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Spherical (t,t)-designs with a small number of vectors

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A B S T R A C T

For $t \in \{1,2,\ldots\}$ fixed, a natural class of spherical designs is given by the vectors $v_1,\ldots,v_n$ in $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$ (not all zero) which give equality in the bound

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j, v_k\rangle|^{2t} \geq c_t(\mathbb{F}^d)\Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t}\Big)^2,$$

where $c_t(\mathbb{F}^d)$ is a known constant. These spherical (t,t)-designs integrate a space of homogeneous polynomials of degree $2t$, and are variously known as real spherical half-designs of order $2t$, complex (projective) t-designs, complex spherical semi-designs, and as tight frames when $t = 1$. Little is known about the minimal number of vectors n for such a design.

Here we report on the results of a numerical search for (t,t)-designs with a minimal number of vectors. In some cases, we obtain the designs explicitly as an orbit of a unitary action of a finite group on the sphere. We also list all the currently known (t,t)-designs. It is shown that many of these belong to a family of designs which we construct from the complex reflection groups. This family includes several new spherical (t,t)-designs with a small number of vectors.

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1. Introduction

Let $\mathbb{S} = \mathbb{S}_{\mathbb{F}}$ be the unit sphere in $\mathbb{F}^d$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and σ be the normalised surface area measure on $\mathbb{S}$. A "spherical design" is a sequence of points $v_1, \ldots, v_n$ in $\mathbb{S}$ for which the integration (cubature) rule

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \frac{1}{n} \sum_{j=1}^{n} p(v_j),$$

holds for all p in some finite dimensional space of polynomials P. For example, when $\mathbb{F} = \mathbb{R}$ and P is the polynomials of degree $\le t$ one has a (**real**) **spherical t-design**. The existence of a spherical design for n sufficiently large was proved in [29].

There are various equivalent conditions to being a spherical design [15], [3]. These include being an integration rule for a subspace of harmonic polynomials, and a variational characterisation. In this paper, we consider (**spherical**) **(t,t)-designs** which are defined to be points (v_j) in $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$ that give equality in the inequality

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle v_j, v_k \rangle|^{2t} \ge c_t(\mathbb{F}^d) \Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t} \Big)^2, \tag{1.1}$$

where

$$c_t(\mathbb{C}^d) := \frac{1}{\binom{d+t-1}{t}}, \qquad c_t(\mathbb{R}^d) := \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{d(d+2)\cdots(d+2(t-1))}. \tag{1.2}$$

We observe that $c_t(\mathbb{R}^d) \ge c_t(\mathbb{C}^d)$, with strict inequality when $t, d > 1$. These designs are determined by the space of polynomials $\mathbb{F}^d \to \mathbb{F}$ given by

$$\Pi^\circ_{t,t}(\mathbb{F}^d) = \mathrm{Hom}(t,t) := \mathrm{span}\{ z \mapsto z^\alpha \overline{z}^\beta : |\alpha| = |\beta| = t \}, \tag{1.3}$$

which are homogeneous of degree t in z and in $\overline{z}$ $(z \in \mathbb{F}^d)$. Equivalently

$$\Pi^\circ_{t,t}(\mathbb{F}^d) = \mathrm{span}\{ z \mapsto |\langle z, v \rangle|^{2t} : v \in \mathbb{F}^d \}. \tag{1.4}$$

We note that $\Pi^\circ_{t,t}(\mathbb{R}^d) = \Pi^\circ_{2t}(\mathbb{R}^d)$, where $\Pi^\circ_k(\mathbb{R}^d)$ is the space of homogeneous polynomials $\mathbb{R}^d \to \mathbb{R}$ of degree k. For unit vectors, these designs are effectively the **t-designs in projective spaces** introduced by [17]. The (t,t)-designs for $\mathbb{R}^d$ are known as **spherical half-designs of order** $2t$ [21]. The (t,t)-designs for $\mathbb{C}^d$ are of interest because of their applications to quantum information theory [16], [27], [35]. They are also known as **complex** (**projective**) **t-designs** [27] and as **complex spherical semi-designs** [22].

The basic theory of spherical (t,t)-designs is developed in [33]. When the vectors $v_1, \ldots, v_n$ in $\mathbb{F}^d$ giving equality in (1.1) are not all zero, then one has the weighted integration rule

$$\int_{\mathbb{S}} p(x)\,d\sigma(x) = \frac{1}{\sum_k \|v_k\|^{2t}} \sum_{j=1}^n p(v_j) = \sum_{\substack{j=1\\ v_j\neq 0}}^n \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}} p(\frac{v_j}{\|v_j\|}), \quad \forall p \in P = \Pi_{t,t}^{\circ}(\mathbb{F}^d),$$

and we will call (v_j) a **weighted** (t,t)-design, with **weights**

$$w_j := \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}} \geq 0, \qquad w_1 + w_2 + \cdots + w_n = 1.$$

By its definition, a (t,t)-design (v_j) is projectively unitarily invariant, i.e., $(c_j U v_j)$ is also a (t,t)-design when $c_j \in \mathbb{F}$, $|c_j| = 1$, and U is unitary. A real spherical t-design has this property if and only if it is **centrally symmetric**, i.e., of the form $(\pm v_j)$.

For t fixed, the set of spherical (t,t)-designs $V = [v_1, \ldots, v_n] \in \mathbb{F}^{d\times n}$ is the algebraic variety given by

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2t} = c_t(\mathbb{F}^d)\Big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Big)^2. \tag{1.5}$$

This variety has been studied in the case $t = 1$ (tight frames) [11]. The purpose of this paper is to explore the algebraic variety of spherical (t,t)-designs for the smallest value of n for which it is nontrivial, i.e., a (t,t)-design of n vectors for $\mathbb{F}^d$ exists. This is done by using the variational characterisation of equality in (1.1) to move towards a nonzero point on the variety (should there be one), for small values of t and d. From these numerical results the smallest value of n is then inferred, and any group orbit structure of the (t,t)-design is identified (using recently developed techniques of [12]). In a number of cases, these *putatively optimal* (t,t)-designs are then used to find an analytic form of what we believe to be a (t,t)-design with the minimal number of vectors.

We also give the results of a search through the *highly symmetric* tight frames given by the complex reflection groups [10] for (t,t)-designs. We find that these include some of the sporadic examples of (t,t)-designs known. This allows us to give a neat listing of all the known spherical (t,t)-designs (with a small number of vectors).

We finish this introduction by giving some examples of spherical (t,t)-designs. In particular, *SICs* and *MUBs*, which are of interest in quantum information theory (where they are viewed as rank one projections giving quantum measurements).

Example 1.1. A real spherical $2t$-design for $\mathbb{R}^d$ is a (t,t)-design for $\mathbb{R}^d$, i.e., a spherical half-design of order $2t$ ([19] give some putatively optimal examples). Conversely, a centrally symmetric (t,t)-design for $\mathbb{R}^d$ is a real spherical $2t$-design for $\mathbb{R}^d$.

Example 1.2. A Euclidean t-design (X,w) for points $X = \cup_j X_j$ on spheres S_j in $\mathbb{R}^d$ of radius r_j and weights $w : X \to \mathbb{R}^+$ is a spherical design satisfying

$$\sum_j \frac{w(X_j)}{|S_j|} \int_{S_j} f\,d\sigma_j = \sum_{x\in X} w(x)f(x), \qquad w(X_j) := \sum_{x\in X_j} w(x),$$

for all polynomials f of degree $\leq t$. We note that both the integral approximated, and the approximation depend on the weights. Taking $f \in \Pi_m^\circ(\mathbb{R}^d)$, i.e., $f(x) = \|x\|^m f(\frac{x}{\|x\|})$, $x \neq 0$, gives

$$\sum_j w(X_j) \int_{\mathbb{S}} f(r_j x)\, d\sigma(x) = \sum_j w(X_j) \int_{\mathbb{S}} r_j^m f(x)\, d\sigma(x) = \sum_{x \in X} w(x) \|x\|^m f\Big(\frac{x}{\|x\|}\Big),$$

which is equivalent to

$$\int_{\mathbb{S}} f(x)\, d\sigma(x) = \sum_{x \in X} \frac{w(x)\|x\|^m}{\sum_y w(y)\|y\|^m} f\Big(\frac{x}{\|x\|}\Big). \tag{1.6}$$

By taking $m = 2t$, we see that a Euclidean $2t$-design gives a (t,t)-design $X^* = (x^*)_{x \in X}$ for $\mathbb{R}^d$, where

$$x^* := w(x)^{\frac{1}{2t}} x, \qquad x \in X. \tag{1.7}$$

Conversely, for a spherical (t,t)-design for $\mathbb{R}^d$, one can associate a constant weight "Euclidean design" with the spheres taken to be those spheres on which the points lie. This satisfies (1.6) for $m = 2t$, and by making it centrally symmetric (if need be) then this Euclidean design integrates all homogeneous polynomials of odd degree. Therefore to satisfy the definition of being a Euclidean design, it must also satisfy (1.6) for $m = 2r$, $1 \leq r < t$, i.e., be a spherical (r,r)-design. This does not follow in general (cf Example 1.7), and so bounds on the number of points in a Euclidean design to not apply. A similar variational condition for a weighted set of points in $\mathbb{R}^d$ to be a Euclidean design is given in [25] (see the discussion after Theorem 1). Nevertheless, some of the spherical (t,t)-designs that we construct do correspond to constant weight Euclidean $2t$-designs (see Example 3.8).

