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Computing orthogonal polynomials on a triangle by degree raising

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ABSTRACT

We give an algorithm for computing orthogonal polynomials over triangular domains in Bernstein–Bézier form which uses only the operator of degree raising and its adjoint. This completely avoids the need to choose an orthogonal basis (or tight frame) for the orthogonal polynomials of a given degree, and hence the difficulties inherent in that approach. The results are valid for Jacobi polynomials on a simplex, and show the close relationship between the Bernstein form of Jacobi polynomials, Hahn polynomials and degree raising.

Key Words: Bernstein–Bézier form, Hahn polynomials, Jacobi polynomials, surface smoothing

AMS (MOS) Subject Classifications: primary 33C45, 65D17, secondary 41A10,

1. Introduction

This paper considers orthogonal polynomials over a triangular (or simplicial) domain. The use of the Bernstein basis for polynomials on triangular domains is well established in CAGD (Computer Aided Geometric Design) since the "Bernstein–Bézier control net" of the Bernstein coefficients closely reflects the shape of the function and simplifies the smoothness conditions across an interface (cf [B87]).

Recently (cf [FGS03], [W06]) there has been interest in the Bernstein(–Bézier) form of orthogonal polynomials on triangular domains. It turns out that the Bernstein coefficients of such orthogonal polynomials can be interpreted as orthogonal polynomials for a discrete inner product; and they can be characterised as the kernel of the adjoint of the degree raising operator for this inner product.

In [FGS03] the Bernstein form of an orthogonal basis for the orthogonal polynomials on a triangle was developed, and [W06] advocated the use of a tight frame invariant under the symmetries of the triangle. From a computational point of view, what one really wants is the matrix representing the orthogonal projection onto the orthogonal polynomials of a given degree with respect to the Bernstein basis. A formula for the entries of this 'projection matrix' was given in [W06]. This involved a multivariate analogue of a ${}_3F_2$ hypergeometric sum, and something of a similar complexity could be developed from [FGS03].

The main result of this paper is a simple formula for this projection matrix. It involves only powers of the degree raising operator and its adjoint, and so can easily be implemented in existing Bernstein–Bézier software. This approach completely circumvents the need choose a basis, or tight frame for the orthogonal polynomials of a given degree.

The rest of the paper is set out as follows. In the next section, we define the Jacobi polynomials on a simplex, and outline how their Bernstein coefficients can be interpreted as orthogonal polynomials for a discrete inner product. This is based on the close relationship between these polynomials and the degree raising operator. In the third and final section, we give the projection matrix and discuss the corresponding algorithm.

The paper uses standard multi–index notation, e.g., for $\alpha, \beta \in \mathbb{Z}_+^d$ and $x \in \mathbb{R}^d$,

$$x^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}, \quad \alpha! := \alpha_1! \cdots \alpha_d!, \quad (\alpha)_\beta := (\alpha_1)_{\beta_1} \cdots (\alpha_d)_{\beta_d},$$

where $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol, and

$$|\alpha| := \alpha_1 + \cdots + \alpha_d, \qquad \Gamma(x) := \prod_j \Gamma(x_j), \quad x_j > 0.$$

2. The Bernstein form and orthogonal polynomials

Throughout $\xi = (\xi_0, \xi_1, \dots, \xi_d)$ will be the barycentric coordinates of a d–simplex $T \subset \mathbb{R}^d$ (the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$) with volume $\mathrm{vol}_d(T)$. For example, the standard triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ has

$$T = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1-x\}, \quad \mathrm{vol}_2(T) = \frac{1}{2}, \qquad \xi(x,y) = (1-x-y, x, y).$$

Let $\Pi_n(V)$ be the polynomials of degree $\le n$ on a d–dimensional affine space V, and $\Pi_n(X)$ their restrictions to a subset $X \subset V$ for which $\Pi_n(V) \to \Pi_n(X) : f \mapsto f|_X$ is invertible. Of particular interest is X given by $\mathbb{R}^d, T$ and the simplex points

$$S_n = S_n(d) := \{\alpha \in \mathbb{Z}_+^{d+1} : |\alpha| = n\}, \qquad \#S_n = \binom{n+d}{d} = \dim(\Pi_n(\mathbb{R}^d)).$$

For example, for $d = 2$, we have $S_0 = \{(0,0,0)\}$, $S_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ and

$$S_2 = \{(2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2)\}.$$

Without confusion, we will refer to $f \in \Pi_n(X)$ as a polynomial of degree $\le n$ in d variables.

