UNIVERSITY OF AUCKLAND, NEW ZEALAND
SCHOOL OF MATHEMATICAL AND INFORMATION SCIENCES

Integral error formulæ for the scale of mean value interpolations which includes Kergin and Hakopian interpolation

Shayne Waldron* (waldron@math.auckland.ac.nz)

Department of Mathematics, University of Auckland
Private Bag 92019, Auckland, New Zealand

Running title: Kergin and Hakopian interpolation

Summary

In this paper, we provide an integral error formula for a certain *scale of mean value interpolations* which includes the multivariate polynomial interpolation schemes of *Kergin* and *Hakopian*. This formula involves only derivatives of order one higher than the degree of the interpolating polynomial space, and from it we can obtain sharp L_∞-estimates. These L_∞-estimates are precisely those that numerical analysts want, to guarantee that a scheme based on such an interpolation has the maximum possible *order*.

AMS (MOS) Subject Classifications: primary 41A05, 41A63, 41A80; secondary 41A10, 41A44, 44A12

Key Words: scale of mean value interpolations, Kergin interpolation, Hakopian interpolation, Lagrange interpolation, Hermite interpolation, Hermite-Genocchi formula, multivariate divided difference, plane wave, lifting, Radon transform

* Supported by the Chebyshev Professorship of Carl de Boor and the Center for the Mathematical Sciences at the University of Wisconsin-Madison.

Integral error formulæ for a certain scale of mean value interpolations which includes Kergin and Hakopian interpolation

1. Introduction

In this paper we study the error in a certain scale of mean value interpolations which includes Kergin and Hakopian interpolation. The literature divides into two different approaches to this problem.

The first is concerned with the convergence of the interpolants as the number of interpolation points increases. Here only Kergin interpolation has been studied. Certain conditions on the position of the interpolation points and the growth of the entire function to be interpolated are given which guarantee that the sequence of interpolants converges uniformly on compact sets. See, e.g., Bloom [2].

We are interested in the second approach, which is to write the error in interpolation as integration against derivatives of high order, much as is done for univariate Hermite interpolation.

There have been several papers in this direction, including Lai and Wang [19] (Hakopian interpolation), [20] (Kergin interpolation), and Gao [12] (mean value interpolation). Each of these gives formulæ for the error, complicated by the spurious use of certain *multivariate divided differences*, involving derivatives of various orders. There seems to be very little correspondence between the degree of the interpolating polynomial space and the order of the derivatives involved. This order can be as low as 0, and as high as twice the degree of the interpolating polynomial space.

In this paper we give an integral error formula for the scale of mean value interpolations that involves only derivatives of order one higher than the degree of the interpolating polynomial space. From this we obtain sharp L_∞-estimates. These estimates imply that a numerical scheme based on mean value interpolation has the highest *order* that its polynomial reproduction allows.

The paper is set out in the following way. To describe the scale of mean value interpolations, we use a certain linear functional $f \mapsto \int_\Theta f$ and the notion of 'lifting' univariate maps. These two notions are studied in requisite detail in Sections 2 and 3 respectively. In Section 4, we define the scale of mean value interpolations and give its Newton form. In Section 5, we give two different integral error formulæ for the scale. In Section 6, from these formulæ, we obtain L_∞-estimates.

Some notation

The space of n-variate polynomials of total degree k will be denoted by $\Pi_k(\mathbb{R}^n)$ and the homogeneous polynomials of degree k by $\Pi_k^0(\mathbb{R}^n)$. The differential operator induced by $g \in \Pi_k(\mathbb{R}^n)$ will be written $g(D)$.

We find it convenient to make no distinction between the matrix $[\theta_1, \ldots, \theta_k]$ and the k-**sequence** $\theta_1, \ldots, \theta_k$ of its columns. So, for example, with B a matrix, $B\Theta$ is interpreted as the product of matrices. Since $[\theta_1, \ldots, \theta_k]f$ is a standard notation for the *divided difference* of f at $\Theta = [\theta_1, \ldots, \theta_k]$, we use for the latter the nonstandard notation

$$\delta_\Theta f = \delta_{[\theta_1,\ldots,\theta_k]} f.$$

Note the special case

$$\delta_{[x]} f = f(x).$$

The notation $\tilde{\Theta} \subset \Theta$ means that $\tilde{\Theta}$ is a subsequence of Θ, $\Theta\backslash\tilde{\Theta}$ denotes the complementary subsequence. The derivative of f in the directions Θ is denoted

$$D_\Theta f := D_{\theta_1} \cdots D_{\theta_k} f.$$

The subsequence consisting of the first j terms of Θ is denoted Θ_j, and

$$x - \Theta := [x - \theta_1, \ldots, x - \theta_k].$$

Thus, with $\Theta := [\theta_1, \ldots, \theta_7]$, we have, for example, that

$$D_{[x-\Theta\backslash\Theta_5, x-\theta_3]} f = D_{x-\theta_6} D_{x-\theta_7} D_{x-\theta_3} f.$$

The diameter and convex hull of a sequence Θ will be that of the corresponding set and will be denoted by $\operatorname{diam}\Theta$ and $\operatorname{conv}\Theta$ respectively. Let $\|\cdot\|$ be the *Euclidean norm.* To measure the size of the k-th derivative of f at $x \in \mathbb{R}^n$, we use the seminorm

$$|D^k f|(x) := \sup_{\substack{u_1,\ldots,u_k \in \mathbb{R}^n \\ \|u_i\| \le 1}} |D_{u_1} \cdots D_{u_k} f(x)|.$$

Notice that

$$|D_{u_1} \cdots D_{u_k} f(x)| \le |D^k f|(x)\, \|u_1\| \cdots \|u_k\|. \tag{1.1}$$

To measure the size of the k-th derivative of f over $K \subset \mathbb{R}^n$, we use

$$|f|_{k,\infty,K} := \sup_{x\in K} |D^k f|(x). \tag{1.2}$$

Because of (1.1), the co-ordinate-independent seminorm $|\cdot|_{k,\infty,K}$ is more appropriate to the analysis that follows than other equivalent seminorms, such as

$$f \mapsto \max_{|\alpha|=k} \|D^\alpha f\|_{L_\infty(K)}.$$

2. The linear functional $f \mapsto \int_\Theta f$

The construction of the maps of Kergin and Hakopian depends intimately on the following linear functional called the **divided difference functional on** $\mathbb{R}^n$ by Micchelli in [23], and analysed there and in [24].