Example 1.3. The $(1,1)$-designs (v_j) for $\mathbb{F}^d$ (with vectors of any lengths) are precisely the finite tight frames [32], [34], i.e., they satisfy the "redundant orthogonal expansion"

$$x = \frac{d}{\sum_{\ell=1}^n \|v_\ell\|^2} \sum_{j=1}^n \langle x, v_j \rangle v_j, \qquad \forall x \in \mathbb{F}^d.$$

Thus the unit-norm (unweighted) $(1,1)$-designs with the minimal number of vectors are the orthonormal bases.

Example 1.4. Three equally spaced unit vectors in $\mathbb{R}^2$ are a $(1,1)$-design for $\mathbb{R}^2$ and $\mathbb{C}^2$. They are a $(2,2)$-design for $\mathbb{R}^2$, but not for $\mathbb{C}^2$.

Example 1.5. A **SIC** (or symmetric informationally complete positive operator valued measure) for $\mathbb{C}^d$, i.e., a set of d^2 unit vectors (v_j) in $\mathbb{C}^d$ with

$$|\langle v_j, v_k\rangle|^2 = \frac{1}{d+1}, \qquad j \neq k,$$

is a $(2,2)$-design of d^2 unit vectors for $\mathbb{C}^d$, with the minimum number of vectors. The existence of a SIC for every dimension d is a problem of great interest [35], [28].

Example 1.6. A set of $d+1$ **MUBs** (mutually unbiased bases) for $\mathbb{C}^d$, i.e., orthogonal bases with

$$|\langle f, g\rangle| = \frac{1}{\sqrt{d}}, \qquad \text{for } f \text{ and } g \text{ in different bases,}$$

gives a $(2,2)$-design of $d(d+1)$ unit vectors for $\mathbb{C}^d$ [23]. This is called a maximal set of MUBs, since there cannot be more than $d+1$ MUBs for $\mathbb{C}^d$.

Example 1.7. In [33], it is shown that if $(v_j)_{j=1}^n$ is a spherical (t,t)-design for $\mathbb{F}^d$, then $(\|v_j\|^{t/r-1}v_j)$ is a spherical (r,r)-design for $\mathbb{F}^d$, $1 \leq r \leq t$, i.e.,

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2r} \|v_j\|^{2(t-r)} \|v_k\|^{2(t-r)} = c_r(\mathbb{F}^d)\Big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Big)^2. \tag{1.8}$$

From the Example 1.7, it follows that the minimal number of vectors in a (t,t)-design for $\mathbb{F}^d$ is an increasing function of t. We now investigate this minimal number.

For spherical designs X with unit vectors, there are Fisher type lower bounds for the number of lines in a projective t-design, which depend on the cardinality of the angle set $A = \{|\langle x, y\rangle| : x, y \in X, x \neq y\}$, that apply. Those projective t-designs meeting these bounds are said to be **tight**. These bounds are rarely met, e.g., one must have $t \leq 5$, $t \neq 4$, and there are only two tight 5-designs [5], [18]. A universally applicable lower bound (see [34] Exercise 6.22, [6]) is that

$$n \geq \binom{t+d-1}{d-1},$$

where n is the number of vectors in a spherical (t,t)-design for $\mathbb{F}^d$.

2. The numerical construction of (t,t)-designs

Let $V = [v_{\alpha\beta}] = [v_1, \ldots, v_n]$, and $p, g : \mathbb{F}^{d\times n} \to \mathbb{R}$ be the homogeneous polynomials given by

$$p(V) := \sum_j \sum_k |\langle v_j, v_k\rangle|^{2t}, \qquad g(V) := \sum_\ell \|v_\ell\|^{2t}. \tag{2.9}$$

Then the spherical (t,t)-designs of n vectors for $\mathbb{F}^d$ (should they exist) are the nontrivial zeros of the nonnegative homogeneous polynomial

$$f(V) := p(V) - c_t(\mathbb{F}^d)g(V)^2 \tag{2.10}$$

of degree $4t$ in the real (and imaginary) parts of entries of $V = [v_{\alpha\beta}] \in \mathbb{F}^{d\times n}$. The minimisers of $p(V) \geq 0$ with $g(V)$ fixed, e.g., $V = [v_j]$ a unit norm sequence, satisfy the Lagrange equations: $\nabla p(V) = \lambda \nabla g(V)$. Moreover, the ones that give spherical (t,t)-designs are minima of f, and so satisfy $\nabla f(V) = 0$, i.e.,

$$\nabla p(V) = 2c_t(\mathbb{F}^d)g(V)\nabla g(V). \tag{2.11}$$

Thus we obtain the following condition for the existence of spherical (t,t)-designs.

Theorem 2.1. *Let $t \geq 1$ and $f : \mathbb{F}^{d\times n} \to \mathbb{R}$ be the nonnegative function given by*

$$f([v_1,\ldots,v_n]) := \sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^{2t} - c_t(\mathbb{F}^d)\big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\big)^2.$$

Then the critical points of f satisfy

$$\sum_j |\langle v_j, v_\beta\rangle|^{2(t-1)}\langle v_\beta, v_j\rangle v_j = c_t(\mathbb{F}^d)\Big(\sum_\ell \|v_\ell\|^{2t}\Big)\|v_\beta\|^{2(t-1)}v_\beta, \quad 1 \leq \beta \leq n.$$

In particular, for $t = 1$, the nonzero critical points of f are the tight frames for $\mathbb{F}^d$, which are all global minima.

Proof. The critical points of f are given by (2.11), where ∇f is the gradient of f viewed as a function of real variables. For $f : \mathbb{C}^d \to \mathbb{R}$ with $f(x_1 + iy_1,\ldots,x_d + iy_d)$ a differentiable function of the real variables $x_1, y_1, \ldots x_d, y_d \in \mathbb{R}$, define a gradient $\nabla f = 2(\overline{\partial}_1 f,\ldots,\overline{\partial}_d f) : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\nabla f := \big(\frac{\partial}{\partial x_j} f(x_1 + iy_1,\ldots,x_d + iy_d) + i\frac{\partial}{\partial y_j} f(x_1 + iy_1,\ldots,x_d + iy_d)\big)_{j=1}^d. \tag{2.12}$$

Then for both $\mathbb{R}^d$ and $\mathbb{C}^d$, we have

$$\nabla(\|\cdot\|^2)(a) = 2a, \qquad \nabla(|\langle\cdot, b\rangle|^2)(a) = 2\langle a, b\rangle b. \tag{2.13}$$

Using these, a calculation shows that the β-columns of $\nabla p(V)$ and $\nabla g(V)$ are

$$4t\sum_j |\langle v_j, v_\beta\rangle|^{2(t-1)}\langle v_\beta, v_j\rangle v_j, \qquad 2t\|v_\beta\|^{2(t-1)}v_\beta.$$

Substituting this into (2.11) gives the desired condition.

For $t = 1$, the $V \neq 0$ which are critical points of $f(V)$ satisfy

$$\sum_j \langle v_\beta, v_j\rangle v_j = \frac{1}{d}\Big(\sum_\ell \|v_\ell\|^2\Big)v_\beta, \quad 1 \leq \beta \leq n,$$

and so, by linearity, (v_j) is tight frame for $\mathcal{H} := \operatorname{span}\{v_\beta\}_{1\le\beta\le n} \subset \mathbb{F}^d$, with frame bound $A = \frac{1}{d}\sum_\ell \|v_\ell\|^2$, and $\dim(\mathcal{H}) = d$, so that (v_j) is a tight frame for $\mathbb{F}^d$. Thus the nonzero critical points of $f(V)$ are precisely the tight frames for $\mathbb{F}^d$. □

Spherical (t,t)-designs can be found *numerically*, by minimising $f(V)$, with $g(V)$ fixed. This can be done by an iterative algorithm which starts at a random V_0, and chooses $V_{k+1} = V_k + W_k$, where W_k is such that $f(V_{k+1}) = f(V_k + W_k) < f(V_k)$. In [7], random directions W_k (of an appropriate size) were considered. Here we take W_k in the direction of maximal decrease (which is more effective close to a minimum). The maximal decrease of f at V is in the direction $W = -\nabla f(V)$, where

$$(\nabla f(V))_{\alpha\beta} = 4t\sum_j |\langle v_j, v_\beta\rangle|^{2(t-1)}\langle v_\beta, v_j\rangle v_{\alpha j} - 4tc_t(\mathbb{F}^d)\Big(\sum_\ell \|v_\ell\|^{2t}\Big)\|v_\beta\|^{2(t-1)}v_{\alpha\beta}.$$

It is also possible to calculate (numerically) the Hessian (second derivative) of f and p at V to investigate the nature of the critical points of f (these are all minima for $t = 1$). The formulas for these Hessians are given in the appendix.

We present the results of our numerical construction of (t,t)-designs in the next two sections (the real and complex cases), together with some explicit constructions motivated by them. We are only aware of two other numerical searches for *putatively optimal* spherical designs: Hardin and Sloane's list of real spherical t-designs in $\mathbb{R}^3$ [19] (for $t \le 12$) and Scott and Grassl's list of SICs (complex spherical $(2,2)$-designs of d^2 vectors for $\mathbb{C}^d$) [28]. We emphasize that the existence of a "numerical" spherical design does not prove that such a design exists (though it may lead to an exact construction), nor does our failure to find a numerical spherical design prove that one cannot exist.

