Metrics Associated to Multivariate Polynomial Inequalities

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Abstract. We discuss the Carathéodory type distance due to Dubiner for a compact set $K \subset I\!R^m$ and introduce a Finsler type distance based on Baran's generalization of the van der Corput - Schaake polynomial inequality. We then discuss the examples of K a sphere, ball, cube and simplex.

§1. Introduction

As is well-known, there is a beautiful, and by now well understood, interplay between univariate polynomial interpolation, classical orthogonal polynomials and complex potential theory. A specific example of this is the fact that all "good" sequences of interpolation points on the interval $I := [-1, 1]$, along with the zeros of the classical orthogonal polynomials on I, must have the asymptotic distribution of the so-called arcsin measure:

$$d\mu_I := \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx.$$

By fortuitous circumstance, this also happens to be the equilibrium measure of complex potential theory (see e.g. the monographs of Szëgo [11] and Saff and Totik [10]).

The Chebyshev polynomials $T_n(x) := \cos(n\cos^{-1}(x))$, $x \in I$, provide a particularly good example of this phenomenom. On the one hand they are orthogonal with respect to the equilibrium measure itself, and on the other hand their zeros are the prototype of a good set of interpolation points. Specifically, these zeros are

$$x_k = \cos(\theta_k), \quad k = 0, 1, \cdots, n-1$$

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where $\theta_k := (2k+1)\pi/n$. Note that the x_k are the projections on to I of the points on the (upper) half-circle $(\cos(\theta_k), \sin(\theta_k))$, $k = 0, 1, \cdots, n-1$. Hence they are *equally spaced* with respect to arclength distance on the circle. For points $a = \cos(\theta_a) \in I$ and $b = \cos(\theta_b) \in I$ this distance is given by

$$\delta_I(a,b) := |\theta_b - \theta_a| = |\cos^{-1}(b) - \cos^{-1}(a)|. \tag{1}$$

A remarkable formula, due to Dubiner, gives a variational form for δ_I in terms of real polynomials $\mathcal{P}(\mathbb{R})$.

Theorem 1 (Dubiner [7]). *For $a, b \in I$, set*

$$\begin{aligned} &d_I(a,b) \\ &:= \sup\{\frac{1}{deg(p)}\delta_I(p(a),p(b)) \,:\, deg(p) \geq 1,\ ||p||_I \leq 1\} \\ &= \sup\{\frac{1}{deg(p)}|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| \,:\, deg(p) \geq 1,\ ||p||_I \leq 1\}. \end{aligned} \tag{2}$$

Then, $\delta_I(a,b) = d_I(a,b)$.

Here, and throughout this note, we use the uniform norm, i.e.,

$$||p||_I := \max_{x \in I} |p(x)|.$$

This variational form is potentially very useful. Consider, for example, the so-called Fekete points (of order n) for I, i.e., those points $\{x_0, \cdots, x_n\} \subset I$ which maximize the Vandermonde determinant

$$\det[x_i^j]_{0 \leq i,j \leq n}.$$

The associated Lagrange polynomials (of degree n) ℓ_i, $0 \leq i \leq n$, are easily seen to have the property that $||\ell_i||_I = 1$, $i = 0, \cdots, n$. Then, by the formula (2), for $i \neq j$, taking $p = \ell_i$,

$$\begin{aligned} d_I(x_i, x_j) &\geq \frac{1}{n}|\cos^{-1}(\ell_i(x_i)) - \cos^{-1}(\ell_i(x_j))| \\ &= \frac{1}{n}|\cos^{-1}(1) - \cos^{-1}(0)| \\ &= \frac{1}{n}|0 - \frac{\pi}{2}| \\ &= \frac{\pi}{2n} \end{aligned}$$

so that the Fekete points must have order $1/n$ spacing with respect to the circle metric δ_I. (Of course, much more is known about the Fekete points for the interval, but we include this example to show how such facts follow *easily* from the Dubiner variational form). Moreover, it this variational form that generalizes easily to higher dimensions. We will come back to this spacing property in §2.

