On the Vandermonde Determinant of Padua-like Points

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Abstract

Recently [1] gave a simple, geometric and explicit construction of bivariate interpolation at points in a square (the so-called Padua points), and showed that the associated norms of the interpolation operator, i.e., the Lebesgue constants, have minimal order of growth of $\mathcal{O}((\log(n))^2)$.

One may observe that these points have the structure of the union of *two* (tensor product) grids, one square and the other rectangular. In this article we give a conjectured formula (in the even degree case) for the Vandermonde determinant of any set of points with exactly this structure. Surprisingly, it factors into the product of two univariate functions. We offer a partial proof that depends on a certain technical lemma (Lemma 1 below) which seems to be true but up till now a correct proof has been elusive.

1 Introduction

Suppose that $K \subset \mathbb{R}^d$ is a compact set with non-empty interior. Let $\Pi(\mathbb{R}^d)$ denote the space of real polynomials in d variables, and $\Pi(K)$ be the restrictions of these polynomials to K. Suppose that we are given some finite dimensional subspace $V \subset \Pi(K)$, of dimension $N := \dim(V)$. Then given N points $\mathcal{A} := \{A_i\}_{i=1}^N \subset K$, the polynomial interpolation problem associated to V and $\mathcal{A}$ is to find for each $f \in C(K)$ (the space of continuous functions on K) a polynomial $p \in V$ such that

$$p(A_j) = f(A_j), \quad j = 1, \cdots, N. \tag{1}$$

If

$$\mathcal{B} := \{p_1, p_2, \cdots, p_N\}$$

is a basis for V so that we may write $p \in V$ as

$$p = \sum_{j=1}^{N} a_j p_j,$$

then the interpolation problem (1) may be expressed in matrix form as

$$[p_j(A_i)]_{1\le i,j\le N}\, \vec{a} = \vec{f} \tag{2}$$

where $\vec{a}$ is the vector of coefficients $\vec{a}_i = a_i$, and $\vec{f}$ denotes the vector of function values $\vec{f}_j = f(A_j)$.

The linear system (2) has a (unique) solution for arbitrary f precisely when the associated determinant, the so-called Vandermonde determinant, is non-zero. Clearly then, this determinant is important for polynomial interpolation.

We will denote the Vandermonde *matrix* by

$$VDM(\mathcal{A};\mathcal{B}) := [p_j(A_i)]_{1\le i,j\le N}$$

and its determinant by

$$vdm(\mathcal{A};\mathcal{B}) := \det(VDM(\mathcal{A};\mathcal{B})).$$

Of course, in one variable, with $V = \Pi_n(\mathbb{R})$, the space of univariate polynomials of degree at most n, and $\mathcal{B} = \{1, x, x^2, \cdots, x^n\}$, (and $N = n+1$) there is the classical formula

$$vdm(\mathcal{A};\mathcal{B}) = \pm \prod_{i<j} (A_j - A_i).$$

In several variables much less is known. There are some special configurations of points for which the Vandermonde determinant may be computed explicity (see e.g. [2]) but the only example in arbitrary dimension is the tensor-product case. We briefly describe this for $d = 2$. Here $N = (n+1)^2$, the set of points is given by a grid

$$\mathcal{A} = \{(x_i, y_j) \mid 0 \le i, j \le n\}$$

and the basis is the tensor-product basis

$$\mathcal{B} = \{x^\alpha y^\beta \mid 0 \le \alpha, \beta \le n\}.$$

It is not difficult to see that then, after re-ordering, the bivariate Vandermonde matrix $[x_i^\alpha y_j^\beta]$ is the tensor product of the univariate Vandermonde matrices $[x_i^\alpha]$ and $[y_j^\beta]$. Hence,

$$\begin{aligned} vdm(\mathcal{A};\mathcal{B}) &= \pm\det\left([x_i^\alpha] \otimes [y_j^\beta]\right) \\ &= \pm\left(\det([x_i^\alpha])\right)^{n+1} \left(\det([y_j^\beta])\right)^{n+1} \\ &= \pm\left(\prod_{i<j}(x_j - x_i)\right)^{n+1} \left(\prod_{i<j}(y_j - y_i)\right)^{n+1}, \end{aligned}$$

using a basic fact about the determinants of the tensor product of matrices (see e.g. [5]).

