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Nice error frames, canonical abstract error groups and the construction of SICs

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ABSTRACT

Nice error bases are generalisations of the Pauli matrices which have applications in quantum information theory. These orthonormal bases for the $d \times d$ matrices $M_d(\mathbb{C})$ also generalise the projective action of the Heisenberg group on $\mathbb{C}^d$. Here we extend nice error bases to *nice error frames*. These are equal-norm tight frames for $M_d(\mathbb{C})$ consisting of $d \times d$ unitary matrices with a group indexing structure. We show that each nice error frame (irreducible faithful projective representation) is associated with a *canonical* abstract error group. This is calculated in number of examples, e.g., for all nice error bases for $d < 14$, which then allows us to investigate which nice error bases might give rise to SICs (symmetric informationally complete positive operator valued measures). These results show that the current catalogue of nice error bases over counts. In particular, we give an explicit example of a SIC for $d = 6$ with a nonabelian index group, and show that the Hoggar lines appear for various nice error bases, some of which are subgroups of the Clifford group. Thus all known SICs appear as orbits of subgroups of the Clifford group.

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1. Introduction

1.1. Motivation

Unitary operator bases were introduced by Schwinger [1] as quantum variables of a physical system. They were specialised to nice error bases by Knill [2,3] to construct quantum error correcting codes, and have been applied to quantum teleportation and dense coding schemes [4]. Our interest stems from their application to the construction of SICs (equiangular lines) [5,6].

Nice error bases are orthonormal bases for the $d \times d$ matrices $M_d(\mathbb{C})$, which generalise the Pauli matrices, and the projective action of the Heisenberg group on $\mathbb{C}^d$. Here we extend nice error bases to *nice error frames*. These are equal-norm tight frames for $M_d(\mathbb{C})$ consisting of $d \times d$ unitary matrices with a group indexing structure. There is growing evidence [7] that nice error frames play a similar role in the construction of complex (projective) spherical t-designs (quantum t-designs) with the minimal number of vectors, as do nice error bases in the special case of SICs.

We show that each nice error frame (irreducible faithful projective representation) is associated with a *canonical* abstract error group. In addition to giving a unique label for nice error frames, this allows us to exhaustively search through all possible groups in the small groups library. The specific structure of a canonical abstract error group makes this feasible, even in the case $d = 8$, where there are $10, 494, 213$ possible groups to consider.

1.2. Outline

In Section 2, we define nice error frames, and show that their matrices can be scaled in such a way that they lie in a canonical (abstract error) group. In Section 3, we outline how all nice error frames (irreducible faithful projective representations) can be constructed from the ordinary representations of abstract groups. This leads to a parallelisable algorithm for calculating nice error frames, which we apply in a number of cases. In Section 4, we give examples of nice error frames, which are not bases. In Section 5, we calculate the canonical abstract error groups for all nice error bases in dimension $d < 14$, and discuss how these relate to the Klappenecker and Rötteler *Catalogue of Nice Error Bases*, and the known SICs. Our classification allows nice error bases to be compared easily, and using it we show that the catalogue over counts. In particular, we show that the Hoggar lines appear for various nice error bases, some of which are subgroups of the Clifford group. Thus all known SICs appear as orbits of subgroups of the Clifford group. In Section 6, for $d = 6$ we give an explicit example of a SIC given by a nice error basis with a nonabelian index group.

2. Nice error frames and canonical abstract error groups

The **Pauli matrices**

$$\sigma_1 = \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

were first used to study spin in quantum mechanics. Together with the identity, they form an orthonormal basis for the 2×2 matrices. More generally, in the field of quantum error correcting codes, an **error operator basis** is an orthonormal basis for the $d\times d$ matrices $M_d(\mathbb{C}) = \mathbb{C}^{d\times d}$, with the (Hilbert–Schmidt/Frobenius) inner product

$$\langle A, B\rangle := \operatorname{trace}(AB^*), \qquad A, B \in M_d(\mathbb{C}).$$

A special class of these was defined by Knill [2,3], as follows.

Definition 2.1. Let G be a group of order d^2. Then unitary matrices $(E_g)_{g\in G}$ in $M_d(\mathbb{C})$ are a **nice** (**unitary**) **error basis** if

1. E_1 is a scalar multiple of the identity I,
2. $E_g E_h = w_{g,h} E_{gh}$, $\forall g, h \in G$, where $w_{g,h} \in \mathbb{C}$,
3. $\operatorname{trace}(E_g) = 0$, $g \neq 1$, $g \in G$ (i.e., they are an error operator basis),

and G is referred to as the **index group**.

In the language of group theory (cf. [8,9]), this is equivalent to the map

$$\rho : g \mapsto E_g$$

being a *unitary irreducible faithful projective representation of* G *of degree* d. Condition 3 ensures that a nice error basis gives the orthogonal expansion

$$A = \frac{1}{d} \sum_{g\in G} \langle A, E_g\rangle E_g, \qquad \forall A \in M_d(\mathbb{C}). \tag{2.2}$$

A motivating example is the Pauli matrices indexed by $\mathbb{Z}_2 \times \mathbb{Z}_2$ as follows

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \mapsto M_2(\mathbb{C}) : (j, k) \mapsto E_{(j,k)} = S^j \Omega^k, \qquad S := \sigma_1,\ \Omega := \sigma_3.$$

A finite sequence of vectors $(f_j)_{j=1}^n$ in a Hilbert space $\mathcal{H}$ is a **tight frame** for $\mathcal{H}$ if there is a $C > 0$, such that

$$C\,\|f\|^2 = \sum_{j=1}^n |\langle f, f_j\rangle|^2, \qquad \forall f \in \mathcal{H}. \tag{2.3}$$

Tight frames generalise orthonormal bases, since (2.3) is equivalent (by the polarisation identity) to the expansion

$$f = \frac{1}{C}\sum_{j=1}^{n}\langle f, f_j\rangle f_j, \qquad \forall f \in \mathcal{H},$$

where $C\dim(\mathcal{H}) = \sum_j \|f_j\|^2$, which is their point of interest. Being a tight frame is equivalent to the variational characterisation (cf. [10])

$$\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle f_j, f_k\rangle|^2 = \frac{1}{\dim(\mathcal{H})}\Big(\sum_{j=1}^{n}\|f_j\|^2\Big)^2. \tag{2.4}$$

Suppose that $\mathcal{H} = M_d(\mathbb{C})$, and $(E_g)_{g\in G}$ are unitary matrices satisfying 1, 2 of Definition 2.1, where G is *any* finite group. These matrices have the equal norms

$$\|E_g\|^2 = \langle E_g, E_g\rangle = \operatorname{trace}(E_g E_g^*) = \operatorname{trace}(I) = d.$$

Since $w_{gh^{-1},h}E_gE_h^{-1} = E_{gh^{-1}}$, we have

$$\langle E_g, E_h\rangle = \operatorname{trace}(E_gE_h^*) = \operatorname{trace}(E_gE_h^{-1}) = \frac{1}{w_{gh^{-1},h}}\operatorname{trace}(E_{gh^{-1}}).$$

Hence the condition (2.4) for $(E_g)_{g\in G}$ to be a tight frame for $M_d(\mathbb{C})$ reduces to

$$\sum_{g\in G}\sum_{h\in G}|\langle E_g, E_h\rangle|^2 = \sum_{g\in G}\sum_{h\in G}|\operatorname{trace}(E_{gh^{-1}})|^2 = |G|\sum_{g\in G}|\operatorname{trace}(E_g)|^2 = \frac{1}{d^2}\Big(|G|d\Big)^2.$$

Thus we arrive at the following definition.