The proof of Theorem 1 is quite simple and depends on a strong form of Bernstein's polynomial inequality due to van der Corput and Schaake:

Theorem 2 (van der Corput - Schaake Inequality [6]). *For $-1 < x < 1$ and p a real polynomial such that $||p||_I \le 1$,*

$$\left| \frac{p'(x)}{\sqrt{1-p^2(x)}} \right| \le \deg(p) \frac{1}{\sqrt{1-x^2}}.$$

Equivalently, we may write

$$\frac{1}{\deg(p)} \left| \frac{d}{dx} \cos^{-1}(p(x)) \right| \le \left| \frac{d}{dx} \cos^{-1}(x) \right|. \tag{3}$$

We note that the trigonometric form of (3) is expressed as

$$\left| \frac{d}{d\theta} \cos^{-1}(T(\theta)) \right| \le \deg(T)$$

where T is now a trigonometric polynomial.

Proof of Theorem 1. Writing

$$\cos^{-1}(p(b)) - \cos^{-1}(p(a)) = \int_a^b \frac{d}{dx} \{\cos^{-1}(p(x))\} dx,$$

we have by (3),

$$\begin{aligned} \frac{1}{\deg(p)} |\cos^{-1}(p(b)) - \cos^{-1}(p(a))| &\le \left| \int_a^b \frac{d}{dx} \cos^{-1}(x) dx \right| \\ &= |\cos^{-1}(b) - \cos^{-1}(a)| \end{aligned}$$

for all polynomials p such that $||p||_I \le 1$. Hence, $d_I(a,b) \le \delta_I(a,b)$. On the other hand, taking the particular case of $p(x) = x$, we have

$$d_I(a,b) \ge \frac{1}{1} \delta_I(a,b) = \delta_I(a,b)$$

and the result follows. □

We also point out that the fact that the Dubiner distance $d_I(a,b)$ equals $\delta_I(a,b)$ not only follows from the van der Corput - Schaake inequality (3), but is equivalent to it. In fact, given that $d_I(a,b) = \delta_I(a,b)$, suppose that $p \in \mathcal{P}(\mathbb{R})$ is such that $||p||_I \le 1$. Then, for any $a = x \in (-1,1)$ and $b = x + h \in I$ (with $h > 0$ sufficiently small)

$$\frac{1}{\deg(p)} \left| \frac{\cos^{-1}(p(x+h)) - \cos^{-1}(p(x))}{h} \right| \le \frac{d_I(x, x+h)}{h} = \frac{\delta_I(x, x+h)}{h}$$

so that letting $h \to 0$, we have

$$\begin{aligned}\frac{1}{\deg(p)}\left|\frac{d}{dx}\cos^{-1}(p(x))\right| &\le \lim_{h\to 0^+}\frac{\delta_I(x,x+h)}{h}\\ &= \lim_{h\to 0^+}\left|\frac{\cos^{-1}(x+h)-\cos^{-1}(x)}{h}\right|\\ &= \left|\frac{d}{dx}\cos^{-1}(x)\right|.\end{aligned}$$

The purpose of this brief note is to discuss the generalization of such metrics to compact subsets $K \subset \mathbb{R}^m$. It should not come as a surprise that multidimensional Markov-Bersnstein inequalities play an import role in this development. Specifically, in §2 we discuss the Dubiner (pseudo) distance d_K. This turns out to be a distance of Carathéodory type. We also introduce a Finsler type metric and distance, b_K, associated to the multivariate generalization of the van der Corput - Schaake inequality due to Baran [2,3]. We refer to b_K as the Baran distance. In §3 we consider explict formulas for these distances on spheres, balls, cubes and simplices.

§2. The Dubiner and Baran Distances

Suppose then that $K \subset \mathbb{R}^m$ is compact. For $a, b \in K$ we define the Dubiner (pseudo) distance to be given in variational (Carathéodory) form by

$$\begin{aligned}&d_K(a,b)\\ &:= \sup\{\frac{1}{\deg(p)}\delta_I(p(a),p(b)) \,:\, \deg(p)\ge 1,\ ||p||_K \le 1\}\\ &= \sup\{\frac{1}{\deg(p)}|\cos^{-1}(p(b))-\cos^{-1}(p(a))| \,:\, \deg(p)\ge 1,\ ||p||_K\le 1\}.\end{aligned} \tag{4}$$

It is also sometimes convenient to think of the definition of d_K in terms of mapping properties:

$$d_K(a,b) := \sup\{\frac{1}{\deg(p)}\delta_I(p(a),p(b)) \,:\, \deg(p)\ge 1,\ p: K\to I\}. \tag{5}$$

Assuming for the moment that this is well-defined, there is an immediate consequence, given by Dubiner, regarding the spacing of the Fekete points for K, completely analogous to the univariate case. To see this we first need to introduce some notation. The polynomials of degree n, when restricted to K, form a vector space which we will denote by $\mathcal{P}_n(K)$. It has a dimension $N_n(K) := \dim(\mathcal{P}_n(K))$, and a basis $\{p_j \,:\, 1 \le j \le N_n(K)\}$.