Each polynomial $f \in \Pi_n(\mathbb{R}^d)$ can be expressed in terms of the Bernstein basis

$$f = \sum_{|\alpha|=n} c_\alpha(f) B_\alpha = \sum_{|\alpha|=n} c_\alpha B_\alpha,$$

where the **Bernstein polynomials** of degree n are defined by

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha = \frac{|\alpha|!}{\alpha!} \xi^\alpha = \frac{n!}{\alpha!} \xi^\alpha, \qquad |\alpha| = n, \quad \alpha \in \mathbb{Z}_+^{d+1}.$$

It is easy to show that $f \in \Pi_n(\mathbb{R}^d)$ has exact degree s if and only if $c(f) \in \Pi_n(S_n)$ has exact degree s. For example, the Bernstein coefficients of $f = 1$ are $c_\alpha(f) = 1$, $\forall \alpha$.

Let μ be some positive measure on X for which

$$\langle f, g \rangle_\mu := \int_X f(x) g(x) \, d\mu(x)$$

defines an inner product on $\Pi_s(X)$ (see [DX01]). Then $\mathcal{P}_s^\mu$ the space of **orthogonal polynomials** of degree s (for the measure μ) consists of all $f \in \Pi_s(X)$ for which

$$\langle f, g \rangle_\mu = 0, \qquad \forall g \in \Pi_{s-1}(\mathbb{R}^d).$$

Note $\dim(\mathcal{P}_s^\mu) = \binom{s+d-1}{d-1} > 0$ for $s > 0$, $d > 1$, and that polynomials in $\mathcal{P}_s^\mu$ need not be orthogonal to each other, even though they are referred to as "orthogonal polynomials".

Let $\nu \in \mathbb{R}^{d+1}$, $\nu_j > 0$ be a fixed parameter. Then

$$\langle f, g \rangle_\nu := \frac{\Gamma(|\nu|)}{\Gamma(\nu)} \frac{1}{d! \operatorname{vol}_d(T)} \int_T f g \, \xi^{\nu-1}, \qquad \nu \in \mathbb{R}^{d+1}, \quad \nu_j > 0. \tag{2.1}$$

defines an inner product on continuous functions on T. The corresponding orthogonal polynomials are called the **Jacobi polynomials** and are denoted by $\mathcal{P}_s^\nu$. They are called **Legendre polynomials** when $\nu_j = 1$, $\forall j$. For the standard triangle (2.1) becomes

$$\langle f, g \rangle_\nu = \frac{\Gamma(\nu_0 + \nu_1 + \nu_2)}{\Gamma(\nu_0)\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 \int_0^{1-x} f(x,y) g(x,y) \, (1-x-y)^{\nu_0} x^{\nu_1} y^{\nu_2} \, dy \, dx.$$

We define an inner product on functions $S_n \to \mathbb{R}$, i.e., polynomials in $\Pi_n(S_n)$, by

$$\langle f, g \rangle_{\nu,n} := \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} f(\alpha) g(\alpha). \tag{2.2}$$

The corresponding orthogonal polynomials of degree s are called **Hahn polynomials**, and we denote them by $\mathcal{P}_s^{\nu,n}$, $0 \le s \le n$. In [W06] it was shown that $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(T)$ is a Jacobi polynomial of degree s if and only if its Bernstein coefficients $c(f) = (c_\alpha)$ are a Hahn polynomial in $\mathcal{P}_s^{\nu,n}$. These coefficients are also closely related to degree raising.

By the multinomial theorem

$$f = \sum_{|\alpha|=n} c_\alpha B_\alpha = \sum_{|\alpha|=n} c_\alpha B_\alpha \Bigl(\sum_{i=0}^{d} \xi_i\Bigr)^j = \sum_{|\alpha|=n+j} (R^j c)_\alpha B_\alpha,$$

where the powers of the **degree raising operator** R are given by

$$(R^j c)_\alpha = \sum_{|\gamma|=j} \binom{j}{\gamma} \frac{(-\alpha)_\gamma}{(-|\alpha|)_j} c_{\alpha-\gamma}, \qquad j = 0, 1, 2, \ldots. \tag{2.3}$$

