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Multivariate Jacobi polynomials with singular weights

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ABSTRACT

First we give a compact treatment of the Jacobi polynomials on a simplex in $\mathbb{R}^d$ which exploits and emphasizes the symmetries that exist. This includes the various ways that they can be defined: via orthogonality conditions, as a hypergeometric series, as eigenfunctions of an elliptic pde, as eigenfunctions of a positive linear operator, and through conditions on the Bernstein–Bézier form. We then consider all aspects of the limiting case when the parameters $\mu = (\mu_0, \ldots, \mu_d)$ of the Jacobi polynomials approach -1, and the weight becomes singular. We show that the orthogonal projection of a continuous function onto the Jacobi polynomials of degree n has a limit as the $\mu_j \to -1$, and give an explicit formula for the corresponding 'orthogonal' expansion. It turns out that this expansion is closely related to the limit of the eigenfunction expansion of the Bernstein operator and a new mean value interpolant.

Key Words: Jacobi polynomials on a simplex, mean value interpolation, multivariate Bernstein operator, singular weight function, tight frame,

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1. Introduction

The Jacobi polynomials for the weight function $(1-x)^\alpha(1+x)^\beta$, $\alpha > -1, \beta > -1$ on the interval $[-1,1]$ are given explicitly by

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (x-1)^{n-k}(x+1)^j. \tag{1.1}$$

If either α or β is set to -1, then this formula makes sense (indeed it is an analytic function of α and β), but the weight is no longer integrable, and the resulting polynomials are not orthogonal polynomials (for some measure). There have been a number of attempts to determine 'orthogonality conditions' for these polynomials: distributional weights [MK78], complex weight functions [R84], [IMR91], Hadamard's finite parts [K97], and orthogonality on a Riemann surface [KMO05]. These works exclude the case $\alpha = \beta = -1$, and have not been extended to multivariate Jacobi polynomials.

In this paper we consider the case $\alpha = \beta = -1$ by taking the limit as $\alpha, \beta \to -1^+$ to obtain the corresponding 'orthogonality conditions'. Our results are given for multivariate Jacobi polynomials. To obtain and understand them, we first give a compact development of basic properties of Jacobi polynomials on a simplex which utilises the symmetry. This includes the various ways they can be defined: via orthogonality conditions, as a hypergeometric series, as eigenfunctions of an elliptic pde, as eigenfunctions of a positive linear operator, and through conditions on the Bernstein–Bézier form.

Next we consider all aspects of the limiting case when the parameters $\mu = (\mu_0, \ldots, \mu_d)$ of the Jacobi polynomials approach -1, and the weight becomes singular. We show that the orthogonal projection of a continuous function onto the Jacobi polynomials of degree n has a limit as the $\mu_j \to -1$, and give an explicit formula for the corresponding 'orthogonal' expansion. This expansion gives rise to a new mean value type interpolation.

Finally, we show that this expansion is closely related to the limit of the eigenfunction expansion of the Bernstein operator and the limiting form of the Bernstein–Durrmeyer operator with Jacobi weights.

2. Jacobi polynomials on a simplex

2.1. Notation

Throughout let $\xi = (\xi_v)_{v \in V}$ be the barycentric coordinates of the d–simplex $T \subset \mathbb{R}^d$ obtained by taking the convex hull of $d+1$ affinely independent points V in $\mathbb{R}^d$, e.g., for $V = \{-1, 1\} \subset \mathbb{R}^1$, T is the interval $[-1,1]$, and

$$\xi_{-1}(x) = \frac{1-x}{2}, \qquad \xi_1(x) = \frac{1+x}{2}.$$

We use standard multiindex notation, e.g., the Jacobi weight is written $\xi^\mu = \prod_{v\in V} \xi_v^{\mu_v}$, where $\mu = (\mu_v)$, $\mu_v > -1$, $\forall v \in V$. For $V = \{-1, 1\}$ and $\mu_{-1} = \alpha$, $\mu_1 = \beta$, this is

$$\xi^\mu(x) = \left(\frac{1-x}{2}\right)^\alpha \left(\frac{1+x}{2}\right)^\beta .$$

We find it convenient (for most formulas) to write the Jacobi parameters $\mu = \nu - 1$, where $\nu > 0$, and consider instead the limit $\nu \to 0^+$ in place of the limit $\mu \to -1^+$. The shorthand notation $1 = (1)$ for the vector of 1's, etc, is used, and causes no confusion as it is easily inferred from the context. By default all multiindices α, β, etc, are in $\mathbb{Z}_+^V$, and we write the v–th coordinate of α as α_v or $\alpha(v)$, whichever is the most convenient. The multiindex which is zero in all coordinates but the v–th where it is 1 is denoted by e_v.

2.2. The inner product

The inner product for the Jacobi weight $\xi^{\nu-1}$, $\nu > 0$ on T is given by

$$\langle f, g\rangle_\nu := \frac{\Gamma(|\nu|)}{\Gamma(\nu)} \frac{1}{d!\,\mathrm{vol}_d(T)} \int_T f g\, \xi^{\nu-1}, \qquad f, g \in C(T), \tag{2.1}$$

where the integral is over T with respect to the Lebesgue measure on $\mathbb{R}^d$, $\mathrm{vol}_d(T)$ is measure of T, and $\Gamma(\nu)$ is the multivariate Gamma function. The normalisation ensures

$$\langle \xi^\alpha, \xi^\beta\rangle_\nu = \frac{(\nu)_{\alpha+\beta}}{(|\nu|)_{|\alpha|+|\beta|}}, \qquad \alpha, \beta \in \mathbb{Z}_+^V, \tag{2.2}$$

where $(\nu)_\alpha$ is the multivariate version of the Pochhammer symbol

$$(x)_n := x(x+1)\cdots(x+n-1), \qquad x \in \mathbb{R}.$$

The associated Hilbert space will be denoted by $L_2(\nu) = L_2(T, \xi^{\nu-1})$, or $L_2(T, \xi^\mu)$.

