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A homothety argument for computing the best constant in some Hardy's inequalities for exterior and punctured domains

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ABSTRACT

An argument based on homothety transformations is used to prove a conjecture about multivariate Hardy's inequalities of Matskewich and Sobolevskii [MS96]. This result and more general inequalities are seen to follow from a multivariate form of Hardy's inequality recently given in Waldron [W97].

Key Words: Hardy's inequality, homothety transformation, integral form of Minkowskii's inequality, Sobolev's embedding theorem, Taylor interpolation

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1. Introduction

Following the notation of Matskewich and Sobolevskii [MS96], for Ω an open subset of $\mathbb{R}^n$, define the 'generalised Hardy's functional'

$$l_p(u, \Omega, m) := \frac{\|D^m u\|_{L_p(\Omega)}}{\|u/(\rho_{\partial\Omega})^m\|_{L_p(\Omega)}}, \qquad 1 \le p \le \infty,\ u \in C_0^\infty(\Omega),\ u \ne 0,\ m = 1, 2, \ldots,$$

where

$$\|D^m u\|_{L_p(\Omega)} := \text{the } L_p(\Omega)\text{–norm of } |D^m u|,$$

with

$$|D^m u|(x) := \sup_{\|\xi_i\|=1} |D_{\xi_1} D_{\xi_2} \cdots D_{\xi_m} u(x)|,$$

and

$$\rho_{\partial\Omega}(x) := \text{the distance of } x \text{ from } \partial\Omega, \text{ the boundary of the domain } \Omega.$$

When $m = 1$ this reduces to $l_p(u, \Omega)$ as in [MS96]. Here $\|\cdot\|$ is the Euclidean norm, and $D_\xi f$ is the directional derivative of f in the (possibly nonunit) direction ξ, defined by

$$D_\xi f(x) := \lim_{t \to 0} \frac{f(x + t\xi) - f(x)}{t}.$$

The main result of this paper is the computation for certain $\Omega \subset \mathbb{R}^n$ of

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \ne 0}} l_p(u, \Omega, m) = C, \qquad (p \ge n),$$

which for $C > 0$ is equivalent to the existence of the sharp *Hardy–type* inequality

$$\|u/(\rho_{\partial\Omega})^m\|_{L_p(\Omega)} \le \frac{1}{C} \|D^m u\|_{L_p(\Omega)}, \qquad \forall u \in C_0^\infty(\Omega). \tag{1.1}$$

We say that Ω is *punctured* (at a) if $\Omega = U \setminus \{a\}$, $a \in U$, where U is an open set, and is an *exterior* domain if $\Omega = \mathbb{R}^n \setminus D$, where D is a nonempty compact set. The precise form of our result is:

Theorem 1.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a punctured open set, or an exterior domain, i.e., $\Omega := \mathbb{R}^n \setminus D$, where $D \subset \mathbb{R}^n$ is a nonempty compact set. Then, for $p \ge n$, $m = 1, 2, \ldots$,*

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \ne 0}} l_p(u, \Omega, m) = \Big(1 - \frac{n}{p}\Big)\Big(2 - \frac{n}{p}\Big) \cdots \Big(m - \frac{n}{p}\Big). \tag{1.3}$$

From this follows:

Corollary 1.4 ([MS96:Th.1]). *Let $\Omega := \mathbb{R}^3 \setminus D$, where $D \subset \mathbb{R}^3$ is a compact set whose boundary is a C^2–smooth, 2–dimensional manifold. For $p \geq 3$,*

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} l_p(u, \Omega) = \inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} l_p(u, \Omega, 1) = 1 - \frac{3}{p}.$$

The result analogous to Corollary 1.4 for $\mathbb{R}^2$ ($n = 2$, $m = 1$) was also proved by Matskewich and Sobolevskii (unpublished).

In Section 3, some other Hardy–type inequalities similar to those of Theorem 1.2 are discussed. In particular, (1.3) can be extended to (3.5).

Fig. 1.1. A punctured open set and an exterior domain

2. Proof of Theorem 1.2

By using *homothety transformations* (*dilations*)

$$x \mapsto \lambda x, \qquad \lambda > 0,$$

we reduce the proof of Theorem 1.2 to the case when $\Omega = \mathbb{R}^n \setminus \{0\}$.

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a punctured open set, or an exterior domain Then, for $1 \leq p \leq \infty$, $m = 1, 2, \ldots$,*

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} l_p(u, \Omega, m) = \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \\ u \neq 0}} l_p(u, \mathbb{R}^n \setminus \{0\}, m) =: I(p, m, n).$$

Proof: In the argument below, we find it convenient to view $C_0^\infty(\Omega)$ as the subspace of $C_0^\infty(\mathbb{R}^n)$ consisting of those functions with compact support contained in Ω.