However, the most common (and most important) choice of V is $V = \Pi_n(\mathbb{R}^d)$, the space of polynomials of degree at most n, (for which $N = \binom{n+d}{d}$) in which case we consider the *total degree* interpolation problem. This has been the subject of much study, but remains incompletely understood. Only recently, [1] gave an explicit example of points, the so-called Pauda points, in the square $K = [-1, 1]^2$ for which the associated Lebesgue function is of optimal growth. To the best of our knowledge, this is the first explicit such example for any such $K \subset \mathbb{R}^d$ with $d > 1$, and is hence an important example that deserves further exploration.

Now, the Padua points (of degree n) are perhaps most easily described as

$$\{(\cos((n+1)t_k), \cos(nt_k) \mid k = 0 \cdots n(n+1)\}$$

where $t_k := k\pi/(n(n+1))$, (see [1] for more details). In Figure 1, below, we display these points for degree $n = 4$.

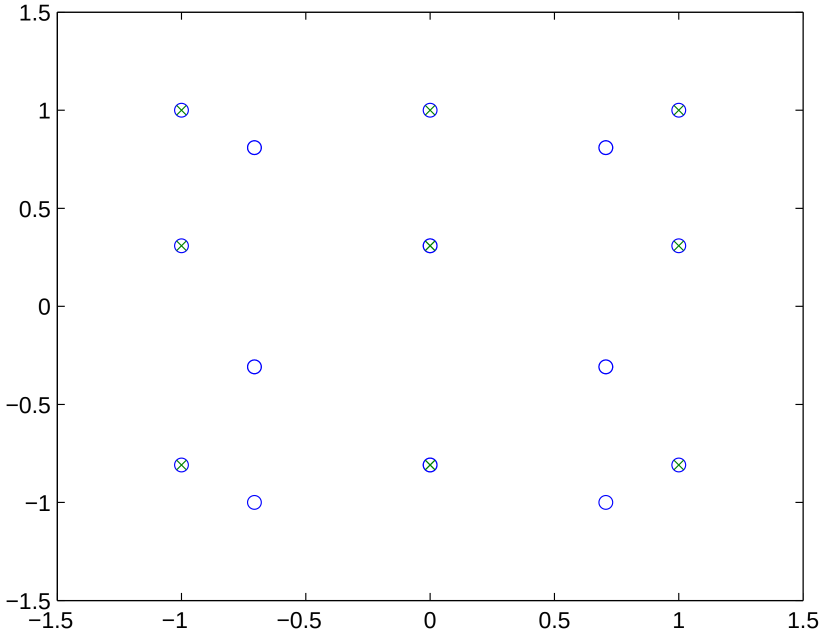

Figure 1 – the Padua points of degree 4

Notice that the $N = \binom{4+2}{2} = 15$ points may be partitioned into grids, the first 3×3 (denoted by $'\otimes'$) and the other 2×3 (denoted by $'\mathrm{O}'$). This is easily seen to always be the case. In fact, the *even* degree n Padua points may be always partitioned into a

$\dfrac{n+2}{2} \times \dfrac{n+2}{2}$ grid and a $\dfrac{n}{2} \times \dfrac{n+2}{2}$ grid, and the the *odd* degree n Padua points may always be partitioned into a $\dfrac{n+1}{2} \times \dfrac{n+1}{2}$ grid and a $\dfrac{n+1}{2} \times \dfrac{n+3}{2}$ grid. We refer to general sets of this form, i.e., that may be partitioned into two grids of these sizes, as *Padua-like points.* For example, for n even, consider

$$\mathcal{A}_n = \mathcal{A}_n^o \cup \mathcal{A}_n^e \tag{3}$$

where

$$\mathcal{A}_n^o := \{(x_{2i+1}, y_{2j+1}) \mid 0 \le i \le n/2,\ 0 \le j \le n/2\}$$

and

$$\mathcal{A}_n^e := \{(x_{2i}, y_{2j}) \mid 1 \le i \le n/2,\ 1 \le j \le (n+2)/2\}.$$

The n odd case is similar.

In this paper we discuss a conjectured formula for the Vandermonde determinant for Padua-like points (3). Surprisingly, it factors into the product of a factor depending only on the x_k and a factor depending only on the y_k, just as does a tensor product Vandermonde, despite *not* being a tensor product set of points. For reasons of simplicity we present only the *even* degree case. The odd degree case is similar.