Definition 2.1. *With* Θ *the sequence* $[\theta_0, \ldots, \theta_k]$ *of* $k+1$ *points in* $\mathrm{I\!R}^n$,

$$f \mapsto \int_\Theta f := \int_0^1 \int_0^{s_1} \ldots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1,$$

with the convention that $\int_{[\,]} f := 0$.

Part of Micchelli's motivation for defining $\int_\Theta f$ was the **Hermite-Genocchi formula**, namely

$$\delta_\Theta f = \int_\Theta D^k f, \quad \forall f \in C^k(\operatorname{conv}\Theta),$$

where Θ is a $(k+1)$-sequence in $\mathrm{I\!R}$.

In this section we outline those properties of $f \mapsto \int_\Theta f$ needed in the remaining sections. Many of these properties are apparent from the following observation of the author.

Observation 2.2. *If* S *is any* k*-simplex in* $\mathrm{I\!R}^m$ *and* $A : \mathrm{I\!R}^m \to \mathrm{I\!R}^n$ *is any affine map taking the* $k+1$ *vertices of* S *onto the* $k+1$ *points in* Θ, *then*

$$\int_\Theta f = \frac{1}{k!\,\mathrm{vol}_k(S)} \int_S f \circ A,$$

with $\mathrm{vol}_k(S)$ *the (*k*-dimensional) volume of* S.

In Definition 2.1

$$A : \mathrm{I\!R}^k \to \mathrm{I\!R}^n : (s_1, \ldots, s_k) \mapsto \theta_0 + s_1(\theta_1 - \theta_0) + \ldots + s_k(\theta_k - \theta_{k-1}),$$

$$S := \{(s_1, \ldots, s_k) \in \mathrm{I\!R}^k : 0 \le s_k \le \cdots \le s_2 \le s_1 \le 1\}.$$

In [24], Micchelli uses a different choice of S and A, namely

$$A : \mathrm{I\!R}^{k+1} \to \mathrm{I\!R}^n : (v_0, \ldots, v_k) \mapsto v_0\theta_0 + \cdots + v_k\theta_k,$$

$$S := \{(v_0, \ldots, v_k) \in \mathrm{I\!R}^{k+1} : v_j \ge 0,\ \sum_{j=0}^k v_j = 1\}.$$

Properties 2.3 (see, e.g., [24], [16]).

(a) *The value of* $\int_\Theta f$ *does not depend on the ordering of the points in* Θ.

(b) *The distribution*

$$M_\Theta : C_0^\infty(\mathrm{I\!R}^n) \to \mathrm{I\!R} : f \mapsto k! \int_\Theta f$$

*is the (***normalised***)* **simplex spline** *with knots* Θ. *It has support* $\operatorname{conv}\Theta$, *and can be represented by the nonnegative bounded function*

$$\operatorname{conv}\Theta \to \mathrm{I\!R} : t \mapsto M(t|\Theta) := \frac{\mathrm{vol}_{k-d}(A^{-1}t \cap S)}{|\det A|\,\mathrm{vol}_k(S)}, \quad d := \dim \operatorname{conv}\Theta,$$

in the sense that

$$M_\Theta f = \int_{\mathrm{conv}\,\Theta} M(\cdot|\Theta) f.$$

Thus, $\int_\Theta f$ *is defined iff* $M(\cdot|\Theta)f \in L_1(\mathrm{conv}\,\Theta)$*.*

(c) *If* $f \in C(\mathrm{conv}\,\Theta)$*, then* $\int_\Theta f$ *is defined and, for some* $\xi \in \mathrm{conv}\,\Theta$*,*

$$\int_\Theta f = \frac{1}{k!} f(\xi).$$

(d) *If* $g : \mathbb{R}^s \to \mathbb{R}$*, and* $B : \mathbb{R}^n \to \mathbb{R}^s$ *is an affine map, then*

$$\int_\Theta (g \circ B) = \int_{B\Theta} g.$$

(e) *If* $f \in C(\mathbb{R}^n)$*, then the map*

$$\Theta \mapsto \int_\Theta f$$

is continuous.

3. Liftable maps

In this section, we discuss univariate maps which may be *lifted* to multivariate ones. These 'liftable' maps are crucial to both the construction and description of the error in a family of linear projectors which includes the Kergin and Hakopian maps. The main papers on 'lifting' are [5], [6], [7] and [17].

We denote the linear functional on $\mathbb{R}^n$, induced by scalar product with $\lambda \in \mathbb{R}^n$, by

$$\lambda^* : \mathbb{R}^n \to \mathbb{R} : x \mapsto \lambda^* x := \sum_{i=1}^n \lambda(i) x(i).$$

A **plane wave** (or **ridge function**) is any map

$$g \circ \lambda^* : \mathbb{R}^n \to \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}^n$. If $g \in C^1(\mathbb{R})$, then we can differentiate $g \circ \lambda^*$, thereby obtaining

$$D_y(g \circ \lambda^*) = (\lambda^* y)\,(Dg) \circ \lambda^*. \tag{3.1}$$

This 'lifts' differentiation to $\mathbb{R}^n$.

In [5] only the lifting of polynomial-valued maps is discussed. To 'lift' the error in such maps, we need a more general definition. The only real difficulty involved in giving such a definition is in choosing the function spaces so that the fundamentality of the plane waves implies the uniqueness of the 'lift'. We propose the following definition which takes care of this.

Definition 3.2. *Let $L : \Xi \mapsto L_\Xi$ associate with each k-sequence Ξ in $\mathbb{R}$ a continuous linear map $L_\Xi : C^s(\mathbb{R}) \to C(\mathbb{R})$. We say that a continuous linear map $\mathcal{L}_\Theta : C^s(\mathbb{R}^n) \to C(\mathbb{R}^n)$ is the* **lift of** L **to** Θ *in* $\mathbb{R}^n$ *if it satisfies*

$$\mathcal{L}_\Theta(g \circ \lambda^*) = (L_{\lambda^*\Theta} g) \circ \lambda^*, \quad \forall \lambda \in \mathbb{R}^n, \ \forall g \in C^s(\mathbb{R}). \tag{3.3}$$

If there exists a lift $\mathcal{L}_\Theta$ of L to each k-sequence Θ in $\mathbb{R}^n$, then we say that L is **liftable** *(***to** $\mathbb{R}^n$*), and call $\mathcal{L} : \Theta \mapsto \mathcal{L}_\Theta$ the* **lift of** L *(***to** $\mathbb{R}^n$*).*

Notice that (3.3) overdetermines the map $\mathcal{L}_\Theta$, and so the use of the definite article in the above definition is justified. Furthermore, by the fundamentality of the polynomial plane waves (which span $\Pi(\mathbb{R}^n)$) in $C^s(\mathbb{R}^n)$, if L can be lifted to $\mathcal{L}_\Theta$, then $\mathcal{L}_\Theta$ is uniquely determined by (3.3). To avoid confusion, we will use calligraphic letters to denote the lift of a univariate map and, from now on, reserve k for the number of points such a map is based on.

