Equations for the overlaps of a SIC

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Abstract

We give a holomorphic quartic polynomial in the overlap variables whose zeros on the torus are precisely the Weyl-Heisenberg SICs (symmetric informationally complete positive operator valued measures). By way of comparison, all the other known systems of equations that determine a Weyl-Heisenberg SIC involve variables and their complex conjugates. We also give a related interesting result about the powers of the projective Fourier transform of the group $G = \mathbb{Z}_d \times \mathbb{Z}_d$.

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1 Introduction

Throughout fix the integer $d \geq 2$, and let ω be the primitive d-th root of unity $\omega := e^{\frac{2\pi}{d} i}$. We think of vectors in $\mathbb{C}^d$ as periodic signals on the group $\mathbb{Z}_d$, and hence index vectors and matrices by elements of $\mathbb{Z}_d$. A set of d^2 unit vectors (v_j) in $\mathbb{C}^d$ (or the lines that they determine) is said to be **equiangular** if

$$|\langle v_j, v_k \rangle|^2 = \frac{1}{d+1}, \qquad j \neq k. \tag{1.1}$$

In quantum information theory, the corresponding rank one orthogonal projections $(v_j v_j^*)$ are said to be a **symmetric informationally complete positive operator valued measure**, or a **SIC** for short. The existence of a SIC for every dimension d is known as *Zauner's conjecture* (from his 1999 thesis, see [Zau10]), or as the *SIC problem*.

There are high precision numerical constructions of SICs [RBKSC04], [SG10], [Sco17], and exact SICs in various dimensions [ACFW18], [GS17]. In all of these constructions, the SIC is a **Weyl-Heisenberg SIC**, i.e., is the orbit $(\rho(g)v)_{g\in G}$ of a *fiducial vector* v under the unitary irreducible projective representation $\rho : G \to \mathcal{U}(\mathbb{C}^{\mathbb{Z}_d})$ of $G = \mathbb{Z}_d \times \mathbb{Z}_d$ with Schur multiplier α given by

$$\rho_{jk} = \rho((j,k)) = S^j \Omega^k, \qquad \alpha((j_1,j_2),(k_1,k_2)) = \omega^{j_2 k_1}, \tag{1.2}$$

where S is the cyclic shift matrix $S_{jk} := \delta_{j,k+1}$ and Ω is the diagonal (modulation) matrix $\Omega_{jk} := \omega^j \delta_{jk}$. In this case, the equiangularity condition (1.1) becomes

$$|\langle S^j \Omega^k v, v \rangle|^2 = \frac{1}{d+1}, \qquad (j,k) \neq (0,0). \tag{1.3}$$

In this paper, we consider equations in the variables

$$c_{jk} = \langle S^j \Omega^k v, v \rangle = \operatorname{trace}(vv^* S^j \Omega^k), \qquad (j,k) \in \mathbb{Z}_d \times \mathbb{Z}_d, \tag{1.4}$$

which determine a (Weyl-Heisenberg) SIC. These variables (or scalar multiples of them) are called the **overlaps** of the SIC. They depend only on the fiducial projector $P = vv^*$. The original attempts to find numerical and exact SIC fiducials (using Groebner basis methods) involved polynomial equations in the variables $v_0, \ldots, v_{d-1}$ and $\overline{v_0}, \ldots, \overline{v_{d-1}}$, such as the equiangularity condition (1.3), the equations (see [BW07], [Kha08], [ADF14])

$$\sum_{r\in\mathbb{Z}_d} v_r \overline{v}_{r+s} \overline{v}_{r+t} v_{r+s+t} = \begin{cases} 0, & s,t \neq 0; \\ \frac{1}{d+1}, & s \neq 0, t = 0, \quad s = 0, t \neq 0; \\ \frac{2}{d+1}, & (s,t) = (0,0), \end{cases} \tag{1.5}$$

and the variational characterisation (used for finding numerical SICs)

$$\frac{1}{d^2} \sum_{(j,k)\in\mathbb{Z}_d^2} |\langle S^j \Omega^k v, v \rangle|^4 = \frac{2}{d(d+1)} \|v\|^4, \qquad \|v\|^2 = 1. \tag{1.6}$$

More recent exact constructions of SICs [ACFW18] have been in the overlap variables c_{jk} (utilising a natural Galois action on them). Clearly the c_{jk} giving a SIC fiducial projector vv^* via (1.4) must satisfy

$$c_{00} = \|v\|^2 = 1, \qquad |c_{jk}|^2 = |\langle S^j\Omega^k v, v\rangle|^2 = \frac{1}{d+1}, \quad (j,k) \neq (0,0), \tag{1.7}$$

and also, by the rule $\Omega^k S^j = \omega^{jk} S^j \Omega^k$,

$$c_{jk} = \overline{\langle v, S^j\Omega^k v\rangle} = \overline{\langle \Omega^{-k} S^{-j} v, v\rangle} = \overline{\langle \omega^{jk} S^{-j}\Omega^{-k} v, v\rangle} = \omega^{-jk}\overline{c_{-j,-k}}. \tag{1.8}$$

These conditions on the overlap variables c_{jk} are not enough to guarantee that they come from a fiducial projector vv^* (and hence prove Zauner's conjecture).

In Section 2, we define a linear operator T, which is an example of the projective Fourier transform, which allows us to reconstruct the fiducial projector as $vv^* = Tc$ from a suitable $c = (c_{jk})$. We prove that in addition to (1.7) and (1.8), the simple condition

$$\operatorname{trace}((Tc)^4) = 1$$

ensures that a c gives a SIC fiducial (Theorem 2.1). We then give some examples, and describe the action of the Clifford group on the SIC fiducials give by overlaps c.

In Section 3, we give some interesting properties of T, i.e., the projective Fourier transform of $G = \mathbb{Z}_d \times \mathbb{Z}_d$. In particular, we show that $(\sqrt{d}T)^{6d} = (-1)^{\frac{1}{2}d(d-1)} I$, and a variant has order $4d$. To our knowledge, this is only the second example of a Fourier transform of finite order, after the (discrete) Fourier transform for a finite abelian group $G = \mathbb{Z}_d$ (which satisfies $F^4 = I$).

In Section 4, we give another system of equations in the overlaps c that determine a SIC. These involve the symbol (z–transform) of the rows of c. The symbols for c giving a SIC turn out to have interesting Riesz-type factorisation properties. We use these to describe the (sporadic) SICs for $d = 3$, which are parametrised by a hypocycloid.

