antiderivatives of functions from the space $\mathrm{PK}[a,b]$ which (we recall) is

$$\mathcal{PK}[a,b] := \{w\in \mathrm{BV}[a,b] : w \text{ is right continuous on } (a,b),\ w(a)=w(b)=0\}$$

viewed as a subspace of $L_\infty[a,b]$. Let

$$\mathrm{PK}^j[a,b] := \{f\in \mathcal{D}'(a,b) : D^j f\in \mathrm{PK}[a,b]\}, \qquad j=0,1,2,\ldots.$$

Then (by Lemma 2.11), $\mathrm{PK}^0[a,b]=\mathrm{PK}[a,b]$, and for $j\ge 1$,

$$\mathrm{PK}^j[a,b] = \{f\in C^{j-1}[a,b] : D^{j-1}f \text{ is absolutely continuous},\ D^j f\in \mathrm{PK}[a,b]\}.$$

Notice that $\mathrm{PK}^j[a,b]\subset W_\infty^j[a,b]\subset W_1^j[a,b]$.

Theorem 3.3. *There is a 1–1 (linear) correspondence between the linear functionals*

$$\lambda\in (C^{k-j}[a,b])^*, \qquad j=0,1,\ldots,k$$

which vanish on Π_k and the functions

$$K\in \mathrm{PK}^j[a,b]$$

that satisfy (for $j\ge 1$) the boundary conditions

$$K(a)=DK(a)=\cdots=D^{j-1}K(a)=0, \qquad K(b)=DK(b)=\cdots=D^{j-1}K(b)=0$$

which is given by the representation

$$\lambda(f) = \int_a^b D^{k+1}f(t)K(t)\,dt, \qquad \forall f \in C^{k+1}[a,b], \tag{3.4}$$

or, equivalently, with $D^jK \in \mathcal{PK}[a,b]$,

$$\lambda(f) = (-1)^{j-1}\int_a^b D^{k-j}f(t)\,d(D^jK)(t), \qquad \forall f \in C^{k-j}[a,b]. \tag{3.5}$$

Further, $\lambda \mapsto \|K\|_{L_1[a,b]}$ *gives an equivalent norm on* $\{\lambda \in (C^{k-j}[a,b])^* : \lambda(\Pi_k) = 0\}$*, and* K *can be computed from*

$$K(t) := \frac{1}{k!}\lambda\left((\cdot - t)_+^k\right), \qquad \forall t \notin J_{\lambda,k}, \tag{3.6}$$

where the countable set $J_{\lambda,k}$ *defined by (2.7) is empty if* $\lambda \in (C^{k-1}[a,b])^*$.

Proof: The proof is by (strong) induction on j and k.
First we prove the result for $j = 0$ and all k. By the example (2.6), there is a 1–1 correspondence between the $\lambda \in (C^k[a,b])^*$ which vanish on Π_{k-1} and the $w \in \mathrm{NBV}[a,b]$ given by

$$\lambda(f) = \int_a^b D^kf(t)\,dw(t), \qquad \forall f \in C^k[a,b], \tag{3.7}$$

where

$$w(t) := w(b) - \frac{1}{k!}\lambda\left((\cdot - t)_+^k\right), \qquad \forall t \notin J_{\lambda,k},$$

with

$$w(b) := \frac{1}{k!}\lambda\left((\cdot)^k + g\right), \qquad \forall g \in \Pi_{k-1}. \tag{3.8}$$

The norm of λ is equivalent to $\|K\|_{L_1[a,b]}$, which is the (total) variation of the function giving the Riemann–Stieltjes measure $K(t)dt$. Since

$$d(D^kf)(t) = D^{k+1}f(t)dt, \quad \forall f \in C^{k+1}[a,b],$$

equation (3.7) can be integrated by parts to obtain

$$\begin{aligned}\lambda(f) &= \int_a^b D^kf(t)\,dw(t)\\ &= D^kf(b)w(b) - D^kf(a)w(a) - \int_a^b D^{k+1}f(t)w(t)\,dt, \qquad \forall f \in C^{k+1}[a,b].\end{aligned} \tag{3.9}$$

