Quaternionic MUBs in $\mathbb{H}^2$ and their reflection symmetries

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Abstract

We consider the primitive quaternionic reflection groups of type P for $\mathbb{H}^2$ that are obtained from Blichfeldt's collineation groups for $\mathbb{C}^4$. These are seen to be intimately related to the maximal set of five quaternionic mutually unbiased bases (MUBs) in $\mathbb{H}^2$, for which they are symmetries. From these groups, we construct other interesting sets of lines that they fix, including a new quaternionic spherical 3-design of 16 lines in $\mathbb{H}^2$ with angles $\{\frac{1}{5}, \frac{3}{5}\}$, which meets the special bound. Some interesting consequences of this investigation include finding imprimitive quaternionic reflection groups with several systems of imprimitivity, and finding a nontrivial reducible subgroup which has a continuous family of eigenvectors.

Key Words: finite tight frames, quaternionic MUBs (mutually unbiased bases), quaternionic reflection groups, representations over the quaternions, Frobenius-Schur indicator, projective spherical t-designs, special and absolute bounds on lines,

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1 Introduction

The (finite irreducible) quaternionic reflection groups, i.e., groups of matrices over the quaternions generated by reflections, were classified by Cohen [Coh80]. There are six rank two primitive quaternionic reflection groups with primitive complexifications, in the families O and P, which were obtained from certain collineation groups for $\mathbb{C}^4$ of Blichfeldt [Bli17]. Here we consider the three groups in the family P, and the small sets of quaternionic lines that they stabilise, which includes the roots of the reflections themselves.

A set of **mutually unbiased bases** (called **MUBs**) for $\mathbb{R}^d$, $\mathbb{C}^d$ or $\mathbb{H}^d$ is a collection of orthonormal bases $\mathcal{B}_1, \ldots, \mathcal{B}_m$ for which vectors v and w in different bases have a fixed common angle, i.e.,

$$|\langle v, w\rangle|^2 = \frac{1}{d}, \qquad v \in \mathcal{B}_j,\ w \in \mathcal{B}_k, \quad j \neq k.$$

Complex MUBs are of interest in quantum information theory as they provide unbiased measurements [Iva81], [WF89]. They are closely related to SICs [ACFW18], [Wal18]. The maximal number of MUBs in $\mathbb{C}^6$ is conjectured to be three [MW24].

For $d = 2$, maximal collections of two and three real and complex MUBs are given by

$$\Bigl\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix}\Bigr\}, \qquad \Bigl\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\end{pmatrix}\Bigr\}. \qquad (1.1)$$

There is a maximal set of five quaternionic MUBs in $\mathbb{H}^2$ given by

$$\Bigl\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm j\end{pmatrix}\Bigr\},\ \Bigl\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm k\end{pmatrix}\Bigr\}. \qquad (1.2)$$

These first appeared in Example 3 of [Hog82] as a "tight 3-design attaining the absolute bound". The Example 4 then extends this to what one might call "nine octonionic MUBs in $\mathbb{O}^2$". We will not consider the general theory of (maximal) quaternionic MUBs, other than remarking that it begins with (1.2), the Example 21 of [Hog82], which gives nine quaternionic MUBs in $\mathbb{H}^4$, and various quaternionic MUB-like configurations [Hog82], [Kan95], [CKM16], [BADL24].

The rest of the paper is set out as follows. We first consider Blichfeldt's original collineation groups for $\mathbb{C}^4$, and construct from them the primitive reflection groups of the type P in [Coh80] (where details were not given). This leads to nice presentations for the groups, which include as an imprimitive reflection group based on the five MUBs extended by adding a single non-monomial reflection matrix.

Next, we observe that the P groups are symmetries of the MUB lines, and consider the associated permutation action on these lines and the MUB pairs. We then consider the roots of the reflections, and the sets of lines stabilised by these reflection groups. In other words, we recognise the reflection groups of type P as symmetry groups of nice (well spaced) configurations of quaternionic lines, such as the five quaternionic MUBs. In particular, we construct a new spherical 3-design of 16 lines in $\mathbb{H}^2$ with angles $\{\frac{1}{5}, \frac{3}{5}\}$, which meets the special bound, and give a nontrivial reducible subgroup which has a continuous family of eigenvectors.

2 The quaternionic reflection groups of type P

We assume some basic familiarity with finite irreducible complex reflection groups and their classification into those which are primitive and imprimitive [ST54], [LT09], and the quaternions $\mathbb{H}$ and matrices over them, see, e.g., [SS95], [Zha97], [CS03], [Voi21].

A quaternionic reflection on $\mathbb{H}^d$ is a nonidentity matrix $g \in M_d(\mathbb{H})$ of finite order which fixes a hyperplane pointwise, equivalently, $\operatorname{rank}(I-g) = d-1$, and a finite reflection group is a finite subgroup of $M_d(\mathbb{H})$ which is generated by reflections. Since finite subgroups of $M_d(\mathbb{H})$ are conjugate to groups of unitary matrices, we suppose henceforth that our reflection groups are unitary. Therefore, a (unitary) reflection g is defined by a **root vector** $a \in \mathbb{H}^d$, and a unit scalar $\xi \in \mathbb{H}$, $\xi \neq 1$, for which

$$gx = x, \quad \forall x \in a^\perp, \qquad ga = a\xi,$$

i.e., the fixed hyperplane is the orthogonal complement of a, and g has order n if and only ξ is a primitive n-th root of unity. If $a \neq 0$ is a vector, then a formula for g is

$$r_{a,\xi} := I - \frac{a(1-\xi)a^*}{\langle a, a\rangle}. \tag{2.3}$$

Throughout, $\mathbb{H}^d$ is considered as a right vector space ($\mathbb{H}$-module), so that linear maps are applied on the left, and we denote the quaternion group by

$$Q_8 := \{1, -1, i, -i, j, -j, k, -k\}.$$

Example 2.1 *By (2.3), the reflection for the root*

$$a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -b^{-1} \end{pmatrix} \in \mathbb{H}^2, \quad |b| = 1,$$

is

$$r_{a,\xi} = \frac{1}{2}\begin{pmatrix} 1+\xi & (1-\xi)b \\ b^{-1}(1-\xi) & b^{-1}(1+\xi)b \end{pmatrix},$$

which is monomial if and only if $\xi = -1$, which gives the reflection of order two

$$r_{a,-1} = \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}.$$

Therefore, the reflections of order two given by the ten MUB vectors of (1.2) are

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}. \tag{2.4}$$

A reflection group G is said to be **imprimitive** if the space on which it acts can be decomposed into proper subspaces which it permutes, a so called system of imprimitivity, otherwise it is said to be **primitive**. For a reflection group (which is unitary) acting on $\mathbb{H}^2$, we can assume the system of imprimitivity is given by the standard basis vectors

$$V_1 = \operatorname{span}_{\mathbb{H}} e_1, \qquad V_2 = \operatorname{span}_{\mathbb{H}} e_2,$$

so that its elements have the (monomial) form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \qquad |a| = |b| = |c| = |d| = 1.$$

Blichfeldt [Bli17] (Chapter VII) classified the irreducible collineation groups for $\mathbb{C}^4$. Collineation groups are groups of matrices defined up to a scalar multiple (in modern terminology linear maps defined on projective spaces). The groups (A), (C), (K) of [Bli17] have orders 60ϕ, 360ϕ, 720ϕ (here ϕ indicates the order of the subgroup of scalar matrices, which is unimportant in the theory), and correspond to the groups of type O in [Coh80] of orders 120, 720, 1440 (see [Wal24]).