The geometric intent of lifting is that the 'lift' of a function which varies in one direction, i.e., a plane wave, should be a plane wave (varying in the same direction) obtained in a natural way from the univariate map to be lifted.

The basic tool for recognising liftable maps and presenting their lifts is to write them as a sum of 'elementary liftable maps', which are defined as follows.

Definition 3.4. *Let $s, m \ge 0$. Fix $a_j \in \mathbb{R}^{k+1}\backslash 0$, $j = 1, \ldots, s$ and $B \in \mathbb{R}^{(k+1)\times(m+1)}$. For each k-sequence Θ in $\mathbb{R}$, let $L_\Theta : C^s(\mathbb{R}) \to C(\mathbb{R})$ be the continuous linear map given by*

$$L_\Theta f(x) := \left(\prod_{j=1}^{s} [x,\Theta] a_j\right) \int_{[x,\Theta]B} D^s f = \int_{[x,\Theta]B} \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j}\right) f. \tag{3.5}$$

We call $L : \Theta \mapsto L_\Theta$ an **elementary** *(*k**-point***)* **liftable map** *(***of order** s*).*

Here and below, in line with our earlier identification of vector sequences and matrices, $[x, \Theta]B$ is the matrix whose j-th column is the vector

$$xB(1, j) + \theta_1 B(2, j) + \cdots + \theta_k B(k + 1, j).$$

In other words, $[x, \Theta]B$ is an $(m + 1)$-sequence.

The equality in (3.5) expresses $L_\Theta f(x)$ in a form which has a natural multivariate analogue. In this way, the definition is tailor-made to make it obvious that such a map is liftable, as we prove next.

Theorem 3.6. *Each elementary liftable map of order s, as in Definition 3.4, is liftable to $\mathbb{R}^n$. Its lift $\mathcal{L} : \Theta \mapsto \mathcal{L}_\Theta$, with $\mathcal{L}_\Theta : C^s(\mathbb{R}^n) \to C(\mathbb{R}^n)$, is given by*

$$\mathcal{L}_\Theta f(x) := \int_{[x,\Theta]B} \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j}\right) f. \tag{3.7}$$

In the special case that $B(1,\cdot)=0$, the range of $\mathcal{L}_\Theta$ is contained in $\Pi_s(\mathbb{R}^n)$.

Proof: The continuity of L_Θ required in Definition 3.4 and the continuity of $\mathcal{L}_\Theta$ asserted in Theorem 3.6, follow from the inequality

$$\|\mathcal{L}_\Theta f\|_{L_\infty(K)} \le \frac{1}{m!}\left(\max_{x\in K}\prod_{j=1}^s \|[x,\Theta]a_j\|\right) |f|_{s,\infty,\mathrm{conv}([x,\Theta]B)},$$

where $K\subset\mathbb{R}^n$ is compact. This is proved by applying, to (3.7), Property 2.3 (c) followed by (1.1) and (1.2).

Given the continuity of the maps L_Θ and $\mathcal{L}_\Theta$, to show that $\mathcal{L}$ is the lift of L, it is sufficient to prove that

$$\mathcal{L}_\Theta(g\circ\lambda^*) = (L_{\lambda^*\Theta}\, g)\circ\lambda^*, \quad \forall\lambda\in\mathbb{R}^n,\ \forall g\in C^s(\mathbb{R}),\ \forall\Theta\in(\mathbb{R}^n)^k.$$

By applying (3.1) s times, it follows that

$$(\mathcal{L}_\Theta(g\circ\lambda^*))(x) = \int_{[x,\Theta]B}\left(\prod_{j=1}^s \lambda^*[x,\Theta]a_j\right)(D^s g)\circ\lambda^*.$$

To the right-hand side of this, we apply Property 2.3 (d) (with λ^* the affine map) and the identity $\lambda^*[x,\Theta]=[\lambda^*x,\lambda^*\Theta]$ to obtain that

$$\int_{[\lambda^*x,\lambda^*\Theta]B}\left(\prod_{j=1}^s[\lambda^*x,\lambda^*\Theta]a_j\right)(D^s g) = (L_{\lambda^*\Theta}\, g)(\lambda^* x).$$

□

The sum (equivalently linear combination) of elementary liftable maps can be lifted to the corresponding sum of the lifts. This is the form in which we will use Theorem 3.6 when lifting (4.5), (5.5) and (5.9).

Example 3.8. In [17] it is shown that (sadly) the divided difference cannot be lifted; however we may lift the following divided difference identity

$$\delta_{[\Theta,v,w]}g = \frac{\delta_{[\Theta,v]}g-\delta_{[\Theta,w]}g}{v-w}, \quad v\ne w. \tag{3.9}$$

By the Hermite-Genocchi formula, (3.9) may be rewritten as

$$(v-w)\int_{[\Theta,v,w]} Df = \int_{[\Theta,v]} f - \int_{[\Theta,w]} f,$$

where $f:=D^k g$ and $k=\#\Theta$. By Theorem 3.6, this lifts to

$$\int_{[\Theta,v,w]} D_{v-w}f = \int_{[\Theta,v]} f - \int_{[\Theta,w]} f, \tag{3.10}$$

for all sufficiently smooth f, where Θ is any finite sequence in $\mathbb{R}^n$ and $v,w\in\mathbb{R}^n$.

An elementary liftable map depends continuously on Θ, in the following sense.

Theorem 3.11. *Let $\mathcal{L}$ be the lift to $\mathbb{R}^n$ of an elementary k-point liftable map of order s. For all $f \in C^s(\mathbb{R}^n)$, the map*

$$(\mathbb{R}^n)^k \to C(\mathbb{R}^n) : \Theta \mapsto \mathcal{L}_\Theta f$$

is continuous.