Definition 2.2. Let G be a group (of order $\geq d^2$). Then unitary matrices $(E_g)_{g\in G}$ in $M_d(\mathbb{C})$ are a **nice** (**unitary**) **error frame** if

1. E_1 is a scalar multiple of the identity I, and no other E_g is,
2. $E_gE_h = w_{g,h}E_{gh}$, $\forall g, h \in G$, where $w_{g,h} \in \mathbb{C}$,
3. $\sum_{g\in G}|\operatorname{trace}(E_g)|^2 = |G|$,

and G is referred to as the **index group**.

As we just observed, a nice error frame is an equal-norm tight frame for $M_d(\mathbb{C})$, i.e.,

$$A = \frac{d}{|G|}\sum_{g\in G}\langle A, E_g\rangle E_g, \qquad \forall A \in M_d(\mathbb{C}). \tag{2.5}$$

Moreover, nice error frames generalise nice error bases, since condition 3 can be written as

$$\sum_{\substack{g\neq 1\\ g\in G}} |\operatorname{trace}(E_g)|^2 = |G| - d^2,$$

which, for $|G| = d^2$, gives

$$\sum_{\substack{g\neq 1\\ g\in G}} |\operatorname{trace}(E_g)|^2 = 0 \quad \Longrightarrow \quad \operatorname{trace}(E_g) = 0, \quad g \neq 1.$$

The conditions 1 and 2 say that

$g \mapsto E_g$ is a unitary faithful projective representation of G of degree d.

It is also *irreducible*, i.e., $\operatorname{span}\{E_g w\}_{g\in G} = \mathbb{C}^d$, $\forall w \neq 0$. To see this, expand the matrix $A = vw^*$, $v \in \mathbb{C}^d$, using (2.5) to obtain

$$vw^* = \frac{d}{|G|}\sum_{g\in G}\langle vw^*, E_g\rangle E_g \quad \Longrightarrow \quad v\|w\|^2 = \frac{d}{|G|}\sum_{g\in G}\langle vw^*, E_g\rangle E_g w.$$

These properties characterise nice error frames (Proposition 2.8).

In general, the matrices $(E_g)_{g\in G}$ of a nice error frame will not have finite order, and hence not generate a finite group. This can be rectified by scaling. Let ω be the d-th root of unity

$$\omega := e^{\frac{2\pi i}{d}}.$$

Since $\det(cA) = c^d \det(A)$, $c \in \mathbb{C}$, $A \in M_d(\mathbb{C})$, we have:

Key observation. *There are* exactly d *scalings of a given E_g which have determinant* 1*, i.e.,*

$$\hat{E}_g = \frac{\omega^j}{\det(E_g)^{1/d}} E_g, \qquad j = 0, 1, \ldots, d-1,$$

where $\det(E_g)^{1/d}$ *is any fixed d-th root of* $\det(E_g)$*.*

Henceforth, let $\hat{E}_g$ denote any of these scalings, so that

$$\det(\hat{E}_g) = 1, \qquad \forall g \in G.$$

Then the $d|G|$ matrices

$$H := \{\omega^j \hat{E}_g : j = 0, \dots, d-1, g \in G\}$$

are distinct. Moreover, they form a group, since after scaling, the condition 2 becomes

$$\hat{E}_g \hat{E}_h = \hat{w}_{g,h} \hat{E}_{gh}, \qquad \forall g, h \in G,$$

and taking determinants of this gives

$$1 = \hat{w}_{g,h}^d \quad \Longrightarrow \quad \hat{w}_{g,h} \in \{1, \omega, \omega^2, \dots \omega^{d-1}\}, \quad \forall g, h \in G.$$

Thus, we arrive at the following definition.

Definition 2.3. Let $(E_g)_{g\in G}$ be a nice error frame for $M_d(\mathbb{C})$. The associated **canonical error group** is

$$H := \{\omega^j \hat{E}_g : j = 0, \dots, d-1,\ g \in G\},$$

and the abstract version of this group is called the **canonical abstract error group**.

We observe that the centre of a canonical error group H is

$$Z(H) = \langle \omega I \rangle \cong \mathbb{Z}_d,$$

since if a matrix commutes with the spanning set $(E_g)_{g\in G}$ for $M_d(\mathbb{C})$, then it commutes with all of $M_d(\mathbb{C})$, and is therefore a scalar matrix. Hence, a group can be a canonical (abstract) error group for at most one dimension d. Further, the index group G of a canonical (abstract) error group H is given by

$$G = \frac{H}{Z(H)}.$$

We will label (abstract) groups according to the “Small Groups Library”, which is used by the computer algebra package `magma`, e.g., the dihedral group of order 8 is

$$D_4 = \texttt{SmallGroup(8,3)} = \texttt{<8,3>}.$$

Example 2.4. The Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ have determinant -1. They generate the group `<16,13>` of order 16. The group generated by just the reflections σ_1 and σ_3 contains $\pm i\sigma_2$, and has order 8. It is the *dihedral group* `<8,3>`. The canonical error group for the nice error basis $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is

$$H = \langle i\sigma_1, i\sigma_2, i\sigma_3 \rangle,$$

which is the *quaternion group* `<8,4>`.

In earlier work (cf. [3,8]), any of the groups `<16,13>`, `<8,3>`, `<8,4>` would have been referred to as an *abstract error group* or *w-covering* of the nice error basis $\{I, \sigma_1, \sigma_2, \sigma_3\}$.

We now specify which nice error frames are considered to be "equivalent".

Definition 2.5. Nice error frames $(E_g)_{g\in G}$ and $(F_h)_{h\in H}$ for $M_d(\mathbb{C})$ are **equivalent** if there is bijection $\sigma : G \to H$ between their index groups, scalars $(c_g)_{g\in G}$ and an invertible $T \in M_d(\mathbb{C})$, such that

$$F_{\sigma g} = c_g T^{-1} E_g T, \qquad \forall g \in G. \tag{2.6}$$

This is more general than *projective equivalence* (cf. [8]) where $G = H$, and reindexing of the elements of $(E_g)_{g\in G}$ is not allowed.

Proposition 2.6. *Equivalent nice error frames have the* same *canonical abstract error group, and (in particular) the same index group.*

Proof. Suppose that nice error frames $(E_g)_{g\in G}$ and $(F_h)_{h\in H}$ for $M_d(\mathbb{C})$ are equivalent. Then (2.6) scales to

$$\hat{F}_{\sigma g} = T^{-1}(\hat{c}_g E_g)T, \qquad \forall g \in G,$$

where $\hat{c}_g \in \{1, \omega, \omega^2, \dots, \omega^{d-1}\}$ (by considering determinants). Thus the canonical error groups are conjugate via T, and so are isomorphic. Since the index group is the abstract error group factored by its centre, the nice error frames also have the same index groups. □

Example 2.7 *(***Heisenberg nice error basis***).* A nice error basis (projective representation) is given by

$$G = \mathbb{Z}_d \times \mathbb{Z}_d \mapsto M_d(\mathbb{C}) : (j,k) \mapsto E_{(j,k)} = S^j \Omega^k,$$

where S is the cyclic shift matrix, and Ω is the modulation matrix, given by

$$(S)_{jk} := \delta_{j,k+1}, \qquad (\Omega)_{jk} = \omega^j \delta_{j,k}, \quad \omega := e^{\frac{2\pi i}{d}}. \tag{2.7}$$

This is the only nice error basis (up to equivalence) for $M_d(\mathbb{C})$ with index group $G = \mathbb{Z}_d \times \mathbb{Z}_d$ (cf. [11]).