Without loss of generality assume that Ω is punctured at 0, or is an exterior domain for which $0 \notin \Omega$. Then, each $u \in C_0^\infty(\Omega)$ has compact support not containing 0, and so $C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n \setminus \{0\})$, giving

$$I(p, m, n) := \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \\ u \neq 0}} l_p(u, \mathbb{R}^n \setminus \{0\}, m) \leq \inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} l_p(u, \Omega, m).$$

Conversely, suppose that $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The change of variables formula shows that the generalised Hardy's functional tranforms under the homothety transformation $x \mapsto \lambda x$ as follows

$$l_p(u, \Omega, m) = l_p(u(\cdot/\lambda), \lambda\Omega, m), \qquad \forall u \in C_0^\infty(\Omega). \tag{2.2}$$

If Ω is punctured at 0, then $u(\cdot/\lambda) \in C_0^\infty(\Omega)$ for sufficiently small $\lambda > 0$, and if Ω is an exterior domain, then $u(\cdot/\lambda) \in C_0^\infty(\Omega)$ for sufficiently large $\lambda > 0$. Hence in either case, by (2.2), we conclude

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} l_p(u, \Omega, m) \leq \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \\ u \neq 0}} l_p(u, \mathbb{R}^n \setminus \{0\}, m) =: I(p, m, n).$$

$\square$

Thus, it is sufficient to compute $I(p,m,n)$. To do this it is convenient to use the *divided difference functional*

$$f \mapsto \int_{[\underbrace{0,\ldots,0}_{m},x]} f := \frac{1}{(m-1)!}\int_0^1 (1-t)^{m-1} f(tx)\,dt. \tag{2.3}$$

(cf equation (2.1.9) of Waldron [W97]). This allows the (univariate) error formula for $T_{<m}$ the operator of Taylor interpolation at 0 by n–variate polynomials of degree $< m$ to be expressed as

$$u(x) - T_{<m}u(x) = \int_{[\underbrace{0,\ldots,0}_{m},x]} D_x^m u, \qquad \forall u \in C^m(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n. \tag{2.4}$$

The *multivariate Hardy's inequality* that for Ω starshaped with respect to 0, and $p > n$

$$\|\ x \mapsto \int_{[\underbrace{0,\ldots,0}_{m},x]} f\ \|_{L_p(\Omega)} \le \frac{1}{(1-n/p)\cdots(m-n/p)}\|f\|_{L_p(\Omega)} \tag{2.5}$$

will be used. This is proved by applying the integral form of Minkowski's inequality to (2.3), and (for $p < \infty$) making the change of variables $y = tx$, $dy = t^n dx$, as follows:

$$\begin{aligned}
\|\ x \mapsto \int_{[\underbrace{0,\ldots,0}_{m},x]} f\ \|_{L_p(\Omega)} &\le \frac{1}{(m-1)!}\int_0^1 (1-t)^{m-1}\|\ x \mapsto f(tx)\ \|_{L_p(\Omega)}\,dt \\
&= \frac{1}{(m-1)!}\int_0^1 (1-t)^{m-1}\left(\int_\Omega |f(tx)|^p\,dx\right)^{1/p} dt \\
&\le \frac{1}{(m-1)!}\int_0^1 (1-t)^{m-1}\left(\int_\Omega \frac{|f(y)|^p}{t^n}\,dy\right)^{1/p} dt \\
&= \frac{1}{(m-1)!}\Big(\int_0^1 (1-t)^{m-1}t^{-n/p}\,dt\Big)\|f\|_{L_p(\Omega)} \\
&= \frac{1}{(m-1)!}\frac{\Gamma(m)\Gamma(1-n/p)}{\Gamma(m+1-n/p)}\|f\|_{L_p(\Omega)} \\
&= \frac{1}{(1-n/p)\cdots(m-n/p)}\|f\|_{L_p(\Omega)}.
\end{aligned}$$

More general forms of this multivariate Hardy's inequality are given in Waldron [W97].

