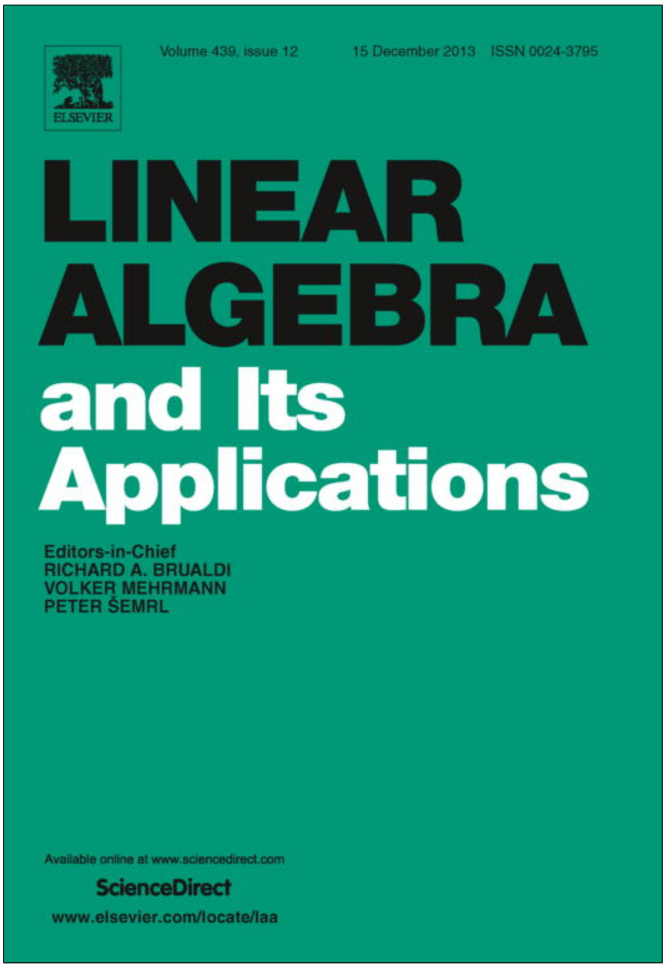
Volume 439, issue 12
15 December 2013 ISSN 0024-3795
ELSEVIER
LINEAR ALGEBRA and Its Applications
Editors-in-Chief
RICHARD A. BRUALDI
VOLKER MEHRMANN
PETER ŠEMRL
Available online at www.sciencedirect.com
ScienceDirect
www.elsevier.com/locate/laa

Linear Algebra and its Applications 439 (2013) 4135–4151

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the construction of highly symmetric tight frames and complex polytopes

Helen Broome, Shayne Waldron *

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

ARTICLE INFO

Article history:
Received 12 September 2013
Accepted 2 October 2013
Available online 31 October 2013
Submitted by R. Brualdi

MSC:
primary 42C15, 51F15, 52B11, 41A10
secondary 20F55, 51M30, 52B15

Keywords:
(Unitary) reflection
Pseudoreflection
(Shephard–Todd) finite reflection group
Regular complex polytopes
Line systems
Symmetry groups
Weyl groups
Harmonic frames
Equal-norm tight frames
Equiangular tight frames
Monster group

ABSTRACT

Many "highly symmetric" configurations of vectors in $\mathbb{C}^d$, such as the vertices of the platonic solids and the regular complex polytopes, are *equal-norm tight frames* by virtue of being the orbit of the irreducible unitary action of their symmetry group. For nonabelian groups there are *uncountably* many such tight frames up to unitary equivalence. The aim of this paper is to single out those orbits which are particularly nice, such as those which are the vertices of a complex polytope. This is done by defining a *finite* class of tight frames of n vectors for $\mathbb{C}^d$ (n and d fixed) which we call the *highly symmetric tight frames*. We outline how these frames can be calculated from the representations of abstract groups using a computer algebra package. We give numerous examples, with a special emphasis on those obtained from the (Shephard–Todd) finite reflection groups. The interrelationships between these frames with complex polytopes, harmonic frames, equiangular tight frames, and Heisenberg frames (maximal sets of equiangular lines) are explored in detail.

1. Introduction

It was observed as early as [4] (cf. [27]) that (effectively) the orbit of a nonzero vector $v \in \mathbb{C}^d$ under the irreducible unitary action of a finite group G is a tight frame for $\mathbb{C}^d$, i.e., provides the "generalised orthogonal expansion"

* Corresponding author.
E-mail address: waldron@math.auckland.ac.nz (S. Waldron).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.10.003

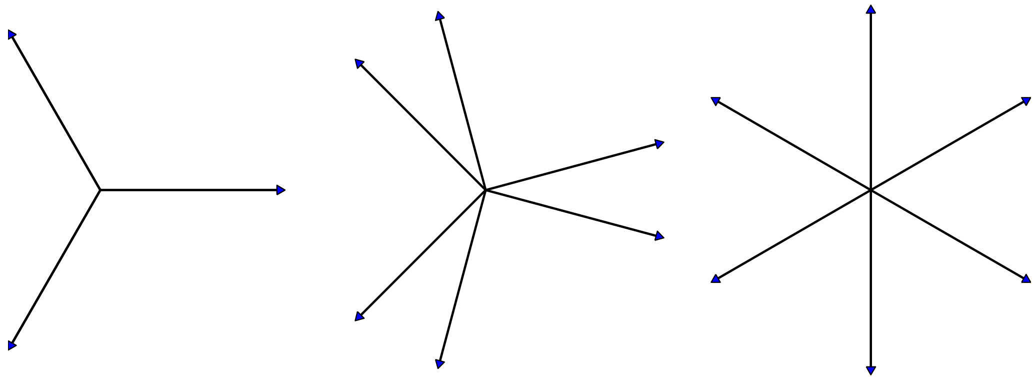

Fig. 1. The unitarily inequivalent tight frames for $v = (\cos\theta, \sin\theta)$, $\theta = \{0, \frac{\pi}{12}, \frac{\pi}{6}\}$.

$$f = \frac{d}{|G|} \frac{1}{\|v\|^2} \sum_{g \in G} \langle f, gv \rangle gv, \quad \forall f \in \mathbb{C}^d.$$

This explains why "highly symmetric" configurations of vectors tend to be tight frames. For finite abelian groups, and possibly reducible actions, there is a finite number of tight frames which can be obtained in this way (see [28]), the so-called *harmonic frames*. However, for nonabelian groups, there are *uncountably* many tight frames (up to unitary equivalence) which can be constructed in this way (see [28,12]). This is easily seen for the unitary action of the dihedral group $D_3 = \langle a, b \rangle$ (the smallest nonabelian group) on $\mathbb{R}^2$ given by a acting as rotation through $\frac{2\pi}{3}$ and b as reflection across the x-axis. For each vector $v = (\cos\theta, \sin\theta)$, $0 \leqslant \theta \leqslant \frac{\pi}{6}$, the tight frames $(gv)_{g \in D_3}$ are unitarily inequivalent (see Fig. 1).

We single out the first tight frame in Fig. 1 of three vectors as being "highly symmetric" since the vector v is fixed by the subgroup of generated by reflection across the x-axis, and so has a *small* orbit under the symmetry group D_3. In the same way, the third tight frame of six equally spaced vectors is also highly symmetric (in relation to a larger symmetry group). In this paper, we formalise these ideas. The key features of the class of *highly symmetric tight frames* that we define are:

- There is a *finite* number of highly symmetric tight frames of n vectors in $\mathbb{C}^d$.
- They can be computed from the representations of abstract groups of order $\leqslant \frac{n!}{(n-d)!}$.
- It is possible to determine whether or not a given tight frame is highly symmetric.
- The vertices of the regular complex polytopes are highly symmetric tight frames.
- Some harmonic frames are highly symmetric tight frames.
- All finite reflection groups give highly symmetric tight frames.

The rest of the paper is set out as follows. Next, we give the basic frame theory and representation theory we require. In Sections 3 and 4, we define *highly symmetric tight frames*, and outline their construction. In essence, we start with the symmetry group as an abstract group, and look for orbits (given by irreducible representations) with a *small* number of vectors. In Section 5, we show that the vertices of the regular complex polytopes are indeed highly symmetric tight frames. In Sections 6 and 7, we describe the highly symmetric tight frames which can be obtained from the imprimitive and primitive (Shephard–Todd) reflection groups. Finally, we consider the connection with Heisenberg frames (sets of equiangular lines), and an example given by the Monster group.