3. Real spherical (t,t)-designs (spherical half-designs)

In Table 1 below, we summarise our numerical results for real spherical (t,t)-designs, i.e., spherical half-designs of order $2t$. This is followed by the other known real spherical (t,t)-designs, including those obtained in §5 (see Tables 4, 5), to give a complete list. We use grey when an analytic form of a putatively optimal design is not known, and give details of those that are known after the table (ST denotes a Shephard Todd group).

With just one exception (Example 3.4), all the currently known optimal spherical half-designs appear in the following way.

Example 3.1. (Tight spherical designs) A spherical $(2t+1)$-design of m vectors for $\mathbb{R}^d$ is said to be **tight** (not to be confused with a tight frame) if it gives equality in the lower bound

$$m \ge 2\binom{d-1+t}{t}$$

Table 1
The minimum numbers n_w and n_e of vectors in a weighted and in a equal-norm spherical (t,t)-design for $\mathbb{R}^d$ (spherical half-design of order $2t$) as calculated numerically.

t	d	n_w	n_e	Comments	
1	d	d	d	orthonormal bases in $\mathbb{R}^d$	(Example 1.3)
t	2	$t+1$	$t+1$	equally spaced lines in $\mathbb{R}^2$	(Example 3.2)
2	3	6	6	equiangular lines in $\mathbb{R}^3$	(Example 3.3)
2	4	11	12	no structure	§5, ST 28, Table 4
2	5	16	20	Example 3.5	no structure
2	6	22	24	group structure	work in progress
2	7	28	28	equiangular lines in $\mathbb{R}^7$	(Example 3.3)
2	8	45	>45	no structure	
3	3	11	16	no structure	possible group structure
3	4	23	>23	group structure	
3	5	41	>41	group structure	
4	3	16	25	Example 3.6	no structure
4	4	43	>43	work in progress	
5	3	24	35	no structure	no structure
Other known real (t,t)-designs with a small number of vectors					
2	6		27	§5, ST 35, Table 5	
Other known optimal real (t,t)-designs					
2	23		276	equiangular lines in $\mathbb{R}^{23}$	(Example 3.3)
3	8		120	§5, ST 37, Table 5	(due to [21])
3	23		2300	tight spherical design	
5	4	60	60	§5, ST 30, Table 4	(Example 3.4)
5	24		98280	tight spherical design	
t	d		$\binom{d-1+t}{t}$	tight spherical $(2t+1)$-designs	

of [15]. A tight spherical $(2t+1)$-design is necessarily centrally symmetric, i.e., of the form $(\pm v_j)$ with $m = 2n$, so that (v_j) is a spherical half-design of order $2t$. This is a 1-1 correspondence [21], and so each tight spherical $(2t+1)$-design of $2n$ vectors gives rise to an optimal spherical (t,t)-design of $n = \binom{d-1+t}{t}$ vectors for $\mathbb{R}^d$ [20].

Optimal spherical half-designs which come from tight spherical designs in this way include orthonormal bases, equally spaced lines, and maximal sets of equiangular lines.

Example 3.2. (Equally spaced lines) The $n = t+1$ equally spaced lines in $\mathbb{R}^2$ given by the vectors

$$(v_j) = \Big\{\Big(\cos\frac{\pi}{n}j, \sin\frac{\pi}{n}j\Big) : j = 0,\ldots,n-1\Big\}$$

are a spherical half-design of order $2t$, i.e., a (t,t)-design.

Example 3.3. (Maximal lines) The unit vectors (v_j) in $\mathbb{R}^d$ (or the lines that they give) are said to be equiangular if they have equal cross-correlation, i.e.,

$$|\langle v_j, v_k\rangle| = \alpha, \quad j \neq k, \qquad \text{for some } \textit{angle } \alpha > 0.$$

The number n of equiangular lines in $\mathbb{R}^d$ satisfies the absolute bound $n \leq \frac{1}{2}d(d+1)$. When this bound is attained, the set of lines has angle $\frac{1}{\sqrt{d+2}}$, and hence is a $(2,2)$-design, by the calculation

$$n\cdot 1+(n^2-n)\Big(\frac{1}{\sqrt{d+2}}\Big)^4=\frac{3}{4}\frac{d(d+1)^2}{d+2}=\frac{1\cdot 3}{d(d+2)}n^2.$$

Such lines can exist only when $d=2,3$ or $d+2$ is the square of an odd integer. Those that appear in Table 1 for $d=2,3,7,23$ are well known. (see §5).

The only known optimal spherical-half design which is not given by a tight spherical design is the following.

Example 3.4. There is a 120-point spherical 11-design for $\mathbb{R}^4$ given by the vertices of the regular four-dimensional polyhedron with the Schläfli symbol $\{3,3,5\}$ [1]. This was proved to be optimal in the class of weighted spherical 11-designs [1], and unique (up to unitary equivalence) in the class of (unweighted) spherical 11-designs [4]. The corresponding 60-vector spherical half-design for $\mathbb{R}^4$ of order 10 is therefore optimal in the class of weighted half-designs for $\mathbb{R}^4$ of order 10 (weighted $(5,5)$-designs). This spherical half-design is a highly symmetric tight frame (see Table 4, ST 30). If it had come from a tight spherical 11-design, then it would have had 56 vectors.

The 21-point spherical half-design for $\mathbb{R}^6$ of order 4 given by a highly symmetric tight frame (see Table 5, ST 35) is a good candidate for a second optimal spherical half-design, since if it corresponded to a tight spherical 5-design, then it would have 21 points.

Motivated by our results, [20] shows that the following spherical half-designs exist.

Example 3.5. There is a weighted spherical $(2,2)$-design of 16 vectors for $\mathbb{R}^5$. This consists of six equiangular lines in $\mathbb{R}^5$ at an angle of $\frac{1}{5}$ (the vertices of a simplex) given by vectors of length $(\frac{20}{21})^{1/4}$, and ten equiangular lines in $\mathbb{R}^5$ at an angle of $\frac{1}{3}$ given by vectors of length $(\frac{36}{35})^{1/4}$, where the angle between lines from different families is $\frac{1}{\sqrt{5}}$. The corresponding normalised weights are

$$\frac{16\big(\frac{20}{21}\big)}{6\big(\frac{20}{21}\big)+10\big(\frac{36}{35}\big)}=\frac{20}{21}\approx 0.9523,\qquad \frac{16\big(\frac{36}{35}\big)}{6\big(\frac{20}{21}\big)+10\big(\frac{36}{35}\big)}=\frac{36}{35}\approx 1.0286.$$

Example 3.6. There is a weighted spherical $(4,4)$-design of 16 vectors for $\mathbb{R}^3$. This can be given explicitly by lines given by the antipodal vertices of the pentakis dodecahedron (a Catalan solid) as follows (the six vertices/lines of the icosahedron are the first six columns)

$$[v_j]:=\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & \tau & 0 & -1 & \tau & 1 & 1 & 1 & 1 & 0 & 0 & \frac{1}{\tau} & \frac{1}{\tau} & \tau & -\tau \\ \tau & 0 & 1 & \tau & 0 & -1 & 1 & 1 & -1 & -1 & \frac{1}{\tau} & \frac{1}{\tau} & \tau & -\tau & 0 & 0 \\ 1 & \tau & 0 & -1 & \tau & 0 & 1 & -1 & 1 & -1 & \tau & -\tau & 0 & 0 & \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix}$$
$$\times\begin{pmatrix} \alpha\Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$$

$$\tau := \frac{1+\sqrt{5}}{2} \text{ (the golden ratio)}, \quad \alpha := \sqrt{\frac{3}{1+\tau^2}}, \quad \Lambda_1 := \left(\frac{20}{21}\right)^{\frac{1}{8}} I_6, \quad \Lambda_2 := \left(\frac{36}{35}\right)^{\frac{1}{8}} I_{10}.$$

Here the weights are the same as in Example 3.5, i.e., $\frac{20}{21} \approx 0.9523$ and $\frac{36}{35} \approx 1.0286$. Compare this with Hardin and Sloan [19], who give evidence for an 8-design for $n = 36, 40, 42, \geq 44$, and of a 9-design for $n = 48, 50, 52, \geq 54$. By taking this $(4,4)$-design and the negatives of its vectors, one has a weighted 9-design of 32 points.

The putatively optimal 16-vector weighted spherical (t,t)-designs of Examples 3.5 and 3.6 are the orbit of two vectors of close to equal norm (under the projective symmetry group of [14]). In both cases, the number of vectors in an optimal (unweighted) spherical (t,t)-design given by a tight spherical design would be $15 = \binom{5-1+2}{2} = \binom{3-1+4}{4}$. This suggests that these weighted spherical half-designs are indeed optimal, and that in certain situations weighted designs are quite natural.

The only other numerical search for putatively optimal real designs is that of [19] for spherical t-designs in $\mathbb{R}^3$. We now compare this with our results for small t.

Example 3.7. There is a minimal 2-design given by the four vertices of the regular tetrahedron (these sum to zero), whilst the minimal $(1,1)$-design is the three vectors of an orthonormal basis (these don't sum to zero). The minimal $(2,2)$-design is given by the six equiangular lines which go through the vertices of the icosahedron. Taking the corresponding 12 vectors (which add to zero) gives the minimal 4-design and 5-design. For the $(3,3)$-design, there is the snubcube of 24 points, which is a minimal 6-design and 7-design. This is not centrally symmetric, and so gives only a 24 point $(3,3)$-design, whilst the mininum numbers of vectors for a $(3,3)$-design calculated are 11 and 16.

Finally, we compare our constructions with some known optimal Euclidean designs.