2 The reconstruction operator

Since the ρ of (1.2) is a unitary irreducible projective representation of dimension d, it follows that $(\rho(g))_{g\in G}$ is a tight frame (called a nice error frame with index group G [CW17]) for the $d \times d$ matrices with the Frobenius inner product

$$\langle A, B\rangle := \operatorname{trace}(AB^*) = \sum_{j,k} a_{jk}\overline{b_{jk}},$$

i.e.,

$$A = \frac{d}{|G|}\sum_{g\in G}\langle A, \rho(g)\rangle\rho(g), \qquad \forall A \in \mathbb{C}^{d\times d}. \tag{2.9}$$

In this particular case, $(\rho(g))_{g\in G} = (S^j\Omega^k)$ is an orthogonal basis. Taking $A = vv^*$ above gives the following formula for reconstruction from the overlaps $c_{jk} = \langle S^j\Omega^k v, v\rangle$

$$vv^* = \frac{d}{d^2}\sum_{j,k}\langle vv^*, S^{-j}\Omega^{-k}\rangle S^{-j}\Omega^{-k} = \frac{1}{d}\sum_{j,k}\omega^{jk}c_{jk}S^{-j}\Omega^{-k} = \frac{1}{d}\sum_{j,k}c_{jk}(S^j\Omega^k)^*,$$

since $\omega^{jk}S^{-j}\Omega^{-k}=(S^j\Omega^k)^*$, and $(S^{-j}\Omega^{-k})^*=\omega^{jk}S^j\Omega^k$ gives

$$\langle vv^*,S^{-j}\Omega^{-k}\rangle=\operatorname{trace}(vv^*(S^{-j}\Omega^{-k})^*)=\operatorname{trace}(v^*\omega^{jk}S^j\Omega^k v)=\omega^{jk}\langle S^j\Omega^k v,v\rangle=\omega^{jk}c_{jk}.$$

Motivated by this, we define a linear map $T:\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}\to\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ by

$$Tc:=\frac{1}{d}\sum_{j,k}c_{jk}(S^j\Omega^k)^*=\frac{1}{d}\sum_{j,k}\omega^{jk}c_{jk}S^{-j}\Omega^{-k}. \tag{2.10}$$

This can be viewed as the α-Fourier transform of [Wal20] (for a Schur multiplier α) which is a map $F_\alpha:\mathbb{C}^G\to\oplus_\rho\mathbb{C}^{d_\rho\times d_\rho}$, where ρ counts over the irreducible projective representations of G with multiplier α (and dimension d_ρ). Here $G=\mathbb{Z}_d\times\mathbb{Z}_d$ has just one such representation, the ρ of (1.2), and F_α of $\nu=c\in\mathbb{C}^G=\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ at the unitary representation ρ is

$$(F_\alpha\nu)_\rho=\sum_{g\in G}\nu(g)\rho(g)^*=\sum_{j,k}c_{jk}(S^j\Omega^k)^*=d(Tc).$$

Thus T is the projective Fourier transform for the group $G=\mathbb{Z}_d\times\mathbb{Z}_d$. For this particular group, we can view the image of a vector in $\mathbb{C}^G$ as being in $\mathbb{C}^G=\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$, and as a result it is natural to consider powers of the Fourier transform. The only other case that we know of where this can be done is for the ordinary representations of a finite abelian group (where the representations give the character group $\hat{G}$, which can be identified with G). In this case the (discrete) Fourier transform has order 4.

We now use T to characterise those vectors (matrices) $c\in\mathbb{C}^G=\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ which give a fiducial projector $vv^*=Tc$.

Lemma 2.1 *Let T be given by* (2.10). *Suppose that $c=(c_{jk})\in\mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ satisfies*

(i) $c_{jk}=\omega^{-jk}\overline{c_{-j,-k}}$

(ii) $c_{00}=1$

(iii) $|c_{jk}|^2=\frac{1}{d+1}$, $(j,k)\neq(0,0)$.

Then Tc is Hermitian, and its eigenvalues $\lambda_1,\dots,\lambda_d$ satisfy

$$\sum_j\lambda_j=1,\qquad\sum_j\lambda_j^2=1,\qquad\sum_{j\neq k}\lambda_j\lambda_k=0,$$

i.e., its characteristic polynomial has the form

$$p_{Tc}(\lambda)=\lambda^d-\lambda^{d-1}+0\lambda^{d-2}+a_{d-3}\lambda^{d-3}+\cdots+a_1\lambda+a_0.$$

Proof: Firstly, observe (i) implies that Tc is Hermitian, since

$$(Tc)^*=\frac{1}{d}\sum_{j,k}\omega^{jk}c_{-j,-k}S^j\Omega^k=\frac{1}{d}\sum_{j,k}\omega^{(-j)(-k)}c_{j,k}S^{-j}\Omega^{-k}=Tc.$$

Since $\mathrm{trace}(S^j\Omega^k) = 0$, $(j,k) \neq (0,0)$, we calculate using (ii) that

$$\sum_j \lambda_j = \mathrm{trace}(Tc) = \frac{1}{d}\sum_{j,k}\omega^{jk}c_{j,k}\,\mathrm{trace}(S^{-j}\Omega^{-k}) = \frac{1}{d}\omega^0 c_{00}\,\mathrm{trace}(I) = c_{00} = 1.$$

The so called 2–trace $\sum_{j\neq k}\lambda_j\lambda_k$ of Tc is equal to $\{(\mathrm{trace}(Tc))^2 - \mathrm{trace}((Tc)^2)\}/2$. Since Tc is Hermitian, $\mathrm{trace}((Tc)^2) = \langle Tc, (Tc)^*\rangle = \langle Tc, Tc\rangle$, and by the orthogonality of the $\rho_{jk} = S^j\Omega^k$, we calculate

$$\langle Tc, Tc\rangle = \frac{1}{d^2}\sum_{j,k}|c_{j,k}|^2\langle \rho_{jk}^*, \rho_{jk}^*\rangle = \frac{1}{d}\sum_{j,k}|c_{jk}|^2.$$

Now by (ii) and (iii),

$$\mathrm{trace}((Tc)^2) = \langle Tc, Tc\rangle = \frac{1}{d}\sum_{j,k}|c_{jk}|^2 = \frac{1}{d}\Big(1 + (d^2-1)\frac{1}{d+1}\Big) = 1,$$

and so $\sum_{j\neq k}\lambda_j\lambda_k = \{(\mathrm{trace}(Tc))^2 - \mathrm{trace}((Tc)^2)\}/2 = (1-1)/2 = 0$. □

Theorem 2.1 *Let T be given by (2.10). Then a matrix $c = (c_{jk}) \in \mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ determines a fiducial projector for a Weyl-Heisenberg SIC by $vv^* = Tc$ if and only if*

(i) $c_{jk} = \omega^{-jk}\overline{c_{-j,-k}}$

(ii) $c_{00} = 1$

(iii) $|c_{jk}|^2 = \dfrac{1}{d+1}$, $(j,k) \neq (0,0)$

(iv) $\mathrm{trace}((Tc)^4) = 1$

Moreover this fiducial satisfies $\langle S^j\Omega^k v, v\rangle = c_{jk}$, and if $v_0 \neq 0$, then v is given by

$$v = \frac{1}{d\overline{v_0}}\overline{c}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.$$