Here we consider the groups 14°, 16°, 18° of orders $10\cdot 16\phi$, $60\cdot 16\phi$, $120\cdot 16\phi$, which correspond to the groups of type P in [Coh80] of orders 320, 1920, 3840. The collineation groups $13°, \ldots, 21°$ of [Bli17] (page 172) are primitive collineation groups generated by an imprimitive group K of order 16ϕ, generated by

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

the element T, which gives the primitive group 13°, and various other elements given by

$$S = \frac{1+i}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \frac{1+i}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix},$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & -i \end{pmatrix}, \quad A = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

These groups, which have K as a normal subgroup, are generated as follows

$$\begin{aligned} &13°: \ K, T, \qquad && 14°: \ K, T, R^2, \qquad && 15°: \ K, T, R, \\ &16°: \ K, T, SB, && 17°: \ K, T, BR, && 18°: \ K, T, A, \\ &19°: \ K, T, B, && 20°: \ K, T, AB, && 21°: \ K, T, S. \end{aligned} \tag{2.5}$$

A group G of complex matrices in $M_{2d}(\mathbb{C})$ gives rise to a group of quaternionic matrices in $M_d(\mathbb{H})$ if it is conjugate to a group of matrices of the **symplectic form**

$$\begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} \in M_{2d}(\mathbb{C}) \quad \Longleftrightarrow \quad A + Bj \in M_d(\mathbb{H}). \tag{2.6}$$

The **Frobenius-Schur indicator** of a complex representation of a finite group G is

$$\iota\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g^2) \in \{-1,0,1\},$$

where χ is the character of the representation. This takes the value -1 if and only if the representation of G corresponds to a quaternionic representation via (2.6) [Gan11]. Blichfeldt's collineation groups $13^\circ,\ldots,21^\circ$ are given as matrix groups over $\mathbb{C}$, with a subgroup of scalar matrices (of order ϕ). Changing the subgroup of scalar matrices (which gives the same collineation group), changes the Frobenius-Schur indicator, and so some care must be taken. Indeed, in view of (2.6), the collineation group must be presented so that its matrices have a real trace, and consequently $\pm I \in M_{2d}(\mathbb{C})$ are the only allowable scalar matrices. It is still quite possible for the group of quaternionic matrices to contain scalar matrices, e.g., $iI, jI \in M_d(\mathbb{H})$ correspond via (2.6) to the nonscalar symplectic matrices

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_{2d}(\mathbb{C}).$$

We first consider the group K, which is an imprimitive normal subgroup of all the collineation groups, together with T. The group generated by A_1, A_2, A_3, A_4 has small group identifier $\langle 32, 49\rangle$ and has Frobenius-Schur indicator 1, and so does not correspond to a group in $M_2(\mathbb{H})$. The trace of T is 1, but it is not in the symplectic form (2.6). Conjugation of T (equivalently $-T$) by the permutation matrices for the permutations $(1\,4)$, $(2\,3)$, $(1\,3\,4\,2)$, $(1\,2\,4\,3)$ gives a matrix of the form (2.6). Calculations show that whatever permutation is taken, the quaternionic groups obtained are identical elementwise (just with different generators). We take

$$P = P_{(1\,4)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which conjugates A_1, iA_2, iA_3, A_4 to the symplectic form (2.6), i.e.,

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} -k & 0 \\ 0 & -k \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

The group generated by A_1, iA_2, iA_3, A_4 has identifier $\langle 32, 50\rangle$, and Schur-Frobenius indicator -1. Here A_2, A_3, which have zero trace, were multiplied by i to obtain the symplectic form. The group K generated by the quaternionic matrices of (2.7) is the imprimitive reflection group generated by the ten MUB reflections of (2.4), which are all of its reflections. This is the group denoted by

$$K = G_{Q_8}(Q_8, C_2) = \mathcal{G}(\{1,i,j,k\},\{\}), \qquad K/\langle -I\rangle = C_2\times C_2\times C_2\times C_2,$$

in [Wal25], which is generated by the reflections

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}. \tag{2.8}$$

The 32 elements of K are

$$\begin{pmatrix} q & 0 \\ 0 & \pm q \end{pmatrix}, \quad \begin{pmatrix} 0 & q \\ \pm q & 0 \end{pmatrix}, \qquad q \in Q_8.$$

Despite four of the reflection pairs for the MUBs (2.4) being nondiagonal matrices, with the other being diagonal, each pair plays an equivalent role. For example, the orbits of the reflections under conjugation by K are of size two, giving the MUB pairs of reflections, and the minimal generating sets of reflection for K are precisely any four reflections, which come from different MUB pairs (orbits). It is also interesting to note that each of the five MUBs of (1.2) are a system of imprimitivity for K. By way of comparison (see [LT09] Theorem 2.16), the only complex reflection groups on $\mathbb{R}^2$ or $\mathbb{C}^2$ which have more than one system of imprimitivity are $G(2,1,2) \cong G(4,4,2)$ and $G(4,2,2)$, which have three systems of imprimitivity given by the three complex MUBs of (1.1).

The conjugate of T by P gives a symplectic matrix of order ten, i.e.,

$$t := \frac{1}{2}\begin{pmatrix} j-k & 1-i \\ -j-k & 1+i \end{pmatrix}, \tag{2.9}$$

which is not a reflection. It maps the lines given by the MUB vectors to themselves, e.g.,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ -i \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ k \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ j \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The group of quaternionic matrices generated by K, i.e., the reflections (2.8), and t of (2.9), contains no further reflections, and hence is not a reflection group.

The other generators from (2.5) which have real traces and conjugate under P to a symplectic matrix are R^2, R, SB, and those which do not have real traces are

$$\operatorname{trace}(BR) = 2i, \quad \operatorname{trace}(A) = \operatorname{trace}(S) = 2\sqrt{2}i, \quad \operatorname{trace}(B) = \sqrt{2}(1+i).$$

After scaling to obtain a real trace, only iA conjugates under P to the symplectic form. After conjugation with P, the matrices R^2, R, SB, iA are in the symplectic form, giving

$$\begin{pmatrix} -1 & 0 \\ 0 & -k \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -i-j & 0 \\ 0 & 1-k \end{pmatrix}, \quad \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}. \tag{2.10}$$

Hence there are primitive quaternionic groups of matrices in $U_2(\mathbb{H})$ corresponding to Blichfeldt's groups $13°, \ldots, 16°$, $18°$, which are generated by the corresponding matrices from (2.8), (2.9), (2.10). A calculation in `magma` shows that three of these are reflection groups, i.e.,

$$G_{14^\circ} = \langle K, t, \begin{pmatrix} -1 & 0 \\ 0 & -k \end{pmatrix}\rangle, \quad G_{16^\circ} = \langle K, t, \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}\rangle, \quad G_{18^\circ} = \langle K, t, \frac{1}{\sqrt{2}}\begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}\rangle.$$