2 A Product Formula for the Vandermonde Determinant

Since we will be manipulating these Vandermonde matrices/determinants we first display their structure somewhat more carefully. For points $\mathcal{A} = \{A_1, A_2, \cdots, A_N\}$ and basis $\mathcal{B} = \{p_1, p_2, \cdots, p_N\}$, then the Vandermonde matrix

$$VDM(\mathcal{A};\mathcal{B}) = \begin{array}{c|cccc|} & p_1 & p_2 & \cdots & p_N \\ A_1 & p_1(A_1) & p_2(A_1) & \cdots & p_N(A_1) \\ A_2 & p_1(A_2) & p_2(A_2) & \cdots & p_N(A_2) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ A_N & p_1(A_N) & p_2(A_N) & \cdots & p_N(A_N) \end{array}.$$

Each *row* of the matrix is indexed by a point in $\mathcal{A}$ and consists of the polynomials in the basis evaluated at that point. Each *column* in the matrix is indexed by a polynomial in $\mathcal{B}$ and consists of that polynomial evaluated at all the points.

Now, our goal is to provide a product formula for $vdm(\mathcal{A};\mathcal{B})$ where $\mathcal{A}$ is the set of Padua-like points (3) and $\mathcal{B}$ is the standard monomial basis for $\Pi_n(\mathbb{R}^2)$,

$$\mathcal{B}_n := \left\{x^\alpha y^\beta \mid \alpha + \beta \le n\right\}. \tag{4}$$

We will also make use of the Tensor Product basis

$$\mathcal{T}_n := \left\{ x^\alpha y^\beta \;\mid\; \max(\alpha,\beta) \le n \right\}. \tag{5}$$

Part of our strategy for computing the Vandermonde determinant is to use a a basis for $\Pi_n(\mathbb{R}^2)$ that is somewhat more convenient than $\mathcal{B}_n$. We remark that, in general, if

$$\mathcal{B} = \{p_1, p_2, \cdots, p_N\}$$

and

$$\mathcal{B}' = \{q_1, q_2, \cdots, q_N\}$$

are two bases for V with transition matrix M, i.e., such that

$$q_j = \sum_{k=1}^{N} M_{jk} p_k,$$

then it is easy to see that

$$vdm(\mathcal{A};\mathcal{B}') = \det(M)vdm(\mathcal{A};\mathcal{B}). \tag{6}$$

In particular, if the transition matrix M is triangular with 1's on the diagonal, then we have

$$vdm(\mathcal{A};\mathcal{B}') = vdm(\mathcal{A};\mathcal{B}).$$

We are going to make use of a particular basis $\mathcal{B}'_n$, constructed as follows. Let

$$a(x) := \prod_{i=0}^{n/2} (x - x_{2i+1})$$

and

$$b(y) := \prod_{j=0}^{n/2} (y - y_{2j+1}).$$

Then set

$$\mathcal{B}'_n := a(x)\mathcal{B}_{\frac{n}{2}-1} \cup b(y)\mathcal{B}_{\frac{n}{2}-1} \cup \mathcal{T}_{\frac{n}{2}}.$$

It is easy to see that $\mathcal{B}'_n$ is indeed a basis of $\Pi_n(\mathbb{R}^2)$ and that the transition matrix between the standard basis $\mathcal{B}_n$ and $\mathcal{B}'_n$ (after a possible re-ordering) is triangular with 1's on the diagonal. Hence

$$vdm(\mathcal{A}_n;\mathcal{B}_n) = \pm vdm(\mathcal{A}_n;\mathcal{B}'_n).$$

Before proceeding, we first make a simplifying notation. Since the space

$$\mathrm{span}\left(a(x)\mathcal{B}_{\frac{n}{2}-1} \cup b(y)\mathcal{B}_{\frac{n}{2}-1}\right)$$

is complementary to span$(\mathcal{T}_{n/2})$ (with respect to $\Pi_n(\mathbb{R}^2)$), we define

$$\mathcal{T}^c_{\frac{n}{2}} := a(x)\mathcal{B}_{\frac{n}{2}-1} \cup b(y)\mathcal{B}_{\frac{n}{2}-1}$$

so that

$$\mathcal{B}'_n = \mathcal{T}_{\frac{n}{2}} \cup \mathcal{T}^c_{\frac{n}{2}}.$$

Notice then that for each point $(x_{2i+1}, y_{2j+1}) \in \mathcal{A}^o_n$ and polynomial $p \in \mathcal{T}^c_{n/2}$, we have $p(x_{2i+1}, y_{2j+1}) = 0$. Hence the Vandermonde matrix has the block form

$$VDM(\mathcal{A}_n; \mathcal{B}'_n) = \begin{array}{c|cc|} & \mathcal{T}_{n/2} & \mathcal{T}^c_{n/2} \\ \hline \mathcal{A}^o_n & \mathrm{A} & 0 \\ \mathcal{A}^e_n & \mathrm{B} & \mathrm{C} \\ \hline \end{array}\,.$$

