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The diagonalisation of the multivariate Bernstein operator

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ABSTRACT

Let B_n be the multivariate Bernstein operator of degree n for a simplex in $\mathbb{R}^s$. In this paper we show that B_n is diagonalisable with the same eigenvalues as the univariate Bernstein operator, i.e.,

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!}\frac{1}{n^k}, \quad k = 1,\ldots,n, \qquad 1 = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0,$$

and we describe the corresponding eigenfuctions and their properties.

Since B_n reproduces only the linear polynomials, these are the eigenspace for $\lambda_1^{(n)} = 1$. For $k > 1$, the $\lambda_k^{(n)}$–eigenspace consists of polynomials of exact degree k, which are uniquely determined by their leading term. These are described in terms of the substitution of the barycentric coordinates (for the underlying simplex) into *elementary eigenfunctions*. It turns out that there are eigenfunctions of every degree k which are common to each B_n, $n \geq k$, for sufficiently large s. The *limiting eigenfunctions* and their connection with orthogonal polynomials of several variables is also considered.

Key Words: multivariate Bernstein operator, diagonalisation, eigenvalues, eigenfunctions, total positivity, Stirling numbers, Jacobi polynomials, semigroup, quasi–interpolant

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1. Introduction

This paper is the multivariate counterpart of Cooper and Waldron [CW00], which gave the spectral decomposition of the univariate Bernstein operator and applications. Here we show that the multivariate Bernstein operator B_n for a simplex in $\mathbb{R}^s$ is also diagonalisable, with the same eigenvalues, namely

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!}\frac{1}{n^k}, \quad k = 1, \ldots, n, \qquad 1 = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0, \tag{1.1}$$

and give an explicit formula for the eigenfunctions. The paper is set out as follows.

In the second half of this section, we define B_n, establish notation and give some technical results. The notation used is based on that of de Boor [B87], which indexes the barycentric coordinates by the vertices they correspond to, rather than imposing some ordering on them. This leads to a compact notation which simplifies the calculations and reveals the underlying geometry.

In Section 2, we give the diagonalisation and describe its symmetries. Since B_n reproduces only the linear polynomials, these give the eigenspace for $\lambda_1^{(n)} = 1$. For $k > 1$, the $\lambda_k^{(n)}$–eigenspace is no longer 1–dimensional, as it is in the univariate case. It consists of polynomials of exact degree k which are uniquely determined by their leading term, and for which an explicit formula is provided.

In Section 3, we show that $B_n f$ takes a simplified form when f is certain ridge–type functions. This is used to describe the eigenfunctions of B_n in terms of the substitution of the barycentric coordinates (for the underlying simplex) into *elementary eigenfunctions*. It turns out that there are eigenfunctions of every degree k which are common to each B_n, $n \geq k$, for sufficiently large s.

In Section 4, we show, as in the univariate case, that the eigenfunctions (with fixed leading term) converge as $n \to \infty$. We describe these *limiting eigenfunctions* and their connection with orthogonal polynomials of several variables.

In Section 5, we give an interesting result about B_n applied to certain shifted factorials, and some consequences of it.

We conclude with some comments about those aspects of the univariate theory which have not been extended to the multivariate setting. The appendix contains a list of the elementary eigenfunctions.

We now give the definitions which will be used throughout.

Definitions

Let V be a set of $s+1$ affinely independent points in $\mathbb{R}^s$, i.e., the vertices of an s–simplex which we denote by T. Denote by $\xi = (\xi_v)_{v \in V}$ the corresponding **barycentric coordinates**, i.e., the unique linear polynomials which satisfy

$$\sum_{v \in V} \xi_v(x) = 1, \qquad \sum_{v \in V} \xi_v(x) v = x, \quad \forall x \in \mathbb{R}^s.$$

We will use standard multi–index notation with indices from $\mathbb{Z}_+^V$ and $\mathbb{Z}_+^s$, so, for example,

$$\xi^\alpha := \prod_{v\in V} \xi_v^{\alpha(v)}, \quad \alpha\in\mathbb{Z}_+^V, \qquad \beta! := \beta_1!\beta_2!\cdots\beta_s!, \quad \beta\in\mathbb{Z}_+^s.$$

The **Bernstein operator** of degree n for the simplex T with vertices V is defined by

$$B_{n,V}f := \sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{Z}_+^V}} \binom{n}{\alpha}\xi^\alpha f(v_\alpha), \qquad \forall f\in C(T), \tag{1.2}$$

where

$$\binom{n}{\alpha} := \frac{n!}{\alpha!(n-|\alpha|)!}, \qquad v_\alpha := \sum_{v\in V}\frac{\alpha(v)}{|\alpha|}v\in T.$$

For $V=\{v_0,v_1,\ldots,v_s\}$, this can be rewritten as

$$B_{n,V}f = \sum_{k=0}^n \sum_{\substack{|\alpha|=k\\ \alpha\in\mathbb{Z}_+^s}} \binom{n}{\alpha}\xi_{v_1}^{\alpha_1}\cdots\xi_{v_s}^{\alpha_s}\xi_{v_0}^{n-k}\, f\Bigl(\frac{\alpha_1v_1+\cdots+\alpha_sv_s+(n-k)v_0}{n}\Bigr). \tag{1.3}$$

If T is the **standard simplex** in $\mathbb{R}^s$, i.e.,

$$V=\{0,e_1,\ldots,e_s\} \qquad (e_i \text{ the standard basis vectors for } \mathbb{R}^s)$$

then (1.3) becomes (cf [L53:(13),p. 51])

$$B_nf(x) = \sum_{k=0}^n\sum_{|\alpha|=k}\binom{n}{\alpha}x_1^{\alpha_1}\cdots x_s^{\alpha_s}(1-x_1-\cdots-x_s)^{n-k}\, f\Bigl(\frac{\alpha_1}{n},\ldots,\frac{\alpha_s}{n}\Bigr). \tag{1.4}$$

It is well known that $B_{n,V}$ maps onto $\Pi_n(\mathbb{R}^s)$ the polynomials of degree n in $\mathbb{R}^s$.

Let S_V be the **symmetry group** of the simplex T with vertices V, i.e., the group of affine transformations which map T onto T. This is (isomorphic to) the symmetric group on the $s+1$ vertices V since an affine map $\mathbb{R}^s\to\mathbb{R}^s$ is uniquely determined by its action on $s+1$ affinely independent points. It follows from (1.2) that

$$B_{n,V}(f\circ A) = (B_{n,V}f)\circ A, \qquad \forall f\in C(T),\ \forall A\in S_V. \tag{1.5}$$

We let S_V act on functions $p\in C(T)$ and linear functionals μ defined on $C(T)$ in the usual way, i.e., for $A\in S_V$

$$A\cdot p := p\circ A^{-1}, \qquad (A\cdot\mu)(f) := \mu(f\circ A), \quad \forall f\in C(T).$$

For $\beta\in\mathbb{Z}_+^V$ with $|\beta|=k$ and $h>0$, define the **multivariate shifted factorial** by

$$[\xi]_h^\beta := \prod_{v\in V}[\xi_v]_h^{\beta(v)}\in\Pi_k, \qquad [\xi_v]_h^{\beta(v)} := \xi_v(\xi_v-h)(\xi_v-2h)\cdots(\xi_v-(\beta(v)-1)h).$$

and the **multivariate Stirling numbers** of the first kind from the univariate ones by

$$S(\beta,\alpha) := \prod_{v\in V} S(\beta(v),\alpha(v)), \qquad \alpha\le\beta. \tag{1.6}$$

These are related by

$$\xi^\beta = \sum_{\alpha\le\beta} S(\beta,\alpha) h^{|\beta-\alpha|}[\xi]_h^\alpha, \qquad \forall\beta\in\mathbb{Z}_+^V,\ \forall h>0, \tag{1.7}$$

which follows from the univariate result

$$\xi_v^{\beta(v)} = \sum_{\alpha(v)=0}^{\beta(v)} S(\beta(v),\alpha(v)) h^{\beta(v)-\alpha(v)}[\xi_v]_h^{\beta(v)-\alpha(v)},$$

by the calculation

$$\begin{aligned}
\xi^\beta &= \prod_{v\in V}\left(\sum_{\alpha(v)=0}^{\beta(v)} S(\beta(v),\alpha(v)) h^{\beta(v)-\alpha(v)}[\xi_v]_h^{\alpha(v)}\right)\\
&= \sum_{\alpha\le\beta}\left(\prod_{v\in V} S(\beta(v),\alpha(v)) h^{\beta(v)-\alpha(v)}[\xi_v]_h^{\alpha(v)}\right)\\
&= \sum_{\alpha\le\beta} S(\beta,\alpha) h^{|\beta-\alpha|}[\xi]_h^\alpha.
\end{aligned}$$

A special case of Theorem 5.2 of Section 5 is that

$$B_{n,V}([\xi]_{1/n}^\beta) = \lambda_k^{(n)}\xi^\beta, \qquad |\beta| = k\le n,$$

where $\lambda_k^{(n)}$ are the eigenvalues defined by (1.1).