It follows from (3.9) that λ vanishes on Π_k if and only if

$$\lambda\left((\cdot)^k/k!\right) = w(b) - w(a) = 0,$$

i.e., if and only if

$$w(b) = w(a) = 0,$$

and (3.9) then gives the 1–1 correspondence (3.4), where

$$K(t) := -w(t) = \frac{1}{k!}\lambda\left((\cdot - t)_+^k\right), \qquad \forall t \notin J_{\lambda,k}.$$

Alternatively, it follows from (3.8) that λ vanishes on Π_k if and only if $w(b) = 0$, and (3.7) gives the 1–1 correspondence (3.5) (with the same definition of K).

Now suppose that the result is true for $j-1$, where $0 \le j-1 \le k-1$. The induction hypothesis gives a 1–1 correspondence between those $\lambda \in (C^{k-1-(j-1)}[a,b])^* = (C^{k-j}[a,b])^*$ which vanish on Π_{k-1} and the functions $w \in \mathrm{PK}^{j-1}[a,b]$ satisfying the boundary conditions

$$w(a) = Dw(a) = \cdots = D^{j-2}w(a) = 0, \qquad w(b) = Dw(b) = \cdots = D^{j-2}w(b) = 0, \quad (3.10)$$

which is given by

$$\lambda(f) = \int_a^b D^k f(t) w(t)\, dt, \qquad \forall f \in C^k[a,b], \tag{3.11}$$

or, equivalently, with $D^{j-1}w \in \mathcal{PK}[a,b]$,

$$\lambda(f) = (-1)^j \int_a^b D^{k-1-(j-1)} f(t)\, d(D^{j-1}w)(t), \qquad \forall f \in C^{k-1-(j-1)}[a,b]. \tag{3.12}$$

Let $K \in \mathrm{PK}^j[a,b]$ be the function defined by (2.14), i.e.,

$$K(t) := -\int_a^t w(\xi)\, d\xi,$$

which satisfies $DK = -w$, and

$$K(a) := -\int_a^a w(\xi)\, d\xi = 0.$$

This K satisfies all the boundary conditions of the theorem except $K(b) = 0$. The linear functional λ defined by (3.11) vanishes on Π_k if and only if

$$K(b) = \int_a^b w(t) = \frac{1}{k!}\lambda\left((\cdot)^k\right) = 0.$$

Since (3.12) can be rewritten as

$$\lambda(f) = (-1)^{j-1} \int_a^b D^{k-j} f(t)\, d(D^j K)(t), \qquad \forall f \in C^{k+1-j}[a,b],$$

where $D^j K = -D^{j-1}w \in \mathcal{PK}[a,b]$, this gives the 1–1 correspondence (3.5). Integrating (3.11) by parts gives (3.4). $\square$

The boundary conditions satisfied by the Peano kernel K above are related to certain moment and orthogonality conditions as detailed in Lemma 2.15. Now we illustrate Theorem 3.3 with some examples.

Example 1. The error in *Simpson's (quadrature) rule*

$$\lambda(f) := \int_a^b f(t)\,dt - \frac{(b-a)}{6}\Big\{f(a) + 4f\big(\frac{a+b}{2}\big) + f(b)\Big\}, \tag{3.13}$$

which defines a $\lambda \in (C[a,b])^*$ that vanishes on Π_3 (the cubics). By Theorem 3.3, the Peano kernel $K \in \mathrm{PK}^3[a,b] \subset W^3_\infty[a,b] \subset C^2[a,b]$ and satisfies the boundary conditions

$$K(a) = DK(a) = D^2K(a) = 0, \qquad K(b) = DK(b) = D^2K(b) = 0,$$

or, equivalently, the moment (orthogonality) conditions

$$\langle DK, \Pi_0\rangle = \langle D^2K, \Pi_1\rangle = \langle D^3K, \Pi_2\rangle = 0.$$

Using (3.6), the kernel K can be computed explicitly as

$$K(t) := \frac{1}{3!}\lambda\left((\cdot - t)_+^3\right) = -\frac{1}{72}\begin{cases}(t-a)^3\left((a+2b)-3t\right), & a \le t \le \frac{a+b}{2}\\ (b-t)^3\left(3t-(2a+b)\right), & \frac{a+b}{2} \le t \le b.\end{cases} \tag{3.14}$$