The **adjoint** of the degree raising operator with respect to (2.2) is defined by

$$\langle Rc, b \rangle_{\nu,n} = \langle c, R_\nu^* b \rangle_{\nu,n-1}, \qquad c : S_{n-1} \to \mathbb{R}, \quad b : S_n \to \mathbb{R}.$$

The powers of the adjoint of R are given by

$$((R_\nu^*)^j b)_\beta = \sum_{|\gamma|=j} \frac{(\beta+\nu)_\gamma}{(|\beta|+1)_j} \binom{j}{\gamma} b_{\beta+\gamma}, \qquad b : S_n \to \mathbb{R}, \quad 0 \le j \le n. \tag{2.4}$$

Theorem 2.5 ([W06]). *Let $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(\mathbb{R}^d)$, $c = (c_\alpha)$ and $0 \le s \le n$. Then the following are equivalent*
(i) $f \in \mathcal{P}_s^\nu$ *(Jacobi polynomials)*
(ii) $c \in \mathcal{P}_s^{\nu,n}$ *(Hahn polynomials)*
(iii) $(R_\nu^*)^{n-s+1} c = 0$.

The association of the (possibly degree raised) Bernstein coefficients of $f \in \mathcal{P}_s^\nu$ with a Hahn polynomial preserves the respective inner products.

Theorem 2.6 ([W06]). *Let $f = \sum_{|\alpha|=n} c_\alpha(f) B_\alpha$, $g = \sum_{|\alpha|=n} c_\alpha(g) B_\alpha$ and $0 \le s \le n$. If f or g belongs to $\mathcal{P}_s^\nu$, then we have*

$$\langle f, g \rangle_\nu = \frac{(n!)^2}{(n-s)!(|\nu|)_{n+s}} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} c_\alpha(f) c_\alpha(g) = \frac{(n!)^2}{(n-s)!(|\nu|)_{n+s}} \langle c(f), c(g) \rangle_{\nu,n}.$$

We will need the following formula for the "commutator" of R and R_ν^*.

Lemma 2.7. *For $c : S_n \to \mathbb{R}$, $n \geq 1$, we have*

$$(|\alpha|+1)^2(R_\nu^* Rc)_\alpha - |\alpha|^2(RR_\nu^* c)_\alpha = (|\nu| + 2|\alpha|)c_\alpha. \tag{2.8}$$

Proof: For $j = 1$, (2.3) and (2.4) become

$$(Rc)_\alpha = \sum_{|\gamma|=1} \frac{(-\alpha)_\gamma}{-|\alpha|} c_{\alpha-\gamma}, \qquad (R_\nu^* c)_\alpha = \sum_{|\gamma|=1} \frac{(\alpha+\nu)_\gamma}{|\alpha|+1} c_{\alpha+\gamma}.$$

Thus we calculate

$$(R_\nu^* Rc)_\alpha = \sum_{|\delta|=1} \frac{(\alpha+\nu)_\delta}{|\alpha|+1} \sum_{|\gamma|=1} \frac{(-\alpha-\delta)_\gamma}{-(|\alpha|+1)} c_{\alpha+\delta-\gamma},$$

$$(RR_\nu^* c)_\alpha = \sum_{|\gamma|=1} \frac{(-\alpha)_\gamma}{-|\alpha|} \sum_{|\delta|=1} \frac{(\alpha-\gamma+\nu)_\delta}{|\alpha|} c_{\alpha-\gamma+\delta},$$

so that

$$(|\alpha|+1)^2(R_\nu^* Rc)_\alpha - |\alpha|^2(RR_\nu^* c)_\alpha = \sum_{|\gamma|=1}\sum_{|\delta|=1} A_{\alpha,\nu,\gamma,\delta}\, c_{\alpha-\gamma+\delta},$$

where $A_{\alpha,\nu,\gamma,\delta} := -(\alpha+\nu)_\delta(-\alpha-\delta)_\gamma + (-\alpha)_\gamma(\alpha-\gamma+\nu)_\delta$. But

$$A_{\alpha,\nu,\gamma,\delta} = -(\alpha+\nu)_\delta(-\alpha)_\gamma + (-\alpha)_\gamma(\alpha+\nu)_\delta = 0, \qquad \gamma \neq \delta,$$

and so only the terms with $\gamma = \delta$ in the above sum are nonzero. For these

$$A_{\alpha,\nu,\gamma,\delta} = -(\alpha+\nu)_\gamma\{(-\alpha)_\gamma - 1\} + (-\alpha)_\gamma\{(\alpha+\nu)_\gamma - 1\} = (\alpha+\nu)_\gamma - (-\alpha)_\gamma = (\nu)_\gamma + 2(\alpha)_\gamma,$$

and we obtain

$$(|\alpha|+1)^2(R_\nu^* Rc)_\alpha - |\alpha|^2(RR_\nu^* c)_\alpha = \sum_{|\gamma|=1} \{(\nu)_\gamma + 2(\alpha)_\gamma\} c_\alpha = (|\nu| + 2|\alpha|)c_\alpha.$$