Proof: By Lemma 2.1, the eigenvalues of the Hermitian matrix Tc satisfy

$$\sum_j \lambda_j = 1, \qquad \sum_j \lambda_j^2 = 1, \qquad \sum_{j\neq k}\lambda_j\lambda_k = 0,$$

so that $0 \leq \lambda_j^2 \leq 1$. Thus $\lambda_j^4 \leq \lambda_j^2$, with equality if and only if $\lambda_j^2 \in \{0,1\}$. But

$$\sum_j \lambda_j^4 = \mathrm{trace}((Tc)^4) = 1 = \mathrm{trace}((Tc)^2) = \sum_j \lambda_j^2,$$

so that $\lambda_j^2 \in \{0,1\}$, $\forall j$, and we must have $\lambda_j = 1$ for some j, and $\lambda_j = 0$ for all others, i.e., Tc is rank one, say

$$Tc = \frac{1}{d}\sum_{j,k} c_{jk}\rho_{jk}^* = vv^*, \qquad v \in \mathbb{C}^{\mathbb{Z}_d}.$$

Since $\{\rho_{jk}\}$ is orthogonal, taking the inner product of the above with $\rho_{jk} = S^j\Omega^k$ gives

$$c_{jk} = \frac{1}{d}c_{jk}\langle\rho_{jk}^*, \rho_{jk}^*\rangle = \langle vv^*, \rho_{jk}^*\rangle = \text{trace}(vv^*S^j\Omega^) = \text{trace}(v^*S^j\Omega^k v) = \langle S^j\Omega^k v, v\rangle.$$

Finally, with e_j the standard basis vectors, we calculate

$$\begin{aligned} j\text{–th entry of } \ \overline{c}\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix} &= \sum_k \overline{c_{jk}} = \sum_k \langle v, S^j\Omega^k v\rangle = \langle v, S^j\Big(\sum_k \Omega^k\Big)v\rangle \\ &= \langle v, S^j \begin{pmatrix} d & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix} v\rangle = \langle v, S^j d v_0 e_0\rangle = d\overline{v_0}\langle v, e_j\rangle = d\overline{v_0}v_j. \end{aligned}$$

□

From the proof, we see that (iv) can be replaced by various equivalent conditions, e.g.,

(iv)′ *The characteristic polynomial of* Tc *has the form* $P_{Tc}(\lambda) = \lambda^d - \lambda^{d-1}$

(iv)″ $\text{trace}((Tc)^j) = 1$, $j = 1, 2, \ldots$

since given (i), (ii), (ii),

(iv)″ $\Longrightarrow$ (iv) $\Longrightarrow$ Tc has eigenvalues $1, 0, \ldots, 0$ $\iff$ (iv)′ $\Longrightarrow$ (iv)″.

By condition (ii), we may set $c_{00} = 1$, to obtain the following characterisation.

Corollary 2.1 *The overlaps of a Weyl-Heisenberg SIC are precisely the zeros of the polynomial* $\text{trace}((Tc)^4) = 1$ *on the torus*

$$|c_{jk}| = \frac{1}{\sqrt{d+1}}, \qquad (j,k) \neq (0,0).$$

The condition (i) allows further variables c_{jk} to be eliminated. When d is odd, half of the (d^2-1) variables c_{jk}, $(j,k) \neq (0,0)$, can be eliminated. For d even, half of the $d^2 - 4$ variables c_{jk}, $(j,k) \notin \{0, \frac{d}{2}\}^2$, can be eliminated, and

$$c_{\frac{d}{2},0} = \overline{c_{\frac{d}{2},0}}, \qquad c_{0,\frac{d}{2}} = \overline{c_{0,\frac{d}{2}}}, \qquad c_{\frac{d}{2},\frac{d}{2}} = (-1)^{\frac{d}{2}}\overline{c_{\frac{d}{2},\frac{d}{2}}}, \tag{2.11}$$

so that $c_{0,\frac{d}{2}}, c_{0,\frac{d}{2}} \in \mathbb{R}$, and $c_{\frac{d}{2},\frac{d}{2}}$ is in $\mathbb{R}$ for $\frac{d}{2}$ even, and is in $i\mathbb{R}$ for $\frac{d}{2}$ odd.

Example 2.1 *For $d=2$, the conditions* (2.11) *of* (i) *give $c_{01}\in\mathbb{R}$, $c_{10}\in\mathbb{R}$, $c_{11}\in i\mathbb{R}$. Hence imposing the conditions* (ii) *and* (iii)*, we have eight possibilities*

$$c_{00}=1,\qquad c_{01}=\pm\frac{1}{\sqrt{3}},\qquad c_{10}=\pm\frac{1}{\sqrt{3}},\qquad c_{11}=\pm i\frac{1}{\sqrt{3}}. \tag{2.12}$$

Taking the '+' choice above gives

$$Tc=T\begin{pmatrix}1 & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}i\end{pmatrix}=\frac{1}{2}\big(I+\frac{1}{\sqrt{3}}(S+\Omega)+\frac{i}{\sqrt{3}}S\Omega\big)=\frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}+1 & 1-i\\ 1+i & \sqrt{3}-1\end{pmatrix},$$

which satisfies $\operatorname{trace}((Tc)^4)=\operatorname{trace}(Tc)=1$, *and so gives a Weyl-Heisenberg SIC*

$$v=\frac{1}{\sqrt{2}\sqrt{3+\sqrt{3}}}\begin{pmatrix}\sqrt{3}+1\\ 1+i\end{pmatrix}.$$

In fact all eight choices give SICs which are equivalent, as we now explain.

The group generated by S and Ω is called the **Heisenberg group**, and its normaliser in the unitary matrices is the **Clifford group**. Indeed, if a $a\in \mathrm{C}(d)$, then

$$a\rho_{jk}a^{-1}=z_a(j,k)\rho_{\psi_a(j,k)},\qquad \forall (j,k)\in\mathbb{Z}_d^2,$$

where ψ_a is matrix multiplication by an element of $SL_2(\mathbb{Z}_d)$. The Clifford group $\mathrm{C}(d)$ maps SIC fiducials to SIC fiducials, via the action

$$a\cdot(vv^*):=(av)(av)^*=a(vv^*)a^{-1},\qquad a\in \mathrm{C}(d).$$

The induced action on the overlaps of the fiducial is given by

$$\begin{aligned}(a\cdot c)_{jk}&=\operatorname{trace}(a(vv^*)a^{-1}S^j\Omega^k)=\langle a^{-1}S^j\Omega^k av,v\rangle=\langle z_{a^{-1}}(j,k)\rho_{\psi_{a^{-1}}(j,k)}v,v\rangle\\ &=z_{a^{-1}}(j,k)c_{\psi_{a^{-1}}(j,k)}.\end{aligned}$$

In [BW19], it is shown that the Clifford group is generated by the scalar matrices, S, Ω, the Fourier transform F and the Zauner matrix Z, where

$$F_{jk}:=\frac{1}{\sqrt{d}}\omega^{jk},\qquad Z_{jk}:=\zeta^{d-1}\mu^{j(j+d)},\quad \mu:=e^{\frac{2\pi}{2d}i},\ \zeta:=e^{\frac{2\pi}{24}i}.$$

For these (see [Wal18])

$$(S^a\Omega^b\cdot c)_{jk}=\omega^{ak-bj}c_{jk},\qquad (F\cdot c)_{jk}=\omega^{-jk}c_{k,-j},\qquad (Z\cdot c)_{jk}=\mu^{j(j+d-2k)}c_{k-j,-j}.$$

When a (Weyl-Heisenberg) SIC fiducial vv^* is known, there is always appears to be one which is given by an eigenvector v of Z (indeed these are often searched for directly). Correspondingly, the overlaps satisfy $Z\cdot c=c$, i.e., the equations

$$\mu^{j(j+d-2k)}c_{k-j,-j}=c_{jk},$$

which allows a further reduction of the variables c_{jk}.