Given a set of $N_n(K)$ points $X_n := \{x_j : 1 \le j \le N_n(K)\} \subset K$, the Vandermonde matrix for this system is defined to be

$$VDM(X_n) := \det[p_i(x_j)].$$

Such an $X_n \subset K$ is said to be a set of Fekete points of degree n for K if they maximize VDM (as a function on $K^{N_n(K)}$). Note, however that unlike for the interval case $K = I$, Fekete points need not be unique; for example, for K a ball, a rotation of a set of Fekete points is also a set of Fekete points. Nevertheless, for any set of Fekete points X_n, we may form the Lagrange polynomials by

$$\ell_i(x) := \frac{VDM((X_n \backslash \{x_i\}) \cup \{x\})}{VDM(X_n)},\ 1 \le i \le N_n(K).$$

These $\ell_i \in \mathcal{P}_n(K)$ are such that $\ell_i(x_j) = \delta_{ij}$ and also, by the maximizing property, that $||\ell_i||_K = 1$, $1 \le i \le N_n(K)$.

Now, just as for the univariate case, take $a = x_i \in X_n$ and $b = x_j \in X_n$, two Fekete points with $i \ne j$ and consider the formula (4) with the specific choice of $p = \ell_i$. Then,

$$\begin{aligned} d_K(x_i, x_j) &\ge \frac{1}{n}|\cos^{-1}(\ell_i(x_i)) - \cos^{-1}(\ell_i(x_j))| \\ &= \frac{1}{n}|\cos^{-1}(1) - \cos^{-1}(0)| \\ &= \frac{1}{n}|0 - \pi/2| \\ &= \frac{\pi}{2n} \end{aligned}.$$

Again, the spacing of the Fekete points, with respect to this Dubiner metric, must be of order $1/n$, for arbitrary compact K!

Now for the Baran metric. As indicated earlier, its definition depends heavily on Baran's generalization of the van der Corput - Schaake inequlaity. We will first need to introduce some terms from Pluripotential Theory (see Klimek[9] for an excellent introduction to this subject).

Definition 1 (Siciak-Zaharjuta Extremal Function). *Suppose that $E \subset \mathbb{C}^m$ is compact. Then for $z \in \mathbb{C}^m$, the function*

$$\Phi_E(z) := \sup\{|p(z)|^{1/\deg(p)} : p \in \mathbb{C}[z_1, \cdots, z_n],\ deg(p) \ge 1,\ ||p||_E \le 1\}$$

is known as the Siciak-Zaharjuta Extremal Function.

If $\Phi_E(z)$ is finite, we may use it to bound polynomials p at points z *outside* E in terms of the norm of p on E, $||p||_E$. Specifically,

$$|p(z)| \le \Phi_E^{\deg(p)}(z)||p||_E.$$

This already makes it a useful object of study.

One of the main facts of Pluripotential Theory is that $\log(\Phi_E)$ is equal to the pluricomplex generalization to several variables of the Green's function with pole at infinity. To state this precisely requires some technical definitions but it is not hard to see that these are exactly what are needed. In fact, $\log(\Phi_E(z)) = \sup\{\frac{1}{\deg(p)}\log|p(z)|\}$ (over suitable polynomials p). In one variable the logarithm of the modulus of an analytic function is generally harmonic, except there is the possibilty that the function has a zero, at which point its logarithm would be $-\infty$, so that we end up with a subharmonic function. In several variables, the logarithm of the modulus of an analytic function need not be subharmonic anymore, but is in general *plurisubharmonic*.