We are now in a position to show that the condition 3 of Definition 2.2 is indeed equivalent to the projective representation $g \mapsto E_g$ being *irreducible*. This generalises Theorem 1 of [8] for nice error bases.

Proposition 2.8 *(Characterisation). Let G be a group and $(E_g)_{g\in G}$ be unitary matrices in $M_d(\mathbb{C})$. Then the following are equivalent:*

1. *$(E_g)_{g\in G}$ is a nice error frame for $M_d(\mathbb{C})$.*
2. *$g \mapsto E_g$ is an irreducible unitary faithful projective representation of G of degree d.*

In this case, the action of the canonical error group H on $\mathbb{C}^d$ is an irreducible special unitary faithful ordinary representation of the canonical abstract error group of dimension d.

Proof. If $g \mapsto E_g$ is an irreducible unitary projective representation of G on $\mathbb{C}^d$, then the canonical error group H can be defined, as above. Its action on $\mathbb{C}^d$ (via multiplication) gives an *irreducible* unitary ordinary representation of H. The corresponding character χ is irreducible, and so its Euclidean inner product with itself is $|H| = d|G|$, which gives

$$\langle \chi, \chi \rangle = \sum_{h\in H} |\operatorname{trace}(h)|^2 = \sum_{j=0}^{d-1} \sum_{g\in G} |\operatorname{trace}(\omega^j E_g)|^2 = d \sum_{g\in G} |\operatorname{trace}(E_g)|^2 = d|G| = |H|,$$

which is the condition 3 of Definition 2.2. Combining this with the previous observations gives the equivalence of 1 and 2 above. □

Example 2.9. For $d = 1$, the only canonical abstract error group is $H = 1$.

Example 2.10. For $d = 2$, a canonical error group is given by the **generalised quaternion group** or **dicyclic group** of order $4n$ $(n > 1)$, which is generated by the matrices

$$\begin{pmatrix} \omega_{2n} & 0 \\ 0 & \omega_{2n}^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \omega_{2n} := e^{\frac{2\pi i}{2n}}.$$

A `magma` calculation (see Table 1 of the appendix) shows that the only other canonical abstract error groups H of order ≤ 200 (with $d = 2$) are

$$H = \texttt{<24,3>},\ G = \texttt{<12,3>}, \qquad H = \texttt{<48,28>},\ G = \texttt{<24,12>}.$$

These turn out to be the canonical abstract error groups obtained from the Shephard–Todd reflection groups numbers 4 and 8, respectively. In view of Proposition 2.8, all canonical abstract error groups for $d = 2$ are given by the *ADE classification* of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ [12].

Example 2.11. Any irreducible group of $d \times d$ matrices which has a finite quotient with its centre gives rise to a canonical abstract error group. In particular, by the N/C theorem, the normaliser of such a group in $GL_d(\mathbb{C})$ also does. As an example, take the Heisenberg group which is generated by the matrices S and Ω of Example 2.7, and its normaliser (often called the **Clifford group**) which is generated by S, Ω and the matrices F and R, given by

$$(F)_{jk} := \frac{1}{\sqrt{d}}\omega^{-jk}, \qquad (R)_{jk} := \mu^{j(j+d)}\delta_{jk}, \qquad \omega := e^{\frac{2\pi i}{d}},\ \mu := e^{\frac{2\pi i}{2d}}. \tag{2.8}$$

Then the canonical abstract error groups for the Heisenberg group and its normaliser are

`<8,4>`, `<48,28>` $(d = 2)$ `<27,3>`, `<648,532>` $(d = 3)$.

For a general d, the Clifford group gives a canonical abstract error group H, with order

$$|H| = d^6 \prod_{p|d}\Big(1 - \frac{1}{p^2}\Big) \qquad (p \text{ the prime factors of } d).$$

For each d, the subgroups of the Clifford group which contain the Heisenberg group give a family of nice error frames. Another way that nice error frames can be constructed (and also deconstructed) is via tensor products.

Proposition 2.12. *Let $(E_{g_1})_{g_1\in G_1}$, $(F_{g_2})_{g_2\in G_2}$ be nice error frames for $M_{d_1}(\mathbb{C})$, $M_{d_2}(\mathbb{C})$. Then their tensor product*

$$(E_{g_1} \otimes F_{g_2})_{(g_1,g_2)\in G_1\times G_2}$$

is a nice error frame for $M_{d_1 d_2}(\mathbb{C})$. In particular, a product of index groups is an index group. Moreover, the canonical error group is

$$H = \{\omega^j(h_1 \otimes h_2) : 0 \le j < d-1, h_1 \in H_1, h_2 \in H_2\}, \qquad \omega := e^{\frac{2\pi i}{d}}, \quad d := d_1 d_2,$$

where H_1, H_2 are the canonical error groups of the nice error frames.

Proof. In view of Proposition 2.8, the first part follows from the theory of (projective) representations. Alternatively, it can be verified directly, e.g., the tensor product satisfies condition 3 of Definition 2.2 since

$$\sum_{(g_1,g_2)\in G_1\times G_2} |\operatorname{trace}(E_{g_1} \otimes F_{g_2})|^2 = \sum_{g_1}\sum_{g_2} |\operatorname{trace}(E_{g_1})\operatorname{trace}(F_{g_2})|^2$$
$$= \Big(\sum_{g_1} |\operatorname{trace}(E_{g_1})|^2\Big)\Big(\sum_{g_2} |\operatorname{trace}(F_{g_2})|^2\Big) = |G|\,|H| = |G \times H|.$$

The tensor product group $H_1 \otimes H_2$ consists of scalar multiples of each $E_{g_1} \otimes E_{g_2}$, with determinant 1, but may not contain all d-roots of unity (if d_1 and d_2 are not coprime), and so we add these. $\square$

Corollary 2.13. *A product of index groups is an index group, and in particular, a product of index groups for nice error bases is an index group for a nice error basis.*

Example 2.14. Let K be a finite abelian group of order d. Since K is a product of cyclic groups, it follows by taking tensor products of the Heisenberg nice error basis (Example 2.7) that $G = K \times K$ is the index group of a nice error basis for $M_d(\mathbb{C})$.

Example 2.15. Taking the tensor product of the two nonabelian index groups for $d = 4$, with the (abelian) index group for $d = 2$, gives two nonabelian index groups for $d = 8$, i.e.,

$$\texttt{<16,3>} \times \texttt{<4,2>} = \texttt{<64,193>}, \qquad \texttt{<16,11>} \times \texttt{<4,2>} = \texttt{<64,261>}.$$

Basic results from character theory imply the following.

Theorem 2.16 *(Abelian index groups). A nice error frame can have an abelian index group only if it is a nice error basis.*

Proof. Let H be the canonical abstract error group of a nice error frame, and $\chi : H \to \mathbb{C}$ be the character of a faithful irreducible representation of degree $\chi(1) = d$.