2. Frames and representations

A sequence of vectors $\Phi = (f_j)_{j \in J}$ is a **frame** for a (real or complex) Hilbert space $\mathcal{H}$ if there exist (**frame bounds**) $A, B > 0$, such that

$$A\|f\|^2 \leqslant \sum_{j \in J} \left|\langle f, f_j \rangle\right|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}, \tag{2.1}$$

and Φ is **tight** if one can choose $A = B$. A tight frame is **normalised** (or **Parseval**) if $A = B = 1$ (which can be achieved by a unique positive scaling). If (f_j) is finite then (2.1) is equivalent to (f_j) spanning $\mathcal{H}$, which in turn is equivalent to the "generalised orthogonal expansion"

$$f = \frac{1}{A}\sum_{j\in J}\langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H},$$

when (f_j) is tight (by the polarisation identity), which of course is the point of interest. Finite frames, particularly *tight* frames, have recently found numerous applications, which include quantum information theory, signal analysis, and orthogonal polynomials of several variables (cf. [19,6,31]). Our motivation is to find tight frames which are distinguished by having a high degree of symmetry, which is known to be advantageous for such applications. Henceforth, we will consider only finite frames (spanning sequences), and, without loss of generality, suppose that $\mathcal{H} = \mathbb{F}^d$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

We now give the basic definitions and results from frame theory (cf. [8,13,32]) and representation theory (cf. [15,17]) that we require.

Finite spanning sequences (f_j) and (g_j) for vector spaces $\mathcal{V}$ and $\mathcal{W}$ are **similar** if there is an invertible linear transformation $Q : \mathcal{V} \to \mathcal{W}$ with $Qf_j = g_j$, $\forall j$. If (f_j) and (g_j) are finite normalised tight frames (for some Hilbert spaces), then they are similar if and only if Q can be taken to unitary, in which case we say they are **unitarily equivalent**. Each finite frame $\Phi = (f_j)$ for a Hilbert space $\mathcal{H}$ is similar to a unique normalised tight frame up to unitary equivalence, namely the **canonical tight frame** (f_j^{can}), which is defined by

$$f_j^{\mathrm{can}} := S^{-\frac{1}{2}} f_j, \quad S : \mathcal{H} \to \mathcal{H} : f \mapsto \sum_j \langle f, f_j\rangle f_j,$$

where the positive linear operator $S = S_\Phi$ above is the **frame operator**. This normalised tight frame is determined (up to unitary equivalence) by its **Gramian matrix**

$$P_\Phi = \mathrm{Gram}\big(\Phi^{\mathrm{can}}\big) := \big(\big\langle f_k^{\mathrm{can}}, f_j^{\mathrm{can}}\big\rangle\big)_{j,k} = \big(\big\langle f_k, S^{-1} f_j\big\rangle\big)_{j,k},$$

which is a projection matrix. Indeed, the sequence of columns of P_Φ gives a canonical copy of this frame (with the Euclidean inner product, and $\mathcal{H}$ the range of the Gramian).

A **representation** of a finite (abstract) group G on the vector space $\mathbb{F}^d$ is a group homomorphism $\rho : G \to \mathcal{GL}_d(\mathbb{F})$, where $\mathcal{GL}_d(\mathbb{F})$ denotes the **general linear group** (of all invertible linear transformations $\mathbb{F}^d \to \mathbb{F}^d$). We say ρ_1 and ρ_2 on $\mathbb{F}^d$ are **equivalent representations** of G if there is an invertible linear map $T : \mathbb{F}^d \to \mathbb{F}^d$ such that

$$\rho_2(g) = T\rho_1(g)T^{-1}, \quad \forall g \in G.$$

When the representation is clear from the context, e.g., if G is a subgroup of $\mathcal{GL}_d(\mathbb{F})$, then we will often abbreviate the linear action $g \cdot v := \rho(g)v$ it induces on $\mathbb{F}^d$ by gv. The stabiliser (isotropy group) of a vector v under this action will be denoted by

$$\mathrm{Stab}(v) = \mathrm{Stab}_G(v) := \{g \in G : gv = v\}.$$

Moreover, to avoid confusion, we reserve the notation M^* for the Hermitian transpose of a linear map (or matrix) M with respect to the Euclidean inner product.

A representation ρ can be "made" to be unitary by defining an inner product on $\mathbb{F}^d$ by

$$\langle x, y\rangle_\rho := \sum_{g\in G}\langle gx, gy\rangle, \tag{2.2}$$

where $\langle gx, gy\rangle$ is the Euclidean (or any other) inner product. We compute

$$\langle x, hy\rangle_\rho = \sum_{g\in G}\big\langle gh\big(h^{-1}x\big), ghy\big\rangle = \big\langle h^{-1}x, y\big\rangle_\rho, \quad \forall h \in G,$$

so that $\rho(h) : \mathbb{F}^d \to \mathbb{F}^d$ is unitary with respect to $\langle\cdot,\cdot\rangle_\rho$. We say that ρ is a **unitary representation** if $\mathbb{F}^d$ is understood to be an inner product space for which the action on $\mathbb{F}^d$ is unitary, i.e., each $\rho(g)$ is a unitary transformation.

It is easy to verify that $\langle x, y\rangle_\rho = \langle x, Ay\rangle$, where $A = A_\rho : \mathbb{F}^d \to \mathbb{F}^d$ is positive definite with respect to the Euclidean inner product (henceforth denoted $\langle\cdot,\cdot\rangle$), and given by

$$A = A_\rho = \sum_{g\in G} \rho(g)^* \rho(g), \tag{2.3}$$

and that $\rho_A := A^{\frac{1}{2}} \rho A^{-\frac{1}{2}}$ is a unitary representation with respect to the Euclidean inner product. In this way, each representation is similar to a unitary representation.

A **group frame** (or G-**frame**) for $\mathbb{F}^d$ (with $\langle\cdot,\cdot\rangle$) is a frame Φ of the form

$$\Phi = (gv)_{g\in G}, \quad gv := \rho(g)v,$$

where $\rho : G \to \mathcal{GL}_d(\mathbb{F})$ is a representation of a finite group G. The canonical tight frame $\Phi^{\text{can}} = (S^{-\frac{1}{2}} gv)_{g\in G}$ is a G-frame, which is given by an equivalent representation, i.e.,

$$\Phi^{\text{can}} = \big(\rho_S(g)w\big)_{g\in G}, \quad \rho_S(g) := S^{-\frac{1}{2}} \rho(g) S^{\frac{1}{2}}, \ w := S^{-\frac{1}{2}} v.$$

A representation is **irreducible** if for every nonzero $v \in \mathbb{F}^d$ the orbit $\{gv\}_{g\in G}$ spans $\mathbb{F}^d$ (i.e., $(gv)_{g\in G}$ is G-frame). Every orbit $(gv)_{g\in G}$ $(v \neq 0)$ of an irreducible unitary action is a *tight* frame, and this characterises irreducible unitary representations [27] (cf. [4]).

If a G-frame Φ is given by an irreducible representation, then the Gramian matrix of its canonical tight frame (which defines Φ^{can} up to unitary equivalence) can be computed without inverting the frame operator $S = S_\Phi$.

Lemma 2.4. *If $\Phi = (gv)_{g\in G}$ is a group frame given by an irreducible representation $\rho : G \to \mathcal{GL}_d(\mathbb{F})$, then the canonical tight frame $\Phi^{\text{can}} = (\phi_g^{\text{can}})_{g\in G}$ satisfies*

$$\big\langle \phi_h^{\text{can}}, \phi_g^{\text{can}} \big\rangle = \big\langle \rho(h)v, S^{-1}\rho(g)v \big\rangle = \frac{d}{|G|} \frac{\langle v, A\rho(h^{-1}g)v\rangle}{\langle v, Av\rangle}, \quad A := \sum_{g\in G} \rho(g)^*\rho(g). \tag{2.5}$$

Proof. We recall that $\langle x, y\rangle_\rho = \langle x, Ay\rangle$, where A is given by (2.5). The G-frame $\Psi = A^{\frac{1}{2}}\Phi = (\rho_A(g)w)_{g\in G}$, $\rho_A(g) := A^{\frac{1}{2}}\rho(g)A^{-\frac{1}{2}}$, $w := A^{\frac{1}{2}}v$ is similar to Φ, and is tight (since it is the orbit under an irreducible unitary action). Since similar frames have canonical tight frames with the same Gramian, and the canonical tight frame of a tight frame is just its normalisation, $\langle \phi_h^{\text{can}}, \phi_g^{\text{can}}\rangle = \langle \rho(h)v, S^{-1}\rho(g)v\rangle$ is equal to

$$\big\langle \psi_h^{\text{can}}, \psi_g^{\text{can}} \big\rangle = \frac{d}{|G|} \frac{\langle A^{\frac{1}{2}}\rho(h)v, A^{\frac{1}{2}}\rho(g)v\rangle}{\langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}v\rangle} = \frac{d}{|G|} \frac{\langle \rho(h)v, A\rho(g)v\rangle}{\langle v, Av\rangle}.$$

Equivalently, one can directly verify the inversion formula: $S^{-1} = \frac{d}{|G|}\frac{1}{\langle v, Av\rangle} A$.