Example 3.8. In [2] Bannai classified all "tight" antipodal Euclidean 5-designs (X,w), $X = X_1 \cup X_2$, supported on two spheres in $\mathbb{R}^d$, i.e., for which the bound

$$n \geq \frac{1}{2}d(d+1)+1,$$

on the number n of antipodal pairs (lines) holds. These give n vector/line spherical $(2,2)$-designs. We now go through the classification.

For $\mathbb{R}^2$, there is a unique such Euclidean 5-design of four lines given by

$$X_1 = \pm\{e_1, e_2\}, \quad X_2 = \pm\{\frac{1}{\sqrt{2}}(r, \pm r)\},\ r \neq 1, \qquad w(x) := \begin{cases} 1, & x \in X_1; \\ \frac{1}{r^4}, & x \in X_2. \end{cases}$$

By (1.7), the corresponding spherical $(2,2)$-design of four lines is given by the vectors

$$x^* = (1)^{\frac{1}{4}} e_j = e_j, \qquad x^* = \left(\frac{1}{r^4}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}}(r, \pm r) = \frac{1}{\sqrt{2}}(1, \pm 1).$$

These give an unweighted spherical $(2,2)$-design of four equally spaced lines in $\mathbb{R}^2$. Three equally spaced lines also give a $(2,2)$-design, which indicates that bounds on the number of points in a Euclidean design do not apply to the corresponding spherical (t,t)-designs.

For $\mathbb{R}^3$, there is a unique such Euclidean 5-design of seven lines given by

$$X_1 = \pm\{e_1, e_2, e_3\}, \quad X_2 = \pm\{\frac{r}{\sqrt{3}}v : |v_j| = 1\}, \qquad w(x) := \begin{cases} 1, & x \in X_1; \\ \frac{9}{8r^4}, & x \in X_2. \end{cases}$$

The corresponding seven vector $(2,2)$-design has vectors with two possible norms (up to a fixed scalar)

$$\|x^*\| = (1)^{\frac{1}{4}}1 = 1, \qquad \|x^*\| = \Big(\frac{9}{8r^4}\Big)^{\frac{1}{4}} r = \Big(\frac{9}{8}\Big)^{\frac{1}{4}}.$$

We observe that this Euclidean design can be chosen to have a constant weight ($r^4 = \frac{9}{8}$), whereas in the previous example this was not possible. Let $n_j = |X_j|$. For the purpose of comparison, we define the normalised weights for the associated (t,t)-design by

$$\hat{w}_j = \hat{w}(x_j) := \frac{n\|x_j^*\|^{2t}}{n_1\|x_1^*\|^{2t} + n_2\|x_2^*\|^{2t}} = \frac{nw(x_j)\|x_1\|^{2t}}{n_1 w(x_1)\|x_1\|^{2t} + n_2 w(x_2)\|x_2\|^{2t}}, \qquad x_j \in X_j.$$

In this case, they are $\hat{w}_1 = \frac{14}{15}$, $\hat{w}_2 = \frac{21}{20}$. The optimal spherical $(2,2)$-design consists of six equiangular lines in $\mathbb{R}^3$.

For $\mathbb{R}^4$, there is no tight Euclidean 5-design of 11 lines, though our numerical calculations indicate that there is an 11 line spherical $(2,2)$-design.

For $\mathbb{R}^5$, there is a tight Euclidean 5-design of $16 = 6 + 10$ lines, with normalised weights $\frac{20}{21}$, $\frac{36}{35}$. This corresponds to the spherical $(2,2)$-design of Example 3.5.

For $\mathbb{R}^6$ there is a $22 = 6 + 16$ line tight Euclidean design with normalised weights $\frac{11}{12}$, $\frac{33}{32}$, which corresponds to the numerical spherical $(2,2)$-design calculated.

There are no further tight antipodal Euclidean 5-designs.

It was later shown that all the tight Euclidean 5-designs are special cases of a general construction of (t,t)-designs as a union of two of lower order [26].

4. Complex spherical (t,t)-designs

In Table 2 below, we give the corresponding results of our numerical search for putatively optimal complex spherical (t,t)-designs.

The orthonormal bases, SICs and MUBs appearing in the table are well studied. A very general construction of weighted $(2,2)$-designs is given in [27]. These are presented as *weighted complex projective t-designs*, and require a function $f : G \to H$ between finite abelian groups with $d = |G| \leq |H|$ satisfying

Table 2
The minimum numbers n_w and n_e of vectors in a weighted and in a equal-norm spherical (t,t)-design for $\mathbb{C}^d$, as calculated numerically.

t	d	n_w	n_e	Comments	
1	d	d	d	orthonormal bases in $\mathbb{C}^d$	(Example 1.3)
2	d	d^2	d^2	SICs (when known to exist)	(Example 1.5)
3	2	6	6	three MUBs for $\mathbb{C}^2$	(Example 1.6)
3	3	22	27	some structure	
3	4	40	40	highly symmetric tight frame	(§5, ST 32, Table 4)
3	5	>100			
4	2	10	12	Example 4.1 (two orbits)	
4	3	47	>47		
4	4	>85	>85		
5	2	12	12	Example 4.2 (one orbit)	
6	2	18	24	some structure	
7	2	22	24	some structure	
8	2	37	>37	some structure	
9	2	44	>44	some structure	
		Other known complex (t,t)-designs with a small number of vectors			
2	d	$d(d+1)$		$d+1$ MUBs for $\mathbb{C}^d$, where d is a prime power	
2	d	$d(\lvert H\rvert+1)$		weighted design, with H abelian of order $\ge d$ [27]	
3	3		36	highly symmetric tight frame	(§5, ST 27, Table 4)
5	4		60	highly symmetric tight frame	(§5, ST 30, Table 4)
3	6		126	highly symmetric tight frame	(§5, ST 34, Table 5)
4	6		672	highly symmetric tight frame	(§5, ST 34, Table 5)

$$f(x+a)-f(x)=b \qquad \text{has at most one solution for each } (a,b)\neq(0,0),$$

to obtain a weighted $(2,2)$-design of $|H|+1$ orthornormal bases for $\mathbb{C}^d$. The 40-vector $(3,3)$-design for $\mathbb{C}^4$ and others are examples of *highly symmetric tight frames*, which are considered in detail in Section 5.

We now give two explicit examples motivated by our calculations.

Example 4.1. (A spherical $(4,4)$-design of 12 lines in $\mathbb{C}^2$) Several unit-norm spherical $(4,4)$-designs of 12 vectors/lines in $\mathbb{C}^2$ were computed numerically. Using the techniques of [14], the projective symmetry group for each was calculated to be the dihedral group of order 10, with the irreducible projective action giving two orbits: one of size 2 (with the vectors orthogonal), and one of size 10. This suggested a $(4,4)$-design of the form

$$\Phi_v=(v,av,a^2v,a^3v,a^4v,bv,abv,a^2bv,a^3bv,a^4bv)\cup(u_1,u_2), \tag{4.14}$$

where $v\in\mathbb{C}^2$ is a unit vector, a (a rotation) and b (a reflection) are generators of the dihedral group and $\{u_1,u_2\}$ is an orthonormal basis. Taking

$$a=\begin{pmatrix}\omega & 0\\ 0 & \overline{\omega}\end{pmatrix},\quad \omega:=e^{\frac{2\pi i}{5}},\qquad b=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad u_1=\begin{pmatrix}1\\0\end{pmatrix},\ u_2=\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.15}$$

and optimising over v to obtain a $(4,4)$-design numerically suggested that the ratio of the components of a suitable v was the *golden ratio* $\frac{\sqrt{5}+1}{2}$, i.e.,

$$v = v_\zeta := \frac{1}{\sqrt{10+2\sqrt{5}}} \begin{pmatrix} (1+\sqrt{5})\zeta \\ 2 \end{pmatrix}, \qquad |\zeta| = 1. \tag{4.16}$$

An elementary calculation shows that (4.14), (4.15), (4.16) define a one-parameter family $\{\Phi_{v_\zeta}\}_{|\zeta|=1}$, of spherical $(4,4)$-designs of 12 unit vectors for $\mathbb{C}^2$.

Somewhat surprisingly, the search for a $(5,5)$-design for $\mathbb{C}^2$ gave a unit-norm one of 12 vectors which is a single orbit. A heuristic explanation for why this was not identified earlier as a $(4,4)$-design, is because there was a one parameter family of such designs and this is an isolated point on the variety.

Example 4.2. A spherical $(5,5)$-design of 12 lines in $\mathbb{C}^2$. Let $\tau := \frac{1}{2}(1+\sqrt{5})$ be the golden ratio, and $G = \langle a, b\rangle$ be the *binary icosahedral group* of order 120 generated by the unitary matrices

$$a = \frac{1}{2}\begin{pmatrix} \tau^{-1} - \tau i & 1 \\ -1 & \tau^{-1} + \tau i \end{pmatrix}, \qquad b = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Then for every unit vector $v \in \mathbb{C}^2$, the G-orbit $(gv)_{g\in G}$ is a $(5,5)$–design of 120 vectors. To show this one must verify that (1.5) holds for $t = 5$. Since G is unitary, this can be simplified to

$$\frac{1}{|G|}\sum_{g\in G} |\langle v, gv\rangle|^{10} = c_5(\mathbb{C}^2)\|v\|^{20}, \qquad \forall v \in \mathbb{C}^2,$$

i.e., if two homogeneous polynomials of degree 20 in the entries of v and $\overline{v}$ are equal. This was done by checking equality at a set of points v on which a polynomial in $\Pi^\circ_{10,10}(\mathbb{C}^2)$ is determined by its values. We observe that a has order 5 and $b^2 = -I$. Hence if v is an eigenvector of a, then $(gv)_{g\in G}$ consists of $120/10 = 12$ lines. From each of these lines we can select a vector to obtain $(5,5)$–design of 12 vectors.