Recall the **centre** of a character $\chi : H \to \mathbb{C}$ is the subgroup

$$Z(\chi) := \{h \in H : |\chi(h)| = \chi(1)\},$$

and that if χ is irreducible

$$\frac{Z(\chi)}{\ker(\chi)} = Z\Big(\frac{H}{\ker(\chi)}\Big), \qquad \ker(\chi) := \{h \in H : \chi(h) = \chi(1)\}.$$

Since the representation is faithful, $\ker(\chi) = 1$, and so this becomes

$$Z(\chi) = Z(H).$$

Thus the index group is $G = H/Z(\chi)$, by Theorem 2.13 of [13], if G is abelian, then

$$|G| = [H : Z(\chi)] = \chi(1)^2 = d^2. \quad \square$$

Remark 2.17. A canonical error group is an example of a *central group frame*, i.e.,

$$\Phi = \big(\rho(g)\big)_{h \in H}, \qquad \text{where } \rho : H \to SL_d(\mathbb{C}) \text{ is a representation}$$

is a tight frame for $M_d(\mathbb{C})$ which satisfies the “symmetry condition”

$$\langle h\phi, g\phi \rangle = \langle h\psi, g\psi \rangle, \qquad \forall g, h \in \rho(H), \quad \forall \phi, \psi \in \Phi.$$

3. Calculations

Finding the centre of a group and its irreducible representations are fast calculations, and a representation can always be made unitary (by an appropriate conjugation). Thus the following characterisation of abstract error groups gives a practical algorithm for their calculation, and hence that of the nice error frames they correspond to.

Proposition 3.1 *(Algorithm). A group H is a canonical abstract error group if and only if:*

1. *Its centre $Z(H)$ is cyclic of order d.*
2. *It has a faithful irreducible ordinary representation ρ of degree d, which is special, i.e.,* $\det(h) = 1$, $\forall h \in H$.

In particular, for $d > 1$ all canonical abstract error groups are nonabelian.

The nice error frame given by such a representation is $(E_g)_{g\in G}$, where

$$G := \frac{H}{Z(H)}, \qquad E_g \in \rho(g).$$

It remains only to determine which of these are equivalent.

Proposition 3.2 *(Equivalence condition). If $\rho : H \to M_d(\mathbb{C})$ is a faithful irreducible special unitary ordinary representation of H, then so is*

$$\rho_\sigma : h \mapsto \rho(\sigma h), \qquad \sigma \in \mathrm{Aut}(H),$$

where $\mathrm{Aut}(H)$ *denotes the automorphisms of H. These give equivalent nice error frames, even though the representations may not be equivalent if σ is an outer automorphism.*

Proof. Since an automorphism σ of H fixes the centre $Z(H)$, it induces an automorphism $\sigma_G \in \mathrm{Aut}(G)$ on the index group $G = H/Z(H)$. Thus a nice error frame $(F_g)_{g\in G}$ for ρ_σ is reindexing of one for ρ, since

$$F_g := \rho_\sigma(g) = \rho(\sigma g) = \rho(\sigma_G(g)), \qquad \forall g \in G.$$

If σ is an inner automorphism, i.e., $\sigma h = k^{-1}hk$, then ρ and ρ_σ are equivalent ordinary representations of H, since

$$\rho_\sigma(h) = \rho(k^{-1}hk)) = \rho(k)^{-1}\rho(h)\rho(k). \quad \square$$

A **monomial** (or **generalised permutation**) **matrix** is a $d \times d$ matrix with exactly one nonzero entry in each row and column. A matrix group or representation is said to be **monomial** if all of its matrices are.

In practice, the action groups of the ordinary representations of H calculated in `magma` with the command

```
AbsolutelyIrreducibleModules(H,Rationals());
```

are often the same monomial group (cf. [14]). When they are not, it is easy to just work will the small number of representations which are not equivalent in this way, rather than try to apply the outer automorphisms to possibly reduce this set.

Next we give results of our calculations, as just outlined.

4. Examples of nice error frames

In Table 1 of the Appendix, we list the first few canonical abstract error groups and the index group for nice error frames (which are not bases) for $2 \leq d \leq 7$. As suggested by this, Example 2.10 and Proposition 2.12, nice error frames are numerous.

Proposition 4.1. *For each $d \geq 2$, there are infinitely many canonical abstract error groups.*

An infinite family of these can be constructed as *monomial* representations. Clearly, a set of monomial matrices with nonzero entries given by m-th roots of unity (m fixed) generates a finite group, and such a group could be enlarged to ensure its action on $\mathbb{C}^d$ is irreducible.

From Table 1, we observe that index groups may be repeated in different dimensions.

Example 4.2 *(Repeated index groups).* A group G may be the index group for nice error frames in more than one dimension d, e.g., $G =$ `<12,3>` is the index group for a nice error frame for $M_2(\mathbb{C})$ ($H =$ `<24,3>`), and also one for $M_3(\mathbb{C})$ ($H =$ `<36,11>`).

There is evidence (see [15–18,7] and the next section) that complex (projective) spherical t-designs (quantum t-designs) with the minimal number of vectors often come as the orbit of a nice error frame.

5. Nice error bases and SICs

A tight frame $\Phi = (\phi_j)$ for $\mathbb{C}^d$ consisting of d^2 unit vectors is **equiangular** if

$$|\langle \phi_j, \phi_k \rangle|^2 = \frac{1}{d+1}, \qquad j \neq k. \tag{5.9}$$

Such a tight frame (or more precisely the corresponding orthogonal projections $P_j = \phi_j\phi_j^*$) is known in the quantum physics literature as a **SIC** or **SIC-POVM** (symmetric informationally complete positive operator valued measure).

All the known constructions of SICs (see [6,19]) are G**-covariant**, i.e., are an orbit

$$(E_g v)_{g\in G}, \qquad v \in \mathbb{C}^d$$

of nice error basis $(E_g)_{g\in G}$ for $M_d(\mathbb{C})$, where G is an abelian group, and $v \in \mathbb{C}^d$ is called a **fiducial vector**. With one exception, $G = \mathbb{Z}_d \times \mathbb{Z}_d$ with the Heisenberg nice error basis. The exception is for $d = 8$, where there is, in addition, Hoggar's construction [20,21], which we will refer to as the *Hoggar lines*. For this, the nice error basis is a triple tensor product of the Heisenberg nice error basis for $d = 2$ (the Pauli matrices), and the index group is

$$G = K \times K, \qquad K := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Klappenecker and Rötteler [8] have computed all the possible nonabelian index groups, and some of the corresponding nice error bases (up to equivalence as projective representations) for $d \le 10$, see the *Catalogue of Nice Error Bases* at

`http://faculty.cs.tamu.edu/klappi/ueb/ueb.html`.

These were used by Renes et al. [22] to construct G-covariant SICs. They found these numerically (to within 10^{-15}) for $G = \mathbb{Z}_d \times \mathbb{Z}_d$ and for one nonabelian group G in dimensions $d = 6, 8, 9$ (not all groups were tested). These SICs with nonabelian index groups are projectively unitarily equivalent to ones obtained from the Heisenberg nice error basis [19]. A proof that SICs do indeed exist for all d is generally referred to as Zauner's conjecture (cf. [23]). As of [6], the conjecture has been proved for

$$d = 2, 3, 4, \ldots, 15, 19, 24, 35, 48$$

by analytic constructions which were motivated by numerical results.