Since ρ is unitary with respect to $\langle\cdot,\cdot\rangle_\rho$, we have

$$\big\langle \rho(h)v, A\rho(g)v \big\rangle = \big\langle \rho(h)v, \rho(g)v \big\rangle_\rho = \big\langle v, \rho\big(h^{-1}g\big)v \big\rangle_\rho = \big\langle v, A\rho\big(h^{-1}g\big)v \big\rangle,$$

which completes the proof. □

Lemma 2.4 shows that if Φ is a G-frame given by an irreducible representation, then the canonical tight frame Φ^{can} is a G-**matrix**, i.e., has the form

$$\big\langle \phi_g^{\text{can}}, \phi_h^{\text{can}} \big\rangle = \nu\big(g^{-1}h\big), \quad \forall g, h \in G, \ \nu : G \mapsto \mathbb{F}. \tag{2.6}$$

This is also true if the representation is reducible (cf. [28]). Thus, for Φ a G-frame, each row (or column) of $\text{Gram}(\Phi^{\text{can}})$ is a permutation of the "angle" sequence $(\nu(g))_{g\in G}$. This motivates the following definition.

The **angles** (or **angle multiset**) of a G-frame Φ for $\mathbb{F}^d$ are the multiset

$$\mathrm{Ang}(\Phi) := \frac{|G|}{d}\{\langle\phi_1^{\mathrm{can}}, \phi_g^{\mathrm{can}}\rangle: g \in G,\ g \neq 1\} = \left\{\frac{\langle v, A\rho(g)v\rangle}{\langle v, Av\rangle}: g \in G,\ g \neq 1\right\},$$

where A is given by (2.3). The angle multiset of a group frame, which is easily calculated, is an invariant of its similarity class, though it need not define it. The set of angles of the line system associated with Φ (cf. [17]) is

$$\Theta(\Phi) := \{|z|: z \in \mathrm{Ang}(\Phi),\ |z| \neq 1\}.$$

If s is the cardinality of $\Theta(\Phi)$, i.e., the number of different moduli of the angles in $\mathrm{Ang}(\Phi)$, then we say Φ is s-**angular** (or **equiangular** when $s = 1$). Let n be the number of vectors in Φ, and k be the order of the group of scalar matrices which map Φ to Φ. The estimate for the number of angles in a line system ([11], see also [16]) implies that

$$n \leqslant b := k \begin{cases} \binom{d+s-1}{s}\binom{d+s-2}{s-1}, & 0 \in \mathrm{Ang}(\Phi); \\ \binom{d+s-1}{s}^2, & 0 \notin \mathrm{Ang}(\Phi). \end{cases} \tag{2.7}$$

If G is abelian, then a tight G-frame is called a **harmonic frame** (whether of not its elements are explicitly indexed by G, of which there may be more than one possibility up to group isomorphism). There is a *finite* number of harmonic frames of n vectors for $\mathbb{C}^d$ up to unitary equivalence, and these can be constructed from the character tables of the abelian groups of order n (see [14,7]).

3. Highly symmetric tight frames

Our definition of a frame being "highly symmetric" is closely tied to the notion of a "symmetry" of a spanning sequence.

The **symmetry group** of a finite spanning sequence $\Phi = (f_j)_{j\in J}$ for a vector space X (see [30]) is the group of permutations on its index set J given by

$$\mathrm{Sym}(\Phi) := \{\sigma \in S_J: \exists L_\sigma \in \mathcal{GL}(X) \text{ with } L_\sigma f_j = f_{\sigma j},\ \forall j \in J\}.$$

Since linear maps are determined by their action on a spanning sequence, it follows that each L_σ above is unique (and can be computed), so that

$$\pi_\Phi : \mathrm{Sym}(\Phi) \to \mathcal{GL}(X) : \sigma \mapsto L_\sigma \tag{3.1}$$

is a representation, which is unitary if Φ is a tight frame for $X = \mathbb{F}^d$. Clearly, similar frames have the *same* symmetry group. We call (3.1) and the associated action

$$\mathrm{Sym}(\Phi) \times X \to X : (\sigma, v) \mapsto L_\sigma v$$

the **representation (action) induced by** (the symmetries of) Φ.

We will be interested in frames of *distinct* vectors, which will be presented as the *set* obtained from a G-orbit. We will not labour the point that these must be thought of as a finite sequence to define the symmetry group, as above (or even to be a frame by our formal definition). However, we do observe that in this case the action of the symmetry group $\mathrm{Sym}(\Phi)$ on Φ is *faithful*, and so can (and will) be thought of as a subgroup of $\mathcal{GL}(X)$.

If Φ is a G-frame, then its symmetry group contains G (via the regular representation), so that $|\mathrm{Sym}(\Phi)| \geqslant |G| = |\Phi|$. The following definition ensures $|\mathrm{Sym}(\Phi)| > |\Phi|$.

Definition 3.2. A finite frame Φ of distinct vectors is **highly symmetric** if the action of its symmetry group $\mathrm{Sym}(\Phi)$ is irreducible, transitive, and the stabiliser of any one vector (and hence all) is a nontrivial subgroup which fixes a space of dimension exactly one.

If Φ is a highly symmetric frame, then $|\mathrm{Sym}(\Phi)| > |\Phi|$ (by the orbit size theorem). The definition implies that there are no highly symmetric frames for 1-dimensional spaces (cf. Example 7). Thus we now suppose $d > 1$.

As discussed in the introduction, the "Mercedes-Benz" frame of three equally spaced vectors in $\mathbb{R}^2$ (the first frame of Fig. 1) is a highly symmetric tight frame, by virtue of the fact that each of the reflections in its (dihedral) symmetry group fixes a vector.

Example 1. The standard orthonormal basis $\{e_1, \dots, e_d\}$ for $\mathbb{F}^d$ is not a highly symmetric tight frame, since its symmetry group fixes the vector $e_1 + \cdots + e_d$. However, the vertices of the regular d-simplex always are (the Mercedes-Benz frame is the case $d = 2$). Since both of these frames are harmonic, we conclude that a highly symmetric tight frame may or may not be harmonic. Moreover, for many harmonic frames of n vectors the symmetry group has order n (cf. [14]), which implies that they are not highly symmetric.

As defined, a highly symmetric frame has *distinct* vectors, and so it may not be a group frame (cf. Example 9). Of course, Φ is a $\mathrm{Sym}(\Phi)$-frame where each of the vectors is repeated a fixed number of times. Since a frame is highly symmetric if and only if the canonical tight frame is, it suffices to consider only the tight highly symmetric frames. Before detailing their construction, we observe that the class of highly symmetric tight frames is finite.

Theorem 3.3. *Fix n, d ($n \geqslant d$). There is a finite number of highly symmetric normalised tight frames of n vectors for $\mathbb{F}^d$ (up to unitary equivalence).*

Proof. Suppose that Φ is a highly symmetric normalised tight frame of n vectors for $\mathbb{F}^d$. Then it is determined, up to unitary equivalence, by the representation induced by $\mathrm{Sym}(\Phi)$, and a subgroup H which fixes only the 1-dimensional subspace spanned by some vector in Φ. There is a finite number of choices for $\mathrm{Sym}(\Phi)$ since its order is $\leqslant |S_n| = n!$, and hence (by Maschke's theorem) a finite number of possible representations. As there is only a finite number of choices for H, it follows that the class of such frames is finite. □

The only other known finite class of tight frames that can be constructed from nonabelian groups is the central G-frames. A G-frame $\Phi = (gv)_{g \in G}$ is said to be **central** if the function $\nu : G \to \mathbb{F}$ of (2.6) is a class function, i.e., is constant on the conjugacy classes of G, which is equivalent to the "symmetry condition"

$$\langle g\phi, h\phi \rangle = \langle g\psi, h\psi \rangle, \quad \forall g, h \in G, \ \forall \phi, \psi \in \Phi.$$

These frames generalise the harmonic frames (when G is abelian). In [29], it was shown how all central G-frames could be constructed from the irreducible representations of G.

4. The construction of highly symmetric tight frames

The proof of Theorem 3.3 can be implemented in a computer algebra package such as `Magma` or `GAP` (cf. [3,1]) to calculate all highly symmetric tight frames (n, d fixed). The calculations here (results to follow) were done in `Magma`. Henceforth, `typewriter` font will refer to `Magma` commands. For those unfamiliar with these packages, we now outline the key features which make this possible:

- All (abstract) groups G of a given small order can be accessed from a list.

For example, A_5, the icosahedron's rotational symmetry group, is the group `<120,35>` (number 35 on the list of groups of order 120), which can be accessed by `G:=SmallGroup(120,35)`.

- All representations of G over $\mathbb{F}$ can be computed.

For example, `AbsolutelyIrreducibleModules(G,Rationals())` calculates all of the irreducible modules (representations) of G for $\mathbb{F}=\mathbb{C}$. So far, it is only implemented for G a solvable (soluble) group, where Schur's algorithm is used. The documentation indicates that the Burnside algorithm may be implemented at some time in the future for G non-solvable. At present, `IrreducibleModules(G,K)` is defined only when K is certain cyclotomic fields (in addition to finite fields).

Since `IrreducibleModules(G,K)` presently does not cover all the cases of interest to us, we focus on those frames coming from the specific (and available) representations given by the finite reflection groups – our original point of interest. The construction of *all* highly symmetric tight frames for small n, d must wait.

We now outline our algorithm, followed by an instructive worked example.

Algorithm. To construct all highly symmetric tight frames Φ of n vectors in $\mathbb{F}^d$.