3 Properties of the projective Fourier transform

Here we consider some properties of $T : \mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d} \to \mathbb{C}^{\mathbb{Z}_d\times\mathbb{Z}_d}$ given by (2.10), i.e., the projective Fourier transform of $G=\mathbb{Z}_d\times\mathbb{Z}_d$. It follows from the Plancherel formula for projective representations [Wal20], or (2.9) that $\sqrt{d}T$ is unitary. Indeed, (2.9) can be written as $I=T\Lambda$, where $\Lambda:\mathbb{C}^G\to\mathbb{C}^G : A\mapsto (\langle A,\rho(g)\rangle)_{g\in G}$ satisfies

$$\langle A,A\rangle = \frac{1}{d}\sum_{g\in G}|\langle A,\rho(g)\rangle|^2 = \frac{1}{d}\langle \Lambda A,\Lambda A\rangle,$$

so that $\frac{1}{\sqrt{d}}\Lambda$ is unitary, and hence $\sqrt{d}T$ is unitary.

We now show that $\sqrt{d}T$ has finite order ($6d$ or $12d$), i.e., the projective Fourier transform for ρ of (1.2) has finite order (Theorem 3.1). To do this, we need a technical lemma (Lemma 3.1), based on the Zauner matrix Z (of order 3), which can be factored

$$Z=\zeta^{d-1}RF, \quad \zeta := e^{\frac{2\pi i}{24}}, \qquad (R)_{jk}=\mu^{j(j+d)}\delta_{jk}, \quad \mu := e^{\frac{2\pi i}{2d}},$$

where F is the Fourier matrix, and R is diagonal. The *strong* form of Zauner's conjecture is that there is a SIC fiducial which is an eigenvector of Z, for every dimension d.

Lemma 3.1 *For any d, we have that*

$$(R^2F)^2 = \zeta^{-6(d-1)}(RF)R^{-2}(RF)^{-1}, \qquad \zeta=e^{\frac{2\pi i}{24}},$$

and, in particular

$$(R^2F)^{2d} = (-1)^{\frac{1}{2}d(d-1)}I.$$

Proof: Write $Z=cRF$, $c=\zeta^{d-1}$. Since $Z^3=I$ and $F^4=I$, we have

$$\begin{aligned}R^2F &= R(RF)^2(RF)^{-1} = R(\overline{c}Z)^2(RF)^{-1} = \overline{c}^2RZ^{-1}(RF)^{-1} = \overline{c}^3RF^{-1}R^{-1}(RF)^{-1}\\ &= \overline{c}^3(RF)(F^2R^{-1})(RF)^{-1}.\end{aligned}$$

Since the permutation matrix F^2 commutes with R (or any power of R), we have

$$(R^2F)^2 = \overline{c}^6(RF)(F^2R^{-1})^2(RF)^{-1} = \overline{c}^6(RF)R^{-2}(RF)^{-1},$$

where $\overline{c}^6=\zeta^{-6(d-1)}$. Since $R^{2d}=I$, we obtain

$$(R^2F)^{2d} = \overline{c}^{6d}(RF)R^{-2d}(RF)^{-1} = \overline{c}^{6d}I, \qquad \overline{c}^{6d}=\zeta^{-6d(d-1)}=(-1)^{\frac{1}{2}d(d-1)},$$

which completes the proof. □

Theorem 3.1 *The reconstruction operator T of (2.10) has finite order, i.e.,*

$$(\sqrt{d}T)^{6d} = (-1)^{\frac{1}{2}d(d-1)} I.$$

Proof: We consider T with respect to the standard basis $E_{jk} = e_j e_k^*$ for matrices, ordered so that the coordinates of c have the block structure $[c] = (c_0, \ldots, c_{d-1})^T$, where c_j is the j-th column of the matrix c (this is the order of matlab's `reshape(c,d^2,1)`).

The (j,k)-block A_{jk} of the (block) matrix representation $[\sqrt{d}T]$ of $\sqrt{d}T$ is given by

$$\begin{aligned} A_{jk}v &= j\text{-th column of } \sqrt{d}T([0\ldots 0, v, 0\ldots 0]) \qquad (v \text{ is the } k\text{-th column}) \\ &= \frac{1}{\sqrt{d}} \sum_{a,b} [0\ldots 0, v, 0\ldots 0]_{ab} (S^a \Omega^b)^* e_j = \frac{1}{\sqrt{d}} \sum_a v_a \Omega^{-k} S^{-a} e_j \\ &= \frac{1}{\sqrt{d}} \sum_a (\Omega^{-k} P_{-1} S^{-j} e_a) v_a = \frac{1}{\sqrt{d}} (\Omega^{-k} P_{-1} S^{-j}) v, \end{aligned}$$

so that

$$A_{jk} = \frac{1}{\sqrt{d}} \Omega^{-k} P_{-1} S^{-j}, \qquad (A_{jk})_{ab} = \frac{1}{\sqrt{d}} \omega^{-ak} \delta_{a,j-b}.$$

The (j,k)-block B_{jk} of $[\sqrt{d}T]^2$ is given by

$$\begin{aligned} (B_{jk})_{ab} &= \Big(\sum_r A_{jr} A_{rk}\Big)_{ab} = \sum_r \sum_t (A_{jr})_{at} (A_{rk})_{tb} = \frac{1}{d} \sum_r \sum_t \omega^{-ar} \delta_{a,j-t} \omega^{-tk} \delta_{t,r-b} \\ &= \frac{1}{d} \omega^{-a(j-a+b)} \omega^{-(j-a)k} = \frac{1}{d} \omega^{a^2-aj+ak-jk-ab}. \end{aligned}$$

The (j,k)-block C_{jk} of $[\sqrt{d}T]^3$ is given by

$$\begin{aligned} (C_{jk})_{ab} &= \Big(\sum_r B_{jr} A_{rk}\Big)_{ab} = \sum_r \sum_t (B_{jr})_{at} (A_{rk})_{tb} = \sum_r \sum_t (B_{jr})_{at} (A_{rk})_{tb} \\ &= \frac{1}{d\sqrt{d}} \sum_r \sum_t \omega^{a^2-aj+ar-jr-at} \omega^{-tk} \delta_{t,r-b} = \frac{1}{d\sqrt{d}} \sum_r \omega^{a^2-aj+ar-jr-a(r-b)} \omega^{-(r-b)k} \\ &= \frac{1}{d\sqrt{d}} \omega^{a^2-aj+ab+bk} \sum_r \omega^{-r(j+k)} = \frac{1}{\sqrt{d}} \omega^{a^2-aj+ab+bk} \delta_{j,-k} = (R^2 \Omega^{-j} F \Omega^k)_{ab} \delta_{j,-k}, \end{aligned}$$

so that

$$C_{jk} = \begin{cases} 0, & k \neq -j; \\ R^2 \Omega^{-j} F \Omega^{-j}, & k = -j. \end{cases}$$