In Tables 2 and 3 of the Appendix, we give all the canonical abstract error groups for nice error bases in dimensions $d < 14$ as calculated using `magma` (which does not have the groups of order $14^3 = 2744$ available). From these it is easy to find all canonical error groups (irreducible faithful special representations). Despite the fact there are 10, 494, 213 groups of order $8^3 = 512$ we were able to search all of them to find the canonical abstract error groups H for $d = 8$. This was possible because only a small number of these groups have a nontrivial cyclic centre.

Our tables are consistent with the Catalogue of Nice Error bases, having exactly the same index groups. The Catalogue over counts nice error bases, e.g., for the index group $G =$ `<16,11>` generators for two nice error bases are given, but the generators for the first, together with the scalar matrix iI generate the second, and so they give the same nice error basis. The Catalogue does seem to be exhaustive.

Our calculations show that there exist nice error bases with the same index group G which are not equivalent.

Example 5.1 *(Inequivalent nice error bases).* For $d = 8$, there are 47 canonical abstract error groups, and only 42 index groups. In particular, there are three canonical abstract error groups for $G =$ `<64,67>`, and hence at least three inequivalent nice error bases with this index group. Moreover, two of these give rise to SICs, one does not.

Using our calculated canonical abstract error groups, we undertook an extensive search for numerical SICs for $d < 14$ using the variational approach of [22]. These results are summarised in Tables 2 and 3. They are consistent with the calculations of [19] §10.5 for $d \leq 9$ using the Catalogue of Nice Error bases, which were done independently.

In [19] it was shown that certain SICs obtained from the Heisenberg nice error basis are also G-covariant for nice error bases which occur as subgroups of the Clifford group. In a similar vein, we determined which nice error bases appear as subgroups of the Clifford group. In particular (see Table 3), we found that 13 nice error bases which give the Hoggar lines (there are 22 in total) appear as subgroups of the Clifford group. Thus all known SICs are obtained from nice error bases which appear as subgroups of the Clifford group.

We now give an explicit example of the Hoggar lines as the orbit of a subgroup of the Clifford group. Recall the Clifford group is generated by the matrices S, Ω, F, R of (2.7) and (2.8), together with the unit scalar matrices. It contains the permutation matrices P_a, which are given by

$$(P_a)_{jk} := \delta_{aj,k}, \qquad a \in \mathbb{Z}_d^*.$$

Example 5.2 *(The Hoggar lines from a subgroup of the Clifford group).* The canonical abstract error group $H =$ `<512,6278298>`, with index group $G =$ `<64,195>` has rank 5, and appears as the subgroup of the Clifford group generated by

$$R^8, \qquad \omega I, \qquad \mu R^2 S^3 P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ 0 & 0 & \mu^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ \mu^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu^3 & 0 \\ 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^3 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega^7 R^6 P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega^2 S^6 P_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix}.$$

A fiducial vector v which gives the Hoggar lines as an orbit under this nice error basis is

$$v := \frac{1}{12}\begin{pmatrix} \sqrt{6} + \left(2\sqrt{3} - \sqrt{6}\right) i \\ 0 \\ 2\sqrt{3}(-1+i) \\ 2\sqrt{6 - 3\sqrt{2}}\, i \\ \sqrt{6} - \left(2\sqrt{3} + \sqrt{6}\right) i \\ 0 \\ 2\sqrt{3}(1-i) \\ 2\sqrt{6 + 3\sqrt{2}} \end{pmatrix}.$$

6. A SIC with a nonabelian index group for $d = 6$

Here we give an analytic construction of a G-covariant SIC for $d = 6$, which is the orbit of a nice error basis with the *nonabelian* index group

$$G = \mathbb{Z}_3 \times A_4 = \texttt{SmallGroup(36,11)},$$

and canonical abstract error group `SmallGroup(216,42)`. Based on our extensive numerical calculations, and those of [22,19], this appears to be the first such example. It was first found numerically (to within 10^{-15}) by Renes et al. [22]. We were unaware that Grassl [24] gave an analytic form. Our presentation is simpler. It turns out [19] that this SIC is in fact generated by the Heisenberg group, and so is of less interest than initially thought.

We define matrices (with 2×2 blocks) by

$$B := \begin{pmatrix} i\sigma_1 & & \\ & i\sigma_2 & \\ & & i\sigma_3 \end{pmatrix}, \quad S^2 := \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

$$A := \begin{pmatrix} I & & \\ & \omega I & \\ & & \omega^2 I \end{pmatrix}, \quad \omega := e^{\frac{2\pi i}{3}}, \tag{6.10}$$

where the Pauli matrices σ_j of (2.1) are normalised to have determinant 1. These generate a nice error basis for $M_6(\mathbb{C})$, as follows.

Proposition 6.1 (**Nice error basis**). *The unitary matrices B, S^2, A of (6.10) generate a group*

$$H := \langle B, S^2, A\rangle \subset SL_6(\mathbb{C}), \qquad |H| = 216 = 6^3$$

which gives a unitary faithful irreducible representation of `SmallGroup(216,42)`, *and has centre*

$$Z(H) = \langle -\omega I\rangle, \qquad |Z(H)| = 6.$$

In particular, taking a matrix E_g from each coset of

$$G := \frac{H}{Z(H)} = \texttt{SmallGroup}(36, 11)$$

gives a nice error basis $(E_g)_{g\in G}$ for $M_6(\mathbb{C})$ with index group G.

Using the variational characterisation (2.4) it is easy to search numerically for fiducial vectors for SICs. For our nice error basis this yielded 864 fiducial vectors, up to normalisation by a scalar, including

$$v = \begin{pmatrix} \alpha r_0 \\ r_0\tau^{63} \\ r_1\xi_1 \\ \alpha r_1\xi_1\tau^{63} \\ r_2\xi_2 \\ \alpha r_2\xi_2\tau^{45} \end{pmatrix}, \qquad \begin{gathered} \alpha \approx 0.5176, \\ r_0 \approx 0.6774,\ r_1 \approx 0.3690,\ r_2 \approx 0.4400, \\ \xi_1 \approx -0.9170 - 0.3988i,\ |\xi_1| = 1, \\ \xi_2 \approx 0.2044 + 0.9789i,\ |\xi_2| = 1, \\ \tau^9 = \tfrac{1+i}{\sqrt{2}}. \end{gathered} \tag{6.11}$$

6.1. The analytic form of the SIC

We observe (numerically) that all 864 solutions v have the ratio of successive pairs of entries given by

$$\frac{v_1}{v_2}, \frac{v_3}{v_4}, \frac{v_5}{v_6} \in \Big\{\alpha^j\Big(\frac{1+i}{\sqrt{2}}\Big)^{1+2k} : j = \pm 1, k = 0, 1, 2, 3\Big\}. \tag{6.12}$$

Moreover, the moduli of the entries are

$$\Big\{\{|v_1|, |v_2|\}, \{|v_3|, |v_4|\}, \{|v_5|, |v_6|\}\Big\} = \Big\{\{r_0, \alpha r_0\}, \{r_1, \alpha r_1\}, \{r_2, \alpha r_2\}\Big\},$$

and the sign of the ratio of entries from different pairs is

$$\Big\{\xi_1, \xi_2, \frac{1}{\xi_1}, \frac{1}{\xi_2}, \frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_1}\Big\}\{\tau^j : 0 \le j < 72\}, \qquad \tau := e^{\frac{2\pi i}{72}}.$$

In effect, all 864 solutions can be constructed from α, r_0, r_1, r_2, ξ_1, ξ_2 and τ.