Hence

$$\begin{aligned} vdm(\mathcal{A}_n; \mathcal{B}'_n) &= \pm \det(A) \times \det(C) \\ &= \pm\, vdm(\mathcal{A}^o_n; \mathcal{T}_{n/2}) \times vdm(\mathcal{A}^e_n; \mathcal{T}^c_{n/2}). \end{aligned} \tag{7}$$

But recall that

$$\mathcal{A}^o_n := \{(x_{2i+1}, y_{2j+1}) \mid 0 \le i \le n/2,\ 0 \le j \le n/2\}$$

is a $(n/2+1) \times (n/2+1)$ grid, so that $vdm(\mathcal{A}^o_n; \mathcal{T}_{n/2})$ is a tensor-product Vandermoncde determinant, and we have

$$vdm(\mathcal{A}^o_n; \mathcal{T}_{n/2}) = \pm \left(\prod_{0\le i<j\le n/2} (x_{2j+1} - x_{2i+1}) \right)^{n/2+1} \left(\prod_{0\le i<j\le n/2} (y_{2j+1} - y_{2i+1}) \right)^{n/2+1}. \tag{8}$$

We therefore proceed to the problem of computing $vdm(\mathcal{A}^e_n; \mathcal{T}^c_{n/2})$.

We begin by forming the $n/2$ subsets of univariate (in y) polynomials

$$Q_k := \{y^j \,|\, 0 \le j \le k-1\} \cup \{y^j b(y) \,|\, 0 \le j \le n/2 - k\}, \quad k = 1, \cdots, n/2. \tag{9}$$

Explicitly

$$\begin{aligned} Q_1 &= \{1, b(y), yb(y), y^2b(y), \cdots, y^{n/2-2}b(y), y^{n/2-1}b(y)\} \\ Q_2 &= \{1, y, b(y), yb(y), y^2b(y) \cdots, y^{n/2-3}b(y), y^{n/2-2}b(y)\} \\ Q_3 &= \{1, y, y^2, b(y), yb(y), y^2b(y) \cdots, y^{n/2-4}b(y), y^{n/2-3}b(y)\} \\ &\ \ \vdots \\ Q_{n/2-1} &= \{1, y, y^2, \cdots, y^{n/2-2}, b(y), yb(y)\} \\ Q_{n/2} &= \{1, y, y^2, \cdots, y^{n/2-1}, b(y)\}. \end{aligned}$$

Each of these $n/2$ sets Q_k has cardinality $n/2+1$, for a total of $(n/2)(n/2+1)$ (repeated) entries.

Now,

$$\mathcal{T}^c_{\frac{n}{2}} = a(x)\mathcal{B}_{\frac{n}{2}-1} \cup b(y)\mathcal{B}_{\frac{n}{2}-1}$$

also consists of

$$2\binom{n/2}{2} = (n/2)(n/2+1)$$

polynomials. We "partition" the members of $\mathcal{T}^c_{n/2}$ among the Q_k as follows: for $k = 1, 2, \cdots, n/2$ set

$$\widetilde{Q}_k := \{x^{k-j-1}y^j a(x) \,|\, 0 \le j \le k-1\} \cup \{x^{k-1}y^j b(y) \,|\, 0 \le j \le n/2-k\}. \qquad (10)$$