□

Corollary 2.9. *For $c : S_n \to \mathbb{R}$, we have*

$$(n+k)^2 R_\nu^* R^k c = n^2 R^k R_\nu^* c + k(|\nu| + 2n + k - 1)R^{k-1}c \qquad n \geq 0, \quad k \geq 0. \tag{2.10}$$

Proof: For $n \geq 1$ the result follows from (2.8) by induction. It also holds for $n = 0$. Since in that case c is the constant c_0, and

$$(R_\nu^* R^k c)_\beta = c_0 (R_\nu^* 1)_\beta = c_0 \sum_{|\gamma|=1} \frac{(\beta+\nu)_\gamma}{(|\beta|+1)_1} \binom{1}{\gamma} = \frac{k-1+|\nu|}{k} c_0, \qquad k \geq 1.$$

□

Corollary 2.11. *Let* $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \mathcal{P}_s^\nu$, $n \geq s$. *For* $j, k \geq 0$ *with* $n + k - j \geq 0$,

$$(R_\nu^*)^j R^k c = \frac{(n-s+k-m+1)_m}{(n+k-m+1)_m^2} (|\nu| + n + s + k - m)_m (R_\nu^*)^{j-m} R^{k-m} c,$$

for any $0 \leq m \leq \min\{j, k\}$.

Proof: First suppose $s = n$. Then $R_\nu^* c = 0$ by (iii) of Theorem 2.5, so that m applications of (2.10) gives

$$\begin{aligned}(R_\nu^*)^j R^k c &= \frac{k(|\nu| + 2n + k - 1)}{(n+k)^2} (R_\nu^*)^{j-1} R^{k-1} c = \cdots \\ &= \frac{(k-m+1)_m (|\nu| + 2n + k - m)_m}{(n+k-m+1)_m^2} (R_\nu^*)^{j-m} R^{k-m} c.\end{aligned}$$

Now suppose $0 \leq s \leq n$, so that $c = R^{n-s} b$. Then by the result just proved

$$\begin{aligned}(R_\nu^*)^j R^k c &= (R_\nu^*)^j R^{n-s+k} b \\ &= \frac{(n-s+k-m+1)_m (|\nu| + 2s + n - s + k - m)_m}{(s+n-s+k-m+1)_m^2} (R_\nu^*)^{j-m} R^{n-s+k-m} b \\ &= \frac{(n-s+k-m+1)_m (|\nu| + s + n + k - m)_m}{(n+k-m+1)_m^2} (R_\nu^*)^{j-m} R^{k-m} c.\end{aligned}$$

□

3. The orthogonal projection matrix

In this section functions $\mathbb{R}^d \to \mathbb{R}$, $T \to \mathbb{R}$ will be equipped with the inner product (2.1), and those $S_n \to \mathbb{R}$ with (2.2). So for example, by the (orthogonal) projection of $f \in \Pi_n(\mathbb{R}^d)$ onto $\mathcal{P}_s^\nu$, $0 \leq s \leq n$ we mean the unique $p \in \mathcal{P}_s^\nu$ with $\langle f - p, g \rangle_\nu = 0$, $\forall g \in \mathcal{P}_s^\nu$.

We denote by c^n the map taking $f \in \Pi_n(\mathbb{R}^d)$ to its coordinates in the Bernstein basis

$$c^n : \Pi_n(\mathbb{R}^d) \to \mathbb{R}^{S_n} : f \mapsto c^n(f) = (c_\alpha^n(f))_{|\alpha|=n}, \qquad f = \sum_{|\alpha|=n} c_\alpha^n(f) B_\alpha,$$

and by B_n its inverse the basis map

$$B_n := [B_\alpha : |\alpha| = n] : \mathbb{R}^{S_n} \to \Pi_n(\mathbb{R}^d) : b \mapsto \sum_{|\alpha|=n} b_\alpha B_\alpha, \qquad B_n = (c^n)^{-1}.$$

Here is the main result.