Proof: By Property 2.3 (e), the map

$$(x, \Theta) \mapsto \mathcal{L}_\Theta f(x)$$

is continuous. □

The literature contains no discussion of the 'continuous' dependence of $\mathcal{L}_\Theta$ on Θ. In [5] it is shown that a *complex regular Birkhoff interpolation* procedure is liftable by writing it as a sum of what we have called here elementary liftable maps. Thus, we have the following.

Corollary 3.12. *Let B be the complex regular Birkhoff interpolation procedure and $\mathcal{B}$ its lift to $\mathbb{R}^n$. For each $f \in C^s(\mathbb{R}^n)$, the map*

$$\Theta \mapsto \mathcal{B}_\Theta f$$

is continuous.

In the case $n = 1$, i.e., when $\mathcal{B}_\Theta = B_\Theta$, this continuity result was proved in [11] by using 'de-coalescence' of the interpolation matrix.

Another immediate consequence is the continuous dependence of the Hermite interpolant on its points of interpolation. However, that is a direct consequence of the well-known continuity of $\Theta \mapsto \delta_\Theta f$.

Related considerations

In [5], [17], there is a discussion about lifting the family of distributions

$$\mathbb{R}^{k+1} \to S : (x, \Theta) \mapsto \delta_{[x]} L_\Theta,$$

where S is some suitable space of distributions, e.g., $C_0^\infty(\mathbb{R})$, or, in our case, $E'^s(\mathbb{R})$ (the space of compactly supported distributions of order s).

Lifting such a family is shown there to be equivalent to inverting its *Radon transform.* Without going too far into details, we mention that, for an elementary liftable map of the form (3.5), its Radon transform $\mathtt{H}$ is given by

$$\mathtt{H}(f) := \int_B \Big(\prod_{j=1}^{s} D_{a_j}\Big) f,$$

and so $\mathcal{L}_\Theta$ may be expressed as

$$\mathcal{L}_\Theta f(x) = \mathtt{H}(f \circ [x, \Theta]).$$

One useful consequence of the Radon transform theory is the following **compatibility condition**: if L is liftable, then $(x, \Theta) \mapsto L_\Theta((\cdot)^i)(x)$ is homogeneous of degree i. Moreover, by Property 2.3 (d), if L is an elementary liftable map and f is a homogeneous polynomial of degree i, then $(x, \Theta) \mapsto \mathcal{L}_\Theta f(x)$ is homogeneous of degree i.

4. The scale of mean value interpolations

In this section we describe a family $H^{(m)}$, $m < k$, of liftable maps that were lifted in [13] to obtain multivariate polynomial interpolation schemes. Special cases of these multivariate schemes, referred to in [3:p203] as *the scale of mean value interpolations*, are the well-known maps of Kergin and Hakopian.

We will need the following facts about linear interpolation.

Linear interpolation

Let F be a finite-dimensional space and Λ a finite-dimensional space of linear functionals defined at least on F. We say that the corresponding **linear interpolation problem**, LIP(F, Λ) for short, is **correct** if for every g upon which Λ is defined there is a unique $f \in F$ which agrees with g on Λ, i.e.,

$$\lambda(f) = \lambda(g), \quad \forall \lambda \in \Lambda.$$

The linear map $L : g \mapsto f$ is called the associated (**linear**) **projector** with **interpolants** F and **interpolation conditions** Λ. Each linear projector with finite-dimensional range F is the solution of a LIP(F, Λ) for some unique choice of the interpolation conditions Λ.

Notice that the correctness of LIP(F, Λ) depends only on the action of Λ on F.

The map $H^{(m)}$

Let $D^{-m} f$ be any function with $D^m(D^{-m} f) = f$. If

$$P : C^s(\mathbb{R}) \to \Pi_n(\mathbb{R})$$

is any linear projector, then for $m \le n$

$$f \mapsto D^m P(D^{-m} f),$$

is a linear projector into $\Pi_{n-m}(\mathbb{R})$ which is defined on $C^{s-m}(\mathbb{R})$.

We are interested in the case where P is H_Θ, which is, by definition, the *Hermite interpolation operator* at Θ, a k-sequence in $\mathbb{R}$.

Definition 4.1. *For $0 \le m < k = \#\Theta$, the* **generalised Hermite map**

$$H^{(m)} : \Theta \mapsto H_\Theta^{(m)}$$

is given by the linear projectors

$$H_\Theta^{(m)} : C^{k-m-1}(\mathrm{I\!R}) \to \Pi_{k-m-1}(\mathrm{I\!R}) : f \mapsto D^m(H_\Theta D^{-m} f).$$

For convenience, $H^{(k)} := 0$.

Observe that $H_\Theta^{(0)} = H_\Theta$, which in part justifies the term 'generalised Hermite map'. The generalised Hermite maps $H_\Theta^{(m)}$ occurred in the approximation theory literature before they were lifted by Goodman in [13]; see e.g., de Boor [1] where they were used to bound spline interpolation.

The interpolants for $H_\Theta^{(m)}$ are $\Pi_{k-m-1}(\mathrm{I\!R})$, and the interpolation conditions are

$$\mathrm{span}\{f \mapsto \int_{\tilde\Theta} D^{\#\tilde\Theta-m-1} f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta \ge m+1\}.$$

For Θ a finite sequence in $\mathrm{I\!R}$, let

$$\omega_\Theta(x) := \prod_{\theta\in\Theta}(x-\theta).$$

Note that if $j \le \#\Theta$, then

$$D^j \omega_\Theta = j! \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta = j}} \omega_{\Theta\setminus\tilde\Theta}. \tag{4.2}$$

If $\Theta = [\theta_1, \ldots, \theta_k]$, then we may write the 'Newton form' of $H_\Theta^{(m)}$ as

$$H_\Theta^{(m)} f(x) = \sum_{j=m+1}^{k} \delta_{\Theta_j}(D^{-m} f)\, D^m \omega_{\Theta_{j-1}}(x), \quad m < k. \tag{4.3}$$

The term 'Newton form' used here is justified not only by the fact that (4.3) is obtained by differentiating the Newton form of $H_\Theta(D^{-m} f)$, but by the observation that

$$H_{\Theta_{k+1}}^{(m)} f = H_{\Theta_k}^{(m)} f + \delta_{\Theta_{k+1}}(D^{-m} f)\, D^m \omega_{\Theta_k}, \quad m < k+1.$$

$\mathcal{H}^{(m)}$ the lift of $H^{(m)}$

We now show that $H^{(m)}$ is liftable to $\mathrm{I\!R}^n$. The lifts $\mathcal{H}^{(m)}$, $m < k$, form what we call, with [3], the **scale of mean value interpolations**.