Definition 2 (cf. *Klimek [9, §2.9]). Suppose that $\Omega \subset \mathbb{C}^m$ is a connected open set and that $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous and not identically $-\infty$. Then u is said to be* plurisubharmonic *(written $u \in PSH(\Omega)$) if u, when restricted to any complex line passing through Ω, is subharmonic, or identically $-\infty$.*

Further, $\frac{1}{\deg(p)}\log|p(z)|$ has growth of order $\log|z|$ at infinity and thus it is natural to introduce:

Definition 3 (cf. *Klimek [9, §5.1]). The psh functions of* minimal growth, *or Lelong class, are*

$$\mathcal{L} := \{u \in PSH(\mathbb{C}^m) \,:\, u(z) \le \log(1 + |z|) + O(1)\}.$$

Definition 4 (Pluricomplex Green's Function). *For $E \subset \mathbb{C}^m$, compact and $z \in \mathbb{C}^m$*

$$V_E(z) := \sup\{u(z) \,:\, u \in \mathcal{L},\, u \le 0 \text{ on } E\}.$$

Theorem 2 (cf. *Klimek [9, Thm. 5.1.7]). Suppose that $E \subset \mathbb{C}^m$ is compact. Then $V_E = \log(\Phi_E)$.*

One important special case of the pluricomplex Green's function is for $E = I = [-1, 1] \subset \mathbb{R}^1 \subset \mathbb{C}^1$. Then

$$V_I(z) = \log|h(z)| \tag{6}$$

where

$$h(z) := z + \sqrt{z^2 - 1} \tag{7}$$

is the so-called Joukovski function.

We are now almost ready to state Baran's generalization of the van der Corput - Schaake inequality. Suppose that $\Omega \subset \mathbb{R}^m$ is open, connected and bounded, so that $E := \overline{\Omega}$ is compact. We note that it is not difficult to see that for $x \in \Omega$ and $y \in \mathbb{R}^m$, the function $y \mapsto V_E(x+iy)$ is Lipschitz in y.

Then, given a purely complex direction iy with $y \in \mathbb{R}^m$, and $x \in \Omega$, we set

$$V_E^o(x;y) := \limsup_{t\to 0+,\, z\to y} \frac{V_E(x+itz)-V_E(x)}{t} = \limsup_{t\to 0+,\, z\to y} \frac{V_E(x+itz)}{t} \tag{8}$$

(since $V_E(x)=0$) to be the *Clarke directional derivative* of V_E in the complex direction iy. As shown in Clarke[5, Prop. 2.1.1], the mapping $y \mapsto V_E^o(x;y)$ (for fixed x) is finite, positively homogeneous and subadditive on $\mathbb{R}^m$. In case $V_E^o(x;y)$ equals an ordinary directional derivative, $D_{iy}V_E(x)$, then V_E is said to be Clarke regular. Bedford and Taylor [4, Thm. 3.2] show that if E is *convex and symmetric* then V_E is indeed Clarke regular (and give an explicit formula for the true directional derivative).

Theorem 3 (Baran[2,3]). *Suppose that Ω and E are as above. Then for all $x \in \Omega$, directions $y \in \mathbb{R}^m$ and polynomials p such that $||p||_E \le 1$,*

$$\frac{1}{\deg(p)}\frac{|D_y p(x)|}{\sqrt{1-p^2(x)}} \le V_E^o(x;y). \tag{9}$$

We note that when $E = I \subset \mathbb{R}^1$, we may use the explicit formula (6) to calculate that

$$V_I^o(x;y) = \frac{|y|}{\sqrt{1-x^2}}$$

so that Baran's inequality (9) does indeed reduce to the van der Corput - Schaake inequality (3).

We will use Baran's inequality to construct a distance as follows. First suppose that $a, b \in \Omega$ and that $\gamma(t)$ is a smooth (C^1) curve, lying entirely in Ω such that $\gamma(0)=a$ and $\gamma(1)=b$. As usual, we consider a polynomial p such that $||p||_E \le 1$. Then,

$$\cos^{-1}(p(b)) - \cos^{-1}(p(a)) = \int_0^1 \frac{d}{dt}\{\cos^{-1}(p(\gamma(t)))dt$$

so that

$$|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| \le \int_0^1 \frac{|D_{\gamma'(t)}p(\gamma(t))|}{\sqrt{1-p^2(\gamma(t))}}dt \le \deg(p)\int_0^1 V_E^o(\gamma(t);\gamma'(t))dt.$$

In particular, it follows that

$$d_E(a,b) \le \int_0^1 V_E^o(\gamma(t);\gamma'(t))dt \tag{10}$$

and it is hence natural to define the Baran distance by

$$b_E(a,b) := \inf_\gamma \int_0^1 V_E^o(\gamma(t);\gamma'(t))dt \tag{11}$$

where the infimum is taken over all piecewise smooth paths $\gamma \subset \Omega$ that connect a to b. This is a *Finsler* type distance with $V_E^o(x;y)$ acting as the associated *Finsler metric.* See [1] and also [8] for further details on such matters.