The corresponding $D^3K \in \mathcal{PK}[a,b]$ in the representation (3.5), i.e.,

$$\lambda(f) = \int_a^b f(t)\,d(D^3K)(t), \qquad \forall f \in C[a,b], \tag{3.15}$$

is given by

$$D^3K(t) = t - a - \frac{b-a}{6}\begin{cases}0, & t = a\\ 1, & a < t < \frac{a+b}{2}\\ 5, & \frac{a+b}{2} \le t < b\\ 6, & t = b.\end{cases} \tag{3.16}$$

Notice that (3.15) is simply a restatement of (3.13). In general, by (2.8), the function $D^jK \in \mathcal{PK}[a,b]$ occuring in (3.5) can be computed from

$$(-1)^{j-1}D^jK(t) = \frac{1}{(k-j)!}\lambda\left((\cdot)^{k-j}\right) - \frac{1}{(k-j)!}\lambda\left((\cdot - t)_+^{k-j}\right), \qquad \forall t \in J_{\lambda,k-j}. \tag{3.17}$$

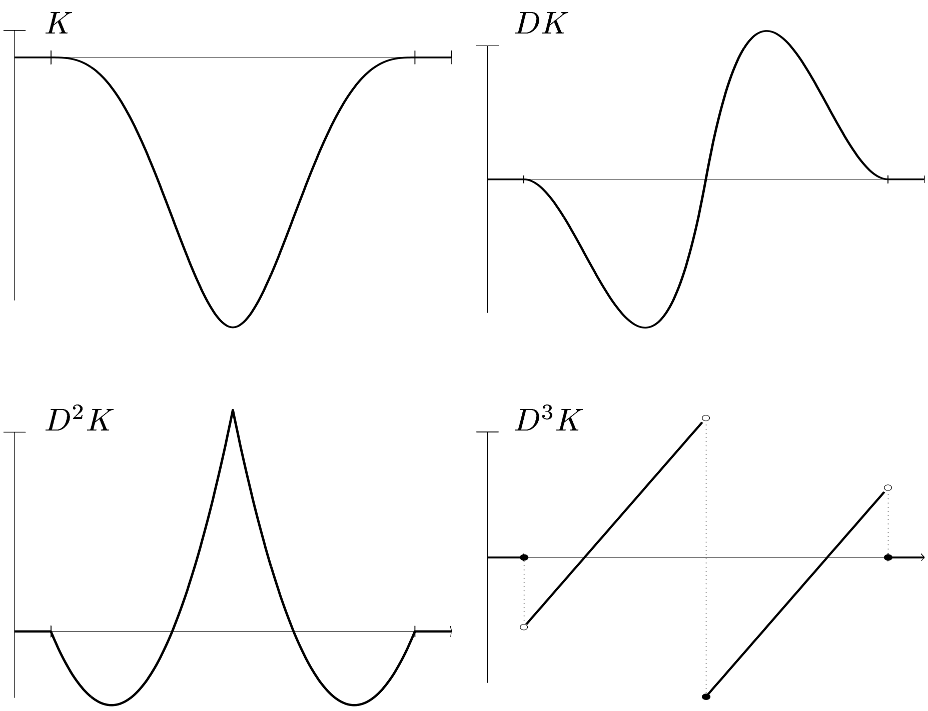

Fig. 3.1. The Peano kernel (and its derivatives) for *Simpson's rule*.
Note that $K(\xi) = DK(\xi) = D^2K(\xi) = 0$, $\xi \in \{a, b\}$, and $D^3K \in \mathcal{PK}[a, b]$.