For $\alpha\in\mathbb{Z}_+^V$ with $\alpha(v_0)=0$ and $h>0$, let Δ_{h,v_0}^α be the **multivariate difference operator** defined by

$$\Delta_{h,v_0}^\alpha f := \sum_{\substack{\beta\le\alpha\\ \beta(v_0)=0\\ \beta\in\mathbb{Z}_+^V}} \binom{\alpha}{\beta}(-1)^{|\alpha-\beta|} f\big(\cdot + h\sum_{v\in V}\beta(v)(v-v_0)\big), \qquad \binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}. \tag{1.8}$$

Let $e_v\in\mathbb{Z}_+^V$ be the multi–index with

$$e_v(w) := \begin{cases} 1, & w=v;\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 1.9. *Let $\alpha \in \mathbb{Z}_+^V$ with $\alpha(v_0) = 0$ and $h > 0$. Then Δ^{α}_{h,v_0} satisfies*

$$\Delta^{e_v}_{h,v_0} f = f(\cdot + h(v - v_0)) - f, \qquad \Delta^{\alpha+\beta}_{h,v_0} f = \Delta^{\alpha}_{h,v_0}(\Delta^{\beta}_{h,v_0} f), \tag{1.10}$$

and

$$\Delta^{\alpha}_{h,v_0}[\xi]^{\beta}_h = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} h^{|\alpha|}[\xi]^{\beta-\alpha}_h, & \alpha \le \beta; \\ 0, & \textit{otherwise.} \end{cases} \tag{1.11}$$

Proof: From the definition (1.8), we have

$$\Delta^{e_v}_{h,v_0} = f(\cdot + h(v - v_0)) - f.$$

Since the first order differences $\Delta^{e_v}_{h,v_0}$ and $\Delta^{e_w}_{h,v_0}$ commute, to prove the second part of (1.10) it is sufficient to show

$$\Delta^{\alpha}_{h,v_0} = \prod_{\substack{v \in V \\ v \ne v_0}} \underbrace{\Delta^{e_v}_{h,v_0} \cdots \Delta^{e_v}_{h,v_0}}_{\alpha(v) \text{ times}}, \tag{1.12}$$

by induction on $|\alpha|$. For $|\alpha| = 0$ we have $\Delta^{\alpha}_{h,v_0} f = f$. Now suppose $\alpha(w) > 0$, for some $w \in V$, then by the inductive hypothesis

$$\begin{aligned}
\prod_{\substack{v \in V \\ v \ne v_0}} \underbrace{\Delta^{e_v}_{h,v_0} \cdots \Delta^{e_v}_{h,v_0}}_{\alpha(v) \text{ times}} f &= \Delta^{e_w}_{h,v_0} \Delta^{\alpha-e_w}_{h,v_0} f \\
&= \Delta^{e_w}_{h,v_0} \sum_{\substack{\beta \le \alpha - e_w \\ \beta(v_0)=0 \\ \beta \in \mathbb{Z}_+^V}} \binom{\alpha - e_w}{\beta} (-1)^{|\alpha - e_w - \beta|} f\bigl(\cdot + h \sum_{v \in V} \beta(v)(v - v_0)\bigr) \\
&= \sum_{\substack{\beta \le \alpha \\ \beta(w) > 0}} \binom{\alpha - e_w}{\beta - e_w} (-1)^{|\alpha - \beta|} f\bigl(\cdot + h \sum_{v \in V} \beta(v)(v - v_0)\bigr) \\
&\quad - \sum_{\substack{\beta \le \alpha \\ \beta(w) < \alpha(w)}} \binom{\alpha - e_w}{\beta} (-1)^{|\alpha - e_w - \beta|} f\bigl(\cdot + h \sum_{v \in V} \beta(v)(v - v_0)\bigr) \\
&= \Delta^{\alpha}_{h,v_0} f,
\end{aligned}$$

which completes the induction.

For distinct points $v_0, v, w \in V$, it can easily be shown that

$$\xi_v(\cdot + h(v - v_0)) = \xi_v + h, \qquad \xi_w(\cdot + h(v - v_0)) = \xi_w,$$

from which it follows that, for $v \ne v_0$,

$$\Delta^{e_v}_{h,v_0}\left([\xi_w]^{\beta(w)}_h\right) = \begin{cases} \beta(w) h [\xi_w]^{\beta(w)-1}_h, & w = v; \\ 0, & w \ne v. \end{cases} \tag{1.13}$$

Substituting (1.13) into (1.12) then gives (1.11). □

2. The diagonalisation

Our diagonalisation is based on the representation of $B_{n,V}f$ in terms of the basis

$$\mathcal{B}_{v_0} := \{\xi^\alpha : \alpha \in \mathbb{Z}_+^V, \alpha(v_0) = 0, |\alpha| \le n\}, \qquad v_0 \in V \tag{2.1}$$

for $\Pi_n(\mathbb{R}^s)$. For the standard simplex and $v_0 = 0$ this is the monomials. The matrix representation of $B_{n,V}$ with respect to $\mathcal{B}_{v_0}$ is block triangular. From this we obtain the eigenvalues and a basis of eigenfunctions.

Lemma 2.2 (Block triangular form for $B_{n,V}$). *Fix $v_0 \in V$. Then*

$$B_{n,V}f = \sum_{\substack{|\alpha|\le n \\ \alpha(v_0)=0 \\ \alpha\in\mathbb{Z}_+^V}} \binom{n}{\alpha} \xi^\alpha \Delta^\alpha_{\frac{1}{n},v_0} f(v_0), \qquad \forall f \in C(T), \tag{2.3}$$

and, in particular,

$$B_{n,V}(\xi^\beta) = \lambda_k^{(n)} \xi^\beta + \frac{n!}{n^k} \sum_{\substack{\alpha<\beta \\ \alpha(v_0)=0}} \frac{S(\beta,\alpha)}{(n-|\alpha|)!} \xi^\alpha, \qquad \forall \beta \in \mathbb{Z}_+^V,\ |\beta| \le n. \tag{2.4}$$

Proof: In definition (1.2), split $\alpha = \gamma + \alpha(v_0)e_{v_0}$ to obtain

$$B_{n,V}f = \sum_{\substack{|\gamma|\le n \\ \gamma(v_0)=0 \\ \gamma\in\mathbb{Z}_+^V}} \binom{n}{\gamma} \xi^\gamma \xi_{v_0}^{n-|\gamma|} f\left(\frac{|\gamma|v_\gamma + (n-|\gamma|)v_0}{n}\right).$$

Since $\sum_{v\in V} \xi_v = 1$, the multinomial theorem gives

$$\xi_{v_0}^{n-|\gamma|} = (1 - \sum_{\substack{v\in V \\ v\ne v_0}} \xi_v)^{n-|\gamma|} = \sum_{\substack{|\beta|\le n-|\gamma| \\ \beta(v_0)=0 \\ \beta\in\mathbb{Z}_+^V}} \binom{n-|\gamma|}{\beta} (-1)^{|\beta|} \xi^\beta.$$

Hence we obtain

$$\begin{aligned} B_{n,V}f &= \sum_{\substack{|\gamma|\le n \\ \gamma(v_0)=0}} \sum_{\substack{|\beta|\le n-|\gamma| \\ \beta(v_0)=0}} \binom{n}{\gamma}\binom{n-|\gamma|}{\beta} (-1)^{|\beta|} \xi^{\gamma+\beta} f\left(\frac{|\gamma|v_\gamma + (n-|\gamma|)v_0}{n}\right) \\ &= \sum_{\substack{|\alpha|\le n \\ \alpha(v_0)=0}} \binom{n}{\alpha} \xi^\alpha c^\alpha_{1/n}(f), \end{aligned}$$

where

$$\binom{n}{\alpha} c^\alpha_{1/n}(f) := \sum_{\gamma+\beta=\alpha} \sum_{\substack{|\gamma|\le n \\ \gamma(v_0)=0}} \sum_{\substack{|\beta|\le n-|\gamma| \\ \beta(v_0)=0}} \binom{n}{\gamma}\binom{n-|\gamma|}{\beta} (-1)^{|\beta|} f\left(\frac{|\gamma|v_\gamma + (n-|\gamma|)v_0}{n}\right).$$

This gives (2.3), via the simplification

$$
\begin{aligned}
c^{\alpha}_{1/n}(f) &= \frac{\alpha!(n-|\alpha|)!}{n!} \sum_{\substack{\gamma\le\alpha\\ \gamma(v_0)=0}} \frac{n!}{\gamma!(n-|\gamma|)!}\frac{(n-|\gamma|)!}{(\alpha-\beta)!(n-|\alpha|)!}(-1)^{|\alpha-\gamma|} \\
&\qquad \times f\left(v_0+\frac{1}{n}(|\gamma|v_\gamma-|\gamma|v_0)\right) \\
&= \sum_{\substack{\gamma\le\alpha\\ \gamma(v_0)=0}} \binom{\alpha}{\gamma}(-1)^{|\alpha-\gamma|} f\left(v_0+\frac{1}{n}\sum_{v\in V}\gamma(v)(v-v_0)\right) \\
&= \Delta^{\alpha}_{\frac{1}{n},v_0} f(v_0).
\end{aligned}
$$

By (1.11) and (1.7)

$$
\begin{aligned}
\Delta^{\alpha}_{h,v_0}\xi^{\beta} &= \Delta^{\alpha}_{h,v_0}\big(\sum_{\gamma\le\beta} S(\beta,\gamma)h^{|\beta-\gamma|}[\xi]^{\gamma}_{h}\big) = \sum_{\gamma\le\beta} S(\beta,\gamma)h^{|\beta-\gamma|}\Delta^{\alpha}_{h,v_0}([\xi]^{\gamma}_{h}) \\
&= \sum_{\alpha\le\gamma\le\beta} S(\beta,\gamma)h^{|\beta-\gamma|}\frac{\gamma!}{(\gamma-\alpha)!}h^{|\alpha|}[\xi]^{\gamma-\alpha}_{h}.
\end{aligned}
$$

Since

$$
[\xi]^{\gamma-\alpha}_{h}(v_0) = \begin{cases} 1, & \gamma=\alpha; \\ 0, & \text{otherwise,} \end{cases}
$$

this implies

$$
(\Delta^{\alpha}_{h,v_0}\xi^{\beta})(v_0) = \begin{cases} \alpha! S(\beta,\alpha)h^{|\beta|}, & \alpha\le\beta; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}
$$

Hence, substituting $f=\xi^{\beta}$, $|\beta|\le n$, into (2.3), and using (2.5), gives

$$
B_{n,V}(\xi^{\beta}) = \sum_{\substack{|\alpha|\le n\\ \alpha(v_0)=0}} \binom{n}{\alpha}\xi^{\alpha}(\Delta^{\alpha}_{\frac{1}{n},v_0}\xi^{\beta})(v_0) = \sum_{\substack{\alpha\le\beta\\ \alpha(v_0)=0}} \frac{n!}{\alpha!(n-|\alpha|)!}\xi^{\alpha}\alpha! S(\beta,\alpha)\left(\frac{1}{n}\right)^{|\beta|},
$$

which can be rewritten as (2.4). □

Let $p_{\uparrow}$ denote the **leading term** of the polynomial p, i.e., the unique homogeneous polynomial that satisfies

$$
\deg(p-p_{\uparrow}) < \deg(p).
$$

Denote by $\Pi^0_k(\mathbb{R}^s)$ the homogeneous polynomials of degree k.