Now, by (10), we automatically have that

$$d_E(a,b) \le b_E(a,b). \qquad (12).$$

We will see that in some cases $d_E = b_E$ but this is *not* true in general.

§3. Some Explicit Examples

The Sphere.

Proposition 1 (Dubiner [7]). *Suppose that $K = S^{m-1} \subset \mathbb{R}^m$ is the unit sphere. Then for $a, b \in S^{m-1}$,*

$$d_{S^{m-1}}(a,b) = \cos^{-1}(a\cdot b),$$

the geodesic distance on the sphere between a and b.

Proof: We show that $d_{S^{m-1}}(a,b) \ge \cos^{-1}(a\cdot b)$ and that $d_{S^{m-1}}(a,b) \le \cos^{-1}(a\cdot b)$. If $a=b$ there is nothing to do, so we suppose that $a \ne b$.

For the lower bound, take $p(x) := a\cdot x$, giving

$$\begin{aligned} d_{S^{m-1}}(a,b) &\ge \frac{1}{1}|\cos^{-1}(a\cdot b) - \cos^{-1}(a\cdot a)| \\ &= |\cos^{-1}(a\cdot b) - \cos^{-1}(1)| \\ &= |\cos^{-1}(a\cdot b) - 0| \\ &= \cos^{-1}(a\cdot b). \end{aligned}$$

For the upper bound, for $||p||_{S^{m-1}} \le 1$, set

$$\tilde{p}(\theta) := p(a\cos(\theta) + \frac{b-(a\cdot b)}{\sqrt{1-(a\cdot b)^2}}\sin(\theta)).$$

Then $\tilde{p}(\theta)$ is a trigonometric polynomial of degree $\deg(\tilde{p}) = \deg(p)$ and it is easy to verify that $\tilde{p}(0) = p(a)$ and $\tilde{p}(\cos^{-1}(a\cdot b)) = p(b)$. Hence the trigonometric form of the van der Corput - Schaake inequality,

$$\left|\frac{d}{d\theta}\cos^{-1}(\theta)(\tilde{p}(\theta))\right| \le \deg(\tilde{p}),$$

yields

$$\begin{aligned}
&\frac{1}{\deg(p)}|\cos^{-1}(p(b)) - \cos^{-1}(p(a))| \\
&= \frac{1}{\deg(\tilde{p})}|\cos^{-1}(\tilde{p}(\cos^{-1}(a\cdot b)) - \cos^{-1}(\tilde{p}(0))| \\
&\le \frac{1}{\deg(\tilde{p})}\sup_{0\le\theta\le\cos^{-1}(a\cdot b)}\left|\frac{d}{d\theta}\cos^{-1}(\tilde{p}(\theta))\right| |\cos^{-1}(a\cdot b) - 0| \\
&\le \frac{1}{\deg(\tilde{p})}\deg(\tilde{p})\cos^{-1}(a\cdot b) \\
&= \cos^{-1}(a\cdot b).
\end{aligned}$$

Thus $d_{S^{m-1}}(a,b) \le \cos^{-1}(a\cdot b)$ and we are done. □

We remark that since S^{m-1} is contained inside the algebraic variety

$$A := \{z = (z_1, \cdots, z_m) \,:\, z_1^2 + \cdots + z_m^2 = 1\},$$

by Definition 4,

$$V_{S^{m-1}}(z) \ge \sup\{\frac{1}{2}\log|z_1^2 + \cdots + z_m^2| + j \,:\, j = 1, 2, 3, \cdots\}$$

so that $V_{S^{m-1}} = +\infty$ on $\mathbb{C}^m \backslash A$. Thus the Baran distance is not well-defined.

The Ball.