Example 2. The *divided difference* at $k+2$ distinct points

$$a \leq x_0 < x_1 < \cdots < x_k < x_{k+1} \leq b$$

given by

$$\lambda(f) := [x_0, x_1, \ldots x_{k+1}]f := \sum_{j=0}^{k+1} \frac{f(x_j)}{\prod_{i \neq j}(x_j - x_i)},$$

which defines a $\lambda \in (C[a, b])^*$ that vanishes on Π_k. The corresponding Peano kernel is

$$K = M := M(\cdot|x_0, \ldots, x_{k+1})$$

the *normalised B–spline* of degree k with knot sequence $x_0 < x_1 < \cdots < x_{k+1}$, namely

$$M(t) := [x_0, \ldots, x_{k+1}]\big((\cdot - t)_+^k\big).$$

Several well known properties of B–splines follow immediately from Theorem 3.3. For example, $M \in \mathrm{PK}^k[a, b] \subset W_\infty^k[a, b] \subset C^{k-1}[a, b]$, its support is contained in $[x_0, x_{k+1}]$, and it satisfies the boundary conditions

$$M(\xi) = DM(\xi) = \cdots = D^{k-1}M(\xi) = 0, \quad \xi \in \{x_0, x_{k+1}\}.$$

Less well known are the orthogonality conditions (see Burchard [Bu73])

$$\langle D^r M, \Pi_{r-1} \rangle = 0, \qquad r = 1, 2, \ldots, k, \tag{3.18}$$

which follow from Lemma 2.15.

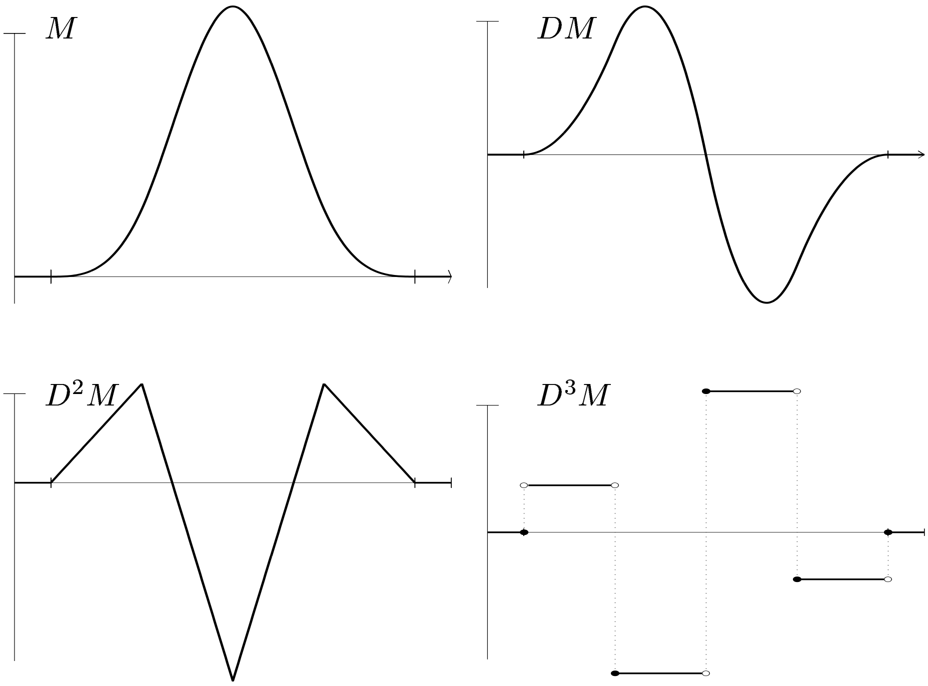

Fig. 3.2. The *cubic B–spline* $M := M(\cdot|x_0,\dots,x_4)$ with equally spaced knots $x_j := a + j(b-a)/4$ (and its derivatives), with $D^3M \in \mathcal{PK}[a,b]$

As a second application of Theorem 2.2 we consider the Peano kernels corresponding to the linear functionals (3.2). Note that $W_q^j[a,b] \subset W_1^j[a,b]$.