Theorem 3.1. *The matrix $A = B_s^{-1} P B_s$ representing P the orthogonal projection of $\Pi_s(\mathbb{R}^d)$ onto $\mathcal{P}_s^\nu$ with respect to the Bernstein basis $(B_\alpha)_{|\alpha|=s}$ is given by*

$$A = \sum_{k=0}^{s} \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k} R^k (R_\nu^*)^k. \tag{3.2}$$

Proof: The matrix A given by (3.2) represents P if and only if $B_s A B_s^{-1} f \in \mathcal{P}_s^\nu$, $f - B_s A B_s^{-1} f \perp B_s A B_s^{-1} f$, $\forall f \in \Pi_s(\mathbb{R}^d)$. Writing $f = \sum_{|\alpha|=s} c_\alpha B_\alpha = B_s c$, this becomes

$$B_s A c \in \mathcal{P}_s^\nu, \qquad B_s c - B_s A c \perp B_s A c, \qquad \forall c : S_s \to \mathbb{R}.$$

By Theorem 2.5, the first of these conditions is equivalent to $R_\nu^*(Ac) = 0$. Given that this is satisfied, we can use Theorem 2.6 to express the second condition as

$$\langle B_s c - B_s A c, B_s A c \rangle_\nu = \frac{(s!)^2}{(|\nu|)_{2s}} \langle c - Ac, Ac \rangle_{\nu,s} = 0.$$

Hence it suffices to show that $R_\nu^*(Ac) = 0$ and $c - Ac \perp Ac$.

By (2.10), the sum $\sum a_k R^k (R_\nu^*)^k c$, $c : S_s \mapsto \mathbb{R}$, is in the kernel of R_ν^* if

$$\begin{aligned}
R_v^* \sum_{k=0}^{s} a_k R^k (R_\nu^*)^k &= \sum_{k=0}^{s} a_k (R_v^* R^k)(R_\nu^*)^k \\
&= \frac{1}{s^2} \sum_{k=0}^{s} a_k \Big((s-k)^2 R^k R_\nu^* + k(|\nu| + 2(s-k) + k - 1) R^{k-1} \Big) (R_\nu^*)^k \\
&= \frac{1}{s^2} \sum_{k=0}^{s} a_k \Big((s-k)^2 R^k (R_\nu^*)^{k+1} + k(|\nu| + 2s - k - 1) R^{k-1} (R_\nu^*)^k \Big) \\
&= \frac{1}{s^2} \sum_{k=0}^{s-1} \Big\{ a_k (s-k)^2 + a_{k+1}(k+1)(|\nu| + 2s - k - 2) \Big\} R^k (R_\nu^*)^{k+1} = 0.
\end{aligned}$$

We satisfy this by solving the recurrence

$$a_k (s-k)^2 + a_{k+1}(k+1)(|\nu| + 2s - k - 2) = 0, \quad 0 \le k < s, \qquad a_0 := 1,$$

to obtain

$$a_k = \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k}, \qquad 0 \le k \le s.$$

Thus $R_\nu^*(Ac) = 0$. Furthermore, since $a_0 = 1$, we have

$$c - Ac = -R \sum_{k=1}^{s} \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k} R^{k-1} (R_\nu^*)^k c,$$

so that $c - Ac$ is in the image of R, and hence is orthogonal to $Ac \in \ker R_\nu^*$. $\square$

This result can be generalised as follows.

Theorem 3.3. *The matrix $B = B_m^{-1} Q B_n$ representing Q the orthogonal projection of $\Pi_n(\mathrm{IR}^d)$ onto $\mathcal{P}_s^\nu$ with respect to the bases $(B_\alpha)_{|\alpha|=n}$ and $(B_\alpha)_{|\alpha|=m}$, $m \geq s$ is given by*

$$B = \frac{(|\nu|+2s-1)}{(n-s)!} \sum_{k=0}^{s} \frac{(s-k+1)_{n-s+k}^2 (-1)^k}{k!(|\nu|+2s-k-1)_{n-s+k+1}} R^{m-s+k} (R_\nu^*)^{n-s+k}. \tag{3.4}$$

Proof: With A, P as in Theorem 3.1, we will prove that

$$B = \lambda R^{m-s} A (R_\nu^*)^{n-s}, \qquad \lambda := \frac{(s+1)_{n-s}^2}{(1)_{n-s}} \frac{1}{(|\nu|+2s)_{n-s}},$$

and hence obtain (3.4) from (3.2). Clearly it suffices to prove this for $m = s$, i.e., with $B := \lambda A (R_\nu^*)^{n-s}$ that $B_s B B_n^{-1} f \in \mathcal{P}_s^\nu$ and $f - B_s B B_n^{-1} f \perp B_s B B_n^{-1} f$, $\forall f \in \Pi_n(\mathrm{IR}^d)$.