Proposition 2. *Supppose that $\Omega = \{x \in \mathbb{R}^m \,:\, |x| < 1\}$ so that $E := \overline{\Omega} = B^m$ is the closed unit ball in $\mathbb{R}^m$. Then for $a, b \in B^m$,*

$$d_{B^m}(a,b) = b_{B^m}(a,b) = \cos^{-1}(\tilde{a}\cdot\tilde{b})$$

where $\tilde{a} := (a, \sqrt{1-|a|^2})$, $\tilde{b} := (b, \sqrt{1-|b|^2})$ are a and b lifted to the surrounding sphere, $S^m \subset \mathbb{R}^{m+1}$.

Proof: Again we may suppose that $a \ne b$. Recall that $d_E(a,b) \le b_E(a,b)$ always. Hence it suffices to show that $d_{B^m}(a,b) \ge \cos^{-1}(\tilde{a}\cdot\tilde{b})$ and that $b_{B^m}(a,b) \le \cos^{-1}(\tilde{a}\cdot\tilde{b})$.

To see that $d_{B^m}(a,b) \geq \cos^{-1}(\tilde{a}\cdot\tilde{b})$, consider the two dimensional plane spanned by $\tilde{a}$, $\tilde{b}$ and $0 \in \mathbb{R}^{m+1}$. If this plane is contained in $\mathbb{R}^m$ (i.e. iff $|a| = |b| = 1$ iff $\tilde{a} = (a,0)$ and $\tilde{b} = (b,0)$) then take $p(x) := a\cdot x$. As in the spherical case, we obtain

$$\begin{aligned} d_{B^m}(a,b) &\geq \cos^{-1}(a\cdot b) \\ &= \cos^{-1}(\tilde{a}\cdot\tilde{b}), \end{aligned}$$

in this case.

If this plane is not contained in $\mathbb{R}^m$, it intersects $\mathbb{R}^m$ in a line through the origin. Let v be the unit direction vector of this line. Specifically,

$$v = k\{-\sqrt{1-|b|^2}a + \sqrt{1-|a|^2}b\}$$

where k is a normalization constant. Then, taking $p(x) := v\cdot x$ yields

$$\begin{aligned} d_{B^m}(a,b) &\geq \frac{1}{1}|\cos^{-1}(v\cdot b) - \cos^{-1}(v\cdot a)| \\ &= \cos^{-1}((v\cdot b)(v\cdot a) + \sqrt{1-(v\cdot b)^2}\sqrt{1-(v\cdot a)^2}) \\ &\quad \text{(by the cosine sum formula)} \\ &= \cos^{-1}(\langle v\cdot a, \sqrt{1-(v\cdot a)^2}\rangle \cdot \langle v\cdot b, \sqrt{1-(v\cdot b)^2}\rangle). \end{aligned}$$

But an elementary (but perhaps tedious) calculation, or else a moment's geometric reflection, shows that the angle between the vectors

$$\langle v\cdot a, \sqrt{1-(v\cdot a)^2}\rangle \text{ and } \langle v\cdot b, \sqrt{1-(v\cdot b)^2}\rangle$$

is the *same* as that between $\tilde{a}$ and $\tilde{b}$ and we are done.

Now to show that $b_{B^m}(a,b) \leq \cos^{-1}(\tilde{a}\cdot\tilde{b})$ (we will actually show that they are equal).

There is an explict formula for the extremal function for the real ball B^m considered as a subset of $\mathbb{C}^m$,

$$V_{B^m}(z) = \frac{1}{2}\log(h(|z|^2 + |\sum_{j=1}^m z_j^2 - 1|)), \quad z \in \mathbb{C}^m$$

where, as before, h is the Joukovski function (7) (cf. Klimek[9, Example 5.4.6]). Hence one may calculate (cf. Baran[3, Example 1.3]) that

$$\begin{aligned} V^o_{B^m}(x;y) &= \frac{\sqrt{(1-|x|^2)|y|^2 + (x\cdot y)^2}}{\sqrt{1-|x|^2}} \\ &= \sqrt{|y|^2 + \left(\frac{x\cdot y}{\sqrt{1-|x|^2}}\right)^2}. \end{aligned}$$

From this, it follows easily that

$$V_{B^m}^o(\gamma(t);\gamma'(t)) = \left|\frac{d}{dt}\langle\gamma(t),\sqrt{1-|\gamma(t)|^2}\rangle\right|,$$

i.e.,

$$V_{B^m}^0(\gamma(t);\gamma'(t)) = |\tilde{\gamma}'(t)|$$

where

$$\tilde{\gamma}(t) := (\gamma(t),\sqrt{1-|\gamma(t)|^2})$$

is the curve $\gamma(t)$ lifted from B^m to the surrounding sphere S^m!