We now give the main result of this section.

Theorem 6.2. *Let $(E_g)_{g\in G}$ be the nice error basis of Proposition 6.1 with the nonabelian index group $G :=$ `SmallGroup(36,11)`. Then the unit vector*

$$v := \begin{pmatrix} \alpha r_0 \\ r_0 \frac{1-i}{\sqrt{2}} \\ r_1\xi_1 \\ \alpha r_1 \xi_1 \frac{1-i}{\sqrt{2}} \\ r_2\xi_2 \\ \alpha r_2\xi_2 \frac{-1-i}{\sqrt{2}} \end{pmatrix}, \tag{6.13}$$

where

$$\alpha := \frac{\sqrt{2}}{1+\sqrt{3}} = \frac{\sqrt{3-\sqrt{3}}}{\sqrt{3+\sqrt{3}}}, \qquad r_1 := \frac{1}{\sqrt{14}}\frac{\sqrt{7-\sqrt{21}}}{\sqrt{3-\sqrt{3}}},$$

$$r_0 := r_+, \quad r_2 := r_-, \qquad r_\pm := \frac{\sqrt{7+\sqrt{21}\pm\sqrt{14}\sqrt{\sqrt{21}-3}}}{2\sqrt{7}\sqrt{3-\sqrt{3}}},$$

$$\xi_1 = \tau^{50}\sqrt[3]{\beta - i\sqrt{1-\beta^2}}, \qquad \xi_2 = \frac{\tau^{31}}{4}\Big(\sqrt{7}-\sqrt{3}-i\sqrt{6+2\sqrt{21}}\Big), \tag{6.14}$$

$$\beta := -\frac{1}{8}\sqrt{46-6\sqrt{21}+6\sqrt{6\sqrt{21}-18}}, \qquad \tau := e^{\frac{2\pi i}{72}}$$

gives a G-covariant SIC $(E_g v)_{g\in G}$ for $\mathbb{C}^6$.

Proof. Motivated by the numerical fiducial vector (6.11), let v have the form (6.13), where

$$\alpha, r_0, r_1, r_2 > 0, \qquad \xi_1, \xi_2 \in \mathbb{C}, \quad |\xi_1| = |\xi_2| = 1.$$

The condition that v have unit norm is

$$(r_0^2 + r_1^2 + r_2^2)(1+\alpha^2) = 1. \tag{6.15}$$

Since $(E_g)_{g\in G}$ is a nice error basis,

$$|\langle E_g v, E_h v\rangle| = |\langle v, E_g^* E_h v\rangle| = |\langle v, E_g^{-1} E_h v\rangle| = |\langle v, E_{g^{-1}h} v\rangle|,$$

and so the angle moduli conditions (5.9) for v to be a fiducial vector reduce to

$$|\langle v, E_g v\rangle|^2 = \frac{1}{7}, \quad \forall g \in G, \ g \neq 1. \tag{6.16}$$

Of the $|G|-1=35$ angle moduli equations, 17 are independent of ξ_1 and ξ_2, which gives as system of eight equations. This system is symmetric in r_0 and r_2, and four of the equations factor, giving

$$\begin{aligned} r_0r_2(\alpha^2+1) &= \frac{1}{\sqrt{7}}, \\ (1-\alpha^2)r_0^2-\sqrt{2}\alpha r_1^2+\sqrt{2}\alpha r_2^2 &= \frac{1}{\sqrt{7}}, \\ \sqrt{2}\alpha r_0^2-\sqrt{2}\alpha r_1^2+(1-\alpha^2)r_2^2 &= \frac{1}{\sqrt{7}}, \\ \sqrt{2}\alpha r_0^2-(1-\alpha^2)r_1^2+\sqrt{2}\alpha r_2^2 &= \frac{1}{\sqrt{7}}. \end{aligned} \tag{6.17}$$

From (6.15) and (6.17), it is easy to solve for α, r_0, r_1, r_2, and then verify that all eight equations and (6.15) hold. To this end, once α has been determined,

$$1+\alpha^2=3-\sqrt{3}, \qquad 1-\alpha^2=\sqrt{2}\alpha=\frac{1}{\sqrt{3}}(3-\sqrt{3}),$$

and so with $R_j := \sqrt{3-\sqrt{3}}\,r_j$, the equations (6.15), (6.17) simplify to

$$(R_0^2+R_2^2)+R_1^2=1, \qquad R_0R_2=\frac{1}{\sqrt{7}}, \qquad (R_0^2+R_2^2)-R_1^2=\frac{\sqrt{3}}{\sqrt{7}}.$$

The remaining 18 angle moduli equations, reduce to three equations, each occurring six times. Using $1+\alpha^2 i=2\alpha\tau^3$ and $\alpha(1-i)=\sqrt{2}\alpha\tau^{-9}$, these can be written as

$$p_j(\xi_1,\xi_2) := \left|2\alpha r_1r_2\tau^3\omega^j\frac{\xi_1}{\xi_2}+2\alpha r_0r_1\omega^{-j}\frac{1}{\xi_1}+\sqrt{2}\alpha r_0r_2\tau^{-9}\xi_2\right|^2=\frac{1}{7}, \qquad j=0,1,2. \tag{6.18}$$

These equations can be solved with a computer algebra package (we used MAPLE). However, the formulas for ξ_1 and ξ_2 so obtained are very complicated, and it could not be verified that they satisfy the original equations (6.18). Thus we spent considerable effort finding simpler formulas that could be easily written down, and which could be verified to give a fiducial vector.

We now briefly outline how this was done, then give the fine details in the following subsections. The equations (6.18) cannot be symmetrised (to find a simpler equation), since

$$\frac{1}{3}(p_0+p_1+p_2)=\frac{1}{7}.$$

However, by the variational characterisation (2.4) of tight frames they can be replaced by the single equation

$$\frac{1}{3}(p_0^2+p_1^2+p_2^2) = \Big(\frac{1}{7}\Big)^2. \tag{6.19}$$

Though this equation has twice the degree of the three original equations, it has a simple form (with many zero terms), i.e.,

$$\begin{aligned}&\frac{64r_0^2r_1^2r_2^2}{(\sqrt{3}+1)^4}\Big\{\sqrt{2}r_0r_1\Big(\frac{\xi_1^3}{\tau^6}+\frac{\tau^6}{\xi_1^3}\Big)+\sqrt{2}r_1r_2\Big(\frac{\tau^{15}\xi_1^3}{\xi_2^3}+\frac{\xi_2^3}{\tau^{15}\xi_1^3}\Big)+r_0r_2\Big(\frac{\xi_2^3}{\tau^{21}}+\frac{\tau^{21}}{\xi_2^3}\Big)\Big\}\\ &\quad+\frac{33-4\sqrt{21}}{441}=\frac{1}{7^2}.\end{aligned} \tag{6.20}$$

Here, τ^6, τ^{15}, τ^{21} appear because

$$\tau^6=\frac{1}{2}\big(\sqrt{3}+i\big),\qquad \tau^{15}=\frac{1}{2\sqrt{2}}\big(\sqrt{3}-1+(1+\sqrt{3})i\big),$$
$$\tau^{21}=\frac{1}{2\sqrt{2}}\big(1-\sqrt{3}+(1+\sqrt{3})i\big).$$