The first condition holds since $B_s A$ maps into $\mathcal{P}_s^\nu$, and so by Theorem 2.6, writing $f = \sum_{|\alpha|=n} c_\alpha B_\alpha = B_n c$ and $B_s B B_n^{-1} f = \lambda B_n R^{n-s} A (R_\nu^*)^{n-s} c$, the second becomes

$$\langle c - \lambda R^{n-s} A (R_\nu^*)^{n-s} c, R^{n-s} A (R_\nu^*)^{n-s} c \rangle_{\nu,n} = 0,$$

which we can rewrite as

$$\langle b - \lambda (R_\nu^*)^{n-s} R^{n-s} A b, Ab \rangle_{\nu,s} = 0, \qquad b := (R_\nu^*)^{n-s} c. \tag{3.5}$$

Now by Corollary 2.11,

$$(R_\nu^*)^{n-s} R^{n-s} A b = \frac{(1)_{n-s}}{(s+1)_{n-s}^2} (|\nu|+2s)_{n-s} A b = \frac{1}{\lambda} A b,$$

so that $\langle \lambda (R_\nu^*)^{n-s} R^{n-s} A b, Ab \rangle_{\nu,s} = \langle Ab, Ab \rangle_{\nu,s}$, and by Theorem 2.6,

$$\begin{aligned} \langle b, Ab \rangle_{\nu,s} &= \langle B_s B_s^{-1} b, B_s P B_s^{-1} b \rangle_{\nu,s} = \frac{(|\nu|)_{2s}}{(s!)^2} \langle B_s^{-1} b, P B_s^{-1} b \rangle_\nu \\ &= \frac{(|\nu|)_{2s}}{(s!)^2} \langle P B_s^{-1} b, P B_s^{-1} b \rangle_\nu = \langle B_s P B_s^{-1} b, B_s P B_s^{-1} b \rangle_{\nu,s} = \langle Ab, Ab \rangle_{\nu,s}. \end{aligned}$$

Hence (3.5) holds and we obtain (3.4). □

If $f \in \Pi_s(\mathrm{IR}^d)$ and q is its projection onto $\Pi_{s-1}(\mathrm{IR}^d)$, then $p := f - q$ is the orthogonal projection of f onto $\mathcal{P}_s^\nu$. This relationship between p and q gives a geometric interpretation of the Bernstein coefficients of the projection onto polynomials of one degree less.

Proposition 3.6. *Let $f = \sum_{|\alpha|=s} b_\alpha B_\alpha \in \Pi_s(\mathrm{IR}^d)$ and q be its projection onto $\Pi_{s-1}(\mathrm{IR}^d)$. Then the Bernstein coefficients of q are given by the orthogonal projection of b onto $\mathcal{P}_{s-1}^{\nu,s}$.*

Proof: By the previous remark, $q \in \Pi_{s-1}(\mathrm{IR}^d)$ is the orthogonal projection of f onto $\Pi_{s-1}(\mathrm{IR}^d)$ if and only if

$$f - q \in \mathcal{P}_s^\nu \iff R_\nu^*(b - Rc^{s-1}(q)) = 0 \iff R_\nu^* R c^{s-1}(q) = R_\nu^* b.$$

The last of these is the normal equations for the least squares solution of $Rc^{s-1}(q) = b$, and so $c^n(q) = Rc^{n-1}(q)$ is the orthogonal projection of b onto $\mathcal{P}_{s-1}^{\nu,s}$. □

Corollary 3.7. *The matrix which represents the orthogonal projection of* $\Pi_s(\mathbb{R}^d)$ *onto* $\Pi_{s-1}(\mathbb{R}^d)$ *with respect to the Bernstein basis* $(B_\alpha)_{|\alpha|=s}$ *is given by*

$$R(R_\nu^* R)^{-1} R_\nu^* = -\sum_{k=1}^{s} \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k} R^k (R_\nu^*)^k. \tag{3.8}$$