Explicitly,

$$\begin{aligned}
\widetilde{Q}_1 &= \{a(x), b(y), yb(y), y^2b(y), \cdots, y^{n/2-2}b(y), y^{n/2-1}b(y)\} \\
\widetilde{Q}_2 &= \{xa(x), ya(x), xb(y), xyb(y), xy^2b(y) \cdots, xy^{n/2-3}b(y), xy^{n/2-2}b(y)\} \\
\widetilde{Q}_3 &= \{x^2a(x), xya(x), y^2a(x), x^2b(y), x^2yb(y), x^2y^2b(y) \cdots, x^2y^{n/2-4}b(y), x^2y^{n/2-3}b(y)\} \\
&\cdot \\
&\cdot \\
\widetilde{Q}_{n/2-1} &= \{x^{n/2-2}a(x), x^{n/2-3}ya(x), x^{n/2-4}y^2a(x), \cdots, y^{n/2-2}a(x), x^{n/2-2}b(y), x^{n/2-2}yb(y)\} \\
\widetilde{Q}_{n/2} &= \{x^{n/2-1}a(x), x^{n/2-2}ya(x), x^{n/2-3}y^2a(x), \cdots, y^{n/2-1}a(x), x^{n/2-1}b(y)\}
\end{aligned}$$

and it is easily verified that indeed,

$$\widetilde{Q}_1 \cup \widetilde{Q}_2 \cup \cdots \cup \widetilde{Q}_{n/2} = \mathcal{T}^c_{n/2}.$$

Further, we re-order the points of $\mathcal{A}^e_n$ according to the "columns" of the grid by setting

$$\begin{aligned}
X_{2i} &:= \{(x,y) \in \mathcal{A}^e_n \,|\, x = x_{2i}\} \\
&= \{(x_{2i}, y_{2j}) \,|\, j = 1 \cdots n/2+1\}, \quad i = 1, 2, \cdots, n/2.
\end{aligned}$$

Note that each set X_{2i} has cardinality

$$|X_{2i}| = \frac{n}{2} + 1 = |\widetilde{Q}_k|.$$

We now form the Vandermonde determinant with this partitioning

$$VDM(\mathcal{A}_n^e;\mathcal{T}_{n/2}^c) = \pm \quad \begin{array}{cc} & \begin{array}{cccccc} \widetilde{Q}_1 & \widetilde{Q}_2 & \cdot & \cdot & \cdot & \widetilde{Q}_{n/2} \end{array} \\ \begin{array}{c} X_2 \\ \\ X_4 \\ \\ \cdot \\ \\ \cdot \\ \\ X_{n/2} \end{array} & \left| \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right| \end{array}$$

which may be regarded as an $(n/2) \times (n/2)$ block matrix of $(n/2+1) \times (n/2+1)$ size blocks consisting of the Vandermonde matrices $VDM(X_{2i};\widetilde{Q}_j)$.

Now, write

$$Q_j = \{f_1(y), f_2(y), \cdots, f_{n/2+1}(y)\}$$

and

$$\widetilde{Q}_j = \{\widetilde{f}_1(x,y), \widetilde{f}_2(x,y), \cdots, \widetilde{f}_{n/2}(x,y)\}$$

where the $f_k(y)$ are given by (9) and the associated $\widetilde{f}_k(x,y)$, by (10). We may, in fact, write

$$\widetilde{f}_k(x,y) = g_k(x) f_k(y) \tag{11}$$

where this relationship defines the $g_k(x)$.

Then the column in the matrix $VDM(X_{2i},\widetilde{Q}_j)$ corresponding to $\widetilde{f}_k(x,y)$ is

$$\begin{bmatrix} \widetilde{f}_k(x_{2i},y_2) \\ \widetilde{f}_k(x_{2i},y_4) \\ \cdot \\ \cdot \\ \widetilde{f}_k(x_{2i},y_{n+2}) \end{bmatrix} = \begin{bmatrix} f_k(y_2)g_k(x_{2i}) \\ f_k(y_4)g_k(x_{2i}) \\ \cdot \\ \cdot \\ f_k(y_{n+2})g_k(x_{2i}) \end{bmatrix} = \begin{bmatrix} f_k(y_2) \\ f_k(y_4) \\ \cdot \\ \cdot \\ f_k(y_{n+2}) \end{bmatrix} g_k(x_{2i}).$$