Theorem 2.6 (Diagonalisation of $B_{n,V}$). *The multivariate Bernstein operator $B_{n,V}$ is diagonalisable, with the same eigenvalues as the univariate Bernstein operator, i.e.,*

$$
\lambda^{(n)}_k := \frac{n!}{(n-k)!}\frac{1}{n^k}, \quad k=1,\ldots,n, \qquad 1=\lambda^{(n)}_1 > \lambda^{(n)}_2 > \cdots > \lambda^{(n)}_n > 0.
$$

Let $P_{k,V}^{(n)}$ denote the $\lambda_k^{(n)}$–eigenspace. Then

$$P_{1,V}^{(n)} = \Pi_1(\mathbb{R}^s), \qquad \forall n. \tag{2.7}$$

For $k > 1$, $P_{k,V}^{(n)}$ consists of polynomials of exact degree k, which are uniquely determined by their leading term, i.e., $P_{k,V}^{(n)}$ is isomorphic to $\Pi_k^0(\mathbb{R}^s)$ via $P_{k,V}^{(n)} \to \Pi_k^0(\mathbb{R}^s) : p \mapsto p_\uparrow$. Let $p_f^{(n)}$, $\deg(f) = k \le n$, denote the $\lambda_k^{(n)}$–eigenfunction with leading term $f_\uparrow$. Then (for $v_0 \in V$ fixed) a basis for $P_{k,V}^{(n)}$ is given by

$$\{p_{\xi^\beta}^{(n)} : \beta \in \mathbb{Z}_+^V, \beta(v_0) = 0, |\beta| = k\}, \qquad p_{\xi^\beta}^{(n)} = \sum_{\alpha \le \beta} c(\alpha, \beta, n)\xi^\alpha, \tag{2.8}$$

where the coefficients can be calculated using the recurrence formula

$$\begin{aligned} c(\beta,\beta,n) &:= 1, \\ c(\alpha,\beta,n) &:= \frac{n^{|\beta|}}{[n-|\alpha|]_1^{|\beta-\alpha|} - n^{|\beta-\alpha|}} \sum_{\alpha<\gamma\le\beta} \frac{S(\gamma,\alpha)}{n^{|\gamma|}} c(\gamma,\beta,n), \quad \alpha < \beta. \end{aligned} \tag{2.9}$$

Let $\mathcal{M}_{k,V}^{(n)}$ denote the dual space to $P_{k,V}^{(n)}$, i.e., those $\mu \in \operatorname{span}\{f \mapsto f(v_\alpha) : |\alpha| = n\}$ for which

$$\mu(P_{j,V}^{(n)}) = \{0\}, \qquad \forall j \ne k \ (j = 1, \ldots, n).$$

The spaces $P_k^{(n)}$ and $\mathcal{M}_k^{(n)}$ are S_V–invariant, i.e., they have the symmetry properties

$$S_V \cdot P_k^{(n)} = P_k^{(n)}, \qquad S_V \cdot \mathcal{M}_k^{(n)} = \mathcal{M}_k^{(n)}.$$

Proof: Since $B_{n,V}$ maps onto $\Pi_n(\mathbb{R}^s)$, it follows from (2.3) that $\mathcal{B}_{v_0}$ of (2.1) is a basis for $\Pi_n(\mathbb{R}^s)$. We now show that the linear operator defined by $\uparrow : p \mapsto p_\uparrow$ takes $\mathcal{B}_{v_0}$ to another basis for $\Pi_n(\mathbb{R}^s)$, i.e., the homogeneous polynomials

$$B_{v_0\uparrow} = \bigcup_{k=0}^{n} \{(\xi^\beta)_\uparrow : \beta \in \mathbb{Z}_+^V, \beta(v_0) = 0, |\beta| = k\} \tag{2.10}$$

are linearly independent. Suppose that

$$\sum_{\substack{|\beta|=k \\ \beta(v_0)=0}} a_\beta (\xi^\beta)_\uparrow = 0 \quad \Longleftrightarrow \quad \sum_{\substack{|\beta|=k \\ \beta(v_0)=0}} a_\beta \xi^\beta = p, \quad \exists p \in \Pi_{k-1}(\mathbb{R}^s).$$

Then for $\delta \in \mathbb{Z}_+^V$ with $|\delta| = k$ and $\delta(v_0) = 0$, (1.7) and (1.11) give

$$\Delta_{h,v_0}^\delta(p) = \Delta_{h,v_0}^\delta\Big(\sum_{\substack{|\beta|=k \\ \beta(v_0)=0}} a_\beta \sum_{\alpha\le\beta} S(\beta,\alpha) h^{|\beta-\alpha|} [\xi]_h^\alpha \Big) = a_\delta = 0,$$

which proves the asserted linear independence.

The linear eigenspace (2.7) is well known. For $k > 1$, we will use (2.3) to show that $B_{n,V}$ has $\lambda_k^{(n)}$–eigenfunctions of the form

$$p_{\xi^\beta}^{(n)} = \sum_{\alpha\le\beta} c(\alpha,\beta,n)\xi^\alpha = \xi^\beta + \sum_{\alpha<\beta} c(\alpha,\beta,n)\xi^\alpha \in \xi^\beta + \Pi_{k-1}(\mathbb{R}^s) \qquad |\beta| = k,\ \beta(v_0) = 0.$$

Since $\uparrow$ maps these to a basis for $\Pi_k^0(\mathbb{R}^s)$, they are the basis of a subspace of $P_{k,V}^{(n)}$ which is isomorphic to $\Pi_k^0(\mathbb{R}^s)$ (via $p \mapsto p_\uparrow$). The dimension count

$$\dim(\Pi_1(\mathbb{R}^s)) + \sum_{k=2}^{n} \dim(\Pi_k^0(\mathbb{R}^s)) = \dim(\Pi_n(\mathbb{R}^s)),$$

shows this subspace is all of $P_{k,V}^{(n)}$, and so $B_{n,V}$ is diagonalisable.

We now show such eigenfunctions exist. Substituting

$$f = \sum_{\alpha\le\beta} c(\alpha,\beta,n)\xi^\alpha, \qquad |\beta| = k,\ \beta(v_0) = 0 \tag{2.11}$$

into the eigenfunction equation $B_{n,V}(f) = \lambda_k^{(n)} f$, and expanding using (2.4) gives

$$\sum_{\gamma\le\beta} c(\gamma,\beta,n) B_{n,V}(\xi^\gamma) = \sum_{\gamma\le\beta} c(\gamma,\beta,n)\frac{n!}{n^{|\gamma|}} \sum_{\alpha\le\gamma} \frac{S(\gamma,\alpha)}{(n-|\alpha|)!}\xi^\alpha = \lambda_k^{(n)} \sum_{\alpha\le\beta} c(\alpha,\beta,n)\xi^\alpha.$$

Equating coefficients of the linearly independent functions ξ^α in the above gives

$$\begin{aligned} \lambda_k^{(n)} c(\alpha,\beta,n) &= \sum_{\alpha\le\gamma\le\beta} \frac{n!}{n^{|\gamma|}} \frac{S(\gamma,\alpha)}{(n-|\alpha|)!} c(\gamma,\beta,n) \\ &= \lambda_{|\alpha|}^{(n)} c(\alpha,\beta,n) + \sum_{\alpha<\gamma\le\beta} \frac{n!}{n^{|\gamma|}} \frac{S(\gamma,\alpha)}{(n-|\alpha|)!} c(\gamma,\beta,n). \end{aligned} \tag{2.12}$$

For $\alpha = \beta$, (2.12) is satisfied for any choice of $c(\beta,\beta,n)$. Suppose that $c(\beta,\beta,n) := 1$. For $\alpha < \beta$, (2.12) can be rewritten as

$$\begin{aligned} c(\alpha,\beta,n) &= \frac{1}{\lambda_k^{(n)} - \lambda_{|\alpha|}^{(n)}} \sum_{\alpha<\gamma\le\beta} \frac{n!}{n^{|\gamma|}} \frac{S(\gamma,\alpha)}{(n-|\alpha|)!} c(\gamma,\beta,n) \\ &= \frac{n^{|\beta|}}{[n-|\alpha|]_1^{|\beta-\alpha|} - n^{|\beta-\alpha|}} \sum_{\alpha<\gamma\le\beta} \frac{S(\gamma,\alpha)}{n^{|\gamma|}} c(\gamma,\beta,n). \end{aligned}$$

This recursively defines $c(\alpha,\beta,n)$, $\alpha < \beta$ from $c(\beta,\beta,n)$, and hence an eigenfunction of the form (2.11) exists and is given by (2.9).