It therefore follows, that $[\sqrt{d}T]^6$ is block diagonal, with diagonal blocks

$$Q_{jj} = C_{j,-j} C_{-j,j} = (R^2 \Omega^{-j} F \Omega^{-j})(R^2 \Omega^j F \Omega^j) = \Omega^{-j} (R^2 F)^2 \Omega^j.$$

Thus $[\sqrt{d}T]^{6d}$ is block diagonal, and, by Lemma 3.1, its diagonal blocks simplify to

$$\Omega^{-j} (R^2 F)^{2d} \Omega^j = \Omega^{-j} (-1)^{\frac{1}{2}d(d-1)} I \Omega^j = (-1)^{\frac{1}{2}d(d-1)} I,$$

i.e., $[(\sqrt{d}T)^{6d}] = [(-1)^{\frac{1}{2}d(d-1)} I]$. □

Since the projective representation (1.2) of $\mathbb{Z}_d \times \mathbb{Z}_d$ is *not* an ordinary representation, there is no canonical presentation of the projective Fourier transform at ρ, as with the Fourier transform for $\mathbb{Z}_d$, which gives F (of order 4), by taking $\alpha = 1$. Indeed, one could take $\tilde{\rho}((j,k)) = b_{jk} S^j \Omega^k$, for any unit scalars b_{jk}, with a corresponding $\tilde{\alpha}$-transform (reconstruction operator)

$$\tilde{T}c := \frac{1}{d} \sum_{j,k} c_{jk} (b_{jk} S^j \Omega^k)^* = \frac{1}{d} \sum_{j,k} c_{jk} \overline{b_{jk}} \omega^{jk} S^{-j} \Omega^{-k}.$$

For a general choice for b_{jk}, $\sqrt{d}\tilde{T}$ is again unitary, but not of finite order. During our investigation, we came across various choices giving operators of finite order, in particular

$$Lc := \frac{1}{d} \sum_{j,k} c_{jk} (\omega^{jk} S^j \Omega^k)^* = \frac{1}{d} \sum_{j,k} c_{jk} S^{-j} \Omega^{-k}. \tag{3.13}$$

It can be shown that L has the compact form

$$Lc = F^*(F \circ (F^* c F^*)), \tag{3.14}$$

where $\circ$ is the Hadamard product. From this, we obtain the following.

Theorem 3.2 *The operator L of (3.13) satisfies*

$$(\sqrt{d}L)^4 c = (F^* R^{-2} F)c \qquad \textit{(matrix multiplication)},$$

and hence, since $R^{2d} = I$, we have

$$(\sqrt{d}L)^{4d} = I.$$

Proof: We first verify the compact form (3.14),

$$(Lc)_{ab} = \frac{1}{d} \sum_{j,k} c_{jk} (S^{-j} \Omega^{-k})_{ab} = \frac{1}{d} \sum_{j,k} c_{jk} \omega^{-kb} \delta_{a,b-j} = \frac{1}{d} \sum_{k} c_{b-a,k} \omega^{-kb},$$

$$\begin{aligned}\big(F^*(F \circ (F^* c F^*))\big)_{ab} &= \sum_{r,t,k} (F^*)_{ar} (F)_{rb} (F^*)_{rt} c_{tk} (F^*)_{kb} \\ &= \frac{1}{d^2} \sum_{r,t,k} \omega^{-ar+rb-rt} c_{tk} \omega^{-kb} = \frac{1}{d} \sum_{t,k} \delta_{t,b-a} c_{tk} \omega^{-kb} = \frac{1}{d} \sum_{k} c_{b-a,k} \omega^{-kb}.\end{aligned}$$

Define the operation $\tilde{A} = P_{1-} A P_{-1}$ of conjugation by the permutation matrix $P_{-1} = F^2$ of order 2. This distributes over matrix multiplication, the Hadamard product, leaving F (and its powers) unchanged, so that $Lc = F^*(F \circ (F\tilde{c}F))$, and

$$\begin{aligned}L^2 c &= F^*(F \circ (F[F^*(F \circ (F\tilde{c}F))]\tilde{}F)) = F^*(F \circ (F[F^*(F \circ (FcF))]F)) \\ &= F^*([F \circ ([F \circ (FcF)]F)]F)F^* = F^* M^2 (FcF) F^*,\end{aligned}$$

where

$$Mc := (F\circ c)F, \qquad (Mc)_{jk} = \sum_t (F\circ c)_{jt}F_{tk} = \frac{1}{d}\sum_t \omega^{jt}c_{jt}\omega^{tk} = \frac{1}{d}\sum_t \omega^{(j+k)t}c_{jt},$$

$$(M^2c)_{jk} = \frac{1}{d}\sum_t \omega^{(j+k)t}(Mc)_{jt} = \frac{1}{d}\sum_t \omega^{(j+k)t}\frac{1}{d}\sum_r \omega^{(j+t)r}c_{jr} = \frac{1}{d}\omega^{-j(j+k)}c_{j,-(j+k)},$$

$$(M^4c)_{jk} = \frac{1}{d^2}\omega^{-j(j+k)}\omega^{-j(-k)}c_{jk} = \frac{1}{d^2}\mu^{-2j(j+d)}c_{jk} = (\frac{1}{d^2}R^{-2}c)_{jk}.$$

Thus, $M^4c = \frac{1}{d^2}R^{-2}c$, which gives

$$(\sqrt{d}L)^4c = d^2F^*M^2(F[F^*M^2(FcF)F^*]F)F^* = d^2F^*M^4(FcF)F^*$$
$$= F^*(R^{-2}FcF)F^* = F^*R^{-2}Fc,$$

and $(\sqrt{d}L)^{4d}c = (F^*R^{-2}F)^dc = F^*R^{-2d}Fc = c$. □

4 Equivalent equations for Heisenberg frames

In this section, we give another condition that ensures Tc has rank one, which leads to a set of equations for c which express in terms of polynomials $p_j(z)$ which are z–transforms of the rows of c. These polynomials $p_j(z)$ have interesting Riesz–type factorisation properties, which we use to find a solution for $d=4$.

We use the following condition which ensures that a matrix $A\in\mathbb{C}^{d\times d}$ has rank one.

Lemma 4.1 *$A = vv^*$ for some $v\in\mathbb{C}^d$ with $v_m\neq 0$ if and only if $a_{mm}>0$ and*

$$A = \frac{1}{a_{mm}}\begin{bmatrix} a_{0m}\\ a_{1m}\\ a_{2m}\\ \vdots\end{bmatrix}\begin{bmatrix} a_{0m}\\ a_{1m}\\ a_{2m}\\ \vdots\end{bmatrix}^*. \tag{4.15}$$

Proof: First suppose that $A=vv^*$ for such a v. Then

$$\overline{v_m}v = Ae_m = \begin{bmatrix} a_{0m}\\ a_{1m}\\ a_{2m}\\ \vdots\end{bmatrix}, \qquad |v_m|^2 = (\overline{v_m}v)_m = a_{mm},$$

so that $a_{mm}>0$, and (4.15) holds since $(\overline{v_m}v)(\overline{v_m}v)^* = |v_m|^2(vv^*)$.