Since K is contained in these groups, adding the last generator is the same as adding the reflections

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1+i \\ -1-i & 0 \end{pmatrix} = r_{(1,\frac{1+i}{\sqrt{2}}),-1}, \tag{2.11}$$

respectively, and since

$$t\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = r_{(1,-1),i},$$

is a reflection (of order 4), we can conclude (by hand) that the above are reflection groups, with

$$G_{14^\circ} = \langle K, r_{(1,0),k}, r_{(1,1),i} \rangle, \tag{2.12}$$

since $r_{(1,1),i} \in G_{14^\circ}$. The 30 reflections in G_{14° are given by the ξ, a pairs

$$\langle i \rangle : \quad \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm k \end{pmatrix}, \quad \langle j \rangle : \quad \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \begin{pmatrix} 1 \\ \pm j \end{pmatrix}, \quad \langle k \rangle : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.13}$$

and the 70 reflections in G_{16° by

$$Q_8 : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \begin{pmatrix} 1 \\ \pm j \end{pmatrix}, \begin{pmatrix} 1 \\ \pm k \end{pmatrix}. \tag{2.14}$$

We observe from above that G_{14° is a subgroup of G_{16°. A calculation shows that there are six conjugates G_{q_1,q_2}, $q_1 \neq q_2$, $q_1, q_2 \in \{i, j, k\}$, of G_{14° in G_{16°, given by the reflections

$$\langle q_1 \rangle : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle q_2 \rangle : \quad \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm q_1 \end{pmatrix}, \quad \langle q_1 q_2 \rangle : \quad \begin{pmatrix} 1 \\ \pm q_2 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm q_1 q_2 \end{pmatrix}. \tag{2.15}$$

In view of (2.12), for G_{q_1,q_2} we can take any generators for K, together with the reflections $r_{(1,0),q_1}$, $r_{(1,1),q_2}$. It turns out these generators alone are sufficient, i.e.,

$$G_{q_1,q_2} = \langle r_{(1,0),q_1}, r_{(1,1),q_2} \rangle = \langle \begin{pmatrix} q_1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+q_2 & -1+q_2 \\ -1+q_2 & 1+q_2 \end{pmatrix} \rangle. \tag{2.16}$$

The scalar ξ for a reflection $r_{a,\xi}$ depends on the particular multiple of the root taken, i.e.,

$$r_{a,\xi} = r_{a\beta,\beta^{-1}\xi\beta}, \qquad \beta \in \mathbb{H}^*.$$

In [Coh80], the group H_a of scalars associated with a root is taken to be the same for all roots a. To change the scalars $\xi = q_2$ to q_1 and $\xi = q_1 q_2$ to q_1 in (2.15), we take $\beta = (1 - q_1 q_2)$ and $\beta = (1 - q_2)$, to obtain

$$\langle q_1 \rangle : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}(1-q_1q_2), \begin{pmatrix} 1 \\ \pm q_1 \end{pmatrix}(1-q_1q_2), \begin{pmatrix} 1 \\ \pm q_2 \end{pmatrix}(1-q_2), \begin{pmatrix} 1 \\ \pm q_1 q_2 \end{pmatrix}(1-q_2).$$

Taking $q_1 = j$, $q_2 = k$ above gives the root system of [Coh80] (Table II), and so we conclude the group given there is $G_{j,k}$, whereas the group given by (2.13) is $G_{k,i}$.

We now consider generators for the groups of type P. For the first G_{14°, (2.16) gives

$$P_1 = H_{320} := G_{i,j} = \langle r_{(1,0),i}, r_{(1,1),j} \rangle = \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+j & -1+j \\ -1+j & 1+j \end{pmatrix} \rangle, \tag{2.17}$$

where $|P_1| = 320$. The comment of (2.11) implies that the next group G_{16° is

$$P_2 = H_{1920} := \langle P_1, r_{(1,0),j}\rangle = \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+j & -1+j \\ -1+j & 1+j \end{pmatrix}\rangle, \tag{2.18}$$

where $|P_2| = 1920$. Both of these groups have the five MUBs as the roots of their reflections. Similar considerations give G_{18° as

$$G_{18^\circ} = \langle P_2, r_{(1,\frac{1+i}{\sqrt{2}}),-1}\rangle = \langle P_2, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1+i \\ -1-i & 0 \end{pmatrix}\rangle.$$

It is easily verified that G_{18° contains the reflection given by the Fourier matrix, i.e.,

$$F := r_{(-1,1+\sqrt{2}),-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in G_{18^\circ}, \tag{2.19}$$

which leads to the generating reflections

$$P_3 = H_{3840} := \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\rangle, \tag{2.20}$$

where $|P_3| = 3840$.

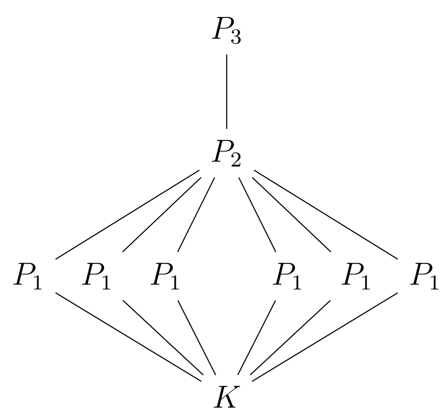

Figure 1: The P groups: $K \lhd P_1, P_2, P_3$ and $P_2 \lhd P_3$. The group P_1 occurs six times as a subgroup of P_2 (a single conjugacy class), i.e., as G_{q_1,q_2}, $q_1 \neq q_2$, $q_1, q_2 \in \{i, j, k\}$.

The reflection group P_3 has a 110 reflections, consisting of the 70 reflections (2.14) of P_2, and 40 reflections of order two which are the orbit of F under the conjugation action of P_2. These are given by the root lines

$$\begin{pmatrix} \sqrt{2} \\ p+q \end{pmatrix}, \quad p+q \neq 0, \quad \{p,q\} \subset Q_8, \qquad \begin{pmatrix} 1+\sqrt{2} \\ q \end{pmatrix}, \begin{pmatrix} q \\ 1+\sqrt{2} \end{pmatrix}, \quad q \in Q_8. \tag{2.21}$$

The first 24 of these give rise to monomial reflections, i.e.,

$$\begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}, \quad b = \frac{p+q}{\sqrt{2}} \neq 0, \quad \{p,q\} \subset Q_8, \tag{2.22}$$

and the last 16 give rise to the non-monomial reflections

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -\overline{q}\\ -q & 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -q\\ -\overline{q} & -1\end{pmatrix}, \quad q \in Q_8. \tag{2.23}$$

Example 2.2 *It is interesting to observe that P_3 is generated by five of the 40 reflections of order two given by (2.21), e.g.,*

$$\begin{pmatrix}0 & b\\ b^{-1} & 0\end{pmatrix}, \quad \sqrt{2}b \in \{1+i, 1-i, 1+j, 1+k\}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

This is a consequence of P_3 being given in [Coh80] via a root system based on (2.21).

Table 1: Generating reflections for K and the P groups. The group P_3 is also generated by five of the 40 reflections of order two given by (2.22) and (2.23) (see Example 2.2).