We split (6.20) into two parts:

$$\sqrt{2}r_0r_1\Big(\frac{\xi_1^3}{\tau^6}+\frac{\tau^6}{\xi_1^3}\Big)+\sqrt{2}r_1r_2\Big(\frac{\tau^{15}\xi_1^3}{\xi_2^3}+\frac{\xi_2^3}{\tau^{15}\xi_1^3}\Big)=\frac{\frac{2}{7}\sqrt{21}-2}{3-\sqrt{3}}, \tag{6.21}$$

$$r_0r_2\Big(\frac{\xi_2^3}{\tau^{21}}+\frac{\tau^{21}}{\xi_2^3}\Big)=\frac{\frac{1}{2}-\frac{3}{14}\sqrt{21}}{3-\sqrt{3}}. \tag{6.22}$$

It follows from the calculation

$$\frac{64r_0^2r_1^2r_2^2}{(\sqrt{3}+1)^4(3-\sqrt{3})}=\frac{14-2\sqrt{21}}{441},$$
$$\frac{14-2\sqrt{21}}{441}\Big(\frac{2}{7}\sqrt{21}-2+\frac{1}{2}-\frac{3}{14}\sqrt{21}\Big)=\frac{4\sqrt{21}-24}{441},$$

that a solution of (6.21) and (6.22) is a solution of (6.20). Thus, to complete the proof, it suffices to show that ξ_1 and ξ_2 defined by (6.14) satisfy the equations (6.21) and (6.22).

The determination of ξ_2. The equation (6.22) can be written as

$$2\Re\Big(\frac{\xi_2^3}{\tau^{21}}\Big)=\frac{\xi_2^3}{\tau^{21}}+\frac{\tau^{21}}{\xi_2^3}=\frac{\sqrt{7}-3\sqrt{3}}{2}, \tag{6.23}$$

from which we obtain

$$\frac{\xi_2^3}{\tau^{21}}=\frac{\sqrt{7}-3\sqrt{3}}{4}+\frac{i}{4}\sqrt{6\sqrt{21}-18}.$$

To avoid taking a cube root, we observe that the minimal polynomial of $z:=\frac{\xi_2}{\tau^{31}}$ over $\mathbb{Q}$

$$1 - x^2 - 3x^4 - x^6 + x^8$$

has a factor

$$x^2 - \frac{\sqrt{7}-\sqrt{3}}{2}x + 1$$

with z as a root, which gives

$$\frac{\xi_2}{\tau^{31}} = \frac{\sqrt{7}-\sqrt{3}}{4} - \frac{i}{4}\sqrt{6+2\sqrt{21}}.$$

We use this to define ξ_2. It follows that this choice satisfies (6.22), since

$$\frac{\xi_2^3}{\tau^{21}} = \left(\frac{\xi_2}{\tau^{31}}\right)^3 = \left(\frac{\sqrt{7}-\sqrt{3}}{4} - \frac{i}{4}\sqrt{6+2\sqrt{21}}\right)^3 = \frac{\sqrt{7}-3\sqrt{3}}{4} + \frac{i}{4}\sqrt{6\sqrt{21}-18}.$$

The determination of ξ_1. The minimal polynomial of both $\frac{\xi_1^3}{\tau^6} + \frac{\tau^6}{\xi_1^3}$ and $\frac{\xi_1^3\tau^{15}}{\xi_2^3} + \frac{\xi_2^3}{\xi_1^3\tau^{15}}$ is

$$16x^8 - 184x^6 + 780x^4 - 1018x^2 + 1,$$

and they are roots of the factor

$$4x^4 + (3\sqrt{21}-23)x^2 - 12\sqrt{21} + 55,$$

which gives

$$\frac{\xi_1^3}{\tau^6} + \frac{\tau^6}{\xi_1^3} = -\frac{1}{4}\sqrt{46 - 6\sqrt{21} + 6\sqrt{6\sqrt{21}-18}}, \tag{6.24}$$

$$\frac{\tau^{15}\xi_1^3}{\xi_2^3} + \frac{\xi_2^3}{\tau^{15}\xi_1^3} = -\frac{1}{4}\sqrt{46 - 6\sqrt{21} - 6\sqrt{6\sqrt{21}-18}}. \tag{6.25}$$

From (6.24), we obtain

$$\frac{\xi_1^3}{\tau^6} = \beta - i\sqrt{1-\beta^2}, \quad \xi_1 = \tau^{50}\sqrt[3]{\beta - i\sqrt{1-\beta^2}},$$
$$\beta := -\frac{1}{8}\sqrt{46 - 6\sqrt{21} + 6\sqrt{6\sqrt{21}-18}}.$$

Since ξ_1 and ξ_2 are defined as solutions of (6.24) and (6.23), we need to check these definitions are consistent with (6.25). This follows by the calculations

$$\begin{aligned}\frac{\tau^{15}\xi_1^3}{\xi_2^3} + \frac{\xi_2^3}{\tau^{15}\xi_1^3} &= 2\Re\left(\frac{\xi_1^3}{\tau^6}\frac{\tau^{21}}{\xi_2^3}\right) = 2\Re\left\{\left(\beta - i\sqrt{1-\beta^2}\right)\left(\frac{\sqrt{7}-3\sqrt{3}}{4} - \frac{i}{4}\sqrt{6\sqrt{21}-18}\right)\right\} \\ &= \beta\frac{\sqrt{7}-3\sqrt{3}}{2} - \frac{\sqrt{1-\beta^2}}{2}\sqrt{6\sqrt{21}-18} < 0,\end{aligned}$$

$$\Big(\beta\frac{\sqrt{7}-3\sqrt{3}}{2}-\frac{\sqrt{1-\beta^2}}{2}\sqrt{6\sqrt{21}-18}\Big)^2=\frac{1}{4^2}\big(46-6\sqrt{21}-6\sqrt{6\sqrt{21}-18}\big).$$

Finally, we verify that our ξ_1 and ξ_2 satisfy (6.21). Substituting

$$\sqrt{2}r_\pm r_1=\frac{\sqrt{28\pm\sqrt{14}(7-\sqrt{21})\sqrt{\sqrt{21}-3}}}{14(3-\sqrt{3})}$$

and (6.24), (6.25) into the LHS of (6.21) gives

$$\begin{aligned}&\frac{-1}{56(3-\sqrt{3})}\Big\{\Big(56+4\sqrt{14}\sqrt{\sqrt{21}-3}-8\sqrt{21}\Big)+\Big(56-4\sqrt{14}\sqrt{\sqrt{21}-3}-8\sqrt{21}\Big)\Big\}\\&=\frac{\frac{2}{7}\sqrt{21}-2}{3-\sqrt{3}},\end{aligned}$$

as required.

Remark. There are other relations between ξ_1 and ξ_2, e.g., the minimal polynomial of $\xi_1/\xi_2^{1/2}$ is

$$16x^{48}-31x^{24}+16,$$

which leads to

$$\frac{\xi_1}{\xi_2^{1/2}}=\tau^{33}\sqrt[24]{\frac{31-3\sqrt{7}i}{32}}.$$

However, it is not possible to verify that ξ_1, ξ_2 developed from this satisfy the angle equations. □

The property (6.12) of the fiducial vectors $v\in\mathbb{C}^6$ we obtained ensures that the subvectors

$$\begin{pmatrix}v_1\\v_2\end{pmatrix},\quad\begin{pmatrix}v_3\\v_4\end{pmatrix},\quad\begin{pmatrix}v_5\\v_6\end{pmatrix}$$

are fiducial vectors for the nice error basis $E=\{I,\sigma_1,\sigma_2,\sigma_3\}$ given by the Pauli matrices. In this way, the our SICs for $\mathbb{C}^6$ are built by "splicing together" those for $\mathbb{C}^2$.