Proof: From the proof of Proposition 3.6, we have $c^{s-1}(q) = (R_\nu^* R)^{-1} R_\nu^* b$. Hence the matrix is given by $R(R_\nu^* R)^{-1} R_\nu^*$, and by the remark, the matrix A of Theorem 3.1 is given by

$$A = I - R(R_\nu^* R)^{-1} R_\nu^*.$$

The second formula in (3.8) is obtained by substituting (3.2) into $R(R_\nu^* R)^{-1} R_\nu^* = I - A$. □

These results allow us to calculate the orthogonal projections onto $\mathcal{P}_s^\nu$ and $\Pi_s(\mathbb{R}^d)$ by applying only the operation R of degree raising and its adjoint. The adjoint R_ν^* is easily calculated. Indeed, with e_α denoting the α–th standard basis vector, we have

$$\frac{(\nu)_\alpha}{\alpha!}(R_\nu^* e_\beta)_\alpha = \langle R_\nu^* e_\beta, e_\alpha \rangle_{\nu,n-1} = \langle e_\beta, R e_\alpha \rangle_{\nu,n} = \frac{(\nu)_\beta}{\beta!}(R e_\alpha)_\beta$$

so that the entries of the matrices representing R_ν^* and R with respect to the standard basis satisfy

$$(R_\nu^*)_{\alpha\beta} = \frac{\alpha!}{(\nu)_\alpha} \frac{(\nu)_\beta}{\beta!} R_{\beta\alpha}.$$

For Legendre polynomials ($\nu = 1$) R_ν^* is simply the 'matrix transpose' of R.

A natural multivariate analogue of the Legendre/Jacobi basis is to express $f \in \Pi_n(\mathbb{R}^d)$ as $f = f_0 + \cdots + f_n$, with $f_s \in \mathcal{P}_s^\nu$. In terms of the Bernstein basis, with $c := c^n(f)$, this is

$$c = \sum_{s=0}^{n} c_s, \quad c_s := c^n(f_s) \in \mathcal{P}_s^{\nu,n},$$

where c_s can be computed by Theorem 3.3. This calculation is well conditioned (cf [F00]) since the matrix B is an orthogonal projection with respect to (2.2), as is the map back to the Bernstein form (adding the c_s). The decomposition $c = \sum_s c_s$ is ideally suited to surface smoothing problems as outlined in [FGS03] (see also [KA00]).

From (3.2) and (3.4) one can (obviously) obtain a formula for the entries of the matrices A and B, e.g., the (α,β)–entry of A is

$$\begin{aligned} a_{\alpha\beta} &= \sum_{k=0}^{s} \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k} (R^k (R_\nu^*)^k \delta_\beta)_\alpha \\ &= \sum_{k=0}^{s} \frac{(s-k+1)_k^2(-1)^k}{k!(|\nu|+2s-k-1)_k} \sum_{|\gamma|=k} \binom{k}{\gamma} \frac{(-\alpha)_\gamma}{(-s)_k} \frac{(\nu)_\beta}{(\nu)_{\alpha-\gamma}} \frac{k!(-1)^{s-k}}{(s-k+1)_k} \frac{(-\beta)_{\alpha-\gamma}}{\beta!}. \end{aligned} \tag{3.9}$$

The following formula for the entries of the matrix B was given in [W06:Th. 3.21]

$$b_{\alpha\beta} = n! \frac{(\nu)_\beta}{\beta!} \frac{(-1)^s}{(s+|\nu|-1)_s} \frac{\binom{m}{s}}{(|\nu|+2s)_{m-s}} \sum_{\substack{\gamma\le\alpha,\beta \\ |\gamma|\le s}} \frac{(s+|\nu|-1)_{|\gamma|}(-\alpha)_\gamma(-\beta)_\gamma(-s)_{|\gamma|}}{(\nu)_\gamma(-m)_{|\gamma|}(-n)_{|\gamma|}\gamma!}. \tag{3.10}$$

For $m = s$ this is a multivariate extension of a ${}_3F_2$ hypergeometric sum. Since the sum in (3.9) is over $\alpha - \beta \le \gamma \le \alpha$, it requires considerable rearrangement to obtain (3.10).

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