G	generating reflections
$K^\dagger$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \begin{pmatrix}0 & j\\ -j & 0\end{pmatrix}, \begin{pmatrix}0 & k\\ -k & 0\end{pmatrix}$
P_1	$\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1+j & -1+j\\ -1+j & 1+j\end{pmatrix}$
P_2	$\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}j & 0\\ 0 & 1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1+j & -1+j\\ -1+j & 1+j\end{pmatrix}$
P_3	$\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}j & 0\\ 0 & 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

† K is generated by any four of these reflections.

Example 2.3 *Consider the reflection group of order* 64 *given by*

$$P_0 = \langle \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \begin{pmatrix}0 & j\\ -j & 0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\rangle,$$

which is generated by K and the Fourier matrix F of (2.19). This was initially assumed to be primitive, and hence a previously unknown such group, but conjugation by the matrices

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & j\\ j & 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & k\\ k & 1\end{pmatrix} \in P_3$$

gives a monomial, and hence imprimitive, reflection group, which has three systems of imprimitivity. It is the imprimitive reflection group $G(4,1,2,2)$ in the family of [Wal25].

There are only four imprimitive complex reflection groups with more than one system of imprimitivity (Theorem 2.16 [LT09]), i.e., two in $\mathbb{C}^2$ (three systems), one in $\mathbb{C}^3$ (four systems), and one in $\mathbb{C}^4$ (three systems). Therefore, given the above example (three systems) and K (five systems), it appears that the number of systems of imprimitivity for quaternionic reflection groups is worthy of some study.

3 The maximal imprimitive reflection subgroups

Our presentations (2.17), (2.18), (2.20) of the P reflection groups (see Table 1) involve just a single non-monomial reflection. Therefore, we can view them as imprimitive (monomial) reflection groups with a single non-monomial reflection added. We now identify these imprimitive reflection groups.

The non-diagonal reflections in a reflection group of rank two have the form (2.22) for a set L of $b \in \mathbb{H}$, called a "reflection system" [Wal25], which satisfy the conditions

1. L generates a finite group $K = \langle L \rangle$.
2. L is closed under the binary operation $(a, b) \mapsto a \circ b := ab^{-1}a$.
3. $1 \in L$.

If L is the closure of $X \subset \mathbb{H}$ under $(a, b) \mapsto a \circ b$, then we say X generates the reflection system L, and we write $L = L(X)$. In view of (2.4), (2.22), the reflection systems for the groups P_1, P_2 are $Q_8 = L(\{1, i, j, k\})$, and for P_3 it is

$$L_{32} := L\Big(\Big\{1, \frac{1+i}{\sqrt{2}}, \frac{1+j}{\sqrt{2}}, \frac{1+k}{\sqrt{2}}\Big\}\Big) = Q_8 \cup \Big\{\frac{p+q}{\sqrt{2}} \neq 0 : \{p, q\} \subset Q_8\Big\}.$$

The 32-element reflection system L_{32} has K the binary octahedral group of order 48 given by

$$\mathcal{O} = \langle \frac{1+i}{\sqrt{2}}, \frac{1+i+j+k}{2} \rangle = \langle \frac{1+i}{\sqrt{2}}, \frac{1+j}{\sqrt{2}} \rangle.$$

It is equivalent to that given in [Wal25], i.e.,

$$L_{32} = \frac{1+i}{\sqrt{2}} L_{32}^{\mathcal{O}}, \qquad L_{32}^{\mathcal{O}} := L\Big(\Big\{1, \frac{1+i}{\sqrt{2}}, \frac{1+i+j+k}{2}\Big\}\Big).$$

The corresponding subreflection systems are

$$L_{20} = L\Big(\Big\{1, \frac{1+i}{\sqrt{2}}, \frac{1+j}{\sqrt{2}}, k\Big\}\Big), \quad L_{18} = L\Big(\Big\{1, \frac{1+i}{\sqrt{2}}, \frac{1+j}{\sqrt{2}}\Big\}\Big), \quad L_{14} = L\Big(\Big\{1, \frac{i-j}{\sqrt{2}}, \frac{i-k}{\sqrt{2}}, \frac{j+k}{\sqrt{2}}\Big\}\Big).$$

The imprimitive quaternionic reflection groups of rank two have a canonical form $G = G_K(L, H)$, where L is a reflection system with $K = \langle L \rangle$, which gives the nondiagonal reflections, and H is normal subgroup of K, which gives the diagonal reflections in G, i.e.,

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \qquad h \in H, \, h \neq 1.$$

The monomial reflections of $G = P_1, P_2, P_3$ generate the following imprimitive reflection groups G_M

$$G_{Q_8}(Q_8, C_4), \quad G_{Q_8}(Q_8, Q_8), \quad G_{\mathcal{O}}(L_{32}^{\mathcal{O}}, Q_8),$$

which have orders 64, 128, 768.

For a reflection group G, the reflections for a given root a together with the identity form a subgroup R_a. The group G acts on the reflection subgroups R_a via conjugation.

We will refer to the orbits of this action as the **reflection type** or **reflection orbits** of G. If the m reflection orbits are given by $R_{a_1}, \ldots, R_{a_m}$, we will often write the reflection type as $n_1 R_{a_1}, \ldots, n_m R_{a_m}$, where n_j is the orbit size, and R_{a_j} is an abstract group.

We can now summarise the structure of the P groups.

Table 2: The P groups and their monomial reflection subgroup G_M (each of these appears five times, corresponding to the five sets of imprimitivity for K).

G	$\|G\|$	refs	ref orbits	G_M	$\|G_M\|$	refs	ref orbits
P_3	3840	110	$10Q_8, 40C_2$	$G_{\mathcal{O}}(L_{32}^{\mathcal{O}}, Q_8)$	768	46	$2Q_8, 8C_2, 24C_2$
P_2	1920	70	$10Q_8$	$G_{Q_8}(Q_8, Q_8)$	128	22	$2Q_8, 8C_2$
P_1	320	30	$10C_4$	$G_{Q_8}(Q_8, C_4)$	64	14	$2C_4, 4C_2, 4C_2$
K	32	10		$G_{Q_8}(Q_8, C_2)$	32	10	$2C_2, 2C_2, 2C_2, 2C_2, 2C_2$

4 MUB symmetries

The P groups map the ten MUB lines of (1.2) to themselves, i.e., are symmetries for them. This is easily seen by action of the generators (2.20) for P_3 on the lines, e.g.,

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix},\quad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix},\quad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ q\end{pmatrix}=\begin{pmatrix}1\\ -q\end{pmatrix}(1+q).$$

Moreover, the columns of each matrix in P_3 are a MUB pair. We expect that $P_3 \subset U_2(\mathbb{H})$ is the (full) symmetry group of the ten MUB lines.