1. *Start with an abstract group G* (this will be $\mathrm{Sym}(\Phi)$ or an appropriate subgroup).
   Since $n<|\mathrm{Sym}(\Phi)|\leqslant n(n-1)\cdots(n-d+1)$, there is a finite number of possibilities.
2. *Take all faithful irreducible representations* $\rho: G\to\mathcal{GL}_d(\mathbb{F})$.
   There is a finite number of these, and we have discussed how they may be computed.
3. *Find, up to conjugacy, all subgroups H of $\rho(G)$ which fix a subspace* $\mathrm{span}\{v\}$, $v\neq 0$.
   The command `Subgroups(G)` gives all such subgroups, each with generators $H=\langle h_j\rangle$. The linear system $h_jv=v$, $\forall j$, is easily solved for v. Then $\{gv\}_{g\in G}$ is a highly symmetric tight frame of $|G|/\mathrm{Stab}(v)$ vectors. No other subgroups of $\mathrm{Stab}(v)$ need be considered.
4. *Determine which of the highly symmetric tight frames obtained are unitarily equivalent.*
   A necessary (but not sufficient) condition for unitary equivalence is that the angles be equal. All other cases can be resolved by considering permutations of the Gram matrices.

We observe that G acts faithfully on the (distinct) vectors of such a highly symmetric tight frame Φ, and so is (isomorphic to) a subgroup of $\mathrm{Sym}(\Phi)$.

Example 2. Let G be the solvable group `<18,3>`, for which `Magma` gives the presentation

$$G=\langle g_1,g_2,g_3\colon g_1^2=g_2^3=g_3^3=1,\ g_1^{-1}g_3g_1=g_3^2\rangle.$$

The representations of G over $\mathbb{C}$ can be computed

```
G := SmallGroup(18, 3);
r := AbsolutelyIrreducibleModules(G, Rationals()).
```

There are six 1-dimensional representations, and three of dimension 2, the first given by

```
rho := Representation(r[7]);    rG := ActionGroup(r[7]);
a := rG.1 = rho(G.1);    b := rG.2;    c := rG.3;    sg := Subgroups(rG);
```

$$a=\rho(g_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad b=\rho(g_2)=\begin{pmatrix}\omega^2&0\\0&\omega^2\end{pmatrix},$$

$$c=\rho(g_3)=\begin{pmatrix}\omega^2&0\\0&\omega\end{pmatrix},\qquad \omega:=e^{\frac{2\pi i}{3}}.$$

The subspace fixed by a (nontrivial) subgroup H given by `sg` can be found by the command `NullspaceMatrix(M-Id)`, where `M` is a block matrix of generators for H and `Id` is the corresponding identity block matrix. Thus, we obtain two highly symmetric tight frames:

$$6\text{ vectors}:\quad v=v_1=(1,0),\qquad \mathrm{Stab}(v_1)=\langle bc\rangle,$$

$$9\text{ vectors}:\quad v=v_2=(1,1),\qquad \mathrm{Stab}(v_2)=\langle a\rangle,$$

which are a cross and a cube (cf. Example 6). These are the only highly symmetric tight frames we obtain, since the eighth representation is not faithful, and $\rho(G)$ is the same for the seventh and ninth.

Example 3. There are no highly symmetric tight frames of five vectors in $\mathbb{C}^3$. Such a tight frame would have a symmetry group of order a multiple of 5, which is at most $5 \cdot 4 \cdot 3 = 60$. A computer search over all groups in this range shows there is no such frame. By way of contrast, the tight frame of five vectors in $\mathbb{C}^3$ with the largest symmetry group is the vertices of a *trigonal bipyramid*, which has symmetry group of order 12 (see [30]).

5. Complex polytopes and finite reflection groups

The notion of a (regular) complex polytope has evolved from the original "conceptual definition" of [22] (cf. [10]) to today where they are mostly studied in an abstract combinatorial setting (cf. [18]). We will follow [20]. The main thrust of the classical theory is that as soon as enough regularity is imposed, the symmetry group is generated by (complex) reflections, which leads to a complete classification via the symmetry group. In this setting, there are interesting recent developments, such as the construction of d^2-equiangular lines in $\mathbb{C}^d$ whose symmetry group is *not* a reflection group (cf. Section 8).

A transformation $g \in \mathcal{GL}_d(\mathbb{F})$ is a **(complex) reflection** (or **pseudoreflection**) if it has finite order m and $\operatorname{rank}(g - I) = 1$, i.e., g fixes a hyperplane H, and maps some $v \mapsto \omega v$ where $v \notin H$ is nonzero and ω is a primitive m-th root of unity. The terminology and geometric motivation comes from $\mathbb{R}^d$ with $\omega = -1$.

A finite subgroup of $\mathcal{GL}_d(\mathbb{F})$ is a **reflection group** if it is generated by its reflections. We have already observed that finite subgroups of $\mathcal{GL}_d(\mathbb{F})$ are unitary for the inner product (2.2). Thus, reflection groups are also called *unitary* reflection groups.

Frames are sequences of vectors (points), whereas polytopes, such as the platonic solids, have points, lines (through points), and faces, etc. The technical definition (to follow), specifies these *j-faces* ($j = 0, 1, \ldots$) as affine subspaces of $\mathbb{F}^d$, together with some combinatorial properties motivated by the case $\mathbb{R}^3$. Of course, such a face is the affine hull of the vertices it contains, and it is convenient to move between the two. For complex spaces, a line (1-face) may contain *more* than two points, which challenges one's intuition.

Definition 5.1. (See [20].) A d-**polytope-configuration** is a finite family $\mathcal{P}$ of affine subspaces of $\mathbb{F}^d$ of dimensions $j = -1, 0, 1, \ldots, d$, called **elements** or j-**faces**, ordered by inclusion $\subset$, which form lattice with the properties:

(i) If $F_{j-1} \subset F_{j+1}$ are $j-1$- and $j+1$-faces, then there are *at least* two j-faces contained between them. (Modified diamond condition)
(ii) If $F \subset G$ are faces, then there is a sequence of faces $F = H_0 \subset H_1 \subset \cdots \subset H_k = G$ with $\dim(H_j) = \dim(F) + j$, $\forall j$. (Connectedness)

For brevity, we call such a $\mathcal{P}$ a **complex polytope**. We now follow the usual practice and translate $\mathcal{P}$ so that the barycentre (average of the vertices) is zero. This allows the vertices to be thought of as vectors, and ensures that the affine maps of the vertices to themselves are linear (and ultimately unitary).

Definition 5.2. The **symmetry group** $\operatorname{Sym}(\mathcal{P})$ of a d-polytope-configuration $\mathcal{P}$ (with barycentre 0) is the group of $g \in \mathcal{GL}_d(\mathbb{F})$ which maps the elements of $\mathcal{P}$ to themselves.

In particular, if $\Phi_{\mathcal{P}}$ is the points (vectors) of $\mathcal{P}$, then $\operatorname{Sym}(\mathcal{P})$ is a subgroup of $\operatorname{Sym}(\Phi_{\mathcal{P}})$ (viewed as linear transformations of $\mathbb{F}^d$).

Definition 5.3. A **flag** of d-polytope-configuration $\mathcal{P}$ is a sequence F of faces with

$$F = (F_{-1}, F_0, F_1, \ldots, F_d), \qquad F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_d, \qquad \dim(F_j) = j, \quad \forall j,$$

and $\mathcal{P}$ is **regular** if $\mathrm{Sym}(\mathcal{P})$ is transitive on the flags of $\mathcal{P}$.

Shephard [22,23] showed the symmetry group of a regular complex polytope is an irreducible reflection group, and classified all such polytopes via their symmetry groups. More precisely, let F be a flag of a regular complex polytope $\mathcal{P}$, and c_j be the centre of the j-face F_j, i.e., the average of its vertices. Then there are generating reflections $r_0, \ldots, r_{d-1}$ for $\mathrm{Sym}(\mathcal{P})$ where r_j fixes $c_0, \ldots, c_{j-1}, c_{j+1}, \ldots, c_d$ and maps F_j to another j-face, i.e., r_j maps F to a flag which differs only in the j-face. The symmetry group of a regular complex polytope is encoded in the **(generalised) Schläfli symbol** (cf. [20])

$$p_0\{q_1\}p_1\{q_2\}p_2 \cdots p_{d-2}\{q_{d-1}\}p_{d-1}.$$

Theorem 5.4. *The vertices of the regular complex polytopes are highly symmetric tight frames. In particular, the vertices of the regular complex polytopes can be constructed from their abstract symmetry groups (which contain the corresponding reflection group).*

Proof. Let $\mathcal{P}$ be a regular complex polytope, and $\Phi = \Phi_{\mathcal{P}}$ be its vertices. View $G = \mathrm{Sym}(\Phi)$ as a subgroup of $\mathcal{GL}_d(\mathbb{F})$. Then $H = \mathrm{Sym}(\mathcal{P})$ is a subgroup of G, which is irreducible and transitive on the flags, and in particular is transitive on the vertices Φ. Thus, Φ will be a highly symmetric (tight) frame provided that $\mathrm{Stab}_H(v) \subset \mathrm{Stab}_G(v)$ fixes a space of dimension exactly one for each $v \in \Phi$.