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Appendix

Table 1
The canonical abstract error groups and index groups for the first few nice error frames, which are not bases, for $2 \le d \le 7$.

$d = 2$		$d = 3$		$d = 4$	
<12,1>	<6,1>	<36,11>	<12,3>	<80,28>	<20,3>
<16,9>	<8,3>	<54,8>	<18,4>	<96,157>	<24,8>
<20,1>	<10,1>	<63,3>	<21,1>	<96,215>	<24,14>
<24,3>	<12,3>	<72,42>	<24,12>	<128,523>	<32,27>
<24,4>	<12,4>	<81,9>	<27,3>	<128,545>	<32,24>
<28,1>	<14,1>	<108,15>	<36,9>	<128,749>	<32,34>
<32,20>	<16,7>	<108,22>	<36,11>	<128,782>	<32,31>
<36,1>	<18,1>	<117,3>	<39,1>	<128,864>	<32,6>
<40,4>	<20,4>	<144,68>	<48,3>	<128,880>	<32,9>
<44,1>	<22,1>	<162,14>	<54,5>	<128,1750>	<32,27>
<48,8>	<24,6>	<171,4>	<57,1>	<128,1799>	<32,28>
<48,28>	<24,12>	<189,8>	<63,3>	<128,2146>	<32,39>
$d = 5$		$d = 6$		$d = 7$	
<250,8>	<50,4>	<252,16>	<42,1>	<392,39>	<56,11>
<275,3>	<55,1>	<288,230>	<48,3>	<686,8>	<98,4>
<375,2>	<75,2>	<288,896>	<48,48>	<1029,12>	<147,5>
<400,213>	<80,49>	<288,982>	<48,49>	<1176,219>	<168,43>
<500,25>	<100,12>	<288,986>	<48,50>	<1372,14>	<196,8>
<625,7>	<125,3>	<324,20>	<54,7>		

Table 2
Nice error bases for $d < 14$, $d \neq 8$. Here H is the canonical abstract error group, G is the index group, and `sic` indicates that a SIC exists numerically.

d	H	G
1	`<1,1>`	`<1,1>` $= \mathbb{Z}_1$ `sic`
2	`<8,4>`	`<4,2>` $= \mathbb{Z}_2^2$ `sic`
3	`<27,3>`	`<9,2>` $= \mathbb{Z}_3^2$ `sic`
4	`<64,19>`	`<16,2>` $= \mathbb{Z}_4^2$ `sic`
	`<64,94>`	`<16,3>`
	`<64,256>`	`<16,11>`
	`<64,266>`	`<16,14>` $= (\mathbb{Z}_2 \times \mathbb{Z}_2)^2$
5	`<125,3>`	`<25,2>` $= \mathbb{Z}_5^2$ `sic`
6	`<216,42>`	`<36,11>` $= \mathbb{Z}_3 \times A_4$ `sic`
	`<216,66>`	`<36,13>`
	`<216,80>`	`<36,14>` $= \mathbb{Z}_6^2$ `sic`
7	`<343,3>`	`<49,2>` $= \mathbb{Z}_7^2$ `sic`
9	`<729,24>`	`<81,2>` $= \mathbb{Z}_9^2$ `sic`
	`<729,30>`	`<81,4>`
	`<729,405>`	`<81,9>` `sic`
	`<729,489>`	`<81,12>`
	`<729,503>`	`<81,15>` $= (\mathbb{Z}_3 \times \mathbb{Z}_3)^2$
10	`<1000,70>`	`<100,15>`
	`<1000,84>`	`<100,16>` $= \mathbb{Z}_{10}^2$ `sic`
11	`<1331,3>`	`<121,2>` $= \mathbb{Z}_{11}^2$ `sic`
12	`<1728,1294>`	`<144,68>` `sic`
	`<1728,2011>`	`<144,92>`
	`<1728,2079>`	`<144,101>` $= \mathbb{Z}_{12}^2$ `sic`
	`<1728,2983>`	`<144,132>`
	`<1728,10718>`	`<144,95>`
	`<1728,10926>`	`<144,100>`
	`<1728,11061>`	`<144,102>`
	`<1728,13457>`	`<144,136>`
	`<1728,20393>`	`<144,170>`
	`<1728,20436>`	`<144,172>`
	`<1728,20556>`	`<144,177>`
	`<1728,20771>`	`<144,179>`
	`<1728,30353>`	`<144,184>`
	`<1728,30562>`	`<144,189>`
	`<1728,30928>`	`<144,193>`
	`<1728,30953>`	`<144,194>`
	`<1728,31061>`	`<144,196>`
	`<1728,31093>`	`<144,197>` $= (\mathbb{Z}_2 \times \mathbb{Z}_6)^2$
13	`<2197,3>`	`<169,2>` $= \mathbb{Z}_{13}^2$ `sic`

Table 3
The nice error bases for $d = 8$. Those which are subgroups of the Clifford group are labelled with an *. All SICs are the Hoggar lines, except for $H =$ <512,451>, $G = \mathbb{Z}_8^2$.

H	G
<512,451>	<64,2> $= \mathbb{Z}_8^2$ sic*
<512,452>	<64,3> sic*
<512,35969>	<64,8> sic*
<512,36083>	<64,10>
<512,59117>	<64,34> *
<512,59133>	<64,35> *
<512,260804>	<64,58> *
<512,261506>	<64,67> sic*
<512,261511>	<64,67> sic*
<512,261518>	<64,67> *
<512,262018>	<64,60> sic*
<512,262052>	<64,62> sic*
<512,265618>	<64,69> sic*
<512,265839>	<64,68> sic*
<512,265911>	<64,71> sic*
<512,266014>	<64,72>*
<512,266267>	<64,73>
<512,266357>	<64,75> sic
<512,266373>	<64,74> sic
<512,266477>	<64,78> sic
<512,266583>	<64,77> sic
<512,266616>	<64,82>
<512,400195>	<64,90> sic*
<512,400223>	<64,90> sic
<512,400443>	<64,123> *
<512,401215>	<64,91> sic*
<512,402896>	<64,128> *
<512,402951>	<64,138> sic
<512,402963>	<64,138>
<512,403139>	<64,162> *
<512,406850>	<64,174> *
<512,406879>	<64,167> *
<512,406902>	<64,179> *
<512,6276980>	<64,192> $= (\mathbb{Z}_2 \times \mathbb{Z}_4)^2$ *
<512,6277027>	<64,193> sic*
<512,6278298>	<64,195> sic*
<512,6279917>	<64,202> sic
<512,6279938>	<64,202> sic
<512,6280116>	<64,203>
<512,6291080>	<64,226>
<512,6339777>	<64,211>
<512,6339869>	<64,207>
<512,6375318>	<64,236>
<512,6376278>	<64,216>
<512,7421157>	<64,242>
<512,10481364>	<64,261>
<512,10494180>	<64,267> $= (\mathbb{Z}_2^3)^2$ sic

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