Example 4.2 can be generalised by taking groups other than the binary icosahedral group. We now consider these so called *highly symmetric tight frames*.

5. Highly symmetric tight frames

Many of the putatively optimal spherical (t,t)-designs presented in the previous sections are the orbit of a single vector/line under the unitary action of a finite group, and have a larger group of symmetries. One way to capture this, is the idea of a *highly symmetric* frame. A finite frame Φ of distinct vectors is **highly symmetric** if the action of its symmetry group $\mathrm{Sym}(\Phi)$ is irreducible, transitive, and the stabiliser of any one vector (and hence all) is a nontrivial subgroup which fixes a space of dimension exactly one.

In [10], all the highly symmetric tight frames with symmetry group a finite (irreducible) complex reflection group were calculated (in a search for equiangular lines), except in a few cases. The stabilisers are the *maximal parabolic subgroups*, and by using the recent `Complements.m` software package of Don Taylor [31], we were able to compute the few remaining highly symmetric tight frames (Table 6). We then checked the highly symmetric tight frames obtained from reflection groups to see what order of (t,t)-designs their set of lines gives (see Tables 3, 4 and 5).

We assume a basic familiarity with complex reflection groups [24], [30]. A linear map $\mathbb{F}^d \to \mathbb{F}^d$ is a **complex reflection** if it has finite order and fixes a hyperplane, i.e., it is diagonalisable with one eigenvalue a nontrivial root of unity and all the others 1. A finite group generated by reflections is called a **complex reflection group**. The complex reflection groups are classified up to similarity, and can be taken to be unitary. We will use the numbering of Shephard-Todd (ST) for the irreducible complex reflection groups, and the notation $\langle n,m\rangle$ for the m-th group of order n in magma's database of small groups.

In Tables 3, 4 and 5, we give n the *number of lines* in the spherical (t,t)-design (v_j), m the *number of vectors*, and s the *number of angles*, i.e., the number of values $|\langle v_j, v_k\rangle|$ which are not equal to 1 (the case when vectors are on the same line). A frame with one angle is equiangular. We also give the projective symmetry group of the n lines [14], and a group of order m whose orbit is the m vectors, should there be one, i.e., the frame is a group frame.

Some of the highly symmetric tight frames given by reflection groups are putatively optimal spherical (t,t)-designs and others appear to have small numbers of vectors (as indicated in Tables 1 and 2). We now highlight some examples.

Example 5.1. Consider the following unitary complex reflections of orders $2,2,4,3$

$$S=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad F=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\quad R=\begin{pmatrix}1&0\\0&-i\end{pmatrix},\quad Z=e^{\frac{2\pi i}{24}}RF.$$

The Shephard-Todd group number 6 has order 48, and small group number $\langle 48,33\rangle$. It is generated by S,R^2,Z. The standard basis vector $v=e_1$ (which is fixed by R) gives a highly symmetric tight frame which is a $(3,3)$-design. Since the line given by e_1 is fixed by R and $-I=(SR^2)^2$ this design is given by $48/(4\cdot 2)=6$ lines (which are a maximal set of MUBs). The vector

$$v=\begin{pmatrix}\sqrt{3}+1\\1-i\end{pmatrix}$$

is fixed by Z, and its orbit gives a $(2,2)$-design of 4 vectors, i.e., a SIC.

Example 5.2. For $d=2$, all the Shephard-Todd groups give spherical (t,t)-designs, where $t=2,3,5$, and many of these are repeated, e.g., a SIC and a maximal set of MUBs. The

Table 3
The spherical (t,t)-designs of n vectors for $\mathbb{F}^d$ given by the highly symmetric tight frames for the Shephard-Todd listing of the primitive complex reflection groups. n = number of lines, m = orbit size (number of vectors), s = number of angles.

ST	Order	d	t	n	s	$\mathbb{F}$	Symmetry group	m	Group frame	Comments
4	24	2	2	4	1	$\mathbb{C}$	$\langle 12,3\rangle$	8	$\langle 8,4\rangle$	SIC
5	72		2	4	1	$\mathbb{C}$	$\langle 12,3\rangle$	24	$\langle 24,3\rangle$, $\langle 24,11\rangle$	SIC
6	48		2	4	1	$\mathbb{C}$	$\langle 12,3\rangle$	16	$\langle 16,13\rangle$	SIC
			3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	24	$\langle 24,3\rangle$	max MUBs
7	144		2	4	1	$\mathbb{C}$	$\langle 12,3\rangle$	48	$\langle 48,47\rangle$, $\langle 48,33\rangle$	SIC
			3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	72	$\langle 72,25\rangle$	max MUBs
8	96		3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	24	$\langle 24,3\rangle$, $\langle 24,1\rangle$	max MUBs
9	192		3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	48	$\langle 48,4\rangle$, $\langle 48,28\rangle$, $\langle 48,29\rangle$	max MUBs
			3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	96	$\langle 96,67\rangle$, $\langle 96,74\rangle$	
10	288		3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	72	$\langle 72,12\rangle$, $\langle 72,25\rangle$	max MUBs
			3	8	3	$\mathbb{C}$	$\langle 24,12\rangle$	96	$\langle 96,54\rangle$, $\langle 96,67\rangle$	
11	576		3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	144	$\langle 144,69\rangle$, $\langle 144,121\rangle$, $\langle 144,122\rangle$	max MUBs
			3	8	3	$\mathbb{C}$	$\langle 24,12\rangle$	192	$\langle 192,876\rangle$, $\langle 192,963\rangle$	
			3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	288	$\langle 288,400\rangle$, $\langle 288,638\rangle$	
12	48		3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	24	$\langle 24,3\rangle$	
13	96		3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	48	$\langle 48,28\rangle$, $\langle 48,29\rangle$	
			3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	48	$\langle 48,28\rangle$, $\langle 48,33\rangle$	max MUBs
14	144		3	8	3	$\mathbb{C}$	$\langle 24,12\rangle$	48	$\langle 48,26\rangle$, $\langle 48,29\rangle$	
			3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	72	$\langle 72,25\rangle$	
15	288		3	8	3	$\mathbb{C}$	$\langle 24,12\rangle$	96	$\langle 96,182\rangle$, $\langle 96,192\rangle$	
			3	12	4	$\mathbb{C}$	$\langle 24,12\rangle$	144	$\langle 144,121\rangle$, $\langle 144,122\rangle$	
			3	6	2	$\mathbb{C}$	$\langle 24,12\rangle$	144	$\langle 144,121\rangle$, $\langle 144,157\rangle$	max MUBs
16	600		5	12	3	$\mathbb{C}$	$\langle 60,5\rangle$	120	$\langle 120,5\rangle$, $\langle 120,15\rangle$	Example 4.2
17	1200		5	12	3	$\mathbb{C}$	$\langle 60,5\rangle$	240	$\langle 240,93\rangle$, $\langle 240,154\rangle$	Example 4.2
			5	30	8	$\mathbb{C}$	$\langle 60,5\rangle$	600	$\langle 600,54\rangle$	
18	1800		5	12	3	$\mathbb{C}$	$\langle 60,5\rangle$	360	$\langle 360,51\rangle$, $\langle 360,89\rangle$	Example 4.2
			5	20	5	$\mathbb{C}$	$\langle 60,5\rangle$	600	$\langle 600,54\rangle$	
19	3600		5	12	3	$\mathbb{C}$	$\langle 60,5\rangle$	720	$\langle 720,420\rangle$, $\langle 720,708\rangle$	Example 4.2
			5	20	5	$\mathbb{C}$	$\langle 60,5\rangle$	1200	$\langle 1200,483\rangle$	
			5	30	8	$\mathbb{C}$	$\langle 60,5\rangle$	1800	$\langle 1800,328\rangle$	
20	360		5	20	5	$\mathbb{C}$	$\langle 60,5\rangle$	120	$\langle 120,5\rangle$	
21	720		5	20	5	$\mathbb{C}$	$\langle 60,5\rangle$	240	$\langle 240,93\rangle$	
			5	30	8	$\mathbb{C}$	$\langle 60,5\rangle$	360	$\langle 360,51\rangle$	
22	240		5	30	8	$\mathbb{C}$	$\langle 60,5\rangle$	120	$\langle 120,5\rangle$	

reason for this is that the design is given by the lines in the orbit, which only depend on the matrices in the group up to a scalar multiple. One way to obtain a canonical group with this orbit, is to ensure that all the matrices have determinant 1, which leads to the notion of a *canonical abstract error group* [13]. For the Shephard-Todd groups of rank 2, there are just three canonical abstract error groups that appear. These are the *binary tetrahedral group* $\mathcal{T}$, the *binary octahedral group* $\mathcal{O}$, and the *binary icosahedral group* $\mathcal{I}$ (see [24]), where the correspondence is

$$\text{ST 4-7:} \qquad \mathcal{T}/\langle -I\rangle = \langle 12,3\rangle \cong A_4,$$

$$\text{ST 8-15:} \qquad \mathcal{O}/\langle -I\rangle = \langle 24,12\rangle \cong S_4,$$

$$\text{ST 16-22:} \qquad \mathcal{I}/\langle -I\rangle = \langle 60,5\rangle \cong A_5.$$

Table 4
The spherical (t,t)-designs of n vectors for $\mathbb{F}^d$ given by the highly symmetric tight frames for the Shephard-Todd listing of the primitive complex reflection groups. n = number of lines, m = orbit size (number of vectors), s = number of angles.