Theorem 3.19. *Let $1 \le p < \infty$, and $q := p^*$ (the conjugate exponent). There is a 1–1 (linear) correspondence between the linear functionals*

$$\lambda \in (W_p^{k+1-j}[a,b])^*, \qquad j = 0,1,\dots,k+1$$

which vanish on Π_k and the functions

$$K \in W_q^j[a,b]$$

that satisfy (for $j \ge 1$) the boundary conditions

$$K(a) = DK(a) = \cdots = D^{j-1}K(a) = 0, \qquad K(b) = DK(b) = \cdots = D^{j-1}K(b) = 0$$

which is given by

$$\lambda(f) = \int_a^b D^{k+1}f(t)K(t)\,dt, \qquad \forall f \in W_p^{k+1}[a,b], \tag{3.20}$$

or, equivalently,

$$\lambda(f) = (-1)^j \int_a^b D^{k+1-j}f(t)D^jK(t)\,dt, \qquad \forall f \in W_p^{k+1-j}[a,b]. \tag{3.21}$$

Further, $\lambda \mapsto \|K\|_{L_q[a,b]}$ *gives an equivalent norm on* $\{\lambda \in (W_p^{k+1-j}[a,b])^* : \lambda(\Pi_k) = 0\}$.

Proof: The proof is by (strong) induction on j and k.
First we prove the result for $j = 0$ and all k. Let $X = W_p^{k+1}[a,b]$, $1 \le p < \infty$, in Theorem 2.2. By Lemma 2.11, $D^{k+1} : W_p^{k+1}[a,b] \to L_p[a,b]$ is a continuous linear map onto $Y = L_p[a,b]$, which (by the open mapping theorem) is an open map. Thus, there is a 1–1 correspondence between the $\lambda \in (W_p^{k+1}[a,b])^*$ that vanish on Π_k and the $Q \in (L_p[a,b])^*$ given by (2.3). The standard representation for continuous linear functionals on $L_p[a,b]$ as integration against $L_q[a,b]$–functions then gives (3.20) and (3.21). The equivalence of norms is (2.5). In fact, from the proof of (2.5) it follows that with the usual norm on $W_p^{k+1}[a,b]$ these norms are equal (since $\|D^{k+1}\| = 1$ and $r = 1$).

Now suppose that the result is true for $j-1$, where $0 \le j-1 \le k$. The induction hypothesis gives a 1–1 correspondence between those $\lambda \in (W_p^{k-(j-1)}[a,b])^* = (W_p^{k+1-j}[a,b])^*$ that vanish on Π_{k-1} and the functions $w \in W_q^{j-1}[a,b]$ satisfying the boundary conditions

$$w(a) = Dw(a) = \cdots = D^{j-2}w(a) = 0, \qquad w(b) = Dw(b) = \cdots = D^{j-2}w(b) = 0, \quad (3.22)$$

which is given by

$$\lambda(f) = \int_a^b D^k f(t) w(t)\, dt, \qquad \forall f \in W_p^k[a,b], \qquad (3.23)$$

or, equivalently,

$$\lambda(f) = (-1)^{j-1} \int_a^b D^{k-(j-1)} f(t) D^{j-1} w(t)\, dt, \qquad \forall f \in W_p^{k-(j-1)}[a,b]. \qquad (3.24)$$

Let $K \in W_q^j[a,b]$ be the function defined by (2.14), which satisfies $DK = -w$, and

$$K(a) := -\int_a^a w(\xi)\, d\xi = 0. \qquad (3.25)$$

By (3.22) and (3.25), this $K \in W_q^j[a,b]$ satisfies all the boundary conditions of the theorem except $K(b) = 0$. The linear functional λ defined by (3.23) vanishes on Π_k if and only if

$$K(b) = \int_a^b w(t) = \lambda\left((\cdot)^k/k!\right) = 0.$$

Since (3.24) can be rewritten as

$$\lambda(f) = (-1)^j \int_a^b D^{k+1-j} f(t) D^j K(t)\, dt, \qquad \forall f \in W_p^{k+1-j}[a,b],$$

this gives the 1–1 correspondence (3.21). Integrating (3.23) by parts gives the 1–1 correspondence (3.20). □

As before, the Peano kernel K above satisfies the moment and orthogonality conditions as described by Lemma 2.15. The only statement of the Peano kernel theorem which strives for the generality of Sobolev spaces that the author is aware of is in the recent monograph [BHS93:p. 28].