2.3. The space of Jacobi polynomials

Let $\Pi_n = \Pi_n(\mathbb{R}^d)$ be the space of polynomials on $\mathbb{R}^d$ of degree $\le n$. The space $\mathcal{P}_{n,\nu}$ of Jacobi polynomials of degree n with respect to $\langle\cdot,\cdot\rangle_\nu$ consists of all $f \in \Pi_n$ which satisfy

$$\langle f, p\rangle_\nu = 0, \qquad \forall p \in \Pi_{n-1}.$$

This space has dimension $\binom{n+d-1}{d-1}$, and is spanned by the polynomials

$$p^\nu_{\xi^\alpha} := \frac{(-1)^n (\nu)_\alpha}{(n+|\nu|-1)_n} \sum_{\beta\le\alpha} \frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!} \in \xi^\alpha + \Pi_{n-1}, \qquad \alpha \in \mathbb{Z}_+^V, |\alpha| = n. \tag{2.3}$$

A simple computation verifies $p^\nu_{\xi^\alpha} \in \mathcal{P}_{n,\nu}$. The monomial basis of [DX01:p.47] is obtained by taking those polynomials with indices satisfying $\alpha_{v_0} = 0$ for some fixed $v_0 \in V$.

2.4. Eigenfunctions of an elliptic pde

Let $D_v f(x)$ denote the derivative of the function f at $x \in \mathbb{R}^d$ in the direction $v \in \mathbb{R}^d$

$$D_v f(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

Let $\mu = \nu - 1$. The Jacobi polynomials $\mathcal{P}_{k,\nu}$ are the eigenfunctions of the second order pde

$$\begin{aligned} L_\mu f &:= \xi^{-\mu} \sum_{\substack{\{v,w\} \subset V \\ v \neq w}} D_{v-w}(\xi_v \xi_w \xi^\mu D_{v-w} f) = \frac{1}{2} \xi^{-\mu} \sum_{v \in V} \sum_{w \in V} D_{v-w}(\xi_v \xi_w \xi^\mu D_{v-w} f) \\ &= \sum_{\substack{\{v,w\} \subset V \\ v \neq w}} \xi_v \xi_w D_{v-w}^2 f + \sum_{\substack{\{v,w\} \subset V \\ v \neq w}} \{(\mu_v + 1)\xi_w - (\mu_w + 1)\xi_v\} D_{v-w} f \end{aligned} \tag{2.4}$$

for the eigenvalue

$$\lambda_k = \lambda_k(L_\mu) = -k(|\mu| + k + d) = -k(k - 1 + |\nu|).$$

This follows from the simple calculation

$$L_\mu(\xi^\beta) = \lambda_k \xi^\beta + \sum_{v \in V} \beta_v(\beta_v + \mu_v) \frac{\xi^\beta}{\xi_v}, \qquad |\beta| = k \geq 0.$$

This operator has been considered by several authors, see, e.g., [BS00], [BJS04], and the remarks therein. It is elliptic on the interior of T, and is self adjoint with respect to $\langle \cdot, \cdot \rangle_\nu$.

2.5. Eigenfunctions of positive linear operator

The Jacobi polynomials $\mathcal{P}_{n,\nu}$ are the eigenfunctions of the **Bernstein–Durrmeyer operator** of degree n

$$M_n^\nu f := \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha \frac{\langle f, \xi^\alpha \rangle_\nu}{\langle 1, \xi^\alpha \rangle_\nu}, \qquad \forall f \in L_2(T, \xi^{\nu-1}), \tag{2.5}$$

for the eigenvalues

$$\lambda_k = \lambda_k(M_n^\nu) := \frac{n!}{(n-k)!} \frac{\Gamma(n + |\nu|)}{\Gamma(n + k + |\nu|)}, \qquad k = 0, 1, \ldots, n.$$

See Derriennic [D85] for further details about this positive self adjoint operator.

2.6. Bernstein–Bézier form

Let $f = \sum_{|\alpha|=n} c_\alpha B_\alpha$ be the Bernstein–Bézier form of $f \in \Pi_n$, where $B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha$. Then $f \in \mathcal{P}_{n,\nu}$ if and only if its Bernstein–Bézier coefficients $c = (c_\alpha)$ satisfy

$$\langle f, \xi^\beta \rangle_\nu = \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!} \frac{(\nu)_{\alpha+\beta}}{(|\nu|)_{|\alpha|+|\beta|}} = 0, \qquad |\beta| < n. \tag{2.6}$$

This can be interpreted as saying that $\alpha \mapsto c_\alpha$ is a Hahn polynomial of degree n (see [W06:Th.2.3]), or rewritten (see [W06:Cor.2.5]) as

$$R_\nu^* c = 0, \qquad (R_\nu^* c)_\beta := \sum_{w\in V} \frac{\beta_w + \nu_w}{|\beta|+1} c_{\beta+e_w}, \quad |\beta| = n-1. \tag{2.7}$$

3. The limiting form

Now we consider the limiting form of the Jacobi polynomials $\mathcal{P}_{n,\nu}$ as $\nu \to 0^+$ (equivalently as $\mu \to -1^+$). Our techniques also apply if only some $\nu_w \to 0^+$, but for simplicity, we will not present this more general case. Since

$$\langle 1, 1\rangle_\nu = 1, \qquad \langle \xi_w, 1\rangle_\nu = \frac{\nu_w}{|\nu|},$$

$\langle \xi_w, 1\rangle_\nu$ does not have a limit as $\nu \to 0^+$, and so it is not possible to define a limiting form of $\mathcal{P}_{n,\nu}$ via a limit weight function. However, with $Q_{n,\nu} f$ denoting the orthogonal projection of $f \in C(T)$ onto $\mathcal{P}_{n,\nu}$, we will show that $Q_{n,\nu} f$ has a limit as $\nu \to 0^+$. To do this, it is convenient to start with (2.3).

3.1. The limit of $\mathcal{P}_{n,\nu}$

Since

$$\frac{(\nu)_\alpha}{(\nu)_\beta} = (\nu+\beta)_{\alpha-\beta}, \qquad \beta \le \alpha,$$

we can take the limit as $\nu \to 0^+$ in (2.3) for $n \ge 2$, to obtain

$$p^*_{\xi^\alpha} := \lim_{\nu\to 0^+} p^\nu_{\xi^\alpha} = \frac{(-1)^n}{(n-1)_n} \sum_{\beta\le\alpha} (n-1)_{|\beta|}(-\alpha)_\beta(\beta)_{\alpha-\beta} \frac{\xi^\beta}{\beta!} \in \xi^\alpha + \Pi_{n-1}, \qquad |\alpha| = n, \tag{3.1}$$

and we let

$$\mathcal{P}_{n,*} = \lim_{\nu\to 0^+} \mathcal{P}_{n,\nu} := \operatorname{span}\{p^*_{\xi^\alpha} : |\alpha| = n\}, \qquad n \ge 2.$$

We will call $\mathcal{P}_{n,*}$ Jacobi polynomials (of degree n) for the singular weight. For $n = 0, 1$, we have

$$p^\nu_1 = 1, \qquad p^\nu_{\xi_w} = \xi_w - \frac{\nu_w}{|\nu|},$$

and so $\mathcal{P}_{1,\nu}$ does not have a limit as $\nu \to 0^+$.