ST	Order	d	t	n	s	$\mathbb{F}$	Symmetry group	m	Group frame	Comments
23	120	3	2	6	1	$\mathbb{R}$	$\langle 60,5\rangle$	12	$\langle 12,3\rangle$	equiangular
			2	10	2	$\mathbb{R}$	$\langle 60,5\rangle$	20	–	two angles
			2	15	4	$\mathbb{R}$	$\langle 60,5\rangle$	30	–	
24	336		2	21	3	$\mathbb{C}$	$\langle 168,42\rangle$	42	$\langle 42,2\rangle$	
			2	28	4	$\mathbb{C}$	$\langle 168,42\rangle$	56	–	
25	648		2	9	1	$\mathbb{C}$	$\langle 216,153\rangle$	27	$\langle 27,3\rangle$, $\langle 27,4\rangle$	SIC
			2	12	2	$\mathbb{C}$	$\langle 216,153\rangle$	72	–	max MUBS
26	1296		2	9	1	$\mathbb{C}$	$\langle 216,153\rangle$	54	$\langle 54,8\rangle$, $\langle 54,10\rangle$, $\langle 54,11\rangle$	SIC
			2	12	2	$\mathbb{C}$	$\langle 216,153\rangle$	72	–	max MUBS
			2	36	4	$\mathbb{C}$	$\langle 216,153\rangle$	216	$\langle 216,88\rangle$	
27	2160		3	36	4	$\mathbb{C}$	$\langle 360,118\rangle$	216	–	
			3	45	5	$\mathbb{C}$	$\langle 360,118\rangle$	270	–	
			3	60	8	$\mathbb{C}$	$\langle 360,118\rangle$	360	–	
28	1152	4	2	12	2	$\mathbb{R}$	$\langle 576,8654\rangle$	24	$\langle 24,1\rangle\langle 24,3\rangle$, $\langle 24,11\rangle$	real MUBs
			2	48	6	$\mathbb{R}$	$\langle 576,8654\rangle$	96	$\langle 96,67\rangle$, $\langle 96,201\rangle$, $\langle 96,204\rangle$	
29	7680		2	20	2	$\mathbb{C}$	$\langle 1920,\cdot\rangle$	80	$\langle 80,30\rangle$	max MUBs
			2	40	3	$\mathbb{C}$		160	–	
			2	80	5	$\mathbb{C}$		320	$\langle 320,1581\rangle$, $\langle 320,1586\rangle$	
			2	160	10	$\mathbb{C}$		640	–	
30	14400		5	60	4	$\mathbb{R}$	$\langle 7200,\cdot\rangle$	120	$\langle 120,5\rangle$, $\langle 120,15\rangle$	
			5	300	15	$\mathbb{R}$		600	$\langle 600,54\rangle$	
			5	360	18	$\mathbb{R}$		720	–	
			5	600	32	$\mathbb{R}$		1200	–	
31	46080		3	60	3	$\mathbb{C}$	$\langle 11520,\cdot\rangle$	240	–	
			3	480	9	$\mathbb{C}$		1920	$\langle 1920,\cdot\rangle$	
			3	960	16	$\mathbb{C}$		3840	–	
32	155520		3	40	2	$\mathbb{C}$	$\langle 25920,\cdot\rangle$	240	–	MUB like
			3	360	6	$\mathbb{C}$		2160	–	
33	51840	5	2	40	2	$\mathbb{C}$	$\langle 25920,\cdot\rangle$	80	–	two angles
			2	45	2	$\mathbb{C}$		270	–	MUB like
			2	216	5	$\mathbb{C}$		432	–	
			2	540	7	$\mathbb{C}$		1080	–	

Example 5.3. (Maximal MUBs) We obtain a maximal set of MUBs in the dimensions

$$d = 2 \quad \text{(ST 6,7,8,9,10,11,13,15)},$$
$$d = 3 \quad \text{(ST 25, 26)},$$
$$d = 4 \quad \text{(ST 29)}.$$

These MUBs are unique [9] (Theorem 6.5), and they can be obtained from an orthogonal decomposition of the special linear Lie algebra $\mathrm{sl}_d(\mathbb{C})$.

Example 5.4. (Real MUBs) For the real Shephard-Todd group ST 28, we obtain a set of three MUBs for $\mathbb{R}^4$. This gives a 12-vector spherical $(2,2)$-design for $\mathbb{R}^4$. This appears to be the maximal number of real MUBs possible [8]. Further, were such a design to come from a tight spherical 5-design, then it would have 10 points (there is no such design), and so we suspect that this spherical $(2,2)$-design is optimal.

Table 5
The spherical (t,t)-designs of n vectors for $\mathbb{F}^d$ given by the highly symmetric tight frames for the Shephard-Todd listing of the primitive complex reflection groups. n = number of lines, m = orbit size (number of vectors), s = number of angles.

ST	Order	d	t	n	s	$\mathbb{F}$	Symmetry group	m	Group frame	Comments
34[†]	39191040	6	3	126	2	$\mathbb{C}$	$\langle 6531840, \cdot\rangle$	756	–	MUB like
			4	672	4	$\mathbb{C}$		4032	–	
			⋮	⋮	⋮	⋮				
35	51840		2	27	2	$\mathbb{R}$	$\langle 51840, \cdot\rangle$	27	$\langle 27,3\rangle$, $\langle 27,4\rangle$	two angles
			2	36	2	$\mathbb{R}$		72	–	MUB like
			2	216	6	$\mathbb{R}$		216	$\langle 216,86\rangle$, $\langle 216,88\rangle$	
			2	360	6	$\mathbb{R}$		720	–	
36[†]	2903040	7	2	28	1	$\mathbb{R}$	$\langle 1451520, \cdot\rangle$	56	$\langle 56,11\rangle$	equiangular
			2	63	2	$\mathbb{R}$		126	–	MUB like
			2	288	3	$\mathbb{R}$		576	–	
			2	378	4	$\mathbb{R}$		756	–	
			2	1008	6	$\mathbb{R}$		2016	–	
			2	2016	7	$\mathbb{R}$		4032	–	
			⋮	⋮	⋮	⋮				
37[†]	696729600	8	3	120	2	$\mathbb{R}$	$\langle 348364800, \cdot\rangle$	240	$\langle 240,89\rangle$	MUB like
			3	1080	4	$\mathbb{R}$		2160	–	
			3	3360	6	$\mathbb{R}$		6720	–	
			⋮	⋮	⋮	⋮				

† For the Shephard-Todd groups 34, 36 and 37, there are other maximal parabolic subgroups which generate highly symmetric tight frames (see Table 6), but the number of lines n is too high to determine any properties about them.

Table 6
Addendum to Table 2 of [10]. The highly symmetric tight frames of n vectors in $\mathbb{C}^d$ given by the reflection groups with Shephard-Todd numbers 34, 36, 37.

ST	d	Order	m[†]	b	s	Group frame
34	6	39191040	756	756	2	–
			4032	95256	4	–
			20412			
			54432			
			30240			
			272160			
			163296			
36	7	2903040	56	98	1	$\langle 56,11\rangle$
			126	392	2	–
			576	14112	3	–
			756	88200	4	–
			2016	1707552	6	–
			4032	5889312	7	–
			10080			
37	8	696729600	240	576	2	$\langle 240,89\rangle$
			2160	217800	4	–
			6720	5889312	6	–
			17280			
			60480			
			69120			
			241920			
			483840			

† In Table 2 of [10] m is labelled as n.

Example 5.5. (MUB like configurations) We will say that a (t,t)-design is **MUB like** if it has two angles, one of which is zero, but it is not a set of MUBs. We have the following MUB like spherical (t,t)-designs

$$\begin{aligned}
&40 \text{ vector } (3,3)\text{-design for } \mathbb{C}^4 \quad (\text{ST } 32, \text{ angles } \frac{1}{\sqrt{3}}, 0),\\
&45 \text{ vector } (2,2)\text{-design for } \mathbb{C}^5 \quad (\text{ST } 33, \text{ angles } \frac{1}{2}, 0),\\
&126 \text{ vector } (3,3)\text{-design for } \mathbb{C}^6 \quad (\text{ST } 34, \text{ angles } \frac{1}{2}, 0),\\
&36 \text{ vector } (2,2)\text{-design for } \mathbb{R}^6 \quad (\text{ST } 35, \text{ angles } \frac{1}{2}, 0),\\
&63 \text{ vector } (2,2)\text{-design for } \mathbb{R}^7 \quad (\text{ST } 36, \text{ angles } \frac{1}{2}, 0),\\
&120 \text{ vector } (2,2)\text{-design for } \mathbb{R}^8 \quad (\text{ST } 37, \text{ angles } \frac{1}{2}, 0).
\end{aligned}$$

We also have the following two angle (t,t)-designs

$$\begin{aligned}
&10 \text{ vector } (2,2)\text{-design for } \mathbb{R}^3 \quad (\text{ST } 23, \text{ angles } \frac{\sqrt{5}}{3}, \frac{1}{3}),\\
&40 \text{ vector } (2,2)\text{-design for } \mathbb{C}^5 \quad (\text{ST } 33, \text{ angles } \frac{1}{3}, \frac{1}{\sqrt{3}}),\\
&27 \text{ vector } (2,2)\text{-design for } \mathbb{R}^6 \quad (\text{ST } 35, \text{ angles } \frac{1}{4}, \frac{1}{2}).
\end{aligned}$$

6. Conclusion

We have shown how numerical techniques can be used to find putatively optimal spherical (t,t)-designs, from which explicit spherical designs can then be found. This process led to many known “tight” spherical designs, SICs and MUBs, as well as some new spherical (t,t)-designs with a high degree of symmetry, which we believe to be optimal. Some further insights into the geometry of the algebraic variety of optimal (t,t)-designs were obtained, e.g., the optimal spherical $(5,5)$-designs in $\mathbb{C}^2$ seem to be a lower dimensional subvariety of the optimal $(4,4)$-designs. We also investigated the spherical (t,t)-designs given by the class of highly symmetric tight frames for a complex reflection group. This gave unified description of many of the putatively optimal spherical (t,t)-designs, as well several MUB like designs with a small number of vectors.