Fix a vertex $v \in \Phi$. Since H is a reflection group, Steinberg's fixed point theorem [25] implies that $\mathrm{Stab}_H(v)$ is the group generated by all the reflections which fix v. If F is a flag with $F_0 = \{v\}$, then the $d-1$ generating reflections $r_1, \ldots, r_{d-1}$ fix v, so the subspace fixed by them all is 1-dimensional (and equal to $\mathrm{span}\{v\}$). Thus $\mathrm{Stab}_G(v)$ fixes only $\mathrm{span}\{v\}$. □

Remark. It is an open question whether the symmetry group of the points of a (regular) complex polytope $\mathcal{P}$ can be larger than $\mathrm{Sym}(\mathcal{P})$, i.e., are there symmetries of the points, which do not map flags to flags (cf. Example 11). For real polytopes these are equal, since the j-faces can be uniquely determined by considering the convex hull of the points.

In [24] (cf. [17]) all finite reflection groups were classified. Essentially, they appear as the symmetry groups of "semi-regular" complex polytopes. In the next sections we outline the highly symmetric tight frames which can be obtained from the (imprimitive and primitive) finite reflection groups and the (discrete) Heisenberg group, as determined by our `Magma` calculations. After completing these calculations, we became aware of the fact that these correspond to *maximal proper parabolic subgroups*.

Definition 5.5. A **parabolic subgroup** of a finite reflection group $G \subset \mathcal{GL}_d(\mathbb{F})$ is the pointwise stabiliser of a subset $V \subset \mathbb{F}^d$.

Steinberg's fixed point theorem [25] says that a parabolic subgroup is a finite reflection subgroup. Using this, the parabolic subgroups have been calculated (see [26]).

Theorem 5.6. *If $G \subset \mathcal{GL}_d(\mathbb{F})$ is an irreducible finite reflection group, then $(gv)_{g\in G}$ is a highly symmetric tight frame for $\mathbb{F}^d$ if and only if $H = \mathrm{Stab}(v)$ is a maximal proper parabolic subgroup.*

Proof. Since the parabolic subgroups are generated by reflections, and each reflection fixes a hyperplane, the set V fixed by a maximal proper parabolic subgroup must be a 1-dimensional subspace $V = \mathrm{span}\{v\}$, $v \neq 0$. □

6. Imprimitive groups (ST 1–3)

A representation of G on $\mathbb{F}^d$ is **imprimitive** if $\mathbb{F}^d$ is a direct sum $\mathbb{F}^d = V_1 \oplus \cdots \oplus V_m$ of nonzero subspaces, such that the action of G on $\mathbb{F}^d$ permutes the V_j, otherwise it is **primitive**.

The Shephard–Todd classification of the *imprimitive* irreducible complex reflection groups consists of three infinite families (ST 1–3) given by the groups $G(m,p,d)$, where $m>1$, $p \mid m$, and

$$\left|G(m,p,d)\right| = m^d d!/p.$$

These are available in Magma via the command `ImprimitiveReflectionGroup(m,p,d)`, and can be constructed (cf. [17]) as a group of unitary transformations

$$\begin{aligned} G(1,1,d) &= \langle r_1, r_2, \ldots, r_{d-1}\rangle, \\ G(m,m,d) &= \langle s, r_1, r_2, \ldots, r_{d-1}\rangle, \\ G(m,1,d) &= \langle t, r_1, r_2, \ldots, r_{d-1}\rangle, \\ G(m,p,d) &= \langle s, t^p, r_1, r_2, \ldots, r_{d-1}\rangle, \quad 1<p<m,\ p \mid m \end{aligned}$$

where r_j interchanges e_j and e_{j+1}, t is the reflection $e_1 \mapsto \omega e_1$, $\omega = e^{\frac{2\pi i}{m}}$, and $s = t^{-1} r_1 t$, i.e.,

$$r_1 = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I \end{bmatrix}, \qquad t = \begin{bmatrix} \omega & & \\ & 1 & \\ & & I \end{bmatrix}, \qquad s = \begin{bmatrix} 0 & \overline{\omega} & \\ \omega & 0 & \\ & & I \end{bmatrix}, \tag{6.1}$$

where I is the identity matrix of size $d-2$. The three infinite families are:

ST 1: $G(1,1,d+1) \approx S_{d+1}$ acting on the d-dimensional subspace of vectors $x \in \mathbb{F}^{d+1}$ with $x_1 + \cdots + x_{d+1} = 0$, i.e., the orthogonal complement of $e_1 + \cdots + e_{d+1}$.
ST 2: $G(m,p,d)$, $m,d>1$, $p \mid m$, $(m,p,d) \neq (2,2,2)$ acting on $\mathbb{C}^d$.
ST 3: $G(m,1,1) \approx \mathbb{Z}_m$ acting on $\mathbb{C}$.

We now give some indicative examples from each family (see [5] for more detail).

Example 4 *(The simplex).* Let $G = G(1,1,d+1) \approx S_{d+1}$ act on $\mathcal{H} = (e_1 + \cdots + e_{d+1})^\perp$, the orthogonal complement of $e_1 + \cdots + e_{d+1}$ in $\mathbb{F}^{d+1}$, and

$$w_k := e_1 + \cdots + e_k - \frac{k}{d+1-k}(e_{k+1} + \cdots + e_{d+1}), \quad 1 \leqslant k \leqslant d.$$

Then $|\mathrm{Stab}(w_k)| = k!(d+1-k)!$, so that $\Phi_k = \{g w_k\}_{g \in G}$ is a highly symmetric tight frame of $\binom{d+1}{k}$ vectors for the d-dimensional space $\mathcal{H}$, with symmetry group G. Our calculations indicate that these are the only possibilities. For $k=1$ we can interpret G as the symmetry group of the simplex with vertices given by Φ_1. The other cases are the barycentres of the $(k-1)$-faces of this simplex (so $k=d$ also gives a simplex). In particular, $k=2$, $d=3$ gives the six vertices of the octahedron, and $k=2$, $d=4$ gives ten vectors in $\mathbb{R}^4$ which is not a harmonic frame and has a symmetry group of order 120.

Example 5 *(28 equiangular lines in $\mathbb{R}^7$).* The special case of Example 4 where $G = G(1,1,8)$ acts on the vector

$$v = 3w_2 = (3,3,-1,-1,-1,-1,-1,-1),$$

gives an orbit of 28 vectors which is an equiangular tight frame for a 7-dimensional space.

Example 6 *(The generalised "cross" and "cube").* Let $G = G(m,1,d)$, $|G| = d!m^d$, and

$$v_k := e_1 + \cdots + e_k, \quad 1 \leqslant k \leqslant d.$$

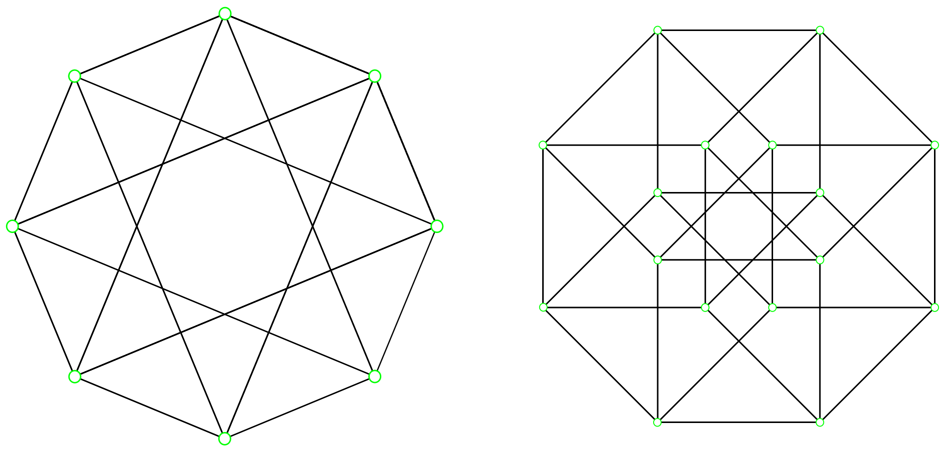

Fig. 2. Symbolic projections of the cross (hexadecachoron) and cube (tesseract) in $\mathbb{R}^4$.

Then v_k has $|\mathrm{Stab}(v_k)| = k!(d-k)!m^{d-k}$, and so its orbit gives a highly symmetric tight frame of $\binom{d}{k}m^k$ vectors. Our calculations suggest that these are all.

The extreme cases are the **(generalised) cross** ($k = 1$) and **cube** ($k = d$), which are regular complex polytopes. These terms originate from the case $m = 2$, $d = 3$, where (see Fig. 2) we have the *octahedron* (6 vertices), *cuboctahedron* (12 vertices), and *cube* (8 vertices), respectively, and $G = G(2, 1, 3)$ is O_h, the *full octahedral group*. For $m = 2$, $d = 4$ (see Fig. 2), the polytopes are the *hexadecachoron* (16-*cell*) (8 vertices), *octaplex* (24-*cell*) (24 vertices), *rectified tesseract* (32 vertices), and *tesseract* (16 vertices).