Conversely, suppose that (4.15) holds with $a_{mm}>0$, then clearly $A=vv^*$ for

$$v := \frac{1}{\sqrt{a_{mm}}}\begin{bmatrix} a_{0m}\\ a_{1m}\\ a_{2m}\\ \vdots\end{bmatrix}, \qquad v_m = \sqrt{a_{mm}}.$$

□

In particular, $Tc = vv^*$ for some $v \in \mathbb{C}^d$ with $v_m \neq 0$ if and only if $(Tc)_{mm} > 0$ and

$$(Tc)_{jk} = \frac{(Tc)_{jm}\overline{(Tc)_{km}}}{(Tc)_{mm}}. \tag{4.16}$$

We now express (4.16) in terms of the following z–transform.

Definition 4.1 *For $j = 0, 1, \ldots, d-1$, the j–th* **symbol** *of c is defined to be the polynomial*

$$p_j(z) := \sum_r c_{-j,r}(\omega^j z)^{-r}.$$

This is the z–transform of the j–th row of the matrix $(\omega^{jk} c_{-j,-k})$, since

$$\sum_k \omega^{jk} c_{-j,-k} z^k = \sum_r \omega^{-jr} c_{-j,r} z^{-r} = \sum_r c_{-j,r}(\omega^j z)^{-r}.$$

We think of $p_j(z)$ as being defined only on $z^d = 1$, since each polynomial of degree d is uniquely determined by its values at the d–th roots of unity. Clearly, we can recover c from the d^2 values $p_j(\omega^k)$, $j, k = 0, \ldots, d-1$. Using (**??**), we calculate

$$p_{j-k}(\omega^k) = \sum_r c_{k-j,r}(\omega^{j-k}\omega^k)^{-r} = \sum_r c_{k-j,r}(\omega^j)^{-r} = d(Tc)_{jk}.$$

Hence (4.16) can be expressed as follows.

Theorem 4.1 *$Tc = vv^*$ for $v \in \mathbb{C}^d$ with $v_m \neq 0$ if and only if the symbols of c satisfy*

$$p_0(\omega^m) > 0, \qquad p_{j-k}(\omega^k) = \frac{p_{j-m}(\omega^m)\overline{p_{k-m}(\omega^m)}}{p_0(\omega^m)}, \quad j, k = 0, \ldots, d-1. \tag{4.17}$$

The symbols corresponding to a solution have interesting Riesz–type factorisation properties, which, for simplicity, we illustrate when $m = 0$.

Corollary 4.1 *$Tc = vv^*$ for $v \in \mathbb{C}^d$ with $v_0 \neq 0$ if and only if the symbols of c satisfy*

$$p_0(1) > 0, \qquad p_{j-k}(\omega^k) = \frac{p_j(1)\overline{p_k(1)}}{p_0(1)}, \quad j, k = 0, \ldots, d-1. \tag{4.18}$$

Moreover, these have the factorisations

$$|p_j(z)|^2 = p_0(z)p_0(\omega^j z), \qquad j = 0, \ldots, d-1, \tag{4.19}$$

and the following invariant

$$\prod_k p_j(\omega^k) = \prod_k p_0(\omega^k), \qquad j = 0, \ldots, d-1. \tag{4.20}$$

Proof: For (4.18) take $m = 0$ in Theorem 4.1. Now re-index to get

$$p_j(\omega^k) = \frac{p_{j+k}(1)\overline{p_k(1)}}{p_0(1)},$$

and take the modulus squared of both sides

$$|p_j(\omega^k)|^2 = \frac{|p_k(1)|^2}{p_0(1)} \frac{|p_{j+k}(1)|^2}{p_0(1)}.$$

But, from $j = 0$ in the first equation,

$$p_0(\omega^k) = \frac{|p_k(1)|^2}{p_0(1)},$$

so that

$$|p_j(\omega^k)|^2 = p_0(\omega^k)p_0(\omega^{j+k}) = p_0(\omega^k)p_0(\omega^j\omega^k),$$

i.e., setting $z = \omega^k$,

$$|p_j(z)|^2 = p_0(z)p_0(\omega^j z).$$

Take the product over k of the re-indexed equation

$$\prod_k p_j(\omega^k) = \frac{1}{(p_0(1))^d} \prod_k p_{j+k}(1)\overline{p_k(1)} = \frac{1}{(p_0(1))^d} \prod_k |p_k(1)|^2,$$

(since each $p_k(1)$ and its conjugate appears exactly *once* in the product). Thus, by (4.19)

$$\prod_k p_j(\omega^k) = \frac{1}{(p_0(1))^d} \prod_{k=0}^{d-1} p_0(1)p_0(\omega^k) = \prod_{k=0}^{d-1} p_0(\omega^k).$$

□

For completeness, we note that the Hermitian condition of Lemma 2.1 can also be succinctly expressed in terms of row symbols.

Lemma 4.2 *Tc is Hermitian if and only if the symbols of c satisfy*

$$\overline{p_j(z)} = p_{-j}(\omega^j z), \qquad j = 0, \ldots, d-1.$$

Proof: From the definition, we calculate

$$\overline{p_j(z)} = \overline{\sum_r c_{-j,r}(\omega^j z)^{-r}} = \sum_r \overline{c_{-j,r}}(\omega^j z)^r = \sum_k \overline{c_{-j,-k}}\omega^{-jk}z^{-k},$$

$$p_{-j}(\omega^j z) = \sum_r c_{j,r}(\omega^{-j}(\omega^j z))^{-r} = \sum_k c_{jk}z^{-k},$$

and so, by equating the coefficients of z^{-k}, the Hermitian condition $c_{jk} = \omega^{-jk}\overline{c_{-j,-k}}$ is equivalent to equality of the above symbols. □

5 The Special Case of $d = 3$

This case already has some interesting geometric features.

Solving the basic equations for the c_{jk}, is also geometrically interesting.