By using (4.2) and the Hermite-Genocchi formula, the 'Newton form' (4.3) may be written as the following sum of elementary liftable maps:

$$H_\Theta^{(m)} f(x) = m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \Big(\prod_{\theta \in \Theta_{j-1} \setminus \tilde\Theta} (x-\theta) \Big) \int_{\Theta_j} D^{j-m-1} f. \tag{4.4}$$

We refer to this as the **Newton form** of $H_\Theta^{(m)}$.

Thus, by Theorem (3.6), the map $H^{(m)}$ can be lifted to $\mathcal{H}^{(m)}$, where

$$\mathcal{H}_\Theta^{(m)} : C^{k-m-1}(\mathbb{R}^n) \to \Pi_{k-m-1}(\mathbb{R}^n),$$

with its **Newton form** given by

$$\mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \int_{\Theta_j} D_{x-\Theta_{j-1}\setminus\tilde\Theta} f. \tag{4.5}$$

This formula (4.5) is due to Goodman [13]. He shows that each $\mathcal{H}_\Theta^{(m)}$ is a linear projector with range $\Pi_{k-m-1}(\mathbb{R}^n)$ and (lifted) interpolation conditions

$$\operatorname{span}\{f \mapsto \int_{\tilde\Theta} g(D) f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta \geq m+1,\ g \in \Pi^0_{\#\tilde\Theta - m - 1}(\mathbb{R}^n)\}. \tag{4.6}$$

Special cases

The map $\mathcal{H}_\Theta^{(0)}$ is the **Kergin map**, see [18] and [24]. The Newton form of Kergin's map,

$$\mathcal{H}_\Theta^{(0)} f(x) = f(\theta_1) + \int_{[\theta_1,\theta_2]} D_{x-\theta_1} f + \cdots + \int_{[\theta_1,\ldots,\theta_k]} D_{x-\theta_1} \cdots D_{x-\theta_{k-1}} f,$$

is given in [24] and [22]. Notice that the interpolation conditions of this map include evaluation at the points Θ. Thus Kergin's map is a multivariate generalisation of Lagrange interpolation.

The map $\mathcal{H}_\Theta^{(1)}$ was introduced in [6] where it was referred to as the **area matching map**. Presumably the term 'area matching' came from the fact that if the points in $\Theta := [\theta_1, \ldots, \theta_k]$ in $\mathbb{R}$ are distinct, then the interpolation conditions of $H_\Theta^{(1)}$ are

$$\operatorname{span}\{f \mapsto \int_{\theta_i}^{\theta_{i+1}} f : i = 1, \ldots, k-1\}.$$

If the $k \geq n$ points in Θ are in general position in $\mathbb{R}^n$, then $\mathcal{H}_\Theta^{(n-1)}$ is the **Hakopian map**, see [14] and [15]. For this map, the interpolation conditions may be written as

$$\operatorname{span}\{f \mapsto \int_{\tilde\Theta} f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta = n\}.$$

Thus, $\mathcal{H}_\Theta^{(n-1)}$ has an extension (the map originally given by Hakopian) to $C(\mathbb{R}^n)$ and interpolants $\Pi_{k-n}(\mathbb{R}^n)$. Though not immediately apparent from (4.6), the interpolation conditions for Hakopian's map include evaluation at the points Θ. Thus it, like Kergin's map, provides a multivariate generalisation of Lagrange interpolation.

For additional discussion on expressing the interpolation conditions for $\mathcal{H}_\Theta^{(m)}$ in terms of derivatives of lower orders than given in (4.6), see [8].

5. Integral error formulæ

Observe that

$$f - H_\Theta^{(m)} f = D^m\big(D^{-m} f - H_\Theta(D^{-m} f)\big). \tag{5.1}$$

Thus, to obtain an error formula for $\mathcal{H}^{(m)}$, one might hope to lift the error formula for Hermite interpolation. In this section, this is done in two ways. The first and more natural way introduces derivatives of higher order than one might like. In the second, this deficiency is remedied by taking advantage of a little-known formula for the derivative of the error in Hermite interpolation.

The first error formula

Using the differentiation rule for divided differences

$$\frac{d^i}{dx^i}\delta_{[x,\Theta]} f = i!\,\delta_{[\underbrace{x,\ldots,x}_{i+1},\Theta]} f, \tag{5.2}$$

the Hermite error formula

$$D^{-m} f(x) - H_\Theta(D^{-m} f)\,(x) = \omega_\Theta(x)\,\delta_{[x,\Theta]}(D^{-m} f) \tag{5.3}$$

can be differentiated (m times) to obtain, by (5.1), that

$$f(x) - H_\Theta^{(m)} f(x) = \sum_{j=0}^{m} \binom{m}{j} D^j \omega_\Theta(x)\,(m-j)!\,\delta_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]}(D^{-m} f). \tag{5.4}$$

Using (4.2) and the Hermite-Genocchi formula, we may write (5.4) as

$$f(x) - H_\Theta^{(m)} f(x) = m! \sum_{j=0}^{m} \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta = j}} \omega_{\Theta\backslash\tilde\Theta}(x) \int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]} (D^{k-j} f), \quad \forall f \in C^k(\mathbb{R}). \tag{5.5}$$

The formula (5.5) expresses the error, $f \mapsto f - H_\Theta^{(m)} f$, as a sum of elementary liftable maps of orders $k-m,\ldots,k$. Thus, using Theorem 3.6, this can be lifted, thereby giving the following.

First error formula. *If $m < k$ and $f \in C^k(\mathbb{R}^n)$, then*

$$f(x) - \mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{j=0}^{m} \sum_{\substack{\tilde\Theta \subset \Theta \\ \#\tilde\Theta = j}} \int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]} D_{x-\Theta\backslash\tilde\Theta} f. \tag{5.6}$$

For Kergin interpolation, i.e., when $m = 0$, this formula reduces to

$$f(x) - \mathcal{H}_\Theta^{(0)} = \int_{[x,\Theta]} D_{x-\Theta} f, \tag{5.7}$$

which was given in Micchelli [24].