The cross and cube are harmonic, generated by the cyclic subgroup $\langle r_1r_2\cdots r_{d-1}t\rangle$, and $\langle q_1, \ldots, q_d\rangle$, where $q_j = (r_1r_2\cdots r_{j-1})^{-1}t(r_1r_2\cdots r_{j-1})$ is the reflection $e_j \mapsto \omega e_j$. The tight frame of 12 vectors for $\mathbb{R}^3$ given by the vertices of the cuboctahedron is *not harmonic*, since it is not generated by any abelian subgroup of its symmetry group O_h, of order 48. According to the list of [14], the only harmonic frames of 12 vectors for $\mathbb{F}^3$ with a symmetry group larger than 36 are three with order 72 and one with symmetry group <384,5557>, which we recognise as the generalised cross $\{i^k e_j\colon\ 1 \leqslant j \leqslant 3,\ 0 \leqslant k \leqslant 3\}$ given by $G(4, 1, 3)$.

Example 7 *(The m-th roots of unity).* Let $G = G(m, 1, 1) \approx \mathbb{Z}_m$ acting on $\mathbb{C}$. Since only the identity stabilises a nonzero vector, no highly symmetric tight frames are obtained from the third Shephard–Todd family.

The imprimitive reflection groups of the ST 2 family can be nested (cf. [17, p. 31]), e.g.,

$$G(m, p, d) \lhd G(m, 1, d), \qquad G(m, p, 2) \lhd G(2m, 2, 2).$$

As a consequence:

- The symmetry group of highly symmetric tight frame obtained from an imprimitive reflection group G may be *larger* than G.
- A highly symmetric tight frame obtained from an imprimitive reflection group G may be a *subset* of one obtained for a larger imprimitive reflection group.

Example 8 *(Nested irreducible reflection groups).* Let

$$G = G(2, 2, d), \quad d > 2, \qquad |G| = 2^{d-1}d! \quad (\text{Coxeter group } D_d).$$

There are highly symmetric tight frames given by the orbits of e_1 and $e_1 + \cdots + e_d$. The first of these is the cross, which has a symmetry group larger than G, namely $G(2, 1, d)$. The second is the *demicube*, a subset of half the vertices of the cube, which has symmetry group $G(2, 1, d)$.

7. Primitive reflection groups (ST 4–37)

There are 34 (exceptional) finite reflection groups in the Shephard–Todd classification. Their numbers and rank (the dimension of the space they act on) are

$$\text{ST 4–22 (rank 2)}, \qquad \text{ST 23–27 (rank 3)}, \qquad \text{ST 28–32 (rank 4)},$$
$$\text{ST 33 (rank 5)}, \qquad \text{ST 34–35 (rank 6)}, \qquad \text{ST 36 (rank 7)}, \qquad \text{ST 37 (rank 8)}.$$

Our `Magma` calculations (see Appendix A) indicate the following behaviour:

- There are highly symmetric tight frames given by each primitive reflection group. In particular, there are ones which are not the vertices of a regular complex polytope.
- These highly symmetric tight frames are not harmonic.
- They may or may not be G-frames (of distinct vectors).
- They have a small number of angle moduli.

The search for highly symmetric tight frames was exhaustive, except for three groups, namely ST 34, 36 and 37, which have large orders (39 191 040, 2 903 040, 696 729 600) and high rank (6, 7, 8). We now highlight a few examples.

Example 9 *(ST 23)*. All highly symmetric tight frames obtained from rank 2 reflection groups are group frames (of distinct vectors). This is not the case in higher dimensions. Let G be the Shephard–Todd group 23, $|G| = 120$, for which `Magma` gives the generators

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{2}(\sqrt{5}+1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & \frac{1}{2}(\sqrt{5}+1) & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

which are *not* unitary matrices. The corresponding matrix A of (2.3) is

$$A = 10 \begin{pmatrix} 16\sqrt{5}+52 & -17\sqrt{5}-33 & 0 \\ -17\sqrt{5}-33 & 16\sqrt{5}+52 & -8\sqrt{5}-26 \\ 0 & -8\sqrt{5}-26 & 16\sqrt{5}+52 \end{pmatrix}.$$

We obtain three highly symmetric tight frames:

$$\text{12 vectors:} \quad v = (\sqrt{5}-1, 0, 2),$$
$$\text{20 vectors:} \quad v = (\sqrt{5}+3, 0, 2),$$
$$\text{30 vectors:} \quad v = (1, 1, 1),$$

which are the vertices of the *icosahedron*, *dodecahedron*, and *icosidodecahedron*. The first of these is a group frame (for `<12,3>`), and the other two are not.

Example 10 *(24 vectors in $\mathbb{C}^2$)*. There are five regular complex polygons with 24 vertices. Their Schläfli symbols and symmetry groups are

3{6}2 `ShephardTodd(6) = <48,33>`, 3{3}2 `ShephardTodd(6) = <48,33>`,

3{4}3 `ShephardTodd(5) = <72,25>`, 4{3}4 `ShephardTodd(8) = <96,67>`,

2{4}12 `ImprimitiveReflectionGroup(12,1,2) = <288,239>`.

The four obtained from the primitive groups are *not* harmonic, by comparison with the symmetry groups of the 33 harmonic frames of 24 vectors (see [14]). The fifth frame is a generalised cross, which is harmonic. In addition to these, there is a highly symmetric tight frame of 24 vectors (which is not a polygon) that can be obtained from the group

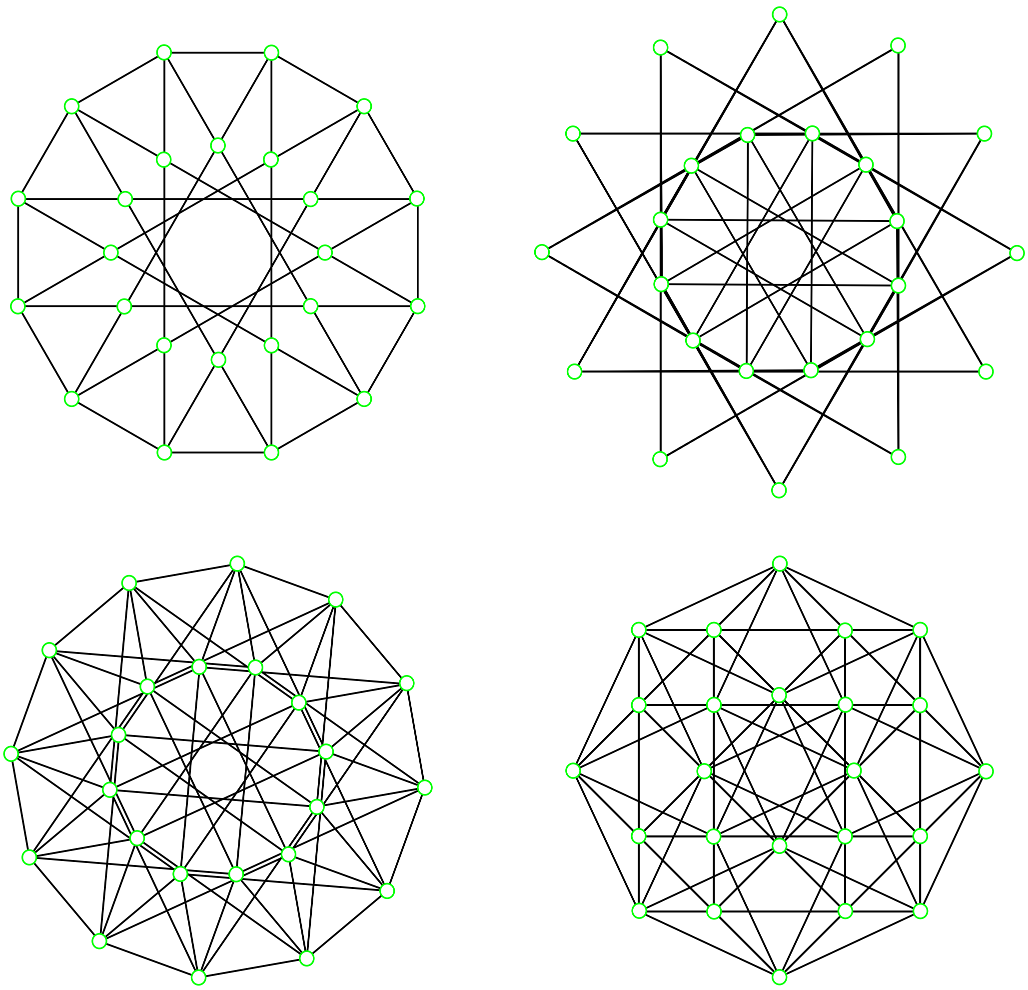

Fig. 3. Symbolic projections of the polygons 3{6}2, 3{3}2, 3{4}3, 4{3}4 with 24 vertices, which are obtained from primitive reflection groups.

$$\texttt{G} := \texttt{ShephardTodd(12)} = \texttt{<48,29>}, \qquad G = \langle g_1, g_2, g_3 \rangle,$$

$$g_1 := \frac{1}{2}\begin{pmatrix} \omega^3 - \omega & -\omega^3 + \omega \\ -\omega^3 + \omega & -\omega^3 + \omega \end{pmatrix}, \qquad g_2 := \frac{1}{2}\begin{pmatrix} \omega^3 - \omega & \omega^3 - \omega \\ \omega^3 - \omega & -\omega^3 + \omega \end{pmatrix},$$

$$g_3 := \begin{pmatrix} 0 & -\omega \\ \omega^3 & 0 \end{pmatrix},$$

and the vector $v = (1, \omega^3)$, where $\omega = e^{\frac{2\pi i}{8}}$. Similarly, this frame is not harmonic (see Fig. 3).