The space $\mathcal{P}_{n,*}$ was considered in [S94], where it was obtained by substituting $\nu = 0$ into the Rodrigues' formula.

Example 1. For $n = 2$, $v \neq w$, we have

$$p^\nu_{\xi_w^2} = \xi_w^2 - 2\frac{1+\nu_w}{2+|\nu|}\xi_w + \frac{\nu_w(\nu_w+1)}{(1+|\nu|)(2+|\nu|)}, \qquad p^*_{\xi_w^2} = \xi_w(\xi_w - 1),$$

$$p^\nu_{\xi_v\xi_w} = \xi_v\xi_w - \frac{1}{2+|\nu|}(\nu_w\xi_v + \nu_v\xi_w) + \frac{\nu_v\nu_w}{(1+|\nu|)(2+|\nu|)}, \qquad p^*_{\xi_v\xi_w} = \xi_v\xi_w.$$

The partial differential operator (2.4) has a limit as $\nu \to 0^+$

$$\sum_{\substack{\{v,w\}\subset V\\ v\neq w}} \xi_v\xi_w D^2_{v-w} f,$$

as does the Bernstein-Durrmeyer operator (2.5), see [W03]. For these, the eigenspace for $\lim_{\nu\to 0^+} \lambda_k$ is $\mathcal{P}_{k,*}$, $k \geq 2$, whilst

$$\lim_{\nu\to 0^+} \lambda_k(L_{\nu-1})) = 0, \quad \lim_{\nu\to 0^+} \lambda_k(M_n^\nu) = 1, (n \geq k) \qquad k = 0, 1,$$

i.e., the λ_0 and λ_1 eigenspaces coalesce to Π_1. It is therefore natural to think of Π_1 as the limit of $\mathcal{P}_{0,\nu} \oplus \mathcal{P}_{1,\nu} = \Pi_1$ as $\nu \to 0^+$. Observe that we have the algebraic direct sum

$$\Pi_n = \bigoplus_{k=1}^n \mathcal{P}_{k,*}, \qquad \mathcal{P}_{1,*} := \Pi_1 \quad (n \geq 1). \tag{3.2}$$

The condition (2.6) on the Bernstein–Bézier coefficients (c_α) of a polynomial $f \in \Pi_n$ that are equivalent to $f \in \mathcal{P}_{n,\nu}$ can be rewritten

$$\frac{(|\nu|)_{n+|\beta|}}{(\nu)_\beta}\langle f, \xi^\beta\rangle_\nu = \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!}(\nu+\beta)_\alpha = 0, \qquad |\beta| < n.$$

We can take the limit of this as $\nu \to 0^+$ to obtain

$$\sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!}(\beta)_\alpha = 0, \qquad |\beta| < n, \tag{3.3}$$

and also of (2.7) to obtain

$$R_0^* c = 0, \qquad (R_0^* c)_\beta := \sum_{w\in V} \frac{\beta_w}{|\beta|+1} c_{\beta+e_w}, \quad |\beta| = n-1. \tag{3.4}$$

It can be shown that $f \in \Pi_n$ is in $\mathcal{P}_{n,*}$, $n \geq 2$ if and only if its Bernstein-Bézier coefficients satisfy either of (3.3) or (3.4). Note for $n = 1$ ($\beta = 0$) the conditions (3.3) and (3.4) degenerate to $0 = 0$.

3.2. A tight frame for $\mathcal{P}_{n,\nu}$

Let $Q_{n,\nu} : C(T) \to \mathcal{P}_{n,\nu}$ be the orthogonal projection onto $\mathcal{P}_{n,\nu}$. This can be written

$$Q_{n,\nu} f = (|\nu|)_{2n} \sum_{|\alpha|=n} \frac{1}{(\nu)_\alpha \alpha!} \langle f, p^\nu_{\xi^\alpha} \rangle_\nu \, p^\nu_{\xi^\alpha}, \qquad \forall f \in C(T), \tag{3.5}$$

see, e.g., [W06:Th.3.5]. The polynomials $\{p^\nu_{\xi^\alpha} : |\alpha| = n\}$ above do not form a basis, and (3.5) is what is termed a tight frame representation. Given that $p^\nu_{\xi^\alpha} \to p^*_{\xi^\alpha}$ as $\nu \to 0^+$, it is natural to try and find a limit of the coefficients in (3.5), and hence of $Q_{n,\nu} f$. This we do next. In principal, a similar calculation could be undertaken with either Appell's biorthogonal system, or Prorial's orthogonal basis, but the formulas obtained are much more complicated (cf the comments in [W06]).

3.3. The limit of $Q_{n,\nu} f$

We need the following linear functional

$$f \mapsto \int_{[\theta_0,\ldots,\theta_k]} f := \frac{1}{k! \operatorname{vol}_k(S)} \int_S f \circ A, \tag{3.6}$$

where S is any k–simplex in $\mathbb{R}^s$ with (k–dimensional) volume $\operatorname{vol}_k(S)$, and $A : \mathbb{R}^s \to \mathbb{R}^d$ is any affine map taking the $k+1$ vertices of S onto the points $\theta_0, \ldots, \theta_k$ in $\mathbb{R}^d$. The change of variables formula shows that (3.6) does not depend on the choice of S and A.

If $\theta_0, \ldots, \theta_k$ are the points of $V = \{v_0, \ldots, v_d\}$ taken with multiplicities $\alpha(v_i) \geq 0$, $\alpha \neq 0$, then a change of variables gives the generalised beta integral

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\alpha(v_d)}]} f = \frac{1}{\Gamma(\alpha_|)} \int_{[\operatorname{supp}(\alpha)]} f\, \xi_|^{\alpha_| - 1}, \qquad \begin{array}{l} \xi_| := (\xi_v)_{v \in \operatorname{supp}(\alpha)}, \\ \alpha_| := \alpha|_{\operatorname{supp}(\alpha)} \end{array} \tag{3.7}$$

where $\operatorname{supp}(\alpha) \subset V$ denotes the support of α. We recall the following.

Lemma 3.8 ([W03:Lem.3.1]). *Let $\beta \in \mathbb{Z}_+^V$, $V = \{v_0, \ldots, v_d\}$ with $|\beta| \geq 1$, then*

$$\lim_{\nu \to 0^+} \frac{\langle f, \xi^\beta \rangle_\nu}{\langle 1, \xi^\beta \rangle_\nu} = (|\beta| - 1)! \int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f. \tag{3.9}$$

We can now compute the limit of $Q_{n,\nu} f$ as $\nu \to 0^+$.