For $p \in P_{k,V}^{(n)}$ and $A \in S_V$, (1.5) gives

$$B_{n,V}(p\circ A) = (B_{n,V}p)\circ A = (\lambda_k^{(n)}p)\circ A = \lambda_k^{(n)}(p\circ A),$$

so that $p\circ A \in P_{k,V}^{(n)}$, and $P_{k,V}^{(n)}$ is S_V–invariant. Similarly, for $\mu \in \mathcal{M}_{k,V}^{(n)}$ and $A \in S_V$,

$$(A\cdot\mu)(P_{j,V}^{(n)}) = \mu(P_{j,V}^{(n)}\circ A) = \mu(P_{j,V}^{(n)}) = \{0\}, \qquad j \ne k\ (j = 1,\ldots,n),$$

and hence $\mathcal{M}_{k,V}^{(n)}$ is S_V–invariant. $\square$

3. Elementary eigenfunctions

If $p_k^{(n)}$ is the $\lambda_k^{(n)}$–eigenfunction ($k > 1$) of the univariate Bernstein operator for the standard simplex $T = [0,1]$, then it can be shown that the $s+1$ polynomials

$$p_k^{(n)} \circ \xi_v, \qquad v \in V$$

are $\lambda_k^{(n)}$–eigenfunctions of $B_{n,V}$ (which are linearly independent for $s > 1$).

The above is a special case of the main result of this section, which effectively says that each eigenfunction of $B_{n,V}$ is also an eigenfunction of all Bernstein operators for higher dimensional simplices when interpreted appropriately. This we describe in terms of the substitution of barycentric coordinates into so called 'elementary' eigenfunctions. The result is based on the following generalisation of the affine change of variables (1.5).

Let $B_n^{\mathbb{R}^d}$ denote the Bernstein operator for S_d the standard simplex in $\mathbb{R}^d$, i.e.,

$$S_d := \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1,\ldots,x_d \geq 0, x_1 + \cdots + x_d \leq 1\}.$$

Lemma 3.1 ($B_{n,V}$ applied to multi–ridge functions). *Let $A : \mathbb{R}^s \to \mathbb{R}^d$ be an affine map onto $\mathbb{R}^d$, with $W := AV$. Then*

$$B_{n,V}(g \circ A) = (B_{n,W}g) \circ A, \qquad \forall g \in C(AT). \tag{3.2}$$

In particular, for $d \leq s$ distinct points $v_1,\ldots,v_d \in V$

$$B_{n,V}(g \circ (\xi_{v_1},\ldots,\xi_{v_d})) = (B_n^{\mathbb{R}^d} g) \circ (\xi_{v_1},\ldots,\xi_{v_d}), \qquad \forall g \in C(S_d), \tag{3.3}$$

where ξ_v are the V–barycentric coordinates, and $(\xi_{v_1},\ldots,\xi_{v_d})$ is the affine map

$$\mathbb{R}^s \to \mathbb{R}^d : x \mapsto (\xi_{v_1}(x),\ldots,\xi_{v_d}(x)).$$

Proof: Since A is affine, $W := AV$ is the set of vertices of a simplex in $\mathbb{R}^d$, and A maps $\{v_\alpha : \alpha \in \mathbb{Z}_+^V, |\alpha| = n\}$ onto $\{w_\beta : \beta \in \mathbb{Z}_+^W, |\beta| = n\}$. Let $\xi = (\xi_v)$ denote the V–barycentric coordinates, and $\eta = (\eta_w)$ the W–barycentric coordinates. Then

$$B_{n,V}(g \circ A) = \sum_{\substack{|\alpha|=n \\ \alpha \in \mathbb{Z}_V^+}} \binom{n}{\alpha} \xi^\alpha g(Av_\alpha) = \sum_{\substack{|\beta|=n \\ \beta \in \mathbb{Z}_W^+}} \sum_{\substack{|\alpha|=n \\ Av_\alpha = w_\beta \\ \alpha \in \mathbb{Z}_V^+}} \binom{n}{\alpha} \xi^\alpha g(w_\beta),$$

and

$$(B_{n,W}g) \circ A = \sum_{\substack{|\beta|=n \\ \beta \in \mathbb{Z}_W^+}} \binom{n}{\beta} (\eta^\beta \circ A)\, g(w_\beta).$$

Hence to prove (3.2), it is sufficient to show that

$$\sum_{\substack{|\alpha|=n \\ Av_\alpha = w_\beta \\ \alpha\in \mathbb{Z}^+_V}} \binom{n}{\alpha}\xi^\alpha = \binom{n}{\beta}\eta^\beta \circ A, \qquad \forall \beta \in \mathbb{Z}^+_W,\ |\beta| = n. \tag{3.4}$$

We now expand the RHS of (3.4) in terms of the basis $\{\xi^\alpha : \alpha \in \mathbb{Z}^V_+, |\alpha| = n\}$ for $\Pi_n(\mathbb{R}^s)$. Observe that $\eta_w \circ A$ is the affine map $\mathbb{R}^s \to \mathbb{R}$ with

$$(\eta_w \circ A)(v) = \begin{cases} 1, & Av = w; \\ 0, & \text{otherwise}, \end{cases} \qquad v \in V,$$

i.e.,

$$\eta_w \circ A = \sum_{\substack{v\in V \\ Av=w}} \xi_v,$$

so that

$$\eta^\beta \circ A = (\prod_{w\in W} \eta_w^{\beta(w)}) \circ A = \prod_{w\in W} (\eta_w \circ A)^{\beta(w)} = \prod_{w\in W} (\sum_{\substack{v\in V \\ Av=w}} \xi_v)^{\beta(w)}.$$

By the multinomial theorem

$$(\sum_{\substack{v\in V \\ Av=w}} \xi_v)^{\beta(w)} = (\sum_{v\in\{v:Av=w\}} \xi_v)^{\beta(w)} = \sum_{\substack{|\delta|=\beta(w) \\ \operatorname{supp}\delta\subset\{v:Av=w\} \\ \delta\in\mathbb{Z}^+_V}} \binom{\beta(w)}{\delta}\xi^\delta,$$

and so we obtain

$$\binom{n}{\beta}\eta^\beta \circ A = \binom{n}{\beta} \prod_{w\in W} (\sum_{\substack{|\delta|=\beta(w) \\ \operatorname{supp}\delta\subset\{v:Av=w\} \\ \delta\in\mathbb{Z}^+_V}} \binom{\beta(w)}{\delta}\xi^\delta) = \sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}^+_V}} c_\alpha \xi^\alpha, \tag{3.5}$$

where the coefficients c_α can be determined by expanding the product. It remains to show that c_α equals the coefficient of ξ^α in LHS of (3.4), i.e.,

$$c_\alpha = \begin{cases} \binom{n}{\alpha}, & Av_\alpha = w_\beta; \\ 0, & \text{otherwise}. \end{cases}$$

Since V is the disjoint union $\cup_{w\in W}\{v : Av = w\}$, the coefficient c_α is zero unless

$$\sum_{\substack{v\in V \\ Av=w}} \alpha(v) = \beta(w),$$

which implies

$$Av_\alpha = \sum_{v\in V} \frac{\alpha(v)}{n} Av = \sum_{w\in W} \sum_{\substack{v\in V\\ Av=w}} \frac{\alpha(v)}{n} w = \sum_{w\in W} \frac{\beta(w)}{n} w = w_\beta.$$

In this case, $\operatorname{supp}\alpha \subset \{v : Av = w\}$, so

$$c_\alpha = \binom{n}{\beta} \prod_{w\in W} \binom{\beta(w)}{\alpha|_{\{v:Av=w\}}} = \frac{n!}{\beta!}\frac{\beta!}{\alpha!} = \binom{n}{\alpha},$$

and we conclude (3.4) holds.

Since each ξ_{v_i} is affine, the map $A := (\xi_{v_1}, \ldots, \xi_{v_d})$ is affine. From

$$Av = \begin{cases} e_i, & v = v_i; \\ 0, & \text{otherwise}, \end{cases} \qquad v \in V,$$

it follows that

$$W := \{Av : v \in V\} = \{0, e_1, \ldots, e_d\} \quad \Longrightarrow \quad AT = S_d,$$

i.e., $B_{n,W} = B_n^{\mathbb{R}^d}$, and (3.3) is proved. □

Definition. *For $k > 1$, the d–variate* **elementary $\lambda_k^{(n)}$–eigenfunction**

$$p^{(n)}_{k_1,\ldots,k_d} : \mathbb{R}^d \to \mathbb{R}, \qquad k := k_1 + \cdots + k_d \le n,\ k_1, \ldots, k_d \ge 1$$

is defined to be the $\lambda_k^{(n)}$–eigenfunction of $B_n^{\mathbb{R}^d}$ with leading term $x_1^{k_1} \cdots x_d^{k_d}$.

These can be computed via

$$p^{(n)}_{k_1,\ldots,k_d}(x) := \sum_{\substack{\alpha\le\beta\\ \alpha\in\mathbb{Z}^d_+}} c(\alpha,\beta,n)x^\alpha, \qquad \beta := (k_1, \ldots, k_d), \tag{3.6}$$

where the coefficients are determined by the recurrence (2.9). This notation is consistent with that of [CW00:(2.6)], where the $\lambda_k^{(n)}$–eigenfunction $p_k^{(n)}$ is precisely the univariate elementary eigenfunction defined above. Observe that changing the ordering of $k_1, \ldots, k_d$ leads to essentially the same elementary eigenfunction (the coordinates are just reordered).