8. Heisenberg frames

The (discrete) **Heisenberg group** $H = \langle S, \Omega \rangle$ is the subgroup of the imprimitive reflection group $G(d, 1, d)$ generated by the (cyclic) shift and modulation operators

$$S = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{d-1} \end{bmatrix}, \qquad \omega := e^{\frac{2\pi i}{d}}.$$

Since S and Ω have order d, and $\Omega^k S^j = \omega^{jk} S^j \Omega^k$, H has order d^3. For $d > 3$, H is not a reflection group, indeed it contains no reflections. The unitary representation given by H acting on $\mathbb{C}^d$ is irreducible, so that every nontrivial orbit is a tight frame for $\mathbb{C}^d$.

The Zauner conjecture (cf. [19,2,21]) is that (for all d) there exists a $v \in \mathbb{C}^d$ whose H-orbit gives a set of d^2-equiangular lines, i.e., $(S^j \Omega^k v)_{0 \leqslant j,k<d}$ is an equiangular tight frame of d^2 vectors for $\mathbb{C}^d$, which is the maximal number allowed by the bound (2.7). This is supported by numerical solutions, and analytic constructions for some values of d. We refer to the resulting equiangular tight frames as *Heisenberg frames*.

We are unable to determine whether Heisenberg frames are highly symmetric, in a projective sense, as there are no effective methods for calculating the (projective) symmetry group of a set of lines (cf. [30]). More, precisely, the vectors $v \in \mathbb{C}^d$ known to give an equiangular tight frame (analytically or numerically) are eigenvectors of a matrix M of order 3 (the "strong" Zauner conjecture). This unitary matrix M normalises H (up to a scalar), and so is a (projective) symmetry of the lines given by the H-orbit of v. However, except for small d, the eigenspace of M which contains v is not 1-dimensional (otherwise Zauner's conjecture would be proved), and so we cannot conclude the equiangular tight frame given by v is highly symmetric. It may be that there are symmetries other than M which fix v. If one could find enough of these additional symmetries, so that only the space spanned by v is fixed by them all, then the corresponding frame would be highly symmetric (in a projective sense), and Zauner's conjecture would be all but proved.

The case $d = 3$ is known to be exceptional. Here there are uncountably many (unitarily inequivalent) Heisenberg frames, and it turns out one of these is highly symmetric.

Example 11 *(Hessian polyhedron).* The *Hessian* is the regular complex polytope with 27 vertices and Schläfli symbol 3{3}3{3}3. Its symmetry group ST 25 (<648,533>) is generated by the following three reflections of order three

$$R_1 = \begin{pmatrix} \omega & & \\ & 1 & \\ & & 1 \end{pmatrix}, \qquad R_2 = \frac{1}{3}\begin{pmatrix} \omega+2 & \omega-1 & \omega-1 \\ \omega-1 & \omega+2 & \omega-1 \\ \omega-1 & \omega-1 & \omega+2 \end{pmatrix}, \qquad R_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega \end{pmatrix},$$

and it has $v = (1, -1, 0)$ as a vertex (cf. [10, p. 119]). These vertices are the H-orbit of v, which is a Heisenberg frame. We observe that H is normal in $G = \langle R_1, R_2, R_3 \rangle$.

The second frame of 72 vectors provides the following point of interest. Multiplication by -1 gives a symmetry which in not in the reflection group G. Thus the symmetry group of this frame (which is not the vertices of a regular complex polytope) is strictly larger than G. There is similar behaviour for the Shephard–Todd groups 13, 15, 25, 33 (third roots), and 35.

We conclude with an example of highly symmetric tight frames with a large number of vectors in a space of high dimension (courtesy of John Duncan).

Example 12 *(The Monster).* The finite simple groups have a similar classification to the finite reflection groups: some infinite families, together with a finite number of exceptional cases (the sporadic groups). The largest sporadic group the Fischer-Griess *Monster* gives rise to highly symmetric tight frames.

Let ρ be the irreducible representation of M in $d = 196\,883$ (the smallest nontrivial dimension). There is a largest conjugacy class of elements of order $2, 3, 4$ in M, which in the *ATLAS* notation are labelled $2A, 3A, 4A$. For any element a in one of these three classes the centraliser of g in M fixes a unique vector v_a (called the *axis* of a) under the action of the 196 883-representation (see [9, §14]). Thus the orbit of v_a under the action given by ρ is a highly symmetric tight frame of $n = |M|/|C_G(a)|$ vectors for $\mathbb{C}^{196\,883}$. Using the *ATLAS of Finite Group Representations*, we calculate the sizes of these frames to be

$$n_{2A} = 97\,239\,461\,142\,009\,186\,000,$$

$$n_{3A} = 214\,577\,690\,036\,031\,541\,739\,520\,000\,000,$$

$$n_{4A} = 97\,145\,685\,362\,919\,706\,207\,382\,495\,808\,000\,000.$$

One can only speculate how many angles these frames might have.

Table 1
The highly symmetric tight frames of n vectors in $\mathbb{C}^d$ given by the primitive reflection groups ST 4–22. Here (P) denotes a non-starry regular complex polytope.

ST	d	Order	n	b	s	Group frame
4	2	$\langle 24, 3\rangle$	8 (P)	8 (4)	1	$\langle 8, 4\rangle$
5		$\langle 72, 25\rangle$	24 (P)	24 (4)	1_2	$\langle 24, 3\rangle$, $\langle 24, 11\rangle$
6		$\langle 48, 33\rangle$	16 (P)	16 (4)	1	$\langle 16, 13\rangle$
			24 (P)	24 (6)	2	$\langle 24, 3\rangle$
7		$\langle 144, 157\rangle$	48	48 (4)	1_2	$\langle 48, 47\rangle$, $\langle 48, 33\rangle$
			72	72 (6)	2	$\langle 72, 25\rangle$
8		$\langle 96, 67\rangle$	24 (P)	24 (6)	2	$\langle 24, 3\rangle$, $\langle 24, 1\rangle$
9		$\langle 192, 963\rangle$	48 (P)	48 (6)	2	$\langle 48, 4\rangle$, $\langle 48, 28\rangle$, $\langle 48, 29\rangle$
			96 (P)	160	4	$\langle 96, 67\rangle$, $\langle 96, 74\rangle$
10		$\langle 288, 400\rangle$	72 (P)	72 (6)	2	$\langle 72, 12\rangle$, $\langle 72, 25\rangle$
			96 (P)	144	3	$\langle 96, 54\rangle$, $\langle 96, 67\rangle$
11		$\langle 576, 5472\rangle$	144	144 (6)	2	$\langle 144, 69\rangle$, $\langle 144, 121\rangle$, $\langle 144, 122\rangle$
			192	288	3	$\langle 192, 876\rangle$, $\langle 192, 963\rangle$
			288	480	4	$\langle 288, 400\rangle$, $\langle 288, 638\rangle$
12		$\langle 48, 29\rangle$	24	40	4	$\langle 24, 3\rangle$
13		$\langle 96, 192\rangle$	48	80	4	$\langle 48, 28\rangle$, $\langle 48, 29\rangle$
			48	48 (6)	2	$\langle 48, 28\rangle$, $\langle 48, 33\rangle$
14		$\langle 144, 122\rangle$	48 (P)	72	3	$\langle 48, 26\rangle$, $\langle 48, 29\rangle$
			72 (P)	120	4	$\langle 72, 25\rangle$
15		$\langle 288, 903\rangle$	96	144	3	$\langle 96, 182\rangle$, $\langle 96, 192\rangle$
			144	240	4	$\langle 144, 121\rangle$, $\langle 144, 122\rangle$
			144	144 (6)	2	$\langle 144, 121\rangle$, $\langle 144, 157\rangle$
16		$\langle 600, 54\rangle$	120 (P)	120 (12)	3	$\langle 120, 5\rangle$, $\langle 120, 15\rangle$
17		$\langle 1200, 483\rangle$	240 (P)	240 (12)	3	$\langle 240, 93\rangle$, $\langle 240, 154\rangle$
			600 (P)	1440	8	$\langle 600, 54\rangle$
18		$\langle 1800, 328\rangle$	360 (P)	360 (12)	3	$\langle 360, 51\rangle$, $\langle 360, 89\rangle$
			600 (P)	900	5	$\langle 600, 54\rangle$
19		$\langle 3600, *\rangle$	720	720 (12)	3	$\langle 720, 420\rangle$, $\langle 720, 708\rangle$
			1200	1800	5	$\langle 1200, 483\rangle$
			1800	4320	8	$\langle 1800, 328\rangle$
20		$\langle 360, 51\rangle$	120 (P)	180	5	$\langle 120, 5\rangle$
21		$\langle 720, 420\rangle$	240 (P)	360	5	$\langle 240, 93\rangle$
			360 (P)	864	8	$\langle 360, 51\rangle$
22		$\langle 240, 93\rangle$	120	288	8	$\langle 120, 5\rangle$

9. Future directions

For the points given by the vectors of the highly symmetric tight frames obtained from the finite reflection groups, it would be natural to find lines, faces, etc., which give a complex polytope. The class of highly symmetric tight frames presented here extends, in the obvious way, to *highly symmetric spanning sets* for a finite dimensional vector space over *any* field. To interpret these as tight frames, one would need to extend the theory of frames (using classical groups).