Theorem 3.10. *Let $Q_{n,\nu} : C(T) \to \Pi_n$ be the $L_2(\nu)$–orthogonal projection onto $\mathcal{P}_{n,\nu}$. Then*

$$\lim_{\nu \to 0^+} (Q_{0,\nu} f + Q_{1,\nu} f) = L_V f := \sum_{v \in V} f(v) \xi_v, \qquad \forall f \in C(T),$$

and, for $n \geq 2$,

$$\lim_{\nu \to 0^+} Q_{n,\nu} f = Q_{n,*} f := \sum_{|\alpha|=n} \lambda_\alpha^*(f)\, p_{\xi^\alpha}^*, \qquad \forall f \in C(T), \tag{3.11}$$

where

$$\lambda_\alpha^*(f) := (2n-1)! \sum_{0<\beta\leq\alpha} \frac{(n-1)_{|\beta|}}{(n-1)_n} \frac{(-1)^{|\alpha-\beta|}}{(\alpha-\beta)!} \frac{1}{\beta!} \int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f. \tag{3.12}$$

The map $Q_{n,*} : C(T) \to \mathcal{P}_{n,*}$ is a linear projector onto $\mathcal{P}_{n,*}$. With $Q_{1,*} := L_V$, we have

$$Q_{j,*} Q_{k,*} = 0, \qquad j \neq k. \tag{3.13}$$

Proof: We write (3.5) as

$$Q_{n,\nu} f = \sum_{|\alpha|=n} \lambda_\alpha^\nu(f)\, p_{\xi^\alpha}^\nu, \qquad \forall f \in C(T),$$

where, by (2.3),

$$\lambda_\alpha^\nu(f) := (|\nu|)_{2n} \frac{\langle f, p_{\xi^\alpha}^\nu \rangle_\nu}{(\nu)_\alpha \alpha!} = \frac{(|\nu|)_{2n}}{\alpha!} \frac{(-1)^n}{(n+|\nu|-1)_n} \sum_{\beta\leq\alpha} \frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\langle f, \xi^\beta \rangle_\nu}{\beta!}.$$

First, observe that

$$Q_{0,\nu} f = \langle f, 1 \rangle_\nu 1 = \sum_{w\in V} \frac{\langle f, 1 \rangle_\nu}{\langle 1, 1 \rangle_\nu} \frac{\nu_w}{|\nu|},$$

and so in view of (3.9), $Q_{0,\nu} f$ does not have a limit as $\nu \to 0^+$. Using $\langle 1, \xi_w \rangle_\nu = \frac{\nu_w}{|\nu|}$, $\sum_w \xi_w = 1$ and $\sum_w \nu_w = |\nu|$, we have

$$\begin{aligned}
Q_{1,\nu} f &= (|\nu|)(|\nu|+1) \sum_{w\in V} \frac{1}{\nu_w} \langle f, \xi_w - \frac{\nu_w}{|\nu|} \rangle_\nu (\xi_w - \frac{\nu_w}{|\nu|}) \\
&= (|\nu|+1) \sum_{w\in V} \left(\frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} - \langle f, 1 \rangle_\nu \right) (\xi_w - \frac{\nu_w}{|\nu|}) \\
&= (|\nu|+1) \sum_{w\in V} \frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} (\xi_w - \frac{\nu_w}{|\nu|}) \\
&= \sum_{w\in V} \frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} (|\nu|\xi_w - \nu_w) + \left(\sum_{w\in V} \frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} \xi_w - \sum_{w\in V} \langle f, \xi_w \rangle_\nu \right) \\
&= \sum_{w\in V} \frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} (|\nu|\xi_w - \nu_w) + \sum_{w\in V} \frac{\langle f, \xi_w \rangle_\nu}{\langle 1, \xi_w \rangle_\nu} \xi_w - \langle f, 1 \rangle_\nu 1.
\end{aligned}$$

Hence, by (3.9),

$$Q_{0,\nu}f + Q_{1,\nu}f = \sum_{w\in V} \frac{\langle f,\xi_w\rangle_\nu}{\langle 1,\xi_w\rangle_\nu}(|\nu|\xi_w - \nu_w) + \sum_{w\in V} \frac{\langle f,\xi_w\rangle_\nu}{\langle 1,\xi_w\rangle_\nu}\xi_w \to \sum_{w\in V} f(w)\xi_w \quad \text{as } \nu\to 0^+.$$

Now suppose $n\ge 2$. Using (2.2), we have

$$\begin{aligned}\lambda_\alpha^\nu(f) &= \frac{(|\nu|)_{2n}}{\alpha!}\frac{(-1)^n}{(n+|\nu|-1)_n}\sum_{\beta\le\alpha}\frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta}\frac{(\nu)_\beta}{(|\nu|)_{|\beta|}}\frac{1}{\beta!}\frac{\langle f,\xi^\beta\rangle_\nu}{\langle 1,\xi^\beta\rangle_\nu}\\ &= \frac{1}{\alpha!}\frac{(-1)^n}{(n+|\nu|-1)_n}\sum_{\beta\le\alpha}(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta(|\nu|+|\beta|)_{2n-|\beta|}\frac{1}{\beta!}\frac{\langle f,\xi^\beta\rangle_\nu}{\langle 1,\xi^\beta\rangle_\nu}.\end{aligned}$$

Since

$$\left|\frac{\langle f,\xi^\beta\rangle_\nu}{\langle 1,\xi^\beta\rangle_\nu}\right| \le \max_{x\in T}|f(x)| =: \|f\|_\infty,$$

the factor $(|\nu|+|\beta|)_{2n-|\beta|}$ ensures the term for $\beta=0$ converges to 0 as $\nu\to 0^+$, and so using (3.9), we obtain

$$\begin{aligned}&\lim_{\nu\to 0^+}\lambda_\alpha^\nu(f)\\ &\qquad = \frac{1}{\alpha!}\frac{(-1)^n}{(n-1)_n}\sum_{0<\beta\le\alpha}(n-1)_{|\beta|}(-\alpha)_\beta(|\beta|)_{2n-|\beta|}\frac{(|\beta|-1)!}{\beta!}\int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f\end{aligned}$$

Using

$$(|\beta|)_{2n-|\beta|}(|\beta|-1)! = (2n-1)!,\qquad \frac{(-\alpha)_\beta}{\alpha!} = \frac{(-1)^{|\beta|}}{(\alpha-\beta)!},\qquad (-1)^n(-1)^{|\beta|} = (-1)^{|\alpha-\beta|},$$

this can be rewritten

$$\lim_{\nu\to 0^+}\lambda_\alpha^\nu(f) = (2n-1)!\sum_{0<\beta\le\alpha}\frac{(n-1)_{|\beta|}}{(n-1)_n}\frac{(-1)^{|\alpha-\beta|}}{(\alpha-\beta)!}\frac{1}{\beta!}\int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_s,\ldots,v_s}_{\beta(v_s)}]} f,$$

and so we obtain (3.11).