Theorem 3.7 (Elementary eigenfunctions). *Let $v_1, \ldots, v_d$ be $d \le s+1$ distinct points in V. Then*

$$p^{(n)}_{k_1,\ldots,k_d}(\xi_{v_1}, \ldots, \xi_{v_d}) := p^{(n)}_{k_1,\ldots,k_d} \circ (\xi_{v_1}, \ldots, \xi_{v_d}) : \mathbb{R}^s \to \mathbb{R}$$

is the $\lambda_k^{(n)}$–eigenfunction of $B_{n,V}$ with leading term $(\xi_{v_1}^{k_1} \cdots \xi_{v_d}^{k_d})_\uparrow$, i.e.,

$$p^{(n)}_{\xi^\beta} = p^{(n)}_{k_1,\ldots,k_d}(\xi_{v_1}, \ldots, \xi_{v_d}), \qquad \beta := (k_1, \ldots, k_d).$$

This has a factor of $\xi_{v_1},\ldots,\xi_{v_d}$. *Indeed, when* $k_1,\ldots,k_m>1$, $k_{m+1}=\cdots=k_d=1$

$$p^{(n)}_{k_1,\ldots,k_d}(\xi_{v_1},\ldots,\xi_{v_d})=\xi_{v_1}\cdots\xi_{v_d}g(\xi_{v_1},\ldots,\xi_{v_m}),\qquad g\in\Pi_{k-d}(\mathbb{R}^m),\tag{3.8}$$

where

$$g(x):=g^{(n)}_{\hat\beta,d}(x)=\sum_{\substack{\delta\le\hat\beta\\ \delta\in\mathbb{Z}^m_+}}a(\delta,\hat\beta,d,n)x^\delta,\qquad \hat\beta=(k_1-1,\ldots,k_m-1),\tag{3.9}$$

can be calculated from $a(\hat\beta,\hat\beta,d,n):=1$, *and for* $\delta<\hat\beta$ *the recurrence*

$$a(\delta,\hat\beta,d,n):=\frac{n^{|\hat\beta|}}{[n-|\delta|-d]_1^{|\hat\beta-\delta|}-n^{|\hat\beta-\delta|}}\sum_{\substack{\delta<\gamma\le\hat\beta\\ \gamma\in\mathbb{Z}^m_+}}\frac{S(\gamma+(1,\ldots,1),\delta+(1,\ldots,1))}{n^{|\gamma|}}a(\gamma,\hat\beta,d,n).\tag{3.10}$$

Moreover, all eigenfunctions of degree ≥ 2 *are zero at each of the vertices* V, *and*

$$p^{(n)}_{k-1,1}(x_1,x_2)=\frac{p^{(n)}_k(x_1)}{x_1-1}x_2.\tag{3.11}$$

Proof: By Lemma 3.1 and the definition of elementary eigenfunctions

$$\begin{aligned}B_{n,V}(p^{(n)}_{k_1,\ldots,k_d}\circ(\xi_{v_1},\ldots,\xi_{v_d}))&=(B^{\mathbb{R}^d}_n p^{(n)}_{k_1,\ldots,k_d})\circ(\xi_{v_1},\ldots,\xi_{v_d})\\&=(\lambda^{(n)}_k p^{(n)}_{k_1,\ldots,k_d})\circ(\xi_{v_1},\ldots,\xi_{v_d})\\&=\lambda^{(n)}_k p^{(n)}_{k_1,\ldots,k_d}\circ(\xi_{v_1},\ldots,\xi_{v_d}),\end{aligned}$$

so that $p^{(n)}_{k_1,\ldots,k_d}(\xi_{v_1},\ldots,\xi_{v_d})$ is a $\lambda^{(n)}_k$–eigenfunction of $B_{n,V}$, and since

$$p^{(n)}_{k_1,\ldots,k_d}(x)=x_1^{k_1}\cdots x_d^{k_d}+\text{lower order powers of }x,$$

its leading term is $(\xi_{v_1}^{k_1}\cdots\xi_{v_d}^{k_d})_\uparrow$.

For $\alpha<(1,\ldots,1)$, i.e., $\alpha_i=0$ for some i, one has $S(\gamma,\alpha)=0$, and so by (2.9) the coefficients in (3.6) satisfy

$$c(\alpha,\beta,n)=0,\qquad \alpha<(1,\ldots,1).$$

This allows us to divide (3.6) by $x^{(1,\ldots,1)}$ to obtain (3.8), with

$$\begin{aligned}g(x)=\frac{p^{(n)}_{k_1,\ldots,k_d}(x)}{x^{(1,\ldots,1)}}&=\sum_{\substack{(1,\ldots,1)\le\alpha\le\beta\\ \alpha\in\mathbb{Z}^d_+}}c(\alpha,\beta,n)x^{\alpha-(1,\ldots,1)}\\&=\sum_{\substack{\alpha-(1,\ldots,1)\le(\hat\beta,0,\ldots,0)\\ \alpha\in\mathbb{Z}^d_+}}c(\alpha,\beta,n)x^{\alpha-(1,\ldots,1)}\\&=\sum_{\substack{\delta\le\hat\beta\\ \delta\in\mathbb{Z}^m_+}}a(\delta,\hat\beta,d,n)x^\delta,\end{aligned}$$

where $\hat{\beta} := (k_1 - 1, \dots, k_m - 1)$, i.e., $\beta = (\hat{\beta}, 0, \dots, 0) + (1, \dots, 1)$, and

$$a(\delta, \hat{\beta}, d, n) = c((\delta, 0, \dots, 0) + (1, \dots, 1), \beta, n).$$

From the recurrence (2.9) we then obtain

$$a(\hat{\beta}, \hat{\beta}, d, n) = c(\beta, \beta, n) := 1,$$

and for $\delta < \hat{\beta}$

$$a(\delta, \hat{\beta}, d, n) := \frac{n^k}{[n{-}|\delta|{-}d]_1^{k-|\delta|-d} - n^{k-|\delta|-d}} \times \sum_{(\delta,0,\dots,0)+(1,\dots,1)<\tilde{\gamma}\leq\beta} \frac{S(\tilde{\gamma}, (\delta, 0, \dots, 0){+}(1, \dots, 1))}{n^{|\tilde{\gamma}|}} c(\tilde{\gamma}, \beta, n).$$

Making the substitution $\tilde{\gamma} = (\gamma, 0, \dots, 0) + (1, \dots, 1)$ in the summation above gives

$$\sum_{\substack{\delta<\gamma\leq\hat{\beta} \\ \gamma\in\mathbb{Z}_+^m}} \frac{S((\gamma, 0, \dots, 1){+}(1, \dots, 1), (\delta, 0, \dots, 0){+}(1, \dots, 1))}{n^{|\gamma|+d}} c((\gamma, 0, \dots, 0) + (1, \dots, 1), \beta, n)$$
$$= \sum_{\substack{\delta<\gamma\leq\hat{\beta} \\ \gamma\in\mathbb{Z}_+^m}} \frac{S(\gamma{+}(1, \dots, 1), \delta{+}(1, \dots, 1))}{n^{|\gamma|+d}} a(\gamma, \hat{\beta}, d, n),$$

and so we obtain

$$a(\delta, \hat{\beta}, d, n) := \frac{n^{k-d}}{[n{-}|\delta|{-}d]_1^{k-|\delta|-d} - n^{k-|\delta|-d}} \times \sum_{\substack{\delta<\gamma\leq\hat{\beta} \\ \gamma\in\mathbb{Z}_+^m}} \frac{S(\gamma{+}(1, \dots, 1), \delta{+}(1, \dots, 1))}{n^{|\gamma|}} a(\gamma, \hat{\beta}, d, n),$$

which is (3.10).

In [CW00:(2.11)], it was shown that $p_k^{(n)}(1) = 0$, $k \geq 2$. This together with the fact

$$\xi_v(w) := \begin{cases} 1, & w = v; \\ 0, & \text{otherwise} \end{cases}$$

implies all the eigenfunctions (3.8) of degree ≥ 2 are zero at the vertices $v \in V$. $\square$

Example 1. For $p^{(n)}_{2,1,\dots,1}$ we have $\hat\beta = 1 \in \mathbb{Z}$ $(d = k-1)$, and g is the univariate linear polynomial $g(x) = x + a(0,1,k-1,n)$, where

$$a(0,1,k-1,n) := \frac{n^1}{[n-(k-1)]_1^1 - n^1}\frac{S((2,1,\dots,1),(1,\dots,1))}{n^1} = -\frac{1}{(k-1)}.$$

Thus

$$p^{(n)}_{2,1,\dots,1}(\xi_{v_1},\dots,\xi_{v_{k-1}}) = \xi_{v_1}\cdots\xi_{v_{k-1}}\left(\xi_{v_1} - \frac{1}{(k-1)}\right)$$

is a $\lambda_k^{(n)}$–eigenfunction of $B_{n,V}$ whenever $2 \le k \le s+2$.

Since this function is independent of n, it follows that there are eigenfuctions of degree k which are shared by all B_n, $k \ge n$, for sufficiently large s.