We order the ten MUB lines as in (1.2) with the $\pm$ entries ordered $+$, $-$. With this labelling, the generators in Table 1 correspond to the following permutations

$$\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix} \longleftrightarrow (3\,6\,4\,5)(7\,9\,8\,10), \qquad \frac{1}{2}\begin{pmatrix}1+j & -1+j\\ -1+j & 1+j\end{pmatrix} \longleftrightarrow (1\,7\,2\,8)(5\,10\,6\,9),$$

$$\begin{pmatrix}j & 0\\ 0 & 1\end{pmatrix} \longleftrightarrow (3\,8\,4\,7)(5\,10\,6\,9), \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \longleftrightarrow (1\,3)(2\,4)(5\,6)(7\,8)(9\,10),$$

and the matrix t of (2.9) to

$$t=\frac{1}{2}\begin{pmatrix}j-k & 1-i\\ -j-k & 1+i\end{pmatrix} \longleftrightarrow (1\,6\,9\,4\,7)(2\,5\,10\,3\,8).$$

The kernel of the action of P_3 on the MUB lines is $\langle -I\rangle$, i.e., $P_3/\langle -I\rangle$ acts faithfully on the ten lines. It is clear from the above permutations that P_3 acts on the five MUB pairs. With these ordered as in (1.2), the permutations corresponding to the above elements are

$$(2\,3)(4\,5), \qquad (1\,4)(3\,5), \qquad (2\,4)(3\,5), \qquad (1\,2),$$

respectively. The kernel of this action on the five MUB pairs is the imprimitive reflection group K, and

$$P_3/K \cong S_5,$$

i.e., any permutation of the five MUB pairs is possible. Similarly, we have

$$P_2/K \cong A_5, \qquad P_1/K \cong D_5 \text{ (dihedral group of order 10).}$$

The quotients of P_j/K above are discussed in the proof of Theorem 4.2 in [Coh80], with the representation for P_1 dating back to Crowe [Cro59]. The groups of type P were introduced in Proposition 4.1 of [Coh80] as

- G (of rank 2) is an extension of a subgroup of S_6 by $\mathbf{D}_2 \circ D_4$ (where $\mathbf{D}_2 \circ D_4 \cong K$).

The imprimitive reflection group P_0 of Example 2.3 can be considered to be of type P, as follows. Its generators (the first three are in K) permute the MUB pairs as follows

$$\begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix},\ b \in \{1, i, j\} \quad \longleftrightarrow \quad (), \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \longleftrightarrow \quad (1\,2),$$

so that

$$P_0/K \cong \langle (1\,2) \rangle = C_2,$$

i.e., P_0 is of type P. The group P_0 is not normal in P_3, having five conjugates.

5 Spherical designs and small sets of invariant lines

We have seen that the orbit of the vector/line $v = e_1$ (or any MUB line) under the action of P_3 (and its subgroups P_1, P_2) is the MUB lines of (1.2). These ten lines are the roots of the reflection groups P_1 and P_2, but not P_3 (which has 40 additional root lines).

These lines are well-spaced, in the sense that the set of angles $|\langle v, w \rangle|^2$ between different lines given by unit vectors $v, w \in \mathbb{H}^2$ is the small set $\{0, \frac{1}{2}\}$. We will give a related notion of being well-spaced, that of being a "spherical design" [DGS77], which corresponds to the lines being a cubature rule for the sphere. In this section, we will use a general method, which does not require the groups $G \subset U_d(\mathbb{H})$ involved be reflection groups, to find small sets of G-invariant lines which provide good spherical designs. This will give the ten MUB lines, and also other interesting configurations (see Table 3).

Let $t \in \{1, 2, \ldots\}$. The set of lines given by n unit vectors (v_j) in $\mathbb{H}^d$ is called a **spherical** (t, t)**-design** for $\mathbb{H}^d$ if they give equality in the inequality

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle v_j, v_k \rangle|^{2t} \geq c_t(\mathbb{H}^d) \Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t} \Big)^2, \qquad c_t(\mathbb{H}^d) := \prod_{j=0}^{t-1} \frac{2+j}{2d+j}, \tag{5.24}$$

i.e.,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle v_j, v_k \rangle|^{2t} = c_t(\mathbb{H}^d) \Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t} \Big)^2, \tag{5.25}$$

see [Wal20] for details. These can be viewed as a cubature rule for the unit sphere in $\mathbb{H}^d$, and are equivalent to the quaternionic **spherical t-designs** of [Hog82]. Therefore, we are interested in large values of t and small numbers of lines n. We note

$$c_1(\mathbb{H}^d) = \frac{1}{d}, \qquad c_2(\mathbb{H}^d) = \frac{3}{d(2d+1)}, \qquad c_3(\mathbb{H}^d) = \frac{6}{d(2d+1)(d+1)}.$$

It follows from (5.25) that the orbit $(gx)_{g\in G}$ of a nonzero vector $x \in \mathbb{H}^d$ under the action of finite group of unitary matrices $G \subset U_d(\mathbb{H})$ is a spherical (t,t)-design if and only if

$$p_G^{(t)}(x) := \frac{1}{|G|}\sum_{g\in G} |\langle x, gx\rangle|^{2t} - c_t(\mathbb{H}^d)\langle x, x\rangle^{2t} = 0. \tag{5.26}$$

Moreover (see [Wal20]), G is irreducible if and only if every orbit is a $(1,1)$-design, i.e.,

$$\frac{1}{|G|}\sum_{g\in G} |\langle x, gx\rangle|^{2} - \frac{1}{d}\langle x, x\rangle^{2} = 0. \tag{5.27}$$

By direct calculation of (5.26) in `magma`, we obtain the following.

Proposition 5.1 *Every orbit of a P group, i.e., P_1, P_2, P_3, is a spherical $(3,3)$-design.*

Proof: We have $p_G^{(3)} = 0$, for $G = P_1, P_2, P_3$. □

In particular, the ten MUB lines are a spherical $(3,3)$-design for $\mathbb{H}^2$. This can be verified directly by evaluating $p_G^{(3)}$ at a unit MUB vector, which gives

$$\frac{1}{10}\Big(1\cdot 1^3 + 1\cdot 0^3 + 8\cdot\Big(\frac{1}{2}\Big)^3\Big) - \frac{6}{2\cdot 5\cdot 3}\cdot 1 = \frac{1}{5} - \frac{1}{5} = 0.$$

We observe that

- The ten MUB lines are a spherical $(3,3)$-design by being the orbit of a P group.
- There is a small number of these vectors, as their stabiliser subgroups are large.

Since the stabiliser group of a line is, by definition, reducible, we can find small sets of lines such as the MUB lines as follows:

- Find the large reducible subgroups of the P groups, and the lines they stabilise.
- The orbit of the stabilised line is then a $(3,3)$-design with a small number of vectors.

A line stabilised by a proper subgroup is called a **fiducial vector** (or line).

The condition (5.27) allows us to identify the reducible subgroups of the P groups. These need not be reflection groups. The only other technical condition is determining those $v \in \mathbb{H}^d$ which give a line stabilised by some $g \in U_d(\mathbb{H})$, i.e.,

$$gv = v\lambda, \qquad \text{for some } \lambda \in \mathbb{H}. \tag{5.28}$$

Nominally, v appears to be an eigenvector for g, but the calculation

$$g(v\alpha) = v\lambda\alpha = v\alpha(\alpha^{-1}\lambda\alpha), \qquad \alpha \in \mathbb{H},$$

shows that there is no natural associated eigenvalue when λ is not real (we will still refer to v as eigenvector). Nevertheless, the condition (5.28) can be verified (without calculating a λ), as the condition which gives equality in the Cauchy-Schwartz inequality, i.e.,

$$|\langle v, gv\rangle|^2 = \langle v, v\rangle^2, \qquad \forall g \in G, \tag{5.29}$$

which gives a quartic polynomial in the 1, i, j, k parts of the coordinates of $v \in \mathbb{H}^d$.