The only other mention of this formula in the literature is for Hakopian interpolation, i.e., when $m = n - 1$, and occurs in the book [3:p200]. There (5.6) is stated incorrectly, and without proof, as

$$f(x) - \mathcal{H}_\Theta^{(n-1)} f(x) = \sum_{j=0}^{n-1} \sum_{\substack{\tilde\Theta \subset \Theta \\ \#\tilde\Theta = j}} \binom{n-1}{j} \int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]} D_{x-\Theta\backslash\tilde\Theta} f.$$

In other words, the constant $\binom{n-1}{j}$ there should be replaced by $(n-1)!$.

The interpolants for $\mathcal{H}_\Theta^{(m)}$ are $\Pi_{k-m-1}(\mathbb{R}^n)$. The error formula (5.6) involves derivatives of orders $k - m, \ldots, k$. For $m > 0$, it would be desirable to not have the higher derivatives $k - m + 1, \ldots, k$ occurring. We now give such a formula.

The second error formula

The higher derivatives in (5.6) are introduced when (5.2) is used to differentiate $x \mapsto \delta_{[x,\Theta]}(D^{-m} f)$ in (5.3). To avoid this problem, we use the following formula for the derivative in Hermite interpolation. It was given independently by Dokken and Lyche [9], [10] and by Wang [26], [27].

Theorem 5.8 ([9],[26]). *If $\Theta = [\theta_1, \ldots, \theta_k]$, $0 \le j < k$ and $f \in C^k(\mathbb{R})$, then*

$$D^j (f - H_\Theta f)(x) = j! \sum_{i=k-j}^{k} \frac{(x-\theta_i)}{(j+i-k)!} D^{j+i-k} \omega_{\Theta_{i-1}}(x)\, \delta_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} f.$$

Applying to (5.1), Theorem 5.8 followed by the Hermite-Genocchi formula, we obtain that, for $f \in C^{k-m}(\mathbb{R})$,

$$f(x) - H_\Theta^{(m)} f(x) = m! \sum_{i=k-m}^{k} \frac{(x-\theta_i)}{(m+i-k)!} D^{m+i-k} \omega_{\Theta_{i-1}}(x) \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D^{k-m} f. \tag{5.9}$$

This formula (5.9) is a sum of elementary liftable maps, each of order $k - m$. Its lift, using Theorem 3.6, gives the following error formula for $\mathcal{H}_\Theta^{(m)}$.

Second error formula. *If $m < k$ and $f \in C^{k-m}(\mathrm{I\!R}^n)$, then*

$$f(x) - \mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{i=k-m}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde\Theta, x-\theta_i]} f. \tag{5.10}$$

This formula involves only derivatives of f of order $k - m$.

Those worried that the formula (5.10) is not symmetric in the points of Θ could, if desired, take the average over all possible orderings for Θ to obtain such a symmetric formula. More to the point, it would be desirable to find the 'simplest' symmetric form of Theorem 5.8.

Derivatives of the error

The univariate identity

$$D^j (H_\Theta^{(m)} f) = H_\Theta^{(m+j)} (D^j f)$$

can be 'lifted' to the following; see, e.g., [3:p205].

Proposition 5.11. *If $m < k$, $j < k - m$, $g \in \Pi_j^0(\mathrm{I\!R}^n)$ and $f \in C^{k-m-1}(\mathrm{I\!R}^n)$, then*

$$g(D)(\mathcal{H}_\Theta^{(m)} f) = \mathcal{H}_\Theta^{(m+j)}(g(D)f).$$

This allows us, in a very natural way, to use an error formula for $\mathcal{H}_\Theta^{(m)}$ to describe the *derivatives of* the error in $\mathcal{H}_\Theta^{(m)}$. In particular, with the second error formula (5.10), we obtain the following.

Theorem 5.12. *If $m < k$, $j < k - m$, $g \in \Pi_j^0$ and $f \in C^{k-m}(\mathrm{I\!R}^n)$, then*

$$g(D)\big(f - \mathcal{H}_\Theta^{(m)} f\big)(x) = (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde\Theta, x-\theta_i]} g(D) f.$$

This formula involves only derivatives of f of order $k - m$.

Proof: By Proposition 5.11,

$$g(D)\big(f - \mathcal{H}_\Theta^{(m)} f\big) = (g(D)f) - \mathcal{H}_\Theta^{(m+j)}(g(D)f).$$

Since $g(D)f \in C^{k-(m+j)}(\mathrm{I\!R}^n)$, we may apply the second error formula (5.10) to the error in $\mathcal{H}_\Theta^{(m+j)}$ at $g(D)f$, thereby obtaining the given formula. □

This theorem is the major result of this paper. It generalises such results as the second error formula (5.10) and Theorem 5.8. It expresses the error in $\mathcal{H}_\Theta^{(m)} f$, and its derivatives, in terms of integration against the derivative of order one higher than the degree of the interpolating polynomial space. This is precisely the estimate that numerical analysts want, to guarantee that their scheme, e.g., a $\mathcal{H}_\Theta^{(m)}$ finite element (see, e.g., [21:p164]), has the maximum possible *order*.

From this Theorem, L_∞-estimates for the error can easily be obtained. This is done in Section 6.

Comparison with the results of Lai-Wang and Gao

The results of [19], [20] and [12] are written in terms of the **multivariate divided differences**

$$[\theta_1, \ldots, \theta_{|\alpha|}]^\alpha f := \int_{[\theta_1,\ldots,\theta_{|\alpha|}]} D^\alpha f, \quad \forall \alpha \in \mathbb{Z}_+^s. \tag{5.13}$$

The simplest of these results to state is the following error formula for Kergin interpolation.

Theorem 5.14 ([20:Th.3.1]). *If $\alpha \in \mathbb{Z}_+^s$ with $|\alpha| \le j < k-1$, then*

$$\begin{aligned} &D^\alpha(f - \mathcal{H}_\Theta^{(0)} f)(x) \\ &= \sum_{r=0}^{|\alpha|} \sum_{\substack{\gamma \le \alpha \\ |\gamma| = r}} \sum_{\substack{\beta \ge \alpha - \gamma \\ |\beta| = j-r}} r! \binom{\alpha}{\gamma} D^{\alpha-\gamma} \omega_\beta(x) \sum_{i=1}^n (x - \theta_{j-r+1})_i \; [\underbrace{x, \ldots, x}_{r+1}, \theta_1, \ldots, \theta_{j-r+1}]^{\beta+\gamma+e^i} f \\ &\quad - \sum_{r=j+1}^{k-1} \sum_{\substack{\gamma \ge \alpha \\ |\gamma| = r}} D^\alpha \omega_\gamma(x) \; [\theta_1, \ldots, \theta_{r+1}]^\gamma f, \end{aligned} \tag{5.15}$$

where

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n},$$

and

$$\omega_\gamma(x) := \sum_{e^{i_1} + \cdots + e^{i_{|\gamma|}} = \gamma} (x - \theta_1)_{i_1} \cdots (x - \theta_{|\gamma|})_{i_{|\gamma|}}.$$

The above uses standard multi-index notation. The i-th component of $x \in \mathbb{R}^n$ is x_i, and e^i is the i-th unit vector in $\mathbb{R}^n$.