Since Π_n is an algebraic direct sum of Π_1 and $\mathcal{P}_{k,*}$, $k=2,\ldots n$, and

$$f = \sum_{k=0}^n Q_{k,\nu}f,\quad \forall f\in\Pi_n \quad\Longrightarrow\quad f = \lim_{\nu\to 0^+}(Q_{0,\nu}f + Q_{1,\nu}f) + \sum_{k=2}^n Q_{k,*}f,\quad \forall f\in\Pi_n$$

we conclude that $Q_{n,*}$ is onto $\mathcal{P}_{n,*}$, $n\ge 2$. Replacing f by $Q_{j,*}f$ in the above, and equating the $\mathcal{P}_{k,*}$ components in the direct sum (3.2), we get

$$Q_{j,*}f = \sum_{k=1}^n Q_{k,*}Q_{j,*}f \quad\Longrightarrow\quad Q_{k,*}Q_{j,*}f = \begin{cases} Q_{j,*}f, & j=k;\\ 0, & k\ne j\end{cases}$$

and so $Q_{n,*}$ is a linear projector which satisifies (3.13). □

The map $L_V = Q_{1,*} : C(T) \to \Pi_1$ above is Lagrange interpolation at the points in V.

Corollary 3.14. *Let $n \geq 1$. For each $f \in C(T)$ there is a unique $p \in \Pi_n$ matching the data*

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f, \qquad 0 < |\beta| \leq n. \tag{3.15}$$

The corresponding linear projector $C(T) \to \Pi_n : f \mapsto p$ is given by $L_n := L_V + \sum_{k=2}^{n} Q_{k,*}$.

Each of the operators L_n interpolates function values at the points in V. They are of a similar type to the interpolation operator of Hakopian [H81]. For Θ a set of $k \geq d$ points in general position in $\mathbb{R}^d$, the Hakopian interpolant is the unique $p \in \Pi_{k-d}(\mathbb{R}^d)$ matching the mean values

$$\int_{[W]} f, \qquad W \subset \Theta, \quad |W| = d.$$

The map L_n does not appear to be liftable (cf [W97]).

Example 2. Consider $n = 2$. For $|\alpha| = 2$, (3.12) gives

$$\lambda^*_\alpha(f) = \frac{6}{\alpha!} \int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)},\ldots,\underbrace{v_s,\ldots,v_s}_{\alpha(v_s)}]} f - 3 \sum_{\substack{v \in V \\ e_v \leq \alpha}} f(v),$$

so that

$$\begin{aligned} Q_{2,*}f &= \sum_{v \in V} \lambda^*_{2e_v}(f)\,(\xi_v^2 - \xi_v) + \sum_{\substack{\{v,w\} \subset V \\ v \neq w}} \lambda^*_{e_v+e_w}(f)\,\xi_v\xi_w \\ &= \sum_{\substack{\{v,w\} \subset V \\ v \neq w}} \left(6 \int_{[v,w]} f - 3f(v) - 3f(w) \right) \xi_v\xi_w. \end{aligned}$$

It is easily verified that the map $L_2 = L_V + Q_{2,*}$ interpolates the function values at the vertices V of T and line integrals over the segments between vertices.

For $d = 1$ and $V = \{0,1\}$, the interpolation conditions (3.15) are $f(0), f(1), \int_0^1 f$, and

$$L_2 f(x) = (1-x)f(0) + xf(1) + \left(6 \int_0^1 f(t)\,dt - 3f(0) - 3f(1) \right) x(1-x).$$

Example 3. The limiting form of the Bernstein–Durrmeyer operator (cf [W03])

$$U_n f := \lim_{\nu \to 0^+} M_n^\mu f = (n-1)! \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha \int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)},\ldots,\underbrace{v_s,\ldots,v_s}_{\alpha(v_s)}]} f, \qquad f \in C(T),$$

which is the operator of [GS91] and [S94], has the diagonal form

$$U_n = \sum_{k=1}^{n} \lambda_k(U_n)\, Q_{k,*}, \qquad \lambda_k(U_n) = \frac{n!}{(n-k)!}\frac{(n-1)!}{(n+k-1)!}.$$

We now consider the 'orthogonality conditions' defining $\mathcal{P}_{n,*}$. Let $\Pi_n(F)$ denote Π_n restricted to some subset $F \subset \mathbb{R}^d$.

Theorem 3.16 (Orthogonality condition). *Let $f \in \Pi_n$, $n \geq 2$. Then f is a Jacobi polynomial for the singular weight, i.e., $f \in \mathcal{P}_{n,*}$, if and only if*

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\alpha(v_d)}]} f = 0. \qquad \forall |\alpha| < n. \tag{3.17}$$

This is equivalent to f being orthogonal to all polynomials of degree less than $n - |W|$ on the convex hull of every nonempty subset $W \subset V$, i.e.,

$$\int_{\mathrm{conv}(W)} f\, p = 0, \qquad \forall p \in \Pi_{n-|W|-1}(\mathrm{conv}(W)). \tag{3.18}$$

Proof: In view of Theorem 3.10, $f \in \mathcal{P}_{n,*}$ if and only if

$$f = \sum_{j=1}^{n} Q_{j,*} f = Q_{n,*} f \quad \Longleftrightarrow \quad Q_{j,*} f = 0, \quad 1 \leq j < n.$$

Since $Q_{j,*} f$ matches the data (3.15), we can have $Q_{j,*} f = 0$ if and only if

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f = 0, \qquad 0 < |\beta| \leq j.$$

Thus $f \in \mathcal{P}_{n,*}$ is equivalent to (3.17).