Example 2. For $p^{(n)}_{3,1,\dots,1}$ we have $\hat\beta = 2$ $(d = k-2)$, and g is the univariate quadratic polynomial $g(x) = x^2 + a(1,2,k-2,n)x + a(0,2,k-2,n)$, where

$$a(1,2,k-2,n) := \frac{n^2}{[n-1-(k-2)]_1^1 - n}\frac{S(3,2)}{n^2} = -\frac{3}{(k-1)},$$

and

$$\begin{aligned}
a(0,2,k-2,n) &:= \frac{n^2}{[n-(k-2)]_1^2 - n^2}\left\{\frac{S(2,1)}{n}\frac{-3}{(k-1)} + \frac{S(3,1)}{n^2}\right\}\\
&= \frac{n^2}{-2nk+3n+k^2-3k+2}\left\{\frac{-3}{n(k-1)} + \frac{1}{n^2}\right\}\\
&= \frac{3n-k+1}{(2nk-3n-k^2+3k-2)(k-1)}.
\end{aligned}$$

Thus

$$\xi_{v_1}\cdots\xi_{v_{k-2}}\left(\xi_{v_1}^2 - \frac{3}{(k-1)}\xi_{v_1} + \frac{3n-k+1}{(2nk-3n-k^2+3k-2)(k-1)}\right)$$

is a $\lambda_k^{(n)}$–eigenfunction of $B_{n,V}$ whenever $3 \le k \le s+3$.

Example 3. For $p^{(n)}_{2,2,1,\dots,1}$ we have $\hat\beta = (1,1)$ $(d = k-2)$, and g is the bivariate quadratic polynomial $g(x) = x_1x_2 + c_1(x_1+x_2) + c_2$, where

$$\begin{aligned}
c_1 = a((1,0),\hat\beta,k-2,n) &= a((0,1),\hat\beta,k-2,n)\\
&= \frac{n^2}{(n-k+1)-n}\frac{S(2,1)}{n^2} = -\frac{1}{(k-1)},
\end{aligned}$$

and

$$\begin{aligned}
c_2 = a((0,0),\hat\beta,k-2,n) &= \frac{n^2}{[n-(k-2)]_1^2 - n^2}\left\{2\frac{S(2,1)}{n}\frac{-1}{(k-1)} + \frac{S((2,2),(1,1))}{n^2}\right\}\\
&= \frac{2n-k+1}{(2nk-3n-k^2+3k-2)(k-1)}.
\end{aligned}$$

Thus, for $k \geq 4$,

$$p^{(n)}_{2,2,1,\ldots,1}(x) = x_1 \cdots x_{k-2} \left(x_1 x_2 - \frac{x_1+x_2}{(k-1)} + \frac{2n-k+1}{(2nk-3n-k^2+3k-2)(k-1)} \right).$$

A list of the elementary eigenfunctions up to degree 5 is provided in the appendix.

4. Limiting eigenfunctions

Here we show that the $\lambda^{(n)}_k$–eigenfunctions $p^{(n)}_f$ converge as $n \to \infty$ to a limit p^*_f. Moreover, the limit of the factor (3.9) of an elementary eigenfunction is a multivariate Jacobi polynomial. This extends Theorems 4.1 and 4.5 of [CW00] to the multivariate setting. Let e_i be the i–th unit vector in $\mathbb{R}^m$, and $(\beta)_\alpha$ the multivariate shifted factorial

$$(\beta)_\alpha := (\beta_1)_{\alpha_1} \cdots (\beta_d)_{\alpha_d}, \qquad (\beta_i)_{\alpha_i} := \beta_i(\beta_i+1)\cdots(\beta_i+\alpha_i-1), \qquad \beta \in \mathbb{R}^d,\ \alpha \in \mathbb{Z}^d_+,$$

with $(2)_\delta := (2)_{\delta_1} \cdots (2)_{\delta_m}$.

Theorem 4.1 (Limiting eigenfunctions). *Express ξ^β, $\beta \in \mathbb{Z}^V_+$, $|\beta| = k$ in the form*

$$\xi^\beta = \xi^{k_1}_{v_1} \cdots \xi^{k_d}_{v_d}, \qquad k = k_1 + \cdots + k_d, \qquad k_1, \ldots, k_m > 1, \quad k_{m+1} = \cdots = k_d = 1,$$

where $v_1, \ldots, v_d$ are $d \leq s+1$ distinct points in V. Then the coefficients of (3.9) satisfy

$$\lim_{n\to\infty} a(\delta, \hat\beta, d, n) = a^*(\delta, \hat\beta, d) := (-1)^{k-d} \frac{k_1! \cdots k_m!}{(k+d-1)_{k-d}} (k+d-1)_{|\delta|} \frac{(-\hat\beta)_\delta}{(2)_\delta} \frac{1}{\delta!}. \tag{4.2}$$

Thus, $p^{(n)}_{\xi^\beta}$ converges uniformly on T to $p^*_{\xi^\beta} = \xi_{v_1} \cdots \xi_{v_d} g(\xi_{v_1}, \ldots, \xi_{v_m})$, where

$$g(x) := g^*_{\hat\beta,d}(x) := \sum_{\substack{\delta \leq \hat\beta \\ \delta \in \mathbb{Z}^m_+}} a^*(\delta, \hat\beta, d)\, x^\delta, \qquad \hat\beta := (k_1-1, \ldots, k_m-1). \tag{4.3}$$

Proof: First we prove by strong induction on $|\hat\beta - \delta|$ that $a(\delta, \hat\beta, d, n)$ converges to a limit $a^*(\delta, \hat\beta, d)$ as $n \to \infty$, which satisfies the recurrence

$$a^*(\delta, \hat\beta, d) = \frac{-1}{|\hat\beta - \delta|(k+d+|\delta|-1)} \sum_{\substack{i=1 \\ \delta_i+1 \leq k_i-1}}^{m} (\delta_i+1)(\delta_i+2) a^*(\delta+e_i, \hat\beta, d), \qquad \delta < \hat\beta. \tag{4.4}$$

Clearly $\lim_{n\to\infty} a(\hat\beta, \hat\beta, d, n) = 1$, which begins the induction. Suppose $\delta < \hat\beta$. Since

$$[n-|\delta|-d]_1^{|\hat\beta-\delta|} - n^{|\hat\beta-\delta|} = \frac{1}{2}|\hat\beta-\delta|(1-k-d-|\delta|)n^{|\hat\beta-\delta|-1} + \text{lower order powers of } n,$$

all the coefficients

$$\frac{n^{|\hat\beta-\gamma|}}{[n-|\delta|-d]_1^{|\hat\beta-\delta|}-n^{|\hat\beta-\delta|}}S(\gamma+(1,\ldots,1),\delta+(1,\ldots,1)),$$

of $a(\gamma,\hat\beta,d,n)$ in (3.10) converge to 0 as $n\to\infty$, except those for $\gamma=\delta+e_i\le\hat\beta$ which converge to

$$\frac{S(\delta+e_i+(1,\ldots,1),\delta+(1,\ldots,1))}{\frac{1}{2}|\hat\beta-\delta|(1-k-d-|\delta|)}=-\frac{(\delta_i+1)(\delta_i+2)}{|\hat\beta-\delta|(k+d+|\delta|-1)}.$$

Using this and the inductive hypothesis, we can take the limit of (3.10) to obtain

$$\lim_{n\to\infty}a(\delta,\hat\beta,d,n)=a^*(\delta,\hat\beta,d),$$

which satisfies (4.4).

The limits $a^*(\delta,\hat\beta,d)$ are uniquely determined by $a^*(\hat\beta,\hat\beta,d)=1$ and (4.4). We now show that the a^* defined in (4.2) satisfies these, and so gives the desired limits. The case $\delta=\hat\beta$ is trivial, and so it suffices to show

$$b(\delta,\hat\beta,d):=(k+d-1)_{|\delta|}\frac{(-\hat\beta)_\delta}{(2)_\delta}\frac{1}{\delta!}$$

satisfies the recurrence (4.4). For $\delta<\hat\beta$, we compute

$$\begin{aligned}
&\sum_{\substack{i=1\\ \delta_i+1\le k_i-1}}^{m}(\delta_i+1)(\delta_i+2)b(\delta+e_i,\hat\beta,d)\\
&\qquad=\sum_{\substack{i=1\\ \delta_i+1\le k_i-1}}^{m}(\delta_i+1)(\delta_i+2)(k+d-1)_{|\delta+e_i|}\frac{(-\hat\beta)_{\delta+e_i}}{(\hat2)_{\delta+e_i}}\frac{1}{(\delta+e_i)!}\\
&\qquad=-(k+d-1)_{|\delta|}\frac{(-\hat\beta)_\delta}{(\hat2)_\delta}\frac{1}{\delta!}\,(k+d+|\delta|-1)\sum_{\substack{i=1\\ \delta_i+1\le\hat\beta_i}}^{m}\hat\beta_i-\delta_i\\
&\qquad=-b(\delta,\hat\beta,d)\,(k+d+|\delta|-1)|\hat\beta-\delta|,
\end{aligned}$$

as required. Since the sequence $p_{\xi^\beta}^{(n)}$ is contained in the finite dimensional space Π_k, it converges to $p^*_{\xi^\beta}\in\Pi_k$ in any norm, and in particular uniformly. $\square$

Let $P^*_{k,V}$ denote the space of limiting eigenfunctions, i.e.,

$$P^*_{1,V} := \Pi_1(\mathrm{I\!R}^s), \qquad P^*_{k,V} := \mathrm{span}\{p^*_{\xi^\beta} : \beta \in \mathbb{Z}^V_+, \beta(v_0) = 0, |\beta| = k\}, \quad k > 1,$$

which is S_V–invariant. It follows immediately that each sequence of eigenfunctions $p^{(n)}_f$, $f_\uparrow \in \Pi^0_k(\mathrm{I\!R}^s)$ converges as $n \to \infty$ to some $p^*_f \in P^*_{k,V}$.