To summarise the classification of this section, we now give a diagram showing the subspaces of $(C^{k+1}[a,b])^*$ considered, together with the smoothness class of the corresponding Peano kernels and the boundary conditions which they satisfy.

Peano kernel classification subspace diagram

Here is a diagram showing subspaces $X^* \subset (C^{k+1}[a,b])^*$, the smoothness class of the corresponding Peano kernels (for $\lambda \in X^*$ vanishing on Π_k), and the boundary conditions satisfied by the Peano kernels K (if any). Here $1 \le p < \infty$, $1 \le p_1 < p_2 < \infty$, with $1 < q \le \infty$, $1 < q_2 < q_1 \le \infty$ the corresponding conjugate indices, and $j = 0, 1, \ldots, k$. All of the inclusions indicated by $\uparrow$ are strict.

Subspaces of $(C^{k+1}[a,b])^*$	Smoothness class of Peano kernels	Boundary conditions satisfied by the corresponding Peano kernels
$(W_p^{k+1}[a,b])^*$	$L_q[a,b]$	
$\uparrow$	$\uparrow$	
$(C^k[a,b])^*$	$\mathrm{PK}[a,b]$	
$\uparrow$		
$(W_p^k[a,b])^*$	$W_q^1[a,b]$	
$\uparrow$	$\uparrow$	$\left.\begin{matrix} K(a)=0 \\ K(b)=0 \end{matrix}\right\}$
$(C^{k-1}[a,b])^*$	$\mathrm{PK}^1[a,b]$	
$\uparrow$		
$\vdots$	$\vdots$	
$\uparrow$		
$(W_{p_2}^{k+1-j}[a,b])^*$	$W_{q_2}^j[a,b]$	
$\uparrow$	$\uparrow$	$\left.\begin{matrix} K(a)=\cdots=D^{j-1}K(a)=0 \\ K(b)=\cdots=D^{j-1}K(b)=0 \end{matrix}\right\}$
$(W_{p_1}^{k+1-j}[a,b])^*$	$W_{q_1}^j[a,b]$	
$\uparrow$	$\uparrow$	
$(C^{k-j}[a,b])^*$	$\mathrm{PK}^j[a,b]$	
$\uparrow$		
$(W_{p_2}^{k-j}[a,b])^*$	$W_{q_2}^{j+1}[a,b]$	
$\uparrow$	$\uparrow$	$\left.\begin{matrix} K(a)=\cdots=D^{j}K(a)=0 \\ K(b)=\cdots=D^{j}K(b)=0 \end{matrix}\right\}$
$(W_{p_1}^{k-j}[a,b])^*$	$W_{q_1}^{j+1}[a,b]$	
$\uparrow$		
$\vdots$	$\vdots$	
$\uparrow$		
$(W_p^2[a,b])^*$	$W_q^{k-1}[a,b]$	
$\uparrow$	$\uparrow$	$\left.\begin{matrix} K(a)=\cdots=D^{k-2}K(a)=0 \\ K(b)=\cdots=D^{k-2}K(b)=0 \end{matrix}\right\}$
$(C^1[a,b])^*$	$\mathrm{PK}^{k-1}[a,b]$	
$\uparrow$		
$(W_p^1[a,b])^*$	$W_q^k[a,b]$	
$\uparrow$	$\uparrow$	$\left.\begin{matrix} K(a)=\cdots=D^{k-1}K(a)=0 \\ K(b)=\cdots=D^{k-1}K(b)=0 \end{matrix}\right\}$
$(C[a,b])^*$	$\mathrm{PK}^k[a,b]$	
$\uparrow$		
$(L_p[a,b])^*$	$W_q^{k+1}[a,b]$	$\left.\begin{matrix} K(a)=\cdots=D^{k}K(a)=0 \\ K(b)=\cdots=D^{k}K(b)=0 \end{matrix}\right\}$

4. Representations of the continuous duals of $C^r[a,b]$ and $W_p^r[a,b]$

In this section we briefly indicate how the Peano kernel representations of Section 3 can be used to represent the continuous dual X^* of

$$X = C^r[a,b], \qquad X = W_p^r[a,b], \quad r \geq 0, \quad 1 \leq p < \infty.$$