Fix $W \subset V$, $W \neq \phi$, and let $\alpha_| := \alpha|_W$, $\xi_| := \xi|_W$. Using (3.7), the conditions in (3.17) for α with $\mathrm{supp}(\alpha) = W$ can be written

$$\int_{[\underbrace{v_0,\ldots,v_0}_{\alpha(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\alpha(v_d)}]} f = \frac{1}{(\alpha_| - 1)!} \int_{[W]} f\, \xi_|^{\alpha_| - 1} = 0,$$

which is equivalent to

$$\int_{\mathrm{conv}(W)} f\, \xi_|^{\beta} = 0, \qquad \forall \beta \in \mathbb{Z}_+^W,\ |\beta| < n - |W|.$$

Since $\{\xi_|^\beta : \beta \in \mathbb{Z}_+^W, |\beta| < n - |W|\}$ spans $\Pi_{n-|W|-1}(\mathrm{conv}(W))$, we therefore obtain the equivalence with (3.18). $\square$

Example 4. The condition (3.17) for $\alpha = e_v$ implies that each function in $\mathcal{P}_{n,*}$ $(n \geq 2)$ vanishes at all the points $v \in V$. This can also be seen by evaluating (3.1) at $v \in V$. Here the only terms which are possibly nonzero are those with $\beta = je_v$, but these are all zero due to the factor $(\beta)_{\alpha-\beta}$, unless $\alpha = ne_v$. For $\alpha = ne_v$, these terms sum to

$$\frac{(-1)^n}{(n-1)_n} \sum_{j=1}^{n} (n-1)_j (-n)_j (j)_{n-j} \frac{1}{j!} = 0.$$

Example 5. For $d = 1$, $V = \{0,1\}$, the first three polynomials $f = p_{\xi_1^2}$, $g = p_{\xi_1^3}$ and $h = p_{\xi_1^4}$ given by (3.1) are

$$f(x) = x(x-1), \quad g(x) = x(x-\frac{1}{2})(x-1), \qquad h(x) = x(x-1)(x^2 - x + \frac{1}{5}).$$

It is easily verified that these satisfy the conditions given by (3.18), i.e.,

$$f(0) = f(1) = 0, \quad g(0) = g(1) = \int_0^1 g = 0, \quad h(0) = h(1) = \int_0^1 h(t)\,dt = \int_0^1 th(t)dt = 0.$$

Example 6. For $n = 3$, the space $\mathcal{P}_{3,*}$ is spanned by the polynomials

$$\xi_v(\xi_v - \frac{1}{2})(\xi_v - 1), \qquad \xi_v\xi_w(\xi_v - 1), \qquad \xi_v\xi_w\xi_u \qquad (v, w, u \text{ distinct points of } V).$$

Clearly, each of these satisfies the conditions of (3.17), i.e.,

$$f(v) = 0, \qquad \int_{[v,w]} f = 0, \quad v \neq w.$$

Example 7. Given the equivalence of (3.17) and (3.18), the linear functionals giving the interpolation conditions (3.15) of the map $L_n = L_V + \sum_{k=2}^{n} Q_{k,*}$ have a basis given by

$$\{f \mapsto \int_{\mathrm{conv}(W)} f\xi^\beta : \phi \neq W \subset V, |W| \leq n, \beta \in \mathbb{Z}_+^W, |\beta| = n - |W|\}.$$

This follows from the dimension count

$$\sum_{j=1}^{d+1} \binom{d+1}{j} \binom{n-j+j-1}{j-1} = \binom{n+d}{d} = \dim(\Pi_n(\mathbb{R}^d)).$$

4. The limit of the diagonal form of the Bernstein operator

The Bernstein operator on the simplex $T = \mathrm{conv}(V)$ is the positive linear operator $B_n : C(T) \to \Pi_n$ given by

$$B_n f = B_{n,V} f := \sum_{\substack{|\alpha|=n \\ \alpha\in\mathbb{Z}_+^V}} \binom{n}{\alpha} \xi^\alpha f(v_\alpha), \qquad v_\alpha := \sum_{v\in V} \frac{\alpha(v)}{|\alpha|} v \in T. \tag{4.1}$$

In [CW02] it was shown that B_n had eigenvalues

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!}\frac{1}{n^k}, \quad k = 1,\ldots,n, \qquad 1 = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0, \tag{4.2}$$

and corresponding eigenvectors of the form

$$p_{\xi^\beta}^{(n)} = \xi^\beta + \text{lower order powers of } \xi, \qquad |\beta| = k.$$

Moreover, $p_{\xi^\beta}^{(n)}$ has a limit as $n \to \infty$. We will show this limit is precisely $p^*_{\xi^\beta} \in \mathcal{P}_{k,*}$, i.e., the Jacobi polynomials for the singular weight are limiting eigenfunctions of the Bernstein operator.

Lemma 4.3. *Let $W = \mathrm{supp}(\alpha)$, $|\alpha| = k \ge 2$ and $\alpha_| := \alpha|_W$. The polynomial $p^*_{\xi^\alpha} \in \mathcal{P}_{k,*}$ can be factored*

$$p^*_{\xi^\alpha} = \Bigl(\prod_{w\in W} \xi_w\Bigr) g(\xi_|), \qquad \xi_| := \xi|_W,$$

where

$$g(\xi_|) := (-1)^{k-|W|} \frac{(k-1)_{|W|}}{(k-1)_k} (2)_{\alpha_|-1} \sum_{\substack{\gamma\in\mathbb{Z}_+^W \\ \gamma\le\alpha_|-1}} (k-1+|W|)_{|\gamma|} \frac{(-(\alpha_|-1))_\gamma}{(2)_\gamma} \frac{\xi_|^\gamma}{\gamma!}. \tag{4.4}$$

Proof: The factor $(\beta)_{\alpha-\beta}$ ensures those terms in (3.1) for $\mathrm{supp}(\beta) \ne W$ are zero, and so

$$\begin{aligned} p^*_{\xi^\alpha} &= \frac{(-1)^k}{(k-1)_k} \sum_{\substack{\beta\le\alpha \\ \mathrm{supp}(\beta)=W}} (k-1)_{|\beta|} (-\alpha)_\beta (\beta)_{\alpha-\beta} \frac{\xi^\beta}{\beta!} \\ &= \frac{(-1)^k}{(k-1)_k} \xi_|^1 \sum_{1\le\beta_|\le\alpha_|} (k-1)_{|\beta_||} (-\alpha_|)_{\beta_|} (\beta_|)_{\alpha_|-\beta_|} \frac{\xi_|^{\beta_|-1}}{\beta_|!}. \end{aligned}$$

Making the substitution $\gamma = \beta_| - 1$, we obtain

$$p^*_{\xi^\alpha} = \frac{(-1)^k}{(k-1)_k} \xi_|^1 \sum_{\substack{\gamma\in\mathbb{Z}_+^W \\ \gamma\le\alpha_|-1}} (k-1)_{|\gamma+1|} (-\alpha_|)_{\gamma+1} (\gamma+1)_{\alpha_|-\gamma-1} \frac{\xi_|^\gamma}{(\gamma+1)!}.$$