The **Lauricella function** $F = F_A$ (see, e.g., [E76:Chap. 2]) with arguments c a scalar, and β, γ, x vectors from $\mathrm{I\!R}^d$ (or $\mathrm{I\!R}^V$) is defined by

$$F(c, \beta; \gamma; x) := \sum_{\alpha \in \mathbb{Z}^d_+} (c)_{|\alpha|} \frac{(\beta)_\alpha}{(\gamma)_\alpha} \frac{x^\alpha}{\alpha!}, \qquad c \in \mathrm{I\!R}, \quad \beta, \gamma, x \in \mathrm{I\!R}^d.$$

Theorem 4.5 (Identification of $p^*_{\xi^\beta}$). *The function g of* (4.3) *can be expressed as*

$$g^*_{\hat\beta,d}(x) = (-1)^{k-d} \frac{k_1! \cdots k_m!}{(k+d-1)_{k-d}} F(k+d-1, -\hat\beta; \hat 2; x), \qquad \hat 2 := (2, \ldots, 2).$$

Proof: From (4.2), (4.3) and the definition of F, we have

$$\begin{aligned} g^*_{\hat\beta,d}(x) &= (-1)^{k-d} \frac{k_1! \cdots k_m!}{(k+d-1)_{k-d}} \sum_{\substack{\delta \le \hat\beta \\ \delta \in \mathbb{Z}^m_+}} (k+d-1)_{|\delta|} \frac{(-\hat\beta)_\delta}{(2)_\delta} \frac{x^\delta}{\delta!} \\ &= (-1)^{k-d} \frac{k_1! \cdots k_m!}{(k+d-1)_{k-d}} F(k+d-1, -\hat\beta; \hat 2; x). \end{aligned}$$

Define d–vectors $\beta := (\hat\beta, 0, \ldots, 0)$, $|\beta| = k-d$, $\kappa := (1, \ldots, 1)$ and $\xi = (\xi_{v_1}, \ldots, \xi_{v_d})$. Then

$$F(k+d-1, -\hat\beta; \hat{;}(\xi_{v_1}, \ldots, \xi_{v_m})) = F(|\beta| + |\kappa| + (d-1), -\beta, \kappa+1, \xi).$$

□

In [W01] it is shown that the factor $g(\xi_{v_1}, \ldots, \xi_{v_m})$, $d \ge 2$ of $p^*_{\xi^\beta}$ is the (multivariate) Jacobi polynomial of degree $k-d$ for the simplex with vertices $\{v_1, \ldots, v_d\}$ and weight $\xi_{v_1} \cdots \xi_{v_d}$ which has leading term $(\xi^{k_1-1}_{v_1} \cdots \xi^{k_m-1}_{v_m})_\uparrow$.

Example 1. Consider the univariate case $T = S_1 := [0,1]$. Here the barycentric coordinates are $\xi_0(x) = 1-x$ and $\xi_1(x) = x$. For $\xi^\beta(x) = x^k$ we have $d = m = 1$, $\hat\beta = k-1$, giving

$$\begin{aligned} p^*_k(x) &= x(-1)^{k-1} \frac{k!}{(k)_{k-1}} F(k, 1-k; 2; x) \\ &= x(-1)^{k-1} \frac{k!(k-1)!}{(2k-2)!} {}_2F_1(1-k, k; 2; x). \end{aligned}$$

Similarly, the leading term of $x^{k-1}(1-x)$ is $-x^k$. So taking $k_1=k-1$, $k_2=1$, $m=1$, $d=2$, $\hat\beta=k-2$ gives

$$\begin{aligned}p_k^*(x)&=-x(1-x)(-1)^{k-2}\frac{(k-1)!}{(k+1)_{k-2}}F(k+1,2-k;2;x)\\&=x(x-1)(-1)^k\frac{k!(k-1)!}{(2k-2)!}{}_2F_1(2-k,k+1;2;x),\end{aligned}$$

and so we have the result of [CW00:Th. 4.5], that

$$p_k^*(x)=\frac{k!(k-2)!}{(2k-2)!}x(x-1)P_{k-2}^{(1,1)}(2x-1),\qquad k\ge 2, \tag{4.6}$$

where $P_j^{(1,1)}$ are the (univariate) Jacobi polynomials which are orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $t\in[-1,1]$.

Example 2. For each $p_{k_1,\ldots,k_d}$, $d\ge 2$,

$$(x_1,\ldots,x_{d-1})\mapsto\frac{p_{k_1,\ldots,k_d}^*(x_1,\ldots,x_{d-1},1-x_1-\cdots-x_{d-1})}{x_1\cdots x_{d-1}(1-x_1-\cdots-x_{d-1})}$$

is a Jacobi polynomial of degree $k-d$ for S_{d-1} with weight $x_1\cdots x_{d-1}(1-x_1-\cdots-x_{d-1})$.

5. $B_{n,V}$ applied to shifted factorials

The following result is of independent interest. In particular, it shows that

$$B_{n,V}([\xi]_{1/n}^\beta)=\lambda_k^{(n)}\xi^\beta,\qquad |\beta|=k\le n, \tag{5.1}$$

which can be used to give an alternative proof of the diagonalisation of $B_{n,V}$.

Theorem 5.2 ($B_{n,V}$ applied to shifted factorials). *Recall for $\beta\in\mathbb{Z}_+^V$ with $|\beta|=k\le n$,*

$$[\xi]_{1/n}^\beta:=\prod_{v\in V}\xi_v\left(\xi_v-\frac{1}{n}\right)\left(\xi_v-\frac{2}{n}\right)\cdots\left(\xi_v-\frac{\beta(v)-1}{n}\right)\in\Pi_k.$$

Then

$$B_{n,V}([\xi]_{1/n}^\beta g)=\lambda_k^{(n)}\xi^\beta B_{n-k,V}(g\left(\tfrac{n-k}{n}\cdot-\tfrac{k}{n}v_\beta\right)),\qquad \forall g\in C(T), \tag{5.3}$$

where the Bernstein polynomials in (5.3) *depend only on the values*

$$\{g(v_\alpha):\alpha\in\mathbb{Z}_+^V,|\alpha|=n,\alpha\ge\beta\}. \tag{5.4}$$

In particular, taking $g=1$ in (5.3) gives (5.1). It can also be shown that

$$B_{n-k,V}\left(g\left(\tfrac{n-k}{n}\cdot-\tfrac{k}{n}v_\beta\right)\right)=(B_{n-k,W}g)\circ\left(\tfrac{n-k}{n}\cdot-\tfrac{k}{n}v_\beta\right),\qquad W:=\tfrac{n-k}{n}V-\tfrac{k}{n}v_\beta. \tag{5.5}$$

Proof: Since v_α is an affine combination of the points in V

$$\xi_v(v_\alpha)=\sum_{w\in V}\frac{\alpha(w)}{n}\xi_v(w)=\frac{\alpha(v)}{n},\qquad \alpha\in \mathbb{Z}_+^V,\ |\alpha|=n,$$

and we have

$$\begin{aligned}([\xi]_{1/n}^\beta)(v_\alpha)&=\prod_{v\in V}\frac{\alpha(v)}{n}\left(\frac{\alpha(v)-1}{n}\right)\left(\frac{\alpha(v)-2}{n}\right)\cdots\left(\frac{\alpha(v)-(\beta(v)-1)}{n}\right)\\&=\frac{1}{n^k}\begin{cases}\alpha!/(\alpha-\beta)!, & \alpha\ge\beta;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{5.6}$$

This implies $B_{n,V}([\xi]_{1/n}^\beta g)$ depends only on (5.4). For $\alpha\ge\beta$, $|\alpha|=n$

$$v_\alpha=\sum_{v\in V}\frac{\alpha(v)v}{n}=\frac{n-k}{n}\sum_{v\in V}\frac{(\alpha(v)-\beta(v))v}{n-k}+\frac{k}{n}\sum_{v\in V}\frac{\beta(v)v}{k}=\frac{n-k}{n}v_{\alpha-\beta}+\frac{k}{n}v_\beta,$$

and hence by (5.6) we obtain

$$\begin{aligned}B_{n,V}([\xi]_{1/n}^\beta g)&=\frac{1}{n^k}\sum_{\substack{|\alpha|=n\\ \alpha\ge\beta}}\binom{n}{\alpha}\xi^\alpha\frac{\alpha!}{(\alpha-\beta)!}g(v_\alpha)\\&=\frac{n!}{(n-k)!}\frac{1}{n^k}\xi^\beta\sum_{\substack{|\alpha-\beta|=n-k\\ \alpha-\beta\ge 0}}\binom{n-k}{\alpha-\beta}\xi^{\alpha-\beta}g\left(\tfrac{n-k}{n}v_{\alpha-\beta}-\tfrac{k}{n}v_\beta\right)\\&=\lambda_k^{(n)}\xi^\beta B_{n-k,V}\left(g\left(\tfrac{n-k}{n}\cdot-\tfrac{k}{n}v_\beta\right)\right).\end{aligned}$$

Applying (3.2) with $A:=\frac{n-k}{n}\cdot-\frac{k}{n}v_\beta$, gives (5.5). □

Let $V_n:=\{v_\alpha:\alpha\in\mathbb{Z}_+^V,|\alpha|=n\}$, then Theorem 5.2 relates the support of the mesh function $f|_{V_n}$ to factors of $B_n(f)$.