Hence

$$\begin{aligned} b_{B^m}(a,b) &= \inf_{\gamma}\{\int_0^1 V_{B^m}^o(\gamma(t);\gamma'(t))dt\} \\ &= \inf_{\tilde{\gamma}}\int_0^1 |\tilde{\gamma}'(t)|dt \\ &= \inf_{\tilde{\gamma}}\{\text{length of } \tilde{\gamma}\} \end{aligned}$$

where $\tilde{\gamma}$ connects $\tilde{a}$ to $\tilde{b}$, on the sphere S^m.

But the geodesics on a sphere are known and so $b_{B^m}(a,b)$ is just the spherical geodesic distance between $\tilde{a}$ and $\tilde{b}$, as claimed. □

The Cube.

Proposition 3. *Suppose now that* $\Omega = (-1,1)^m$ *so that* $E := \overline{\Omega} = [-1,1]^m = I^m$ *is the closed unit cube. Then for* $a,b \in I^m$,

$$\begin{aligned} d_{I^m}(a,b) = b_{I^m}(a,b) &= \max_{1\le j\le m}|d_I(a_j,b_j)| \\ &= \max_{1\le j\le m}|\cos^{-1}(b_j)-\cos^{-1}(a_j)|. \end{aligned}$$

Proof: We may suppose that $a \neq b$. As before, it suffices to show that

$$d_{I^m}(a,b) \ge \max_{1\le j\le m}|d_I(a_j,b_j)| \text{ and } b_{I^m}(a,b) \le \max_{1\le j\le m}|d_I(a_j,b_j)|.$$

To see that $d_{I^m}(a,b) \ge \max_{1\le j\le m}|d_I(a_j,b_j)|$ follows easily by considering $p_j(x) := x_j$, $1 \le j \le m$.

Now to show that $b_{I^m}(a,b) \le \max_{1\le j\le m}|d_I(a_j,b_j)|$.

There is again an explict formula for

$$V_{I^m}(z) = \max_{1\le j\le m}\{\log|h(z_j)|\}, \quad z \in \mathbb{C}^m$$

(cf. Klimek[9, Cor. 5.4.5]). Hence we may compute

$$V^o_{I^m}(x;y) = \max_{1\le j\le m} \frac{|y_j|}{\sqrt{1-x_j^2}}.$$

Consider the curve

$$\gamma(t) := \cos(t\cos^{-1}(b) + (1-t)\cos^{-1}(a))$$

so that the jth component is $\gamma_j(t) = \cos(t\cos^{-1}(b_j) + (1-t)\cos^{-1}(a_j))$. Then

$$\begin{aligned} b_{I^m}(a,b) &\le \int_0^1 V^o_{I^m}(\gamma(t);\gamma'(t))dt \\ &= \int_0^1 \max_{1\le j\le m} \left\{ \frac{|\gamma_j'(t)|}{\sqrt{1-\gamma_j^2(t)}} \right\} dt. \end{aligned}$$

But an easy calculation reveals that

$$\frac{|\gamma_j'(t)|}{\sqrt{1-\gamma_j^2(t)}} = |\cos^{-1}(b_j) - \cos^{-1}(a_j)|$$

so that

$$\begin{aligned} b_{I^m}(a,b) &\le \int_0^1 \max_{1\le j\le m} |\cos^{-1}(b_j) - \cos^{-1}(a_j)|\, dt \\ &= \max_{1\le j\le m} |\cos^{-1}(b_j) - \cos^{-1}(a_j)| \end{aligned}$$

as claimed. □

The Simplex.