This can be rearranged using

$$(-\alpha_|)_{\gamma+1}(\gamma+1)_{\alpha_|-\gamma-1} = (-\alpha_|)_1(-\alpha_|+1)_\gamma(\gamma+1)_{\alpha_|-\gamma-1} = (-1)^{|W|}(-(\alpha_|-1))_\gamma(\gamma+1)_{\alpha_|-\gamma},$$

$$\frac{(\gamma+1)_{\alpha_|-\gamma}}{(\gamma+1)!} = \frac{(\gamma+2)_{\alpha_|-\gamma-1}}{\gamma!} = \frac{(2)_{\alpha_|-1}}{(2)_\gamma}\frac{1}{\gamma!}, \quad (k-1)_{|\gamma+1|} = (l-1)_{|W|}(k-1+|W|)_{|\gamma|},$$

to get

$$\begin{aligned} p^*_{\xi^\alpha} &= \frac{(-1)^k}{(k-1)_k}\xi_|^1 \sum_{\substack{\gamma\in\mathbb{Z}_+^W \\ \gamma\le\alpha_|-1}} (k-1)_{|W|}(k-1+|W|)_{|\gamma|}(-1)^{|W|}(-(\alpha_|-1))_\gamma \frac{(2)_{\alpha_|-1}}{(2)_\gamma}\frac{\xi_|^\gamma}{\gamma!} \\ &= \xi_|^1(-1)^{k-|W|}\frac{(k-1)_{|W|}}{(k-1)_k}(2)_{\alpha_|-1} \sum_{\substack{\gamma\in\mathbb{Z}_+^W \\ \gamma\le\alpha_|-1}} (k-1+|W|)_{|\gamma|}\frac{(-(\alpha_|-1))_\gamma}{(2)_\gamma}\frac{\xi_|^\gamma}{\gamma!} = \xi_|^1\, g(\xi_|), \end{aligned}$$

which gives the desired factorisation. □

In view of (2.3), the factor $g(\xi_|)$ above can be interpreted as a Jacobi polynomial of degree $k-|W|$ on the simplex with vertices W for the parameter $\mu=\nu-1=1$.

Corollary 4.5. *Let $p^{(n)}_{\xi^\beta}$, $2\le|\beta|=k\le n$, be the eigenfunction of Bernstein operator B_n of the form*

$$p^{(n)}_{\xi^\beta} = \xi^\beta + \textit{lower order powers of } \xi,$$

then

$$\lim_{n\to\infty} p^{(n)}_{\xi^\beta} = p^*_{\xi^\beta} \in \mathcal{P}_{k,*}.$$

Proof: In [CW02:Th.4.2] this limit was calculated. The formula given there is precisely the factored form (4.4). □

Now we consider the univariate Bernstein operator on the interval $T=[0,1]$, which is given by

$$B_n f(x) := \sum_{k=0}^n \binom{n}{k} x^k(1-x)^{n-k} f\Big(\frac{k}{n}\Big).$$

With $p_k^{(n)}(x)$ denoting the $\lambda_k^{(n)}$–eigenfunction of B_n with leading term x^k, the diagonal form of B_n can be written

$$B_n f(x) = L_V f(x) + \sum_{k=2}^n \mu_k^{(n)}(f)\, p_k^{(n)}(x), \qquad \forall f\in C[0,1] \tag{4.6}$$

where $\mu_k^{(n)}(f)$ are the dual functionals to the $p_k^{(n)}$. In [CW00:Th.4.2] it was shown that $\mu_k^{(n)}(f)$ has a limit as $n\to\infty$ for f a polynomial. We now show that this limit is (3.12).

Lemma 4.7. *For $k \geq 2$, the operator $Q_{k,*}$ can be expressed*

$$Q_{k,*}f = (2k-1)! \sum_{\substack{W\subset V\\ W\neq\phi}} \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!} p^*_{\xi^\alpha} \int_{[W]} \Big(\sum_{\substack{\beta\\ \mathrm{supp}(\beta)=W}} \frac{(k-1)_{|\beta|}}{(k-1)_k} \frac{(-\alpha)_\beta}{\beta!} \frac{\xi_|^{\beta_|-1}}{(\beta_|-1)!} f \Big), \quad (4.8)$$

where $\beta_| = \beta|_W$, $W = \mathrm{supp}(\beta)$, $\xi_| = \xi|_W$.

Proof: We have

$$Q_{k,*}f = (2k-1)! \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!} \sum_{\beta\neq 0} \frac{(k-1)_{|\beta|}}{(k-1)_k} \frac{(-\alpha)_\beta}{\beta!} p^*_{\xi^\alpha} \int_{[\underbrace{v_0,\ldots,v_0}_{\beta(v_0)},\ldots,\underbrace{v_d,\ldots,v_d}_{\beta(v_d)}]} f.$$

By summing over those β with $\mathrm{supp}(\beta) = W \subset V$, and applying (3.7), we obtain

$$\begin{aligned} Q_{k,*}f &= (2k-1)! \sum_{\substack{W\subset V\\ W\neq\phi}} \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!} \sum_{\substack{\beta\\ \mathrm{supp}(\beta)=W}} \frac{(k-1)_{|\beta|}}{(k-1)_k} \frac{(-\alpha)_\beta}{\beta!} p^*_{\xi^\alpha} \frac{1}{\Gamma(\beta_|)} \int_{[W]} f \xi_|^{\beta_|-1} \\ &= (2k-1)! \sum_{\substack{W\subset V\\ W\neq\phi}} \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!} p^*_{\xi^\alpha} \int_{[W]} \Big(\sum_{\substack{\beta\\ \mathrm{supp}(\beta)=W}} \frac{(k-1)_{|\beta|}}{(k-1)_k} \frac{(-\alpha)_\beta}{\beta!} \frac{\xi_|^{\beta_|-1}}{(\beta_|-1)!} f \Big), \end{aligned}$$

as claimed. $\square$

Theorem 4.9. *Let $d = 1$ and $V = \{0, 1\}$. Then for $k \geq 2$,*

$$Q_{k,*}f = \nu^*_k(f)\, p^*_k, \qquad f \in C[0,1],$$

where $p^*_k := p^*_{\xi_1^k}$, and

$$\nu^*_k(f) = \frac{1}{2}\binom{2k}{k} \Big\{ f(1) + (-1)^k f(0) - k \int_0^1 f(t) P^{(1,1)}_{k-2}(2t-1)\, dt \Big\}. \quad (4.10)$$