Declaration of competing interest

There is no completing interest.

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Appendix A

Here we calculate the Hessian of the function $f : \mathbb{F}^{d\times n} \to \mathbb{R}$ of (2.10) whose critical points with value zero are the spherical (t,t)-designs.

We write each entry of $V = [v_1, \ldots, v_n] = [v_{jk}] \in \mathbb{F}^{d\times n}$ in the Cartesian form

$$v_{jk} = \begin{cases} x_{jk} + iy_{jk}, & \mathbb{F} = \mathbb{C}; \\ x_{jk}, & \mathbb{F} = \mathbb{R}, \end{cases}$$

and let

$$X = \{x_{\alpha\beta}\} \cup \{y_{\alpha\beta}\} \quad \text{for } \mathbb{F} = \mathbb{C}, \qquad X = \{x_{\alpha\beta}\} \quad \text{for } \mathbb{F} = \mathbb{R}.$$

We will refer to X as the real variables of a function $f : \mathbb{F}^{d\times n} \to \mathbb{R} : V \mapsto f(V)$, and define H_f the Hessian of f to be the Hessian of $X \mapsto f(V)$. Thus the Hessian of f at V is the $X \times X$ real symmetric matrix with (r,s)–entry given by

$$H_f(V)_{rs} = \frac{\partial^2 f}{\partial r \partial s}(V), \qquad r, s \in X.$$

Proposition A.1. *Let $f : \mathbb{F}^{d\times n} \to \mathbb{R}$ be the polynomial $f := p - c_t(\mathbb{F}^d)g^2$, where p and g are given by (2.9). Then the (r,s)-entry of the Hessian matrix of f is given by*

$$\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial^2 p}{\partial r \partial s} + 2c_t(\mathbb{F}^d)\Big\{g\frac{\partial^2 g}{\partial r \partial s} + \frac{\partial g}{\partial r}\frac{\partial g}{\partial s}\Big\}, \qquad r, s \in X,$$

where

$$\frac{\partial g}{\partial x_{\alpha\beta}}(V) + i\frac{\partial g}{\partial y_{\alpha\beta}}(V) = 2t\|v_\beta\|^{2(t-1)} v_{\alpha\beta},$$

$$\frac{\partial^2 g}{\partial x_{ab}\partial x_{\alpha\beta}}(V) + i\frac{\partial^2 g}{\partial y_{ab}\partial x_{\alpha\beta}}(V) = t\|v_\beta\|^{2(t-1)}\delta_{a\alpha}\delta_{b\beta} + 2t(t-1)\|v_\beta\|^{2(t-2)}\Re(v_{\alpha\beta})\delta_{b\beta}v_{a\beta},$$

$$\frac{\partial^2 g}{\partial x_{ab}\partial y_{\alpha\beta}}(V) + i\frac{\partial^2 g}{\partial y_{ab}\partial y_{\alpha\beta}}(V)$$
$$= it\|v_\beta\|^{2(t-1)}\delta_{a\alpha}\delta_{b\beta} + 2t(t-1)\|v_\beta\|^{2(t-2)}\operatorname{Im}(v_{\alpha\beta})\delta_{b\beta}v_{a\beta},$$

$$\frac{\partial^2 p}{\partial x_{ab}\partial x_{\alpha\beta}}(V)+i\frac{\partial^2 p}{\partial y_{ab}\partial x_{\alpha\beta}}(V)$$
$$=2t\sum_j|\langle v_j,v_\beta\rangle|^{2(t-1)}\big(\delta_{bj}v_{a\beta}v_{\alpha j}+\langle v_j,v_\beta\rangle\delta_{a\alpha}\delta_{bj}+\delta_{b\beta}v_{aj}\overline{v_{\alpha j}}\big)$$
$$+4t(t-1)\sum_j\Re\big(\langle v_\beta,v_j\rangle v_{\alpha j}\big)|\langle v_j,v_\beta\rangle|^{2(t-2)}\big(\langle v_j,v_\beta\rangle\delta_{bj}v_{a\beta}+\delta_{b\beta}v_{aj}\langle v_\beta,v_j\rangle\big),$$
$$\frac{\partial^2 p}{\partial x_{ab}\partial y_{\alpha\beta}}(V)+i\frac{\partial^2 p}{\partial y_{ab}\partial y_{\alpha\beta}}(V)$$
$$=2it\sum_j|\langle v_j,v_\beta\rangle|^{2(t-1)}\big(\langle v_j,v_\beta\rangle\delta_{a\alpha}\delta_{bj}+\delta_{b\beta}v_{aj}\overline{v_{\alpha j}}-\delta_{bj}v_{a\beta}v_{\alpha j}\big)$$
$$+4t(t-1)\sum_j\operatorname{Im}\big(\langle v_\beta,v_j\rangle v_{\alpha j}\big)|\langle v_j,v_\beta\rangle|^{2(t-2)}\big(\langle v_j,v_\beta\rangle\delta_{bj}v_{a\beta}+\delta_{b\beta}v_{aj}\langle v_\beta,v_j\rangle\big).$$

Proof. The first equation follows from the product rule, i.e.,

$$\frac{\partial^2}{\partial r\partial s}(g^2)=\frac{\partial}{\partial r}\Big(2g\frac{\partial g}{\partial s}\Big)=2g\frac{\partial^2 g}{\partial r\partial s}+2\frac{\partial g}{\partial r}\frac{\partial g}{\partial s}.$$

To find the entries of the Hessians of p and g, we use the Wirtinger calculus:

$$\frac{\partial}{\partial\overline{z}}:=\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big),\qquad\frac{\partial}{\partial\overline{z}}(z)=0,\quad\frac{\partial}{\partial\overline{z}}(\overline{z})=1,$$
$$\frac{\partial h}{\partial x}=2\Re\Big(\frac{\partial h}{\partial\overline{z}}\Big),\qquad\frac{\partial h}{\partial y}=2\operatorname{Im}\Big(\frac{\partial h}{\partial\overline{z}}\Big)\qquad\text{(for } h \text{ real-valued).}$$

We have

$$\frac{\partial g}{\partial\overline{v_{\alpha\beta}}}(V)=\frac{\partial}{\partial\overline{v_{\alpha\beta}}}\Big(\sum_\ell\big(\|v_\ell\|^2\big)^t\Big)=\sum_\ell t\|v_\ell\|^{2(t-1)}\frac{\partial}{\partial\overline{v_{\alpha\beta}}}\sum_j v_{j\ell}\overline{v_{j\ell}}=t\|v_\beta\|^{2(t-1)}v_{\alpha\beta}.$$

Since

$$\frac{\partial}{\partial\overline{v_{\alpha\beta}}}\langle v_j,v_k\rangle=\frac{\partial}{\partial\overline{v_{\alpha\beta}}}\sum_s v_{sj}\overline{v_{sk}}=\delta_{k\beta}v_{\alpha j},$$

we have

$$\frac{\partial p}{\partial\overline{v_{\alpha\beta}}}(V)=\sum_j\sum_k t|\langle v_j,v_k\rangle|^{2(t-1)}\frac{\partial}{\partial\overline{v_{\alpha\beta}}}\Big(\langle v_j,v_k\rangle\langle v_k,v_j\rangle\Big)$$
$$=\sum_j\sum_k t|\langle v_j,v_k\rangle|^{2(t-1)}\Big(\delta_{k\beta}v_{\alpha j}\langle v_k,v_j\rangle+\langle v_j,v_k\rangle\delta_{j\beta}v_{\alpha k}\Big)$$
$$=\sum_j t|\langle v_j,v_\beta\rangle|^{2(t-1)}v_{\alpha j}\langle v_\beta,v_j\rangle+\sum_k t|\langle v_\beta,v_k\rangle|^{2(t-1)}\langle v_\beta,v_k\rangle v_{\alpha k}$$

$$= 2t \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} \langle v_\beta, v_j \rangle v_{\alpha j}.$$