Proposition 4. *Suppose that*

$$\Omega = \{x \in \mathbb{R}^m \,:\, x_j > 0,\ \sum_{j=1}^m x_j < 1\}$$

so that $E := \overline{\Omega}$ is the standard unit simplex. For $a \in E$ let $\tilde{a}$ denote the point on the unit sphere $S^m \subset \mathbb{R}^m$ given by

$$\tilde{a} := (\sqrt{a_1}, \sqrt{a_2}\cdots, \sqrt{a_m}, \sqrt{1-a_1-\cdots-a_m}).$$

Then, for $a, b \in E$,

$$b_E(a,b) = 2d_{S^m}(\tilde{a},\tilde{b}) = 2\cos^{-1}(\tilde{a}\cdot\tilde{b}).$$

Proof: There is again an explict formula for the extremal function of a simplex:

$$V_E(z) = \log(h(|z_1| + \cdots + |z_m| + |z_1 + \cdots + z_m - 1|)), \quad z \in \mathbb{C}^m$$

(cf. Klimek[9, Example 5.4.7]). Using this we may calculate

$$V_E^o(x;y) = \sqrt{\sum_{j=1}^m \frac{y_j^2}{x_j} + \frac{(\sum_{j=1}^m y_j)^2}{1 - \sum_{j=1}^m x_j}}.$$

Now, given a curve $\gamma(t)$ connecting a and b in E, let $\tilde{\gamma}(t)$ deonte the "lifting" of γ to S^m under the same mapping as $\tilde{a}$. Specifically, we set

$$\tilde{\gamma}(t) := \left(\sqrt{\gamma_1(t)}, \sqrt{\gamma_2(t)}, \cdots, \sqrt{\gamma_m(t)}, \sqrt{1 - \sum_{j=1}^m \gamma_j(t)}\right).$$

Then,

$$\tilde{\gamma}'(t) := \frac{1}{2}\left(\frac{\gamma_1'(t)}{\sqrt{\gamma_1(t)}}, \frac{\gamma_2'(t)}{\sqrt{\gamma_2(t)}}, \cdots, \frac{\gamma_m'(t)}{\sqrt{\gamma_m(t)}}, -\frac{\sum_{j=1}^m \gamma_j'(t)}{\sqrt{1 - \sum_{j=1}^m \gamma_j(t)}}\right)$$

and it is easy then to see that, remarkably,

$$V_E^o(\gamma(t);\gamma'(t)) = 2|\tilde{\gamma}'(t)|.$$

Hence,

$$\begin{aligned} b_E(a,b) &= \inf_\gamma\{\int_0^1 V_E^o(\gamma(t);\gamma'(t))dt \\ &= \inf_{\tilde{\gamma}} 2\int_0^1 |\tilde{\gamma}'(t)|dt \\ &= \inf_{\tilde{\gamma}} \{\text{twice the length of } \tilde{\gamma}\}. \end{aligned}$$

We again appeal to the known geodesics on a sphere. □

We point out that for the simplex, a non-symmetric convex set, it is *not* the case that $d_E(a,b) = b_E(a,b)$ for all pairs of points a and b (it is true for certain pairs). Here is an example. Consider the simplex in $\mathbb{R}^2$ and take $a := (0, 1/2)$ and $b := (1/2, 1/2)$.

We claim that $d_E(a,b)=\pi/2$. To see this, note that for $p(x_1,x_2):=2x_1-1$, we have $||p||_E=1$ and so with this particular p,

$$\begin{aligned} d_E(a,b) &\geq \frac{1}{1}|\cos^{-1}(p(1/2,1/2))-\cos^{-1}(p(0,1/2))| \\ &= |\cos^{-1}(0)-\cos^{-1}(-1)| \\ &= |\frac{\pi}{2}-\pi| \\ &= \frac{\pi}{2.} \end{aligned}$$

Further, if $\deg(p)\geq 2$, then

$$\frac{1}{\deg(p)}|\cos^{-1}(p(b))-\cos^{-1}(p(a))|\leq\frac{\pi}{2}$$

and so we need only further analyze polynomials of degree *one*. This is a finite dimensional problem that can be easily completely worked out to verify our claim. We suppress the details of this calculation.

On the other hand, $\tilde{a}=(0,1/\sqrt{2},1/\sqrt{2})$ and $\tilde{b}=(1/\sqrt{(}2),1/\sqrt{2},0)$ so that

$$b_E(a,b)=2\cos^{-1}(\tilde{a}\cdot\tilde{b})=2\cos^{-1}(1/2)=2\pi/3>\pi/2=d_E(a,b).$$

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