Corollary 5.7. *Choose $\beta\in\mathbb{Z}_+^V$ with $|\beta|=k\le n$, and define*

$$V_\beta:=\{v_\alpha\in V_n:\alpha\ge\beta\}.$$

Then the following are equivalent

$$\operatorname{supp}(f|_{V_n})\subset V_\beta\quad\Longleftrightarrow\quad [\xi]_{1/n}^\beta\;\Big|\;(f|_{V_n})\quad\Longleftrightarrow\quad \xi^\beta\;\Big|\;B_{n,V}(f),\quad \forall f\in C(T).$$

When this holds

$$B_n(f) = \lambda_k^{(n)} \xi^\beta (B_{n-k,W}(f/[\xi]_{1/n}^\beta)) \circ \left(\tfrac{n-k}{n} \cdot - \tfrac{k}{n} v_\beta\right), \qquad W := \tfrac{n-k}{n} V - \tfrac{k}{n} v_\beta.$$

Proof: From (5.6), we have

$$V_\beta = \{v_\alpha \in V_n : [\xi]_{1/n}^\beta (v_\alpha) \neq 0\},$$

and so

$$[\xi]_{1/n}^\beta \,\Big|\, (f|_{V_n}) \iff f(v_\alpha) = 0,\ \forall v_\alpha \in V_n \setminus V_\beta \iff \operatorname{supp}(f|_{V_n}) \subset V_\beta.$$

By Theorem 5.2,

$$\begin{aligned} [\xi]_{1/n}^\beta \,\Big|\, (f|_{V_n}) &\iff f|_{V_n} = ([\xi]_{1/n}^\beta)|_{V_n} (f/[\xi]_{1/n}^\beta)|_{V_n} \\ &\iff B_{n,V}(f) = B_{n,V}([\xi]_{1/n}^\beta (f/[\xi]_{1/n}^\beta)) \\ &\iff B_{n,V}(f) = \lambda_k^{(n)} \xi^\beta (B_{n-k,W}(f/[\xi]_{1/n}^\beta)) \circ \left(\tfrac{n-k}{n} \cdot - \tfrac{k}{n} v_\beta\right). \end{aligned}$$

Now suppose that $\xi^\beta \,\Big|\, B_{n,V}(f)$, then

$$\begin{aligned} B_{n,V}(f)/\xi^\beta = \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^{\alpha-\beta} f(v_\alpha) \in \Pi_{n-k}(\mathrm{IR}^s) &\implies f(v_\alpha) = 0,\ \forall \alpha - \beta \not\geq 0 \\ &\implies \operatorname{supp} f|_{V_n} \subset V_\beta. \end{aligned}$$

□

6. Concluding remarks

We conclude with some comments about those parts of [CW00] which have not been generalised here.

The common zeroes of the eigenspaces $P_{k,V}^{(n)}$ do not have the rich structure of the univariate situation (k real zeros in $[0,1]$ with estimates on their location). Indeed, by considering the factored form of elementary eigenfunction $p_{k-s-1,1,\ldots,1}^{(n)}$ it follows that the common zeros are just the vertices V.

By choosing a basis of eigenfunctions for $B_{n,V}$, say that of (2.8) with $p_\alpha^{(n)} := p_{\xi^\alpha}^{(n)}$, one can write down a diagonal form

$$B_{n,V} f = \sum_{k=0}^{n} \lambda_k^{(n)} \sum_{\substack{|\alpha|=k \\ \alpha(v_0)=0}} p_\alpha^{(n)} \mu_\alpha^{(n)}(f), \qquad \forall f \in C(T), \tag{6.1}$$

where the dual functionals $\mu_\alpha^{(n)} \in \mathcal{M}_{k,V}^{(n)}$ can be found explicitly by solving the linear system obtained from

$$\mu_\alpha^{(n)}(p_\beta^{(n)}) = \delta_{\alpha,\beta}, \qquad \forall \alpha, \beta.$$

None of the formulæ so obtained are nice enough to be worth recording. Recent results of [WX01] using (tight) frames to represent Jacobi polynomials on a simplex indicate that it might be more profitable to consider a redundant, but more symmetric, representation of the form

$$B_{n,V} f = \sum_{k=0}^{n} \lambda_k^{(n)} \sum_{|\alpha|=k} p_\alpha^{(n)} \mu_\alpha^{(n)}(f), \qquad \forall f \in C(T),$$

where the inner sum involves all of the S_V–invariant spanning set $\{p_\alpha^{(n)} : \alpha \in \mathbb{Z}_+^V, |\alpha| = k\}$.

In [CW00] it was shown that dual functionals such as $\mu_\alpha^{(n)}$ in (6.1) have a limit as $n \to \infty$ (as functionals on the polynomials). The argument given relied on the fact that dividing $p_k^{(n)}$, $k \geq 2$ by the product of the barycentric coordinates (for the interval) gave a sequence of Jacobi polynomials for which an orthogonal expansion could be used. In the multivariate case this is no longer possible (not all eigenfunctions are divisable by each barycentric coordinate). It is believed that such limits do exist, and that they might be found by an appropriate orthogonal expansion (possibly involving Sobolev orthogonality).

There has been some work on iterates of the bivariate Bernstein operator by [LiP87] and [CF93] generalising the methods of [KR67] (see the comments in [CW00:Sect. 5]). By setting the eigenvalues in (6.1) to 1 we obtain, similarly to the univariate case, the operator

$$L_{n,V} f = \sum_{k=0}^{n} \sum_{\substack{|\alpha|=k \\ \alpha(v_0)=0}} p_\alpha^{(n)} \mu_\alpha^{(n)}(f), \qquad \forall f \in C(T)$$

of Lagrange interpolation from Π_n at the 'simplex points' $\{v_\alpha : |\alpha| = n\}$. The classes of Bernstein quasi–interpolant operators proposed in [CW00] can then be defined in the obvious way. The eigenstructure of the multivariate Kantorovich operator can be deduced in the same was as in the univariate case, see, e.g., [LiS96] (and references therein) for a discussion of the properties of this operator.

7. Appendix

List of the elementary eigenfunctions for $k = 2, \ldots, 5$

degree 2, i.e., $\lambda_2^{(n)} = \dfrac{n!}{(n-2)!n^2}$

$$p_2^{(n)}(x) = x_1(x_1 - 1)$$
$$p_{1,1}^{(n)}(x) = x_1 x_2$$

degree 3, i.e., $\lambda_3^{(n)} = \dfrac{n!}{(n-3)!n^3}$

$$p_3^{(n)}(x) = x_1(x_1 - 1/2)(x_1 - 1)$$
$$p_{2,1}^{(n)}(x) = x_1 x_2 (x_1 - 1/2)$$
$$p_{1,1,1}^{(n)}(x) = x_1 x_2 x_3$$

degree 4, i.e., $\lambda_4^{(n)} = \dfrac{n!}{(n-4)!n^4}$

$$p_4^{(n)}(x) = x_1(x_1 - 1)\left(x_1^2 - x_1 + \frac{n-1}{5n-6}\right)$$
$$p_{3,1}^{(n)}(x) = x_1 x_2 \left(x_1^2 - x_1 + \frac{n-1}{5n-6}\right)$$
$$p_{2,2}^{(n)}(x) = x_1 x_2 \left(x_1 x_2 - \frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{2n-3}{3(5n-6)}\right)$$
$$p_{2,1,1}^{(n)}(x) = x_1 x_2 x_3 (x_1 - 1/3)$$
$$p_{1,1,1,1}^{(n)}(x) = x_1 x_2 x_3 x_4$$

degree 5, i.e., $\lambda_5^{(n)} = \dfrac{n!}{(n-5)!n^5}$

$$p_5^{(n)}(x) = x_1(x_1 - 1/2)(x_1 - 1)\left(x_1^2 - x_1 + \frac{n-1}{7n-12}\right)$$
$$p_{4,1}^{(n)}(x) = x_1 x_2 (x_1 - 1/2)\left(x_1^2 - x_1 + \frac{n-1}{7n-12}\right)$$
$$p_{3,2}^{(n)}(x) = x_1 x_2 \left(x_1^2 x_2 - \frac{1}{4}x_1^2 - \frac{3}{4}x_1 x_2 + \frac{3(n-2)}{2(7n-12)}x_1 + \frac{3n-4}{4(7n-12)}x_2 - \frac{n-2}{4(7n-12)}\right)$$
$$p_{3,1,1}^{(n)}(x) = x_1 x_2 x_3 \left(x_1^2 - \frac{3}{4}x_1 + \frac{3n-4}{4(7n-12)}\right)$$
$$p_{2,2,1}^{(n)}(x) = x_1 x_2 x_3 \left(x_1 x_2 - \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{n-2}{2(7n-12)}\right)$$
$$p_{2,1,1,1}^{(n)}(x) = x_1 x_2 x_3 x_4 (x_1 - 1/4)$$
$$p_{1,1,1,1,1}^{(n)}(x) = x_1 x_2 x_3 x_4 x_5$$

Formulæ for the elementary eigenfunctions of degree k

$$
\begin{aligned}
p^{(n)}_{1,\ldots,1}(x) &= x_1 \cdots x_k \\
p^{(n)}_{2,1,\ldots,1}(x) &= x_1 \cdots x_{k-1} \left(x_1 - \frac{1}{(k-1)} \right) \\
p^{(n)}_{3,1,\ldots,1}(x) &= x_1 \cdots x_{k-2} \left(x_1^2 - \frac{3}{(k-1)} x_1 + \frac{3n-k+1}{(2nk-3n-k^2+3k-2)(k-1)} \right) \\
p^{(n)}_{2,2,1,\ldots,1}(x) &= x_1 \cdots x_{k-2} \left(x_1 x_2 - \frac{x_1+x_2}{(k-1)} + \frac{2n-k+1}{(2nk-3n-k^2+3k-2)(k-1)} \right) \\
p^{(n)}_{k-1,1}(x) &= \frac{p^{(n)}_k(x_1)}{x_1-1} x_2
\end{aligned}
$$

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