Proposition 5.1 *For $d = 3$, c generates a Heisenberg frame with $v_0 \neq 0$ if and only if*

(a) $p_0(z) = 1 + \overline{c_{01}}z + c_{01}z^2$ *and* $p_2(z) = \overline{p_1(\omega^2 z)}$ *(Hermitian conditions)*

(b) $|p_1(z)|^2 = p_0(z)p_0(\omega z)$ *(Riesz factorization)*

(c) $\displaystyle\prod_{k=0}^{2} p_1(\omega^k) = \prod_{k=0}^{2} p_0(\omega^k)$ *(invariant condition)*

(d) $|c_{01}| = |c_{1k}| = \dfrac{1}{2}, \quad k = 0, 1, 2.$

Proof: By Lemma 4.2, the conditions for Tc to be Hermitian are

$$\overline{p_0(z)} = p_0(z), \qquad \overline{p_1(z)} = p_2(\omega z), \qquad \overline{p_2(z)} = p_1(\omega^2 z).$$

Since $p_0(z) = c_{00} + c_{01}z^2 + c_{02}z$, the first equation is satisfied provided

$$\overline{c_{00}} + \overline{c_{01}}z + \overline{c_{02}}z^2 = c_{00} + c_{02}z + c_{01}z^2 \quad \iff \quad c_{00} \in \mathbb{R}, \quad c_{02} = \overline{c_{01}}.$$

The second and third are equivalent, since substituting ωz for z in the third gives

$$\overline{p_2(\omega z)} = p_1(\omega^2(\omega z)) = p_1(z).$$

Hence (a) is equivalent to Tc being Hermitian with $c_{00} = 1$, and implies $c_{02} = \overline{c_{01}}$.

By Corollary 4.1, (a),(b),(c), (d) hold for a Heisenberg frame with $v_0 \neq 0$. For the converse, suppose that (a),(b),(c), (d) hold. Then by (a), Tc is Hermitian with $c_{00} = 1$, and $c_{02} = \overline{c_{01}}$, so that (d) gives $|c_{jk}| = \frac{1}{2}$, $(j,k) \neq (0,0)$. Hence Lemma 2.1, gives

$$P_{Tc}(\lambda) = \lambda^3 - \lambda^2 + 0\lambda + a_0, \qquad a_0 = -\det(Tc).$$

In view of Theorem 2.1, with condition (iv)$'$, we need only show that $\det(Tc) = 0$.

Since $d(Tc)_{jk} = p_{j-k}(\omega^k)$, condition (a) gives

$$3(Tc) = \begin{bmatrix} p_0(1) & p_2(\omega) & p_1(\omega^2) \\ p_1(1) & p_0(\omega) & p_2(\omega^2) \\ p_2(1) & p_1(\omega) & p_0(\omega^2) \end{bmatrix} = \begin{bmatrix} p_0(1) & \overline{p_1(1)} & p_1(\omega^2) \\ p_1(1) & p_0(\omega) & \overline{p_1(\omega)} \\ \overline{p_1(\omega^2)} & p_1(\omega) & p_0(\omega^2) \end{bmatrix}.$$

Since $\overline{p_0(z)} = p_0(z)$, the invariant condition gives $\prod_k p_1(\omega^k) = \prod_k \overline{p_1(\omega^k)} = \prod_k \overline{p_0(\omega^k)}$, and we calculate and so

$$\begin{aligned} 3\det(Tc) &= p_0(1)\{p_0(\omega)p_0(\omega^2) - |p_1(\omega)|^2\} - \overline{p_1(1)}\{p_1(1)p_0(\omega^2) - \overline{p_1(\omega^2)p_1(\omega)}\} \\ &\quad + p_1(\omega^2)\{p_1(1)p_1(\omega) - \overline{p_1(\omega^2)}p_0(\omega)\} \\ &= 3\prod_k p_0(\omega^k) - p_0(1)|p_1(\omega)|^2 - p_0(\omega^2)|p_1(1)|^2 - p_0(\omega)|p_1(\omega^2)|^2. \end{aligned}$$

Applying the Riesz factorisation to the last three terms we then obtain

$$3\det(Tc) = 3\prod_k p_0(\omega^k) - p_0(1)p_0(\omega)p_0(\omega^2) - p_0(\omega^2)p_0(1)p_0(\omega) - p_0(\omega)p_0(\omega^2)p_0(1) = 0.$$

(need to check $v_0 \neq 0$!) □

We now use Proposition 5.1 to find the solutions for $d = 3$. First we consider the Riesz–type factorisation $|p_1(z)|^2 = p_0(z)p_0(\omega z)$. Note that $1 + \omega + \omega^2 = 0$, and the variable z of our symbols satisfies $z^3 = 1$, $\overline{z} = z^2$. Hence multiplying out gives

$$\begin{aligned} |p_1(z)|^2 = p_1(z)\overline{p_1(z)} &= (c_{20} + c_{21}\omega^2 z^2 + c_{22}\omega z)(\overline{c_{20}} + \overline{c_{21}}\omega z + \overline{c_{22}}\omega^2 z^2) \\ &= (\textstyle\sum_k |c_{2k}|^2) + (c_{20}\overline{c_{21}} + c_{21}\overline{c_{22}} + c_{22}\overline{c_{20}})\omega z + (c_{20}\overline{c_{22}} + c_{21}\overline{c_{20}} + c_{22}\overline{c_{21}})\omega^2 z^2, \end{aligned}$$

and

$$\begin{aligned} p_0(z)p_0(\omega z) &= (1 + \overline{c_{01}}z + c_{01}z^2)(1 + \overline{c_{01}}\omega z + c_{01}\omega^2 z^2) \\ &= 1 - |c_{01}|^2 + \omega^2(c_{01}^2 - \overline{c_{01}})z + \omega(\overline{c_{01}}^2 - c_{01})z^2. \end{aligned}$$

Hence, equating the coefficients of $1, z, z^2$, gives

$$\begin{aligned} |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 &= 1 - |c_{01}|^2, \\ c_{20}\overline{c_{21}} + c_{21}\overline{c_{22}} + c_{22}\overline{c_{20}} &= (c_{01}^2 - \overline{c_{01}})\omega, \\ c_{20}\overline{c_{22}} + c_{21}\overline{c_{20}} + c_{22}\overline{c_{21}} &= (\overline{c_{01}}^2 - c_{01})\omega^2. \end{aligned}$$

Since $|c_{01}|^2 = |c_{1k}|^2 = \frac{1}{4}$, the first equation is automatically satisfied. Further, the second and third are conjugates of each other, and so we have only one equation (for the Riesz–type factorisations)

$$c_{20}\overline{c_{22}} + c_{21}\overline{c_{20}} + c_{22}\overline{c_{21}} = (\overline{c_{01}}^2 - c_{01})\omega^2.$$

Setting $z_{jk} := c_{jk}/|c_{jk}| = 2c_{jk}$, this becomes

$$\frac{1}{4}(z_{20}\overline{z_{22}} + z_{21}\overline{z_{20}} + z_{22}\overline{z_{21}}) = (\frac{1}{4}\overline{z_{01}}^2 - \frac{1}{2}z_{01})\omega^2.$$

Since $\overline{z_{01}}^2 = z_{01}^{-2}$, this can be rewritten as

$$z_{20}\overline{z_{22}} + z_{21}\overline{z_{20}} + z_{22}\overline{z_{21}} = \big(\frac{1}{z_{01}^2} - 2z_{01}\big)\omega^2 = \frac{1 - 2z_{01}^3}{z_{01}^2}\omega^2 = \frac{1 - 2(z_{01}/\omega)^3}{(z_{01}/\omega)^2}.$$

Now we set $z := z_{01}/\omega$, so that our equation becomes

$$z_{20}\overline{z_{22}} + z_{21}\overline{z_{20}} + z_{22}\overline{z_{21}} = \frac{1 - 2z^3}{z^2}. \tag{5.21}$$

We proceed to analyze both sides of this equation.