In other words, the Vandermonde matrix

$$VDM(X_{2i};\widetilde{Q}_j) = VDM(\{y_2, y_4, \cdots, y_{n+2}\}; Q_j)\mathrm{diag}(g_1(x_{2i}), g_2(x_{2i}, \cdots, g_{n/2+1})),$$

i.e., $VDM(X_{2i};\widetilde{Q}_j)$ is the product of the *univariate* Vandermonde matrix

$$VDM(\{y_2, y_4, \cdots, y_{n+2}\}; Q_j),$$

depending only on the y-coordinates of the points, and a *diagonal* matrix. Hence, we have the following block structure:

$$VDM(\mathcal{A}_n^e;\mathcal{T}_{n/2}^c) = \begin{bmatrix} M_1D_{11} & M_2D_{12} & \cdot & \cdot & M_{n/2}D_{1,n/2} \\ M_1D_{21} & M_2D_{22} & \cdot & \cdot & M_{n/2}D_{2,n/2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ M_1D_{n/2,1} & M_2D_{n/2,2} & \cdot & \cdot & M_{n/2}D_{n/2,n/2} \end{bmatrix} \tag{12}$$

where

$$M_j := VDM(\{y_2, y_4, y_6, \cdots, y_{n+2}\}; Q_j) \in \mathbb{R}^{(n/2+1)\times(n/2+1)} \quad (13)$$

and D_{ij} is the diagaonal matrix

$$D_{ij} := \mathrm{diag}(g_1(x_{2i}), g_2(x_{2i}), \cdots, g_{n/2+1}(x_{2i})) \in \mathbb{R}^{(n/2+1)\times(n/2+1)} \quad (14)$$

and the g_k are defined (depending on $\widetilde{Q}_j$) as in (11).

Notice that the jth column has a constant (block) factor M_j. However, these are left-factors and hence it is *not* true in general that $\det(M_j)$ is a factor of this block matrix. Remarkably, this seems to be true for our special matrix.

Lemma 1 *We have*

$$\begin{aligned} vdm(\mathcal{A}_n^e; \mathcal{T}_{n/2}^c) &= \pm\left(\prod_{j=1}^{n/2} det(M_j)\right) \begin{vmatrix} D_{11} & D_{12} & \cdot\ \cdot & D_{1,n/2} \\ D_{21} & D_{22} & \cdot\ \cdot & D_{2,n/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ D_{n/2,1} & D_{n/2,2} & \cdot\ \cdot & D_{n/2,n/2} \end{vmatrix} \\ &= \left(\prod_{j=1}^{n/2} vdm(\{y_2, y_4, \cdots, y_{n+2}\}; Q_j)\right) \begin{vmatrix} D_{11} & D_{12} & \cdot\ \cdot & D_{1,n/2} \\ D_{21} & D_{22} & \cdot\ \cdot & D_{2,n/2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ D_{n/2,1} & D_{n/2,2} & \cdot\ \cdot & D_{n/2,n/2} \end{vmatrix}. \end{aligned}$$

Proof. This is precisely the gap in our overall proof. If a proof of this lemma can be provided then the formula we give below will be proven. We tested the formula numerically with the following Matlab code:

```
n=4;  % the degree; must be even.
x=2*sort(rand(1,n+1))-1; %the x coordinates of the grid
y=2*sort(rand(1,n+2))-1; %the y coordiantes of the grid
m=(n/2)*(n/2+1); % the size of the matrix
A=zeros(m,m);
B=zeros(n/2+1,n/2+1); % put block in here
for k=1:(n/2) % form the matrix by Q_k blocks
    for i=1:(n/2)
        u=x(2*i);
        for s=1:(n/2+1)
            v=y(2*s);
            t=0;
            for j=0:(k-1)
                t=t+1;
                B(s,t)=v^j*u^(k-j-1)*a(u,x);
```

```
            end
            for j=0:(n/2-k)
                t=t+1;
                B(s,t)=u^(k-1)*v^j*b(v,y);
            end
        end
        B;
        A(((i-1)*(n/2+1)+1):(i*(n/2+1)),((k-1)*(n/2+1)+1):(k*(n/2+1)))=B;
    end
end
%
% now compute the proposed formula for det(A)
%
det1=1;
% first the n/2 blocks (in the y's) determined by the Q_k
C=zeros(m,m); %put blocks on the diagonal of C
for k=1:(n/2)
    for s=1:(n/2+1)
        v=y(2*s);
        t=0;
        for j=0:(k-1)
            t=t+1;
            B(s,t)=v^j;
        end
        for j=0:(n/2-k)
            t=t+1;
            B(s,t)=v^j*b(v,y);
        end
    end
    C(((k-1)*(n/2+1)+1):(k*(n/2+1)),((k-1)*(n/2+1)+1):(k*(n/2+1)))=B;
    det1=det1*det(B);
end

%
% now form the matrix of diagonal blocks
%
D=A;
for k=1:(n/2)
    for i=1:(n/2)
        A1=A(((i-1)*(n/2+1)+1):(i*(n/2+1)),((k-1)*(n/2+1)+1):(k*(n/2+1)));
        M=C(((k-1)*(n/2+1)+1):(k*(n/2+1)),((k-1)*(n/2+1)+1):(k*(n/2+1)));
        D(((i-1)*(n/2+1)+1):(i*(n/2+1)),((k-1)*(n/2+1)+1):(k*(n/2+1)))=inv(M)*A1;
    end
end
det(A) % this is the determinant on the left side of Lemma 1
```

```
det1*det(D) % this is the determinant on the right side of Lemma 1
det(A)/(det1*det(D)) % this ratio should be pm 1
```