We now outline our computations in `magma`, as detailed above, for $G = P_1$.

- We will take the "large" reducible subgroups of G, to be those which are maximal.
- The corresponding systems of lines will be of minimal size.
- The lines for the proper subgroups of the maximal reducible subgroups (which are automatically reducible) will either be lines for the maximal reducible subgroup, or a larger set of lines.

We observe that a vector $v^\perp$ orthogonal to the line given by a $v \in \mathbb{H}^2$ with real first component is given by the formula

$$v = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a \in \mathbb{R}, \qquad v^\perp = \begin{pmatrix} -\overline{b} \\ a \end{pmatrix}, \tag{5.30}$$

e.g., the roots of (2.21) appear as orthogonal pairs, and the MUB pairs can be written

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ j \end{pmatrix}, \begin{pmatrix} j \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ k \end{pmatrix}, \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

The subgroups of $G \subset M_d(\mathbb{H})$ can be computed in `magma` using the command `Subgroups(G)`, and the lattice by `SubgroupLattice(G)`, the latter only working for complex (symplectic) presentations of the group. For the group P_1 of order 320, the reducible subgroups and the lengths of their conjugacy classes and number of lines are

order	number	length	lines
32	1	5	10
20	1	16	16
16	3	5	20
	4	10	20
10	1	16	32
8	5	5	40
	6	10	40

order	number	length	lines
5	1	16	64
4	3	5	80
	2	10	80
	1	20	80
2	1	1	160
2	1	10	160
1	1	1	320

Of these, there are four are maximal irreducible subgroups, i.e.,

$$H = \langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \rangle, \qquad \text{(Order 32, 10 lines)}, \tag{5.31}$$

$$H = \langle \begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i-j & i-j \\ i-j & -i+j \end{pmatrix} \rangle, \qquad \text{(Order 20, 16 lines)}, \tag{5.32}$$

$$H = \langle \begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix}, \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \rangle, \qquad \text{(Order 16, 20 lines)}, \tag{5.33}$$

$$H = \langle \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i-k & 1-j \\ 1+j & -i-k \end{pmatrix} \rangle, \qquad \text{(Order 16, 20 lines)}. \tag{5.34}$$

None of these are reflection groups. The first is diagonal, and it is easy to see that it stabilises the lines given by standard basis vectors, and no others. The five conjugates of this group in P_1 stabilise the five MUB pairs, and no other lines. Thus the P_1-orbit of any line fixed by the reducible subgroup of order 32 is the ten MUB lines, i.e., we arrive at the MUB lines without using the fact that P_1 is a reflection group.

We now consider the irreducible subgroup of order 20, which gives 16 lines. Since the line given by e_2 is not fixed, we can suppose that a fiducial (stabilised) vector has the form

$$v = \begin{pmatrix} 1 \\ x_1 + x_2 i + x_3 j + x_4 k \end{pmatrix}, \qquad x_1, x_2, x_3, x_4 \in \mathbb{R}, \tag{5.35}$$

so that (5.30) gives

$$v^\perp = \begin{pmatrix} -x_1 + x_2 i + x_3 j + x_4 k \\ 1 \end{pmatrix}.$$

Taking g in (5.29) to be the two generators of (5.32), gives the two quartic equations

$$(x_1 - x_2)^2 + (x_3 - x_4)^2 = 0, \qquad \Longrightarrow \qquad x_2 = x_1,\ x_4 = x_3,$$

$$(x_1^4 + x_2^4 + x_3^4 + x_4^4 + 2x_1^2x_2^2 + 2x_1^2x_3^2 + 2x_1^2x_4^2 + 2x_2^2x_3^2 + 2x_2^2x_4^2 + 2x_3^2x_4^2) \\ + (4x_1^3 + 4x_1x_2^2 + 4x_1x_3^2 + 4x_1x_4^2) + (2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_4^2 - 8x_2x_3 - 4x_1) + 1 = 0.$$

The Gröbner for these equations provided by `GroebnerBasis(I)` in `magma` is involved, and we were unable to automate the calculation of fiducials. The set of equations (5.29) given by all elements of H of (5.32), which are not $\pm I$, consists of the first equation and six of a similar complexity to the second. The Gröbner basis provided by `magma` for the ideal given by taking these seven equations is nicer, with the following equation for x_4

$$(4x^2 + 2x - 1)^3 = 0 \qquad \Longrightarrow \qquad x_4 = \frac{-1 \pm \sqrt{5}}{4}.$$

This leads to two fiducials, which are orthogonal, i.e.,

$$w = \begin{pmatrix} 1 + \sqrt{5} \\ 1 + i + j + k \end{pmatrix}, \qquad w^\perp = \begin{pmatrix} -1 + i + j + k \\ 1 + \sqrt{5} \end{pmatrix}. \tag{5.36}$$

The orbits of w and $w^\perp$ under P_1 and P_2 are 16 lines with angles $\{\frac{1}{5},\frac{3}{5}\}$, given by

$$w: \qquad \begin{pmatrix} 1+\sqrt{5} \\ q(1+i+j+k)\end{pmatrix}, \begin{pmatrix} q(1+i+j+k) \\ 1+\sqrt{5}\end{pmatrix}, \quad q\in Q_8, \tag{5.37}$$

$$w^\perp: \qquad \begin{pmatrix} 1+\sqrt{5} \\ q(-1+i+j+k)\end{pmatrix}, \begin{pmatrix} q(-1+i+j+k) \\ 1+\sqrt{5}\end{pmatrix}, \quad q\in Q_8, \tag{5.38}$$

and the orbit of w or $w^\perp$ under P_3 gives 32 lines, which are the union of the two sets, having angles $\{0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}$.

The spherical 3-design of 16 lines constructed above meets the special bound of [Hog78], [Hog82] for the number of lines in $\mathbb{H}^2$ with exactly two nonzero angles.

Example 5.1 *(Special bound) Hoggar [Hog78] provides two bounds on the number n of vector/lines in $\mathbb{H}^d$ with a finite angle set (indeed having a finite number of angles implies a set of lines is finite). If there are two nonzero angles $A=\{\alpha,\beta\}$, then there is a special bound (depending on the angles), and an absolute bound (not depending on the angles), given by*

$$n\le \frac{d(2d+1)(1-\alpha)(1-\beta)}{3-(2d+1)(\alpha+\beta)+d(2d+1)\alpha\beta}, \qquad n\le \frac{1}{3}d^2(4d^2-1).$$

For $A=\{\frac{1}{5},\frac{3}{5}\}$ $(d=2)$, the special bound gives $n\le 16$, i.e., 3-design of 16 lines in $\mathbb{H}^2$ given by (5.37) or (5.38) meets the special bound. The only other known cases where this special bound is met is for a 2-design of 15 lines in $\mathbb{H}^2$ with angles $\{\frac{1}{4},\frac{5}{8}\}$ obtained from a reflection group of type O [Wal24], and for a 2-design of 64 lines in $\mathbb{H}^4$ with angles $\{\frac{1}{9},\frac{1}{3}\}$ obtained from a quaternionic polytope (Example 22, [Hog82]).