Proof of Theorem 1.2: By Lemma 2.1, it is sufficient to consider $\Omega := \mathbb{R}^n\setminus\{0\}$. For this choice $\rho_{\partial\Omega}(x) = \|x\|$, and each $u \in C_0^\infty(\Omega)$ can be extended to a $u \in C_0^\infty(\mathbb{R}^n)$ by letting $u(0) := 0$. Since u is zero in a neighbourhood of 0, it follows from (2.4) that

$$\frac{u(x)}{(\rho_{\partial\Omega}(x))^m} = \frac{u(x)}{\|x\|^m} = \int_{[\underbrace{0,\ldots,0}_{m},x]} D^m_{\frac{x}{\|x\|}} u, \tag{2.6}$$

giving

$$\left|\frac{u(x)}{(\rho_{\partial\Omega}(x))^m}\right| \le \int_{[\underbrace{0,\ldots,0}_{m},x]} |D^m u|. \tag{2.7}$$

To this apply the multivariate Hardy's inequality (2.5), with $f = |D^m u|$, to obtain that for $1 - n/p > 0$ ($p > n$)

$$\begin{aligned} \|u/(\rho_{\partial\Omega})^m\|_{L_p(\Omega)} &\le \frac{1}{(1-n/p)\cdots(m-n/p)}\| \, |D^m u| \, \|_{L_p(\Omega)} \\ &= \frac{1}{(1-n/p)\cdots(m-n/p)}\|D^m u\|_{L_p(\Omega)}, \end{aligned} \tag{2.8}$$

which, cf (1.1), proves

$$I(p,m,n) \ge \Bigl(1-\frac{n}{p}\Bigr)\Bigl(2-\frac{n}{p}\Bigr)\cdots\Bigl(m-\frac{n}{p}\Bigr), \qquad p > n. \tag{2.9}$$

The reverse inequality of (2.9) (for $p = n$ also) follows since the sharpness of the multivariate Hardy's inequality as used in (2.8) is for functions of the form

$$f = |D^m u| = \|\cdot\|^{-n/p+\varepsilon}\chi_B \text{ as } \varepsilon \to 0$$

(with the relevant feature of f being its behaviour at 0). These functions do not belong to $C_0^\infty(\Omega)$, and so we must use some mollification argument. Here are the details.

Since the map $u \mapsto l_p(u,\Omega,m)$, $u \ne 0$, is continuous when $C_0^\infty(\Omega)$ is given the norm

$$\|u\|^* := \|u/(\rho_{\partial\Omega})^m\|_{L_p(\Omega)} + \|D^m u\|_{L_p(\Omega)},$$

it is sufficient to consider functions $u \ne 0$ from the Sobolev–type space

$$S_p^m(\Omega) := \|\cdot\|^*\text{–closure of } C_0^\infty(\Omega)$$

(cf Maz'ja [M85:Ch.1]), for which $\|D^m u\|_{L_p(\Omega)}$, $\|u/(\rho_{\partial\Omega})^m\|_{L_p(\Omega)}$, and $l_p(u,\Omega,m)$ are defined (and coincide with the extensions given by the density of $C_0^\infty(\Omega)$ in $S_p^m(\Omega)$).

Let $u := u_\varepsilon$, $\varepsilon > 0$ be the radial function

$$u(x) := \begin{cases} \|x\|^{m-n/p+\varepsilon}, & \|x\| \le 1; \\ g_\varepsilon(\|x\|), & \|x\| > 1, \end{cases}$$

where g_ε is chosen so that u is C^m on Ω, i.e., $|\cdot|^{m-n/p+\varepsilon}$ and g_ε have a C^m join at 1. Then, $u \in S_p^m(\Omega)$ and

$$|D^m u|(x) = \begin{cases} (1-n/p+\varepsilon)\cdots(m-n/p+\varepsilon)\|x\|^{-n/p+\varepsilon}, & \|x\| \le 1; \\ |D^m g_\varepsilon|(\|x\|), & \|x\| > 1, \end{cases}$$

giving

$$\begin{aligned} l_p(u,\Omega,m) &\le \frac{\|D^m u\|_{L_p(B)} + \|D^m u\|_{L_p(\mathbb{R}^n\setminus B)}}{\|u/(\rho_{\partial\Omega})^m\|_{L_p(B)}} \\ &= \frac{(1-n/p+\varepsilon)\cdots(m-n/p+\varepsilon)\|(\|\cdot\|^{-n/p+\varepsilon})\|_{L_p(B)} + \|D^m u\|_{L_p(\mathbb{R}^n\setminus B)}}{\|(\|\cdot\|^{m-n/p+\varepsilon}/\|\cdot\|^m)\|_{L_p(B)}} \\ &= (1-n/p+\varepsilon)\cdots(m-n/p+\varepsilon) + \frac{\|D^m u\|_{L_p(\mathbb{R}^n\setminus B)}}{\|(\|\cdot\|^{-n/p+\varepsilon})\|_{L_p(B)}}, \end{aligned}$$