We now consider the second partials. Since

$$\frac{\partial g}{\partial x_{\alpha\beta}}(V) = t\|v_\beta\|^{2(t-1)}(v_{\alpha\beta} + \overline{v_{\alpha\beta}}), \qquad \frac{\partial g}{\partial y_{\alpha\beta}}(V) = t\|v_\beta\|^{2(t-1)} i(\overline{v_{\alpha\beta}} - v_{\alpha\beta}),$$

we have

$$\frac{\partial^2 g}{\partial \overline{v_{ab}} \partial x_{\alpha\beta}}(V) = t\|v_\beta\|^{2(t-1)} \delta_{a\alpha}\delta_{b\beta} + t(t-1)\|v_\beta\|^{2(t-2)}(v_{\alpha\beta} + \overline{v_{\alpha\beta}})\delta_{b\beta} v_{a\beta},$$

$$\frac{\partial^2 g}{\partial \overline{v_{ab}} \partial y_{\alpha\beta}}(V) = it\|v_\beta\|^{2(t-1)} \delta_{a\alpha}\delta_{b\beta} + t(t-1)\|v_\beta\|^{2(t-2)} i(\overline{v_{\alpha\beta}} - v_{\alpha\beta})\delta_{b\beta} v_{a\beta}.$$

Since

$$\frac{\partial p}{\partial x_{\alpha\beta}}(V) = 2t \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} (\langle v_\beta, v_j \rangle v_{\alpha j} + \overline{\langle v_\beta, v_j \rangle v_{\alpha j}}),$$

we have

$$\frac{\partial^2 p}{\partial \overline{v_{ab}} \partial x_{\alpha\beta}}(V) = 2t \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} \frac{\partial}{\partial \overline{v_{ab}}} \big(\langle v_\beta, v_j \rangle v_{\alpha j} + \langle v_j, v_\beta \rangle \overline{v_{\alpha j}}\big)$$

$$+ 2t \sum_j 2\Re\big(\langle v_\beta, v_j \rangle v_{\alpha j}\big)(t-1)|\langle v_j, v_\beta \rangle|^{2(t-2)} \frac{\partial}{\partial \overline{v_{ab}}} \big(\langle v_j, v_\beta \rangle \langle v_\beta, v_j \rangle\big)$$

$$= 2t \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} \big(\delta_{bj} v_{a\beta} v_{\alpha j} + \langle v_j, v_\beta \rangle \delta_{a\alpha}\delta_{bj} + \delta_{b\beta} v_{aj} \overline{v_{\alpha j}}\big)$$

$$+ 4t(t-1) \sum_j \Re\big(\langle v_\beta, v_j \rangle v_{\alpha j}\big) |\langle v_j, v_\beta \rangle|^{2(t-2)} \big(\langle v_j, v_\beta \rangle \delta_{bj} v_{a\beta} + \delta_{b\beta} v_{aj} \langle v_\beta, v_j \rangle\big).$$

Similarly, since

$$\frac{\partial p}{\partial y_{\alpha\beta}}(V) = 2t \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} i(\overline{\langle v_\beta, v_j \rangle v_{\alpha j}} - \langle v_\beta, v_j \rangle v_{\alpha j}),$$

we have

$$\frac{\partial^2 p}{\partial \overline{v_{ab}} \partial y_{\alpha\beta}}(V) = 2it \sum_j |\langle v_j, v_\beta \rangle|^{2(t-1)} \big(\langle v_j, v_\beta \rangle \delta_{a\alpha}\delta_{bj} + \delta_{b\beta} v_{aj} \overline{v_{\alpha j}} - \delta_{bj} v_{a\beta} v_{\alpha j}\big)$$

$$+ 4t(t-1) \sum_j \operatorname{Im}\big(v_{\alpha j} \langle v_\beta, v_j \rangle\big) |\langle v_j, v_\beta \rangle|^{2(t-2)} \big(\langle v_j, v_\beta \rangle \delta_{bj} v_{a\beta} + \delta_{b\beta} v_{aj} \langle v_\beta, v_j \rangle\big). \qquad \square$$

References

[1] N.N. Andreev, A minimal design of order 11 on the three-dimensional sphere, Mat. Zametki 67 (4) (2000) 489–497.

[2] Etsuko Bannai, On antipodal Euclidean tight $(2e+1)$-designs, J. Algebraic Comb. 24 (4) (2006) 391–414.

[3] Eiichi Bannai, Etsuko Bannai, A survey on spherical designs and algebraic combinatorics on spheres, Eur. J. Comb. 30 (6) (2009) 1392–1425.

[4] P. Boyvalenkov, D. Danev, Uniqueness of the 120-point spherical 11-design in four dimensions, Arch. Math. (Basel) 77 (4) (2001) 360–368.

[5] E. Bannai, S.G. Hoggar, Tight t-designs and squarefree integers, Eur. J. Comb. 10 (2) (1989) 113–135.

[6] Peter Boyvalenkov, Linear programming bounds for spherical (k,k)-designs, arXiv e-prints, page, arXiv:2004.00659, April 2020.

[7] J. Bramwell, On the Existence of Spherical (t,t)–Designs. Honours Project, vol. 1, University of Auckland, 2011.

[8] P.O. Boykin, M. Sitharam, M. Tarifi, P. Wocjan, Real mutually unbiased bases, eprint, arXiv: quant-ph/0502024, February 2005.

[9] P. Oscar Boykin, Meera Sitharam, Pham Huu Tiep, Pawel Wocjan, Mutually unbiased bases and orthogonal decompositions of Lie algebras, Quantum Inf. Comput. 7 (4) (2007) 371–382.

[10] Helen Broome, Shayne Waldron, On the construction of highly symmetric tight frames and complex polytopes, Linear Algebra Appl. 439 (12) (2013) 4135–4151.

[11] Jameson Cahill, Dustin G. Mixon, Nate Strawn, Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames, SIAM J. Appl. Algebra Geom. 1 (1) (2017) 38–72.

[12] Tuan-Yow Chien, Shayne Waldron, A characterization of projective unitary equivalence of finite frames and applications, SIAM J. Discrete Math. 30 (2) (2016) 976–994.

[13] Tuan-Yow Chien, Shayne Waldron, Nice error frames, canonical abstract error groups and the construction of SICs, Linear Algebra Appl. 516 (2017) 93–117.

[14] Tuan-Yow Chien, Shayne Waldron, The projective symmetry group of a finite frame, N.Z. J. Math. 48 (2018) 55–81.

[15] P. Delsarte, J.M. Goethals, J.J. Seidel, Spherical codes and designs, Geom. Dedic. 6 (3) (1977) 363–388.

[16] Yonina C. Eldar, G. David Forney Jr., Optimal tight frames and quantum measurement, IEEE Trans. Inf. Theory 48 (3) (2002) 599–610.

[17] S.G. Hoggar, t-designs in projective spaces, Eur. J. Comb. 3 (3) (1982) 233–254.

[18] S.G. Hoggar, Tight 4- and 5-designs in projective spaces, Graphs Comb. 5 (1) (1989) 87–94.

[19] R.H. Hardin, N.J.A. Sloane, McLaren's improved snub cube and other new spherical designs in three dimensions, Discrete Comput. Geom. 15 (4) (1996) 429–441.

[20] Daniel Hughes, Shayne Waldron, Spherical half-designs of high order, Involve 13 (2) (2020) 193–2030.

[21] N.O. Kotelina, A.B. Pevnyi, The Venkov inequality with weights and weighted spherical half-designs, J. Math. Sci. (N.Y.) 173 (6) (2011) 674–682, Problems in mathematical analysis. No. 55.

[22] N.O. Kotelina, A.B. Pevnyĭ, Complex spherical semidesigns, Izv. Vysš. Učebn. Zaved., Mat. 5 (2017) 54–60.

[23] Andreas Klappenecker, Martin Rötteler, Constructions of mutually unbiased bases, in: Finite Fields and Applications, in: Lecture Notes in Comput. Sci., vol. 2948, Springer, Berlin, 2004, pp. 137–144.

[24] Gustav I. Lehrer, Donald E. Taylor, Unitary Reflection Groups, Australian Mathematical Society Lecture Series, vol. 20, Cambridge University Press, Cambridge, 2009.

[25] Tsuyoshi Miezaki, Makoto Tagami, On Euclidean designs and potential energy, Electron. J. Comb. 19 (1) (2012) 2.

[26] Mozhgan Mohammadpour, Shayne Waldron, Constructing high order spherical designs as a union of two of lower order, arXiv e-prints, page, arXiv:1912.07151, December 2019.

[27] Aidan Roy, A.J. Scott, Weighted complex projective 2-designs from bases: optimal state determination by orthogonal measurements, J. Math. Phys. 48 (7) (2007) 072110.

[28] A.J. Scott, M. Grassl, Symmetric informationally complete positive-operator-valued measures: a new computer study, J. Math. Phys. 51 (4) (2010) 042203.

[29] P.D. Seymour, Thomas Zaslavsky, Averaging sets: a generalization of mean values and spherical designs, Adv. Math. 52 (3) (1984) 213–240.

[30] D.E. Taylor, Reflection subgroups of finite complex reflection groups, J. Algebra 366 (2012) 218–234.

[31] D.E. Taylor, Normaliser complements for the parabolic subgroups of finite unitary reflection groups, Notes 12 (2017).

[32] Shayne Waldron, Generalized Welch bound equality sequences are tight frames, IEEE Trans. Inf. Theory 49 (9) (2003) 2307–2309.

[33] Shayne Waldron, A sharpening of the Welch bounds and the existence of real and complex spherical t-designs, IEEE Trans. Inf. Theory 63 (11) (2017) 6849–6857.

[34] Shayne F.D. Waldron, An Introduction to Finite Tight Frames. Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2018.

[35] Gerhard Zauner, Quantum Designs: Foundations of a non-commutative Design Theory, PhD thesis, University of Vienna, 2010, University of Vienna, 1, English translation of 1999 Doctorial thesis including a new preface.