The matrix $VDM(\mathcal{A}_n^e;\mathcal{T}_{n/2}^c)$ is computed as A in the code for random x and y nodes in the interval $[-1,1]$. At the very end, $\det(A) = vdm(\mathcal{A}_n^e;\mathcal{T}_{n/2}^c)$ is computed directly and then the value of the proposed formula in the lemma is given. Finally, the ratio of the two is given. For example, one run with $n=10$, (and random points) gave $\det(A) = -1.10320324074189e-098$ and the proposed formula as $-1.10320324074182e-098$. $\square$

Assuming, for the moment, that Lemma 1 is true, we proceed to compute the rest of the formula for the overall Vandermonde determinant. Since the D_{ij} are diagonal, and hence commute one with another, it is an easy matter to compute $\det[D_{ij}]$.

Lemma 2 *We have*

$$\begin{vmatrix} D_{11} & D_{12} & \cdot & \cdot & D_{1,n/2} \\ D_{21} & D_{22} & \cdot & \cdot & D_{2,n/2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ D_{n/2,1} & D_{n/2,2} & \cdot & \cdot & D_{n/2,n/2} \end{vmatrix} = \pm \prod_{k=1}^{n/2+1} det(C^k)$$

where $C^k \in \mathbb{R}^{(n/2)\times(n/2)}$ *is defined by*

$$(C^k)_{ij} := (D_{ij})_{kk}, \quad k = 1,2,\cdots n/2+1.$$

Proof. In fact, by interchanging rows and columns we may reduce

$$\begin{bmatrix} D_{11} & D_{12} & \cdot & \cdot & D_{1,n/2} \\ D_{21} & D_{22} & \cdot & \cdot & D_{2,n/2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ D_{n/2,1} & D_{n/2,2} & \cdot & \cdot & D_{n/2,n/2} \end{bmatrix} \text{ to } \begin{bmatrix} C^1 & 0 & \cdot & \cdot & 0 \\ 0 & C^2 & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdot & \cdot & C^{n/2+1} \end{bmatrix}$$

and the result follows. $\square$

We now proceed to caluculate the determinants of the C^k. But from the definiton of the g_k, (11), it is easy to see that

$$(C^k)_{ij} = \begin{cases} x_{2i}^{j-k}a(x_{2i}) & : \ j \ge k \\ x_{2i}^{j-1} & : \ j < k \end{cases}$$

for $1 \le k \le n/2+1$ and $1 \le i,j \le n/2$.

The cases $k = 1$ and $k = n/2 + 1$ are slightly special. In particular, for $k = 1$, $j < k$ does not occur and we have

$$(C^1)_{ij} = x_{2i}^{j-1} a(x_{2i}), \quad 1 \le i, j \le n/2.$$

Hence,

$$\begin{aligned} \det(C^1) &= \left(\prod_{i=1}^{n/2} a(x_{2i}) \right) \det[x_{2i}^{j-1}] \\ &= \left(\prod_{i=1}^{n/2} a(x_{2i}) \right) vdm(\{x_2, x_4, x_6, \cdots, x_n\}; \{1, x, x^2, \cdots, x^{n/2-1}\}) \\ &= \left(\prod_{i=1}^{n/2} a(x_{2i}) \right) \prod_{1 \le i < j \le n/2} (x_{2j} - x_{2i}). \end{aligned} \tag{15}$$