The basic argument is that if $\mu_0, \ldots, \mu_k \in X^*$ are linearly independent over $\Pi_k \subset X$, i.e., there exists a (unique) linear projector onto Π_k of the form

$$P = \sum_{i=0}^{k} p_i \mu_i : X \to \Pi_k, \qquad (p_i \in \Pi_k),$$

then the map

$$\mathcal{I} : X^* \to \mathrm{I\!R}^{k+1} \times X^* \cap \mathrm{Ann}(\Pi_k) : \lambda \mapsto \big((\lambda p_i)_{i=0}^{k}, \lambda - \lambda \circ P \big) \tag{4.1}$$

is an *isomorphism* between Banach spaces (i.e., a linear bijection which is a homeomorphism), and for short we write

$$X^* \cong \mathrm{I\!R}^{k+1} \times X^* \cap \mathrm{Ann}(\Pi_k).$$

Here $X^* \cap \mathrm{Ann}(\Pi_k)$ denotes the closed subspace of X^* which annihilates Π_k. It is easily shown that $\mathcal{I}$ is a 1–1, onto, and continuous linear map between Banach spaces. The continuity of $\mathcal{I}^{-1}$ follows from the open mapping theorem.

First let $X = C^r[a,b]$, and choose $k \geq r \geq 0$. Then

$$(C^r[a,b])^* \cong \mathrm{I\!R}^{k+1} \times (C^r[a,b])^* \cap \mathrm{Ann}(\Pi_k), \tag{4.2}$$

and using the representation for $(C^r[a,b])^* \cap \mathrm{Ann}(\Pi_k)$ given by Theorem 3.3 one obtains:

Theorem 4.3 (Representation of $(C^r[a,b])^*$). *Let*

$$\mu_0, \ldots, \mu_k \in (C^r[a,b])^*, \qquad k \geq r \geq 0,$$

be linearly independent over Π_k, with $p_i \in \Pi_k$ the dual polynomials, and $P := \sum_{i=0}^{k} p_i \mu_i$. Then each linear functional $\lambda \in (C^r[a,b])^*$ has a unique representation of the form

$$\lambda(f) = \sum_{i=0}^{k} c_i \mu_i(f) + (-1)^{k-r-1} \int_a^b D^r f(t)\, d(D^{k-r} K)(t), \qquad \forall f \in C^r[a,b], \tag{4.4}$$

where $c_j \in \mathrm{I\!R}$, $K \in \mathrm{PK}^{k-r}[a,b]$ (with $D^{k-r}K \in \mathcal{PK}[a,b]$ above), and K satisfies (for $k - r \geq 1$) the boundary conditions

$$K(a) = \cdots = D^{k-r-1}K(a) = 0, \qquad K(b) = \cdots = D^{k-r-1}K(b) = 0.$$

Further, $\|\lambda\|$ is equivalent to

$$\sum_{i=0}^{k} |c_i| + \int_a^b |K(t)|\,dt = \sum_{i=0}^{k} |c_i| + \|K\|_{L_1[a,b]}. \tag{4.5}$$

In this representation

$$\begin{aligned} c_i &= \lambda(p_i), \qquad i = 0,\ldots,k, \\ K(t) &= \frac{1}{k!}(\lambda - \lambda\circ P)\left((\cdot - t)_+^k\right), \qquad \forall t \notin J_{\lambda-\lambda\circ P,k}, \end{aligned}$$

and (4.4) *can be replaced by*

$$\lambda(f) = \sum_{i=0}^{k} c_i \mu_i(f) + \int_a^b D^{k+1} f(t) K(t)\,dt, \tag{4.6}$$

whenever $f \in C^{k+1}[a,b]$.