The examples from the O and P groups disprove the following conjecture of [Hog82] (for the special bound):

Conjecture 1: Whenever a special or absolute bound is attained, each nonzero $\alpha\in A$ has the form $1/p$ $(p\in\mathbb{N})$, or is irrational.

This could still be true for the roots of a reflection group, which were the bulk of cases considered in [Hog82].

Similar calculations for the maximal reducible subgroups H of P_1, with order 16, given by (5.33) and (5.34), yield a single fiducial vector for each, i.e.,

$$\begin{pmatrix}\sqrt{2}\\ 1+i\end{pmatrix}, \qquad \begin{pmatrix}\sqrt{2}\\ 1+j\end{pmatrix},$$

with the corresponding P_1-orbits of 20 lines, with angles $\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\}$, given by

$$\begin{pmatrix}\sqrt{2}\\ i^m\alpha\end{pmatrix}, \begin{pmatrix}1+\sqrt{2}\\ i^m j\end{pmatrix}, \begin{pmatrix}i^m j\\ 1+\sqrt{2}\end{pmatrix}, \qquad \alpha\in\{1+i,i+j,i-j\}, \quad m=0,1,2,3, \tag{5.39}$$

$$\begin{pmatrix}\sqrt{2}\\ i^m\alpha\end{pmatrix}, \begin{pmatrix}1+\sqrt{2}\\ i^m\end{pmatrix}, \begin{pmatrix}i^m\\ 1+\sqrt{2}\end{pmatrix}, \qquad \alpha\in\{1+j,1-j,j-k\}, \quad m=0,1,2,3. \tag{5.40}$$

We observe that (5.39), (5.40) is a partition of the 40 root lines of P_3 given by (2.21).

If s is the number of angles in a spherical t-design, and $t \geq s-1$, then it is a *regular scheme* (see [Hog84]). Hence, the 3-designs of 10, 16, 20 vectors/lines that we have constructed are regular schemes, since each satisfies $s \leq 4$.

Example 5.2 *The maximal reducible subgroups of P_2 have orders 192, 120, 48, 24, which correspond to systems of 10, 16, 40, 80 lines. The first two of these groups have (5.31) and (5.32) as subgroups, respectively, and so give the sets 10 and 16 lines obtained from P_1. The third group has (5.33) and (5.34) as subgroups, and so fixes the line of both w and $w^\perp$ of (5.36), with either of them being a fiducial for the set of 40 lines consisting of the union of their P_1-orbits (5.37) and (5.38).*

The irreducible subgroup of order 24, which gives 80 lines, is

$$H = \langle \frac{1}{2}\begin{pmatrix} 0 & 1-i-j+k \\ 1+i+j+k & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i+j & i-j \\ i-j & i+j \end{pmatrix} \rangle. \qquad (5.41)$$

This fixes the lines given by the orthogonal vectors

$$\begin{pmatrix} \sqrt{3} \\ 1+i+j \end{pmatrix}, \qquad \begin{pmatrix} -1+i+j \\ \sqrt{3} \end{pmatrix}, \qquad (5.42)$$

and the P_2-orbit of either of these fiducials is the same set of 80 lines at angles $\{0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\}$.

The P_1-orbits of the fiducials of (5.42) give 40 lines (a partition of the 80) with the same angles, i.e., $\{0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\}$. Since this set of 40 lines has not yet appeared, its stabiliser in P_1, which has order 8, must not be a maximal reducible subgroup of P_1. This implies that it fixes a line in one of the sets of 10, 16, 20 lines obtained from the maximal reducible subgroups of P_1. Our direct verification of this fact below, leads to an intriguing example of a "continuous family" of eigenvectors for a matrix group (over the quaternions).

Example 5.3 *Let G be the stabiliser in P_1 of the lines given by the fiducial vectors of (5.42), i.e.,*

$$G := H \cap P_1 = \langle \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i+k & 1+j \\ -1+j & i-k \end{pmatrix} \rangle, \qquad |G| = 8,$$

where H is given by (5.41). The Gröbner basis for the equations (5.29) for a fixed line given by a $v \in \mathbb{H}^2$ of the form (5.35) include $x_4^3 = 0$ and $(x_1 - x_3)^2 = 0$, and reduce to

$$x_3 = x_1, \qquad x_4 = 0, \qquad 2x_1^2 + x_2^2 - 1 = 0.$$

Given that $x_2^2 = 1 - 2x_1^2$, we may solve these equations to get

$$x_1 = x_3 = t, \quad t^2 \leq \frac{1}{2}, \qquad x_2 = \pm\sqrt{1-2t^2}, \qquad x_4 = 0,$$

and hence obtain a continuous family of fiducials (eigenvectors)

$$v = \begin{pmatrix} 1 \\ t(1+j) \pm \sqrt{1-2t^2}i \end{pmatrix}, \quad t \in [-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}].$$

We observe that for the special cases $t = 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$, *we obtain fiducial vectors for* P_1 *giving* 10, 16, 40 *lines, i.e.,*

$$t=0: \quad \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \qquad t = \tfrac{1}{\sqrt{2}}: \quad \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}}(1+j) \end{pmatrix}, \qquad t = \tfrac{1}{\sqrt{3}}: \quad \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}}(1 \pm i + j) \end{pmatrix}.$$

Thus G *is a proper subgroup of the maximal reducible subgroups of* P_1 *given by*

$$\mathrm{Stab}(P_1, \begin{pmatrix} 1 \\ i \end{pmatrix}), \qquad \mathrm{Stab}(P_1, \begin{pmatrix} \sqrt{2} \\ 1+j \end{pmatrix}).$$

Example 5.4 *Of the* 106 *irreducible subgroups of* P_3*, five are maximal, and one of these*

$$H = \langle \frac{1}{\sqrt{2}} \begin{pmatrix} i+j & 0 \\ 0 & i+j \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \rangle \qquad |H| = 48,$$

fixes the orthogonol vectors

$$\begin{pmatrix} 3 \\ 1+i+j \end{pmatrix}, \qquad \begin{pmatrix} -1+i+j \\ 3 \end{pmatrix},$$

each of which is a fiducial vector for a set of 80 *lines with angles* $\{0, \frac{1}{8}, \frac{2}{8}, \ldots, \frac{7}{8}\}$.

The other four maximal reducible subgroups give sets of 10, 32, 40, 80 *lines already obtained (they have a larger symmetry group* P_3*).*

The behaviour uncovered in the Example 5.3, i.e., that a nonscalar 2×2 matrix over the quaternions can have a continuous family of right eigenvectors, is completely different from the complex case (the eigenvalues are uniquely defined and there are at most two eigenvector lines in $\mathbb{C}^2$), and hence of some interest (see [Zha97], [FWZ11]). We now give a variant which illustrates some of the mechanics of this phenomenon.