Formula (5.15) of Theorem 5.14 involves derivatives of f of orders $j+1, \ldots, k-1$; whereas the formula (5.10) involves only derivatives of order k.

Also, in the case of greatest interest for this formula, namely when $j+1 = k-1$ and $\alpha = 0$, formula (5.15) reduces, in the univariate case, to

$$f(x) - H_\Theta f(x) = \omega_{\Theta_{k-1}}(x) \int_{[x,\theta_1,\ldots,\theta_{k-1}]} D^{k-1} f - \omega_{\Theta_{k-1}}(x) \int_{[\theta_1,\ldots,\theta_k]} D^{k-1} f. \tag{5.16}$$

Since formula (5.16) is a sum of elementary liftable maps, and follows from one application of (3.10) to the Hermite error formula

$$f(x) - H_\Theta f(x) = (x-\theta_1)\cdots(x-\theta_k)\int_{[x-\theta_1,\cdots,x-\theta_k]} D^k f,$$

we obtain at once the case $j+1 = k-1$ and $\alpha = 0$ of Theorem 5.14 by lifting (5.16), and in the following form:

$$f(x) - \mathcal{H}_\Theta^{(0)} f(x) = \int_{[x,\theta_1,\ldots\theta_{k-1}]} D_{x-\theta_1}\cdots D_{x-\theta_{k-1}} f - \int_{[\theta_1,\ldots\theta_k]} D_{x-\theta_1}\cdots D_{x-\theta_{k-1}} f. \quad (5.17)$$

If one now expands (5.17) in multivariate divided differences, then one obtains (5.15) for this case. However, it is not clear what has been gained in the process.

Similar considerations, can, and should, be given to other formulas in [19], [20] and [12].

Additional comments

The only justification for the term 'multivariate divided difference' for (5.13) that the author can see, is the identity (3.10), which is due to Micchelli (see [24:Th.6]), and (in its many guises) pervades the multivariate spline literature. With that justification, the term might as well be applied to any linear combination of functionals

$$f \mapsto \int_\Theta g(D) f, \quad \Theta \in (\mathbb{R}^n)^k,\ g \in \Pi_j(\mathbb{R}^n),$$

that can be expressed as a linear combination of other such functionals involving lower order derivatives of f.

6. L_∞-estimates

In this final section, we obtain L_∞-estimates from the formulæ of Section 5. Our choice of the seminorm $|\cdot|_{k,\infty,K}$ defined in (1.2) makes this a straight-forward task. Let

$$h_{x,\Theta} := \max_{\theta\in\Theta} \|x-\theta\| \le \operatorname{diam}[x,\Theta].$$

From the first error formula (5.6), we obtain the following L_∞-estimate.

Proposition 6.1. *If $m < k$ and $f \in C^k(\mathbb{R}^n)$, then*

$$|f(x) - \mathcal{H}_\Theta^{(m)} f(x)| \le \sum_{j=0}^{m} \mathrm{const}_{j,k,m} (h_{x,\Theta})^{k-j} |f|_{k-j,\infty,\mathrm{conv}[x,\Theta]},$$

where

$$\mathrm{const}_{j,k,m} := \frac{m!}{(k+m-j)!}\binom{k}{j}.$$

Proof: To the first error formula (5.6), apply Property 2.3 (c), then use (1.1) and (1.2) to obtain

$$|f(x) - \mathcal{H}_\Theta^{(m)} f(x)| \le m! \sum_{j=0}^{m} \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta = j}} \frac{1}{(k+m-j)!} (h_{x,\Theta})^{k-j} |f|_{k-j,\infty,\mathrm{conv}[x,\Theta]}.$$

Lastly, observe that

$$\#\{\tilde\Theta \subset \Theta : \#\tilde\Theta = j\} = \binom{k}{j}.$$

□

From Theorem 5.12, we obtain the main result of this section.

Theorem 6.2. *If $m < k$, $j < k - m$ and $f \in C^{k-m}(\mathrm{I\!R}^n)$, then*

$$|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x) \le \frac{1}{(k-m-j)!} (h_{x,\Theta})^{k-m-j} |f|_{k-m,\infty,\mathrm{conv}[x,\Theta]}. \tag{6.3}$$

The constant is the best possible in the sense that if $\Theta = [\theta, \ldots, \theta]$, then it cannot be improved.

Proof: To prove the inequality, begin as in the proof of Proposition 6.1, then use the identity:

$$\frac{(m+j)!}{k!} \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} = \frac{1}{(k-m-j)!}.$$

Suppose $\Theta = [\theta, \ldots, \theta]$. By (4.6) we have that $\mathcal{H}_\Theta^{(m)} f$ is the *Taylor interpolant* from $\Pi_{k-m-1}(\mathrm{I\!R}^n)$ to f at θ. Let $u := (x-\theta)/\|x-\theta\|$. Note that $h_{x,\Theta} = \|x-\theta\|$. Then for the plane wave

$$f := (\cdot - u^*\theta)^{k-m} \circ u^* \in \Pi_{k-m}(\mathrm{I\!R}^n),$$

$\mathcal{H}_\Theta^{(m)} f = 0$, and we have, by (3.1), that

$$\begin{aligned} \frac{|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x)}{|f|_{k-m,\infty,\mathrm{conv}[x,\Theta]}} &\ge \frac{|D_u^j f(x)|}{(k-m)!} = \frac{(k-m)\cdots(k-m-j+1)}{(k-m)!} (\cdot - u^*\theta)^{k-m-j} \circ (u^*x) \\ &= \frac{1}{(k-m-j)!} (h_{x,\Theta})^{k-m-j}. \end{aligned}$$

Thus, in the case $\Theta = [\theta, \ldots, \theta]$, the constant is the best possible. □

When $m = 0$, Proposition 6.1 and Theorem 6.2 (with $j = 0$) reduce to

$$|f(x) - \mathcal{H}_\Theta^{(0)} f(x)| \leq \frac{1}{k!}(h_{x,\Theta})^k |f|_{k,\infty,\mathrm{conv}[x,\Theta]},$$

which was given in [24]. For $m > 0$, none of the above L_∞-estimates are in the literature.