Proof: Since $\lambda - \lambda\circ P$ vanishes on Π_k, and

$$(\lambda - \lambda\circ P)f = \lambda(f) - \sum_{i=0}^{k} \lambda(p_i)\mu_i(f) = \lambda(f) - \sum_{i=0}^{k} c_i\mu_i(f),$$

Theorem 3.3 can be applied to $\lambda - \lambda\circ P$ to obtain (4.4) and (4.6). The equivalence of norms (4.5) follows from the isomorphism (4.2). Indeed, if $C^r[a,b]$ is given the (equivalent) norm

$$\|f\| := \max\{|\mu_i(f)| : i = 0,\ldots,k, \|D^r f\|_{L_\infty[a,b]}\},$$

then $\|\lambda\|$ is precisely (4.5) (cf Conway [C85:Ex. 5,p. 80]). □

The representation for $(C^r[a,b])^*$ that is obtained by instead using the representation of $(C^r[a,b])^* \cap \mathrm{Ann}(\Pi_k)$ given by (2.6), i.e.,

$$(C^{k+1}[a,b])^* \cong \mathbb{R}^{k+1} \times \mathrm{NBV}[a,b],$$

is well known (see, e.g., [C85:Ex. 6,p. 80] or [S63:p. 139]).

In a similar way, by choosing $X = W_p^r[a,b]$, $1 \le p < \infty$, $k+1 \ge r \ge 0$, it follows that

$$(W_p^r[a,b])^* \cong \mathbb{R}^{k+1} \times (W_p^r[a,b])^* \cap \mathrm{Ann}(\Pi_k), \tag{4.7}$$

and the representation for $(W_p^r[a,b])^* \cap \mathrm{Ann}(\Pi_k)$ of Theorem 3.19 gives:

Theorem 4.8 (Representation of $(W_p^r[a,b])^*$). *Let*

$$\mu_0, \ldots, \mu_k \in (W_p^r[a,b])^*, \qquad 1 \le p < \infty, \; k+1 \ge r \ge 0,$$

be linearly independent over Π_k, with $p_i \in \Pi_k$ the dual polynomials, and $P := \sum_{i=0}^k p_i \mu_i$. Then each linear functional $\lambda \in (W_p^r[a,b])^*$ has a unique representation of the form

$$\lambda(f) = \sum_{i=0}^{k} c_i \mu_i(f) + (-1)^{k+1-r} \int_a^b D^r f(t) D^{k+1-r} K(t)\, dt, \qquad \forall f \in W_p^r[a,b], \tag{4.9}$$

where $c_j \in \mathrm{I\!R}$, and $K \in W_q^{k+1-r}[a,b]$ ($q := p^*$) satisfies (for $k+1-r \ge 1$)

$$K(a) = \cdots = D^{k-r}K(a) = 0, \qquad K(b) = \cdots = D^{k-r}K(b) = 0.$$

Further, $\|\lambda\|$ is equivalent to

$$\left(\sum_{i=0}^{k} |c_i|^q + \int_a^b |K(t)|^q \, dt \right)^{1/q}. \tag{4.10}$$

In this representation

$$c_i = \lambda(p_i), \qquad i = 0, \ldots, k,$$

and K is the Peano kernel for $\lambda - \lambda \circ P$ (in terms of D^{k+1}).

Proof: As for Theorem 4.8, but using (4.7). If $W_p^r[a,b]$ is given the (equivalent) norm

$$\|f\| := \left(\sum_{i=0}^{k} |\mu_i(f)|^p + \int_a^b |D^r f(t)|^p \, dt \right)^{1/p},$$

then $\|\lambda\|$ is precisely (4.10) (cf [C85:Ex. 4, p. 80]). □

Taking $r = k+1$ in Theorem 4.8 gives

$$(W_p^{k+1}[a,b])^* \cong \mathrm{I\!R}^{k+1} \times L_q[a,b].$$

In Adams [A75:Th. 3.8, p. 48] there is a representation of $(W_p^r(\Omega))^*$, for Ω a domain in $\mathrm{I\!R}^n$. This nonconstructive result is of a different nature to that of (4.9).

Concluding remarks

The use of distribution theory allows additional statements about the smoothness of Peano kernels to be made. For instance, if $(c,d) \subset (a,b)$ does not intersect the support of λ, then $K|_{(c,d)}$ is a polynomial of degree $\le k$. The techniques developed here could also be used to classify the Peano kernels for other spaces X, in particular those of the form Y^j, $Y \subset L_1[a,b]$, together with the corresponding representations of $(Y^j)^*$ (cf this section).

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