Example 5.5 *The reducible subgroup of order* 8 *given by*

$$H = \langle \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \rangle,$$

has a continuous family of eigenvectors given by

$$\begin{pmatrix} 1 \\ t \pm \sqrt{1-t^2}i \end{pmatrix}, \qquad -1 \le t \le 1.$$

The corresponding eigenvalues can be determined by verifying this, e.g.,

$$\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \begin{pmatrix} 1 \\ t \pm \sqrt{1-t^2}i \end{pmatrix} = \begin{pmatrix} tj \pm \sqrt{1-t^2}ji \\ j \end{pmatrix} = \begin{pmatrix} tj \mp \sqrt{1-t^2}k \\ j \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ j(-tj \pm \sqrt{1-t^2}k) \end{pmatrix} (tj \mp \sqrt{1-t^2}k) = \begin{pmatrix} 1 \\ t \pm \sqrt{1-t^2}i \end{pmatrix} (tj \mp \sqrt{1-t^2}k).$$

Table 3: The line systems given by the maximal reducible subgroups of the P groups

G	$\lvert H\rvert$	fiducial	angles	lines	comment
$P_1^\dagger, P_2^\dagger, P_3^\dagger$	$32, 192, 384$	$\begin{pmatrix}1\\0\end{pmatrix}$	$0, \frac{1}{2}$	10	(1.2) MUBs
$P_1^\dagger, P_2^\dagger$	$20, 120$	$\begin{pmatrix}1+\sqrt{5}\\1+i+j+k\end{pmatrix}$	$\frac{1}{5}, \frac{3}{5}$	16	(5.37)
		$\begin{pmatrix}-1+i+j+k\\1+\sqrt{5}\end{pmatrix}$	$\frac{1}{5}, \frac{3}{5}$	16	(5.38)
$P_3^\dagger$	120	$\begin{pmatrix}1+\sqrt{5}\\1+i+j+k\end{pmatrix}$	$0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	32	(5.37), (5.38)
$P_1^\dagger$	16	$\begin{pmatrix}\sqrt{2}\\1+i\end{pmatrix}$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	20	(5.39)
$P_1^\dagger$	16	$\begin{pmatrix}\sqrt{2}\\1+j\end{pmatrix}$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	20	(5.40)
$P_2^\dagger, P_3^\dagger$	$48, 96$	$\begin{pmatrix}\sqrt{2}\\1+i\end{pmatrix}$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	40	(5.39), (5.40)
P_1	8	$\begin{pmatrix}\sqrt{3}\\1+i+j\end{pmatrix}$	$0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$	40	Example 5.3
$P_2^\dagger, P_3^\dagger$	$24, 48$	$\begin{pmatrix}\sqrt{3}\\1+i+j\end{pmatrix}$	$0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$	80	Example 5.2
$P_1, P_2, P_3^\dagger$	$4, 24, 48$	$\begin{pmatrix}3\\1+i+j\end{pmatrix}$	$0, \frac{1}{8}, \frac{2}{8}, \ldots, \frac{7}{8}$	80	Example 5.4

† The stabiliser H of the fiducial vector in P_j is a maximal reducible subgroup of $G = P_j$.

Thus the second matrix can be diagonalised in multiple ways, e.g.,

$$M^{-1}\begin{pmatrix}0 & j\\ j & 0\end{pmatrix}M = \begin{pmatrix}j & 0\\ 0 & k\end{pmatrix}, \begin{pmatrix}j & 0\\ 0 & \frac{j+k}{\sqrt{2}}\end{pmatrix}, \qquad M = \begin{pmatrix}1 & 1\\ 1 & -i\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & \frac{1-i}{\sqrt{2}}\end{pmatrix}.$$

Related to the projective stabiliser group of a line, is the (pointwise) stabiliser group of a vector (or set of vectors), which is reducible. It was shown in [BST23], [Sch23] that for a quaternionic reflection group, these so called *parabolic* subgroups are reflection groups (this is a classical result of Steinberg for complex reflection groups). For the fiducial vectors of Table 3, we calculated the pointwise stabiliser group (a subgroup of the projective stabiliser). These were all trivial, except for the cases

$$\begin{pmatrix}1\\0\end{pmatrix}^{P_1} = \begin{pmatrix}1 & 0\\ 0 & \langle i\rangle\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}^{P_2}, \begin{pmatrix}1\\0\end{pmatrix}^{P_3} = \begin{pmatrix}1 & 0\\ 0 & Q_8\end{pmatrix}, \qquad \begin{pmatrix}\sqrt{2}\\1+i\end{pmatrix}^{P_3} = \langle\begin{pmatrix}0 & \frac{1-i}{\sqrt{2}}\\ \frac{1+i}{\sqrt{2}} & 0\end{pmatrix}\rangle.$$

6 Concluding remarks

The spherical 3-designs with a small number of lines that we obtained as orbits of the P groups are summarised in Table 3. The construction used is essentially that of "highly symmetric tight frames" (see [BW13], [IJM20], [Gan25]). The key idea is to go from a quaternionic representation of an abstract group to finitely many associated nice sets of lines. In the spirit of Hoggar, we offer a conjecture informed by our calculations (see Example 5.3 and Table 3).

Conjecture 1 *For every quaternionic reflection group (or every finite irreducible group of $d \times d$ matrices over $\mathbb{H}$), the maximal reducible subgroups fix a finite number of lines.*

If this holds, then for a given group, taking those lines given by the maximal reducible subgroups gives a *finite* class of "highly symmetric tight frames for $\mathbb{H}^d$".

In Table 1, we gave nice generators for the P groups (none seem to appear in the literature, see [Hog82] and [Coh91]). Here is sample `magma` code for their construction, and the fiducial w of (5.36) which gives 16 lines.

```
F:=CyclotomicField(120);
PR<t>:=PolynomialRing(F);
rt2:=Roots(t^2-2)[1][1]; rt3:=Roots(t^2-3)[1][1]; rt5:=Roots(t^2-5)[1][1];

Q<i,j,k>:=QuaternionAlgebra<F|-1,-1>;

a:=Matrix(Q,2,2,[i,0,0,1]); b:=Matrix(Q,2,2,[j,0,0,1]);
c:=1/2*Matrix(Q,2,2,[1+j,-1+j,-1+j,1+j]);
d:=1/rt2*Matrix(Q,2,2,[1,1,1,-1]);

P1:=MatrixGroup<2,Q|a,c>;
P2:=MatrixGroup<2,Q|a,b,c>;
P3:=MatrixGroup<2,Q|a,b,d>;

w:=Matrix(Q,2,1,[1+rt5,1+i+j+k]);
```

Here is code for the Hermitian transpose (and hence the inner product and angles).

```
// This gives the 1,i,j,k parts of a matrix or polynomial over Q
HtoRparts := function(q);
  x1:=1/4*(q-i*q*i-j*q*j-k*q*k); x2:=1/4/i*(q-i*q*i+j*q*j+k*q*k);
  x3:=1/4/j*(q+i*q*i-j*q*j+k*q*k); x4:=1/4/k*(q+i*q*i+j*q*j-k*q*k);
  return [x1,x2,x3,x4];
end function;

HermTranspose := function(A);
  c:=HtoRparts(A);
  B:=c[1]-c[2]*i-c[3]*j-c[4]*k;
  return Transpose(B);
end function;
```

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