Similarly, for $k = n/2 + 1$, the case $j \ge k$ does not occur, so that

$$(C^{n/2+1})_{ij} = x_{2i}^{j-1}, \quad 1 \le i, j \le n/2.$$

Hence,

$$\begin{aligned} \det(C^{n/2+1}) &= \det[x_{2i}^{j-1}] \\ &= vdm(\{x_2, x_4, x_6, \cdots, x_n\}; \{1, x, \cdots, x^{n/2-1}\}) \\ &= \prod_{1 \le i < j \le n/2} (x_{2j} - x_{2i}). \end{aligned} \tag{16}$$

In the intermediate cases, $2 \le k \le n/2$, we easily determine that

$$\begin{aligned} \det(C^2) &= vdm(\{x_2, x_4, x_6, \cdots, x_n\}; \{1, a(x), xa(x), x^2 a(x), \cdots, x^{n/2-2} a(x)\}), \\ \det(C^3) &= vdm(\{x_2, x_4, x_6, \cdots, x_n\}; \{1, x, a(x), xa(x), x^2 a(x), \cdots, x^{n/2-3} a(x)\}), \\ \det(C^4) &= vdm(\{x_2, x_4, x_6, \cdots, x_n\}; \{1, x, x^2, a(x), xa(x), x^2 a(x), \cdots, x^{n/2-4} a(x)\}) \text{ etc..} \end{aligned}$$

If we define, in analogy with the Q_k defined in (9) (with $n/2$ replaced by $n/2 - 1$),

$$P_k := \{x^j \,|\, 0 \le j \le k-1\} \cup \{x^j a(x) \,|\, 0 \le j \le n/2-1-k\}, \quad k = 1, 2, \cdots, n/2-1, \tag{17}$$

then we may write,

$$\det(C^k) = vdm(\{x_2, x_4, x_6, \cdots, x_n\}; P_{k-1}), \quad k = 2, 3, \cdots, n/2. \tag{18}$$

Combining (15), (16) and (18) we have shown that

Lemma 3 *We have*

$$\begin{vmatrix} D_{11} & D_{12} & \cdot & \cdot & D_{1,n/2} \\ D_{21} & D_{22} & \cdot & \cdot & D_{2,n/2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ D_{n/2,1} & D_{n/2,2} & \cdot & \cdot & D_{n/2,n/2} \end{vmatrix} = \left(\prod_{i=1}^{n/2} a(x_{2i})\right)\left(\prod_{1\le i<j\le n/2}(x_{2j}-x_{2i})\right)^2 \times \prod_{k=2}^{n/2} vdm(\{x_2,x_4,x_6,\cdots,x_n\};P_{k-1}).$$

Lemmas 1 and 3 combine to give us

Proposition 1 *We have*

$$\begin{aligned} vdm(\mathcal{A}_n^e,\mathcal{T}_{n/2}^c) &= \left(\prod_{j=1}^{n/2} vdm(\{y_2,y_4,\cdots,y_{n+2}\};Q_j)\right) \\ &\quad\times\left(\prod_{i=1}^{n/2} a(x_{2i})\right)\left(\prod_{1\le i<j\le n/2}(x_{2j}-x_{2i})\right)^2 \\ &\quad\times\prod_{k=2}^{n/2} vdm(\{x_2,x_4,x_6,\cdots,x_n\};P_{k-1}). \end{aligned}$$

Finally, (7), (8) and Proposition 1 give us our desired formula

Theorem 2 *We have*

$$\begin{aligned} vdm(\mathcal{A}_n;\mathcal{B}_n) &= \pm\left(\prod_{0\le i<j\le n/2}(y_{2j+1}-y_{2i+1})\right)^{n/2+1} \\ &\quad\times\left(\prod_{j=1}^{n/2} vdm(\{y_2,y_4,\cdots,y_{n+2}\};Q_j)\right) \\ &\quad\times\left(\prod_{0\le i<j\le n/2}(x_{2j+1}-x_{2i+1})\right)^{n/2+1} \\ &\quad\times\left(\prod_{i=1}^{n/2} a(x_{2i})\right)\left(\prod_{1\le i<j\le n/2}(x_{2j}-x_{2i})\right)^2 \\ &\quad\times\prod_{k=2}^{n/2} vdm(\{x_2,x_4,x_6,\cdots,x_n\};P_{k-1}). \end{aligned}$$

We remark that the determinants that appear in the above formula do in general have further factorizations, but they do not seem to be particulary simple or enlightening.

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