Proof: For $\alpha = (j, k-j)$, comparing powers of x gives

$$p^*_{\xi^\alpha}(x) = (1-x)^j x^k + \text{lower order terms} = (-1)^j p^*_{\xi_1^k} = (-1)^j p^*_k,$$

so that (4.8) reduces to

$$Q_{k,*}f = (2k-1)! \sum_{\substack{W\subset V\\ W\neq\phi}} \sum_{j=0}^{k} \frac{(-1)^k(-1)^j}{j!(k-j)!} p^*_k \int_{[W]} \Big(\sum_{\substack{\beta\\ \mathrm{supp}(\beta)=W}} \frac{(k-1)_{|\beta|}}{(k-1)_k} \frac{(-(j,k-j))_\beta}{\beta!} \frac{\xi_|^{\beta_|-1}}{(\beta_|-1)!} f \Big).$$

The binomial identity gives

$$\begin{aligned}
\sum_{j=0}^{k} \frac{(-1)^k(-1)^j}{j!(k-j)!}(-(j,k-j))_\beta &= \sum_{\beta_0 \le j \le k-\beta_1} (-1)^k(-1)^j \frac{(-j)_{\beta_0}}{j!} \frac{(-(k-j))_{\beta_1}}{(k-j)!} \\
&= \sum_{\beta_0 \le j \le k-\beta_1} (-1)^k(-1)^j \frac{(-1)^{\beta_0}}{(j-\beta_0)!} \frac{(-1)^{\beta_1}}{(k-j-\beta_1)!} \\
&= \frac{(-1)^{k+\beta_0+\beta_1}}{(k-\beta_0-\beta_1)!} \sum_{r=0}^{k-\beta_0-\beta_1} (-1)^{r+\beta_0} \binom{k-\beta_0-\beta_1}{r} \\
&= \frac{(-1)^{k+\beta_0+\beta_1}}{(k-\beta_0-\beta_1)!} (-1)^{\beta_0}(1-1)^{k-\beta_0-\beta_1} \\
&= \begin{cases} (-1)^{\beta_0}, & |\beta| = \beta_0+\beta_1 = k; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

so that

$$\frac{Q_{k,*}f}{p_k^*} = (2k-1)! \sum_{\substack{W \subset V \\ W \ne \phi}} \int_{[W]} \Big(\sum_{\substack{|\beta|=k \\ \mathrm{supp}(\beta)=W}} \frac{(-1)^{\beta_0}}{\beta!} \frac{\xi_|^{\beta_| - 1}}{(\beta_| - 1)!} f \Big).$$

There is just one term in $(Q_{k,*}f)/p_k^*$ for $W = \{0\}$ and $W = \{1\}$, namely

$$(2k-1)! \frac{(-1)^k}{k!} \frac{f(0)}{(k-1)!} = \frac{1}{2}\binom{2k}{k}(-1)^k f(0), \qquad (2k-1)! \frac{(-1)^0}{k!} \frac{f(1)}{(k-1)!} = \frac{1}{2}\binom{2k}{k} f(0).$$

The remaining terms ($W = V = \{0,1\}$) are

$$(2k-1)! \int_{[0,1]} \Big(\sum_{\substack{|\beta|=k \\ \beta \ge 1}} \frac{(-1)^{\beta_0}}{\beta!} \frac{\xi^{\beta-1}}{(\beta-1)!} f \Big) = (2k-1)! \int_{[0,1]} \Big(\sum_{|\gamma|=k-2} \frac{(-1)^{\gamma_0+1}}{(\gamma+1)!} \frac{\xi^\gamma}{\gamma!} f \Big).$$

Letting $\gamma = (k-2-j, j)$ the above can be written

$$(2k-1)! \int_0^1 \Big(\sum_{j=0}^{k-2} \frac{(-1)^{k-2-j+1}}{(k-j-1)!(j+1)!} \frac{(1-t)^{k-2-j} t^j}{(k-2-j)! j!} f(t) \Big)\, dt. \tag{4.11}$$

Substituting $n = k-2$ and $x = 2t-1$ into (1.1) gives

$$\begin{aligned}
P_{k-2}^{(1,1)}(2t-1) &= 2^{-(k-2)} \sum_{j=0}^{k-2} \binom{k-2+1}{j} \binom{k-2+1}{k-2-j} (2t-2)^{k-2-j}(2t)^j \\
&= \sum_{j=0}^{k-2} (-1)^{k-j} \frac{(k-1)!(k-1)!}{(k-1-j)!(j+1)!} \frac{(1-t)^{k-2-j} t^j}{(k-2-j)! j!}.
\end{aligned}$$

Thus (4.11) is equal to

$$-\frac{(2k-1)!}{(k-1)!(k-1)!} \int_0^1 P_{k-2}^{(1,1)}(2t-1) f(t)\, dt = -\frac{k}{2}\binom{2k}{k} \int_0^1 P_{k-2}^{(1,1)}(2t-1) f(t)\, dt.$$

Thus $(Q_{k,*}f)/p_k^*$ is given by the formula (4.10). □

Corollary 4.12. *Let $\mu_k^{(n)}$, $2 \le k \le n$ be the dual linear functionals in the eigenfunction expansion* (4.6) *of the Bernstein operator B_n. Then for f a polynomial*

$$\lim_{n\to\infty} \mu_k^{(n)}(f) = \nu_k^*(f),$$

where ν_k^* is defined in (4.10).

Proof: In [CW02:Th.4.20] this limit was calculated. The formula $\mu_k^*(f)$ given there for it is precisely (4.10). □

Conclusion

The multivariate Bernstein operator has the diagonal form

$$B_n = \lambda_1^{(n)} P_1^{(n)} + \lambda_2^{(n)} P_2^{(n)} + \cdots + \lambda_n^{(n)} P_n^{(n)}, \qquad n \ge 1,$$

where $P_k^{(n)} : C(T) \to \Pi_k$ is the projection onto the $\lambda_k^{(n)}$–eigenspace. Corollary 4.5 and Corollary 4.12 together imply that for the univariate Bernstein operator

$$\lim_{n\to\infty} P_k^{(n)} f = Q_{k,*} f, \tag{4.13}$$

for all polynomials f.

We conjecture that (4.13) holds for all continuous functions f, and for Bernstein operators B_n in any dimension. To prove this directly would seem to require a tractable expansion of $P_k^{(n)} f$ in terms of the values $\{f(v_\alpha) : |\alpha| = n\}$. Such an expansion is not yet known – even in the univariate case. Still, after all the years of study of the Bernstein operator, the finer detail of its spectral properties have not been fully resolved.

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