Lemma 5.1 *The curve*

$$\theta \mapsto \frac{1 - 2z^3}{z^2}, \quad z = -e^{i\theta}$$

is a 3–cusped hypocycloid (or deltoid).

Proof: Recall that the standard parametric equations for a hypocycloid with radii a and b with $a > b > 0$ are (see, e.g. [Wik23])

$$x(\theta) = (a-b)\cos(\theta) + b\cos(\frac{a-b}{b}\theta),$$
$$y(\theta) = (a-b)\sin(\theta) - b\sin(\frac{a-b}{b}\theta),$$

and if $n = a/b$ is an integer, it is n–cusped. Now

$$w := \frac{1-2z^3}{z^2} = -2z + z^{-2} = 2e^{i\theta} + e^{-2i\theta},$$

which has Cartesian coordinates

$$\Re(w) = 2\cos(\theta) + \cos(2\theta), \qquad \mathrm{Im}(w) = 2\sin(\theta) - \sin(2\theta),$$

and so $w(\theta)$ is a 3–cusped hypocycloid with radii $a = 3$ and $b = 1$. □

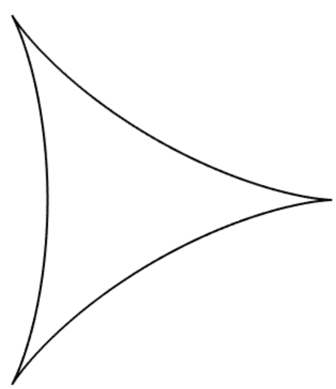

Figure 1: The Hypocycloid

For the left side, note that the product of the three terms

$$z_{20}\overline{z_{22}} \cdot z_{21}\overline{z_{20}} \cdot z_{22}\overline{z_{21}} = 1.$$

Lemma 5.2 *The set of complex numbers*

$$\{z_1 + z_2 + z_3 \,:\, z_1z_2z_3 = 1,\, |z_j| = 1\}$$

is the interior and boundary of the 3*–cusped hypocycloid given by the right side, i.e.,*

$$\theta \mapsto \frac{1-2z^3}{z^2}, \quad z = -e^{i\theta}.$$

In particular, points on the boundary have the form

$$z_1 = z_3 = e^{-i\frac{\phi}{2}}, \quad z_2 = e^{i\phi} \qquad (\theta = -\tfrac{\phi}{2}),$$

$$z_1 = z_2 = e^{-i\frac{\phi}{2}}, \quad z_3 = e^{i\phi},$$

or

$$z_2 = z_3 = e^{-i\frac{\phi}{2}}, \quad z_1 = e^{i\phi}.$$

Proof: Since $z_1 z_2 z_3 = 1$, we can write a point w in the set as

$$w = z_1 + z_2 + z_3, \qquad z_1 := e^{it}, \quad z_2 := e^{i\phi}, \quad z_3 := e^{-i(t+\phi)}.$$

Now fix ϕ, and let t vary. Let A and B be the points on the hypocycloid for $\theta = -\frac{\phi}{2}$ and $\theta = \pi - \frac{\phi}{2}$, i.e.,

$$A = 2e^{-i\frac{\phi}{2}} + e^{i\phi}, \qquad B = -2e^{-i\frac{\phi}{2}} + e^{i\phi}.$$

We claim (cf. Figure 2) that as t varies w traces out the line segment connecting A and B, precisely

$$w = e^{it} + e^{i\phi} + e^{-i(t+\phi)} = \lambda A + (1-\lambda)B, \qquad \lambda = \frac{\cos(t + \frac{\phi}{2}) + 1}{2} \in [0,1],$$

which we verify by multiplying out

$$\begin{aligned} \lambda A + (1-\lambda)B = \lambda(A-B) + B &= \frac{e^{i(t+\frac{\phi}{2})} + e^{-i(t+\frac{\phi}{2})} + 2}{4} 4e^{-i\frac{\phi}{2}} - 2e^{-i\frac{\phi}{2}} + e^{i\phi} \\ &= (e^{it} + e^{-i(t+\phi)} + 2e^{-i\frac{\phi}{2}}) - 2e^{-i\frac{\phi}{2}} + e^{i\phi} = e^{it} + e^{i\phi} + e^{-i(t+\phi)}. \end{aligned}$$

Further we note that this line segment is tangent to the point where $\theta = \phi$, i.e.,

$$C = 2e^{i\phi} + e^{-2i\phi}.$$

Indeed the tangent to the hypocyloid at this point is

$$\frac{d}{d\theta}\big(2e^{i\theta} + e^{-2i\theta}\big)\big|_{\theta=\phi} = 2e^{i\phi} - 2e^{-2i\phi} = i\sin(\tfrac{3}{2}\phi)(A-B),$$

which is collinear with the line segment, except when $\phi = 0, \pm\frac{2}{3}\pi$, the three cusps of the hypocycloid.

At the cusp corresponding to $\phi = 0$, $A = 3$ and $B = -1$, and thus the line segment connecting A and B is also "tangent" at that cusp. The other cusps are handled similarly. □

Thus, from equation (5.21), it follows that $z_{20}\overline{z_{22}}$, $z_{21}\overline{z_{20}}$ and $z_{22}\overline{z_{21}}$ are *boundary* points of the hypocycloid. Solutions may be obtained as follows. Pick one of the boundary solutions, e.g., where $z_1 = z_3$, so that

$$z_{20}\overline{z_{22}} = e^{-i\frac{\phi}{2}}, \qquad z_{21}\overline{z_{20}} = e^{i\phi}, \qquad z_{22}\overline{z_{21}} = e^{-i\frac{\phi}{2}}, \qquad z = -e^{-i\frac{\phi}{2}}.$$

These can be solved using one of them as a free parameter, i.e.,

$$z_{22} = e^{i\frac{\phi}{2}} z_{20} \text{ and } z_{21} = e^{i\phi} z_{20}, \quad |z_{20}| = 1.$$

In this way we arrive at a continuum of parameterized solutions for the overlaps of a SIC in dimension $d = 3$.

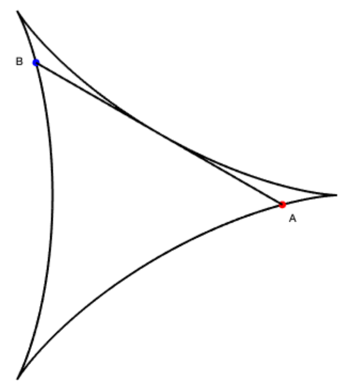

Figure 2: Points A and B on the Hypocycloid

6 Closing Comment

It has sometimes been remarked that the overlaps are zeros of a self-reciprocal polynomial ($z^n p(1/z) = p(z)$) with integer coefficients. The fact that the coefficients are integers is notable and perhaps important. However being self-reciprocal is *not*. Indeed if they are roots of a polynomial $p(z)$ of degree n, then they are also automatically roots of $q(z) := p(z) \times z^n p(1/z)$ and this latter polynomial is self-reciprocal.

7 Acknowledgement

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