Remark 6.4. In [4:Th.2.5] Bos gives the following estimate for Kergin interpolation on the disc. Let Θ consist of k points equally spaced on the disc $\{x \in \mathbb{R}^2 : \|x\| = h\}$, where $h > 0$. Then for $f \in C^k(\mathbb{R}^2)$

$$\max_{\|x\| \leq h} |f(x) - \mathcal{H}_\Theta^{(0)} f(x)| \leq \frac{1}{k!} \frac{4}{2^k} h^k |f|_{k,\infty,\{x:\|x\|\leq h\}}.$$

This indicates that it may be possible to reduce the size of the constant in (6.3) for restricted values of $h_{x,\Theta}$. However, in view of the sharpness for the case of *Taylor interpolation* (when $\Theta = [\theta, \ldots, \theta]$) and the continuity of $\Theta \mapsto \mathcal{H}_\Theta^{(m)} f$ (by Theorem 3.11), for unrestricted values of $h_{x,\Theta}$ the constant is the best possible in all cases.

It is not possible to apply Properties 2.3 (c) to the integral error formulæ of this paper to obtain L_p-estimates for $1 \leq p < \infty$. A partial solution to this impasse, which uses a multivariate form of Hardy's inequality, is given by the author in [25].

Acknowledgement

The author would like to thank Ming-Jun Lai for his many helpful correspondences.

References

[1] de Boor, C.(1975): On bounding spline interpolation. J. Approx. Theory **14**, 191–203

[2] Bloom, T.(1981): Kergin interpolation of entire functions on $\mathbb{C}^n$. Duke Math. J. **48(1)**, 69–83

[3] Bojanov, B. D., H. A. Hakopian, and A. A. Sahakian(1993): Spline functions and multivariate interpolations. Kluwer Academic Publishers;,

[4] Bos, L.(1983): On Kergin interpolation in the disk. J. Approx. Theory **37**, 251–261

[5] Cavaretta, A. S., C. A. Micchelli, and A. Sharma(1980): Multivariate interpolation and the Radon transform. Math. Z. **174**, 263–279

[6] Cavaretta, A. S., C. A. Micchelli, and A. Sharma(1980): Multivariate interpolation and the Radon transform, part II: Some further examples. In: R. DeVore and K. Scherer, eds, Quantitative Approximation, 49–61. Academic Press, New York

[7] Cavaretta, A. S., T. N. T. Goodman, C. A. Micchelli, and A. Sharma(1983): Multivariate interpolation and the Radon transform, part III: Lagrange representation. Can. Math. Soc. Conf. Proc. **3**, 37–50

[8] Dahmen, W. A. and C. A. Micchelli(1983): On the linear independence of multivariate B-splines II: complete configurations. Math. Comp. **41(163)**, 143–163

[9] Dokken, T. and T. Lyche(1978): A divided difference formula for the error in numerical differentiation based on Hermite interpolation. Research Report 40, Institute of informatics, Univ. Oslo

[10] Dokken, T. and T. Lyche(1979): A divided difference formula for the error in Hermite interpolation. BIT **19**, 540–542

[11] Dyn, N., G. G. Lorentz, and S. D. Riemenschneider(1982): Continuity of Birkhoff interpolation. SIAM J. Numer. Anal. **19(3)**, 507–509

[12] Gao, J. B.(1988): Multivariate quasi-Newton interpolation. J. Math. Res. Exposition (in Chinese) **8(3)**, 447–453

[13] Goodman, T. N. T.(1983): Interpolation in minimum semi-norm, and multivariate B-splines. J. Approx. Theory **37**, 212–223

[14] Hakopian, H.(1981): Les differences divisées de plusieurs variables et les interpolations multidimensionnelles de types Lagrangien et Hermitien. C. R. Acad. Sci. Paris Ser. I **292**, 453-456

[15] Hakopian, H.(1982): Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type. J. Approx. Theory **34**, 286–305

[16] Höllig, K.(1986): Multivariate splines. In: C. de Boor, ed, Approximation Theory, Proc. Symp. Appl. Math. **36**, 103–127. Amer.Math.Soc., Providence

[17] Höllig, K. and C. A. Micchelli(1987): Divided differences, hyperbolic equations, and lifting distributions. Constr. Approx. **3**, 143–156

[18] Kergin, P.(1980): A natural interpolation of C^k functions. J. Approx. Theory **29**, 278–293

[19] Lai, Mingjun and Xinghua Wang(1984): A note to the remainder of a multivariate interpolation polynomial. Approx. Theory Appl. **1(1)**, 57–63

[20] Lai, Mingjun and Xinghua Wang(1986): On multivariate Newtonian interpolation. Sci. Sinica Ser. A **29(1)**, 23–32

[21] Lorentz, R. A.(1992): Multivariate Birkhoff interpolation. Springer-Verlag;,

[22] Micchelli, C. A. and P. Milman(1980): A formula for Kergin interpolation in $\mathbb{R}^k$. J. Approx. Theory **29**, 294–296

[23] Micchelli, C. A.(1979): On a numerically efficient method for computing multivariate B-splines. In: W. Schempp and K. Zeller, eds, Multivariate Approximation Theory, 211–248. Birkhäuser, Basel

[24] Micchelli, C. A.(1980): A constructive approach to Kergin interpolation in $\mathbb{R}^k$: multivariate B-splines and Lagrange interpolation. Rocky Mountain J. Math. **10**, 485–497

[25] Waldron, S.(199x): A multivariate form of Hardy's inequality and L_p-error bounds for multivariate Lagrange interpolation schemes. SIAM J. Math. Anal. **xx**, xxx–xxx

[26] Wang, Xinghua(1978): The remainder of numerical differentiation formulæ. Hang Zhou Da Xue Xue Bao (in Chinese) **4(1)**, 1–10

[27] Wang, Xinghua(1979): On remainders of numerical differentiation formulas. Ke Xue Tong Bao (in Chinese) **24(19)**, 869–872