where B is the unit ball in $\mathbb{R}^n$. The quantity

$$\frac{\|D^m u\|_{L_p(\mathbb{R}^n\setminus B)}}{\|(\|\cdot\|^{-n/p+\varepsilon})\|_{L_p(B)}} = \frac{\|\,|\cdot|^{(n-1)/p} D^m g_\varepsilon\,\|_{L_p(1,\infty)}}{\|\,|\cdot|^{-n/p+\varepsilon+(n-1)/p}\,\|_{L_p(0,1)}} = (\varepsilon p)^{1/p}\|\,|\cdot|^{(n-1)/p} D^m g_\varepsilon\,\|_{L_p(1,\infty)}$$

can be made arbitrarily small by appropriate choice of g_ε, and so

$$\begin{aligned} I(p,m,n) &\le \inf\{(1-n/p+\varepsilon)\cdots(m-n/p+\varepsilon) : \varepsilon > 0\} \\ &= \Big(1-\frac{n}{p}\Big)\Big(2-\frac{n}{p}\Big)\cdots\Big(m-\frac{n}{p}\Big), \qquad p \ge n. \end{aligned} \tag{2.10}$$

This completes the proof of Theorem 1.2. □

3. Additional Hardy–type inequalities

The *classical Hardy's inequality* is the sharp inequality that for $p > 1$

$$\|\, x \mapsto \frac{1}{x}\int_0^x f \,\|_{L_p(0,\infty)} \le \frac{p}{p-1}\|f\|_{L_p(0,\infty)}, \qquad \forall f \in L_p(0,\infty), \tag{3.1}$$

or equivalently, the 'differential version'

$$\|\, u/|\cdot| \,\|_{L_p(0,\infty)} \le \frac{p}{p-1}\|Du\|_{L_p(0,\infty)}, \qquad \forall u \in C_0^\infty(0,\infty), \tag{3.2}$$

and its variations.

The multivariate Hardy's inequality of [W97] is a generalisation of the form (3.1), while the Theorem 1.2 proved from it is a generalisation of the form (3.2). There are many other multivariate 'differential versions' of Hardy's inequality that can be proved using variations of this proof, e.g., Theorem 4.3.1 of [W97]. Let us give another example, which follows from almost exactly the same proof as that for Theorem 1.2.

Similarly to (2.4), for $j = 0, 1, \cdots, m-1$, the j–th order derivatives of the error in Taylor interpolation can be expressed as

$$D_{\xi_1} \cdots D_{\xi_j} u(x) - D_{\xi_1} \cdots D_{\xi_j}(T_{<m} u)(x) = \int_{\underbrace{[0,\ldots,0,x]}_{m-j}} D_x^{m-j} D_{\xi_1} \cdots D_{\xi_j} u.$$

If all derivatives of u of order $< m$ vanish at 0, i.e., $T_{<m} u = 0$, then this allows (2.7) to be generalised to

$$\frac{|D^j u|(x)}{\|x\|^{m-j}} \leq \int_{\underbrace{[0,\ldots,0,x]}_{m-j}} |D^m u|. \tag{3.3}$$

By applying the multivariate Hardy's inequality (2.5) to (3.3), as in (2.8), we then obtain:

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^n$ which is starshaped with respect to 0. Then, for $p > n$, $j = 0, \ldots m-1$, $m = 1, 2, \ldots$, there is the sharp inequality that for $\forall u \in W_p^m(\Omega)$ with derivatives of order $< m$ vanishing at 0, i.e., $T_{<m} u = 0$,*

$$\| \, |D^j u| / \| \cdot \|^{m-j} \, \|_{L_p(\Omega)} \leq \frac{1}{(1 - n/p) \cdots (m - j - n/p)} \|D^m u\|_{L_p(\Omega)}.$$

Or, with the same conditions as in Theorem 1.2, and $j = 0, \ldots m-1$, we can extend (1.3) to:

$$\inf_{\substack{u \in C_0^\infty(\Omega) \\ u \neq 0}} \frac{\|D^m u\|_{L_p(\Omega)}}{\| \, |D^j u| / (\rho_{\partial\Omega})^{m-j} \, \|_{L_p(\Omega)}} = \Big(1 - \frac{n}{p}\Big)\Big(2 - \frac{n}{p}\Big) \cdots \Big(m - j - \frac{n}{p}\Big). \tag{3.5}$$

Note, by Sobolev's embedding theorem, the condition $p > n$ in Theorem 3.4 ensures that $u \in W_p^m(\Omega)$ is C^{m-1} at 0, and so $T_{<m} u$ is defined.

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