Acknowledgements

We thank Egon Schulte for some very useful discussions about (regular) complex polytopes, which today are mostly studied in an abstract (combinatorial) setting. We also thank Don Taylor for pointing out the connection between maximal parabolic subgroups and the highly symmetric tight frames for the finite reflection groups, and to John Duncan for information on the monster group.

The symbolic projections of the regular complex polytopes given here were generated by the LaTeX software of Jean-Gabriel and Manuel Luque.

Appendix A

Tables 1 and 2 in this appendix list the highly symmetric tight frames of n vectors in $\mathbb{C}^d$ that can obtained from the 34 primitive reflection groups. As already discussed, it is complete except for

Table 2
The highly symmetric tight frames of n vectors in $\mathbb{C}^d$, $3 \leqslant d \leqslant 8$ given by the primitive reflection groups ST 23–37.

ST	d	Order	n	b	s	Group frame
23	3	120	12 (P)	18	1	$\langle 12, 3\rangle$
			20 (P)	72	2	–
			30	300	4	–
24		336	42	120	3	$\langle 42, 2\rangle$
			56	450	4	–
25		648	27 (P)	27 (9)	1	$\langle 27, 3\rangle$, $\langle 27, 4\rangle$
			72	108	2	–
26		1296	54 (P)	54 (9)	1	$\langle 54, 8\rangle$, $\langle 54, 10\rangle$, $\langle 54, 11\rangle$
			72 (P)	108	2	–
			216	1350	4	$\langle 216, 88\rangle$
27		2160	216	1350	4	–
			270	1890	5	–
			360	9720	8_2	–
28	4	1152	24 (P)	80	2_2	$\langle 24, 1\rangle$, $\langle 24, 3\rangle$, $\langle 24, 11\rangle$
			96	9408	62	$\langle 96, 67\rangle$, $\langle 96, 201\rangle$, $\langle 96, 204\rangle$
29		7680	80	160	2	$\langle 80, 30\rangle$
			160	800	3	–
			320	7840	5_2	$\langle 320, 1581\rangle$, $\langle 320, 1586\rangle$
			640	251 680	10	–
30		14 400	120 (P)	1400	4	$\langle 120, 5\rangle$, $\langle 120, 15\rangle$
			600 (P)	1 109 760	15	$\langle 600, 54\rangle$
			720	3 032 400	18	–
			1200	78 330 560	32	–
31		46 080	240	800	3	–
			1920	145 200	9	$\langle 1920, *\rangle$
			3840	3 162 816	16	–
32		155 520	240 (P)	240 (40)	2	–
			2160	28 224	6	–
33	5	51 840	80	450	2	–
			270	450	2	–
			432	31 752	5	–
			1080	138 600	7	–
34	6	39 191 040	756	*	*	*
			*	*	*	*
35		51 840	27	441	2	$\langle 27, 3\rangle$, $\langle 27, 4\rangle$
			72	252	2	–
			216	213 444	6	$\langle 216, 86\rangle$, $\langle 216, 88\rangle$
			720	232 848	6	–
36	7	2 903 040	126	*	*	*
			*	*	*	*
37	8	696 729 600	240	*	*	*
			*	*	*	*

* These calculations were not possible due to the large size of the group.

the three groups with the largest orders, where only one frame is given. All calculations were exact, except for the determination of the number of angles in a frame, where for $z \in \mathrm{Ang}(\Phi)$ the numerical approximations

```
C := ComplexField(20);    a := Modulus(C!z);    a := ComplexField(17)!a;
```

to $a = |z|$ were compared.

The first three columns of each table gives the Shephard–Todd number of the group, its rank d, and its abstract group number (or order). The others give the following information about each of the highly symmetric tight frames which can be constructed from it:

- n: the number of vectors, (P) denotes that they are the vertices of a non-starry regular complex polytope.
- b: the bound of (2.7) on the number of vectors, (ℓ) denotes the number of lines when it is sharp.
- s: the number of angles in the frame, s_2 indicates there were two conjugacy classes of subgroups giving frames with exactly the same angles (we did not try to resolve whether these give unitarily equivalent frames or not).
- Group frame: whether the frame is a group frame via a subgroup H of the primitive group or not ($-$). If so, then the abstract group number of H is given. We note that none of these groups are abelian.

References

[1] H.U. Besche, B. Eick, E.A. O'Brien, A millennium project: constructing small groups, Internat. J. Algebra Comput. 12 (2002) 623–644.

[2] L. Bos, S. Waldron, Some remarks on Heisenberg frames and sets of equiangular lines, New Zealand J. Math. 36 (2007) 113–137.

[3] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997) 235–265.

[4] R. Brauer, H.S.M. Coxeter, A generalization of theorems of Schönhardt and Mehmke on polytopes, Trans. R. Soc. Canada Sect. III 34 (1940) 29–34.

[5] H. Broome, Highly symmetric tight frames (Honours project report), University of Auckland, 2010.

[6] A. Chebira, J. Kovačević, Life beyond bases: the advent of frames (Part I), IEEE Signal Process. Mag. 24 (2007) 86–104.

[7] T. Chien, S. Waldron, A classification of the harmonic frames up to unitary equivalence, Appl. Comput. Harmon. Anal. 30 (2011) 307–318.

[8] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[9] J.H. Conway, A simple construction for the Fischer–Griess monster group, Invent. Math. 79 (1985) 513–540.

[10] H.S.M. Coxeter, Regular Complex Polytopes, Cambridge University Press, Cambridge, 1991.

[11] P. Delsarte, J.M. Goethals, J.J. Seidel, Bounds for systems of lines, and Jacobi polynomials, Philips Res. Rep. 30 (1975).

[12] D. Han, Classification of finite group-frames and super-frames, Canad. Math. Bull. 50 (2007) 85–96.

[13] D. Han, K. Kornelson, D. Larson, E. Weber, Frames for Undergraduates, Amer. Math. Soc., Providence, RI, 2007.

[14] N. Hay, S. Waldron, On computing all harmonic frames of n vectors in $\mathbb{C}^d$, Appl. Comput. Harmon. Anal. 21 (2006) 168–181.

[15] G. James, M. Liebeck, Representations and Characters of Groups, Cambridge University Press, Cambridge, 1993.

[16] T.H. Koornwinder, A note on the absolute bound for systems of lines, Indag. Math. 38 (1976) 152–153.

[17] G.I. Lehrer, D.E. Taylor, Unitary Reflection Groups, Cambridge University Press, Cambridge, 2009.

[18] P. McMullen, E. Schulte, Abstract Regular Polytopes, Cambridge University Press, Cambridge, 2002.

[19] J.M. Renes, R. Blume-Kohout, A.J. Scott, C.M. Caves, Symmetric informationally complete quantum measurements, J. Math. Phys. 45 (2004) 2171–2180.

[20] E. Schulte, Symmetry of polytopes and polyhedra, in: J.E. Goodman, J. O'Rourke (Eds.), Handbook of Discrete and Computational Geometry, second ed., Chapman and Hall, Boca Raton, 2004, pp. 431–454.

[21] A.J. Scott, M. Grassl, SIC-POVMs: A new computer study, arXiv:0910.5784v2, 2010.

[22] G.C. Shephard, Regular complex polytopes, Proc. London Math. Soc. 2 (1952) 82–97.

[23] G.C. Shephard, Unitary groups generated by reflections, Canad. J. Math. 5 (1953) 364–383.

[24] G.C. Shephard, J.A. Todd, Finite unitary reflection groups, Canad. J. Math. 6 (1954) 274–304.

[25] R. Steinberg, Differential equations invariant under finite reflection groups, Trans. Amer. Math. Soc. 112 (1964) 392–400.

[26] D.E. Taylor, Reflection subgroups of finite complex reflection groups, J. Algebra 366 (2012) 218–234.

[27] R. Vale, S. Waldron, The vertices of the platonic solids are tight frames, in: M. Neamtu, E.B. Saff (Eds.), Advances in Constructive Approximation, Vanderbilt, 2003, Nashboro Press, Brentwood, TN, 2004, pp. 495–498.

[28] R. Vale, S. Waldron, Tight frames and their symmetries, Constr. Approx. 21 (2005) 83–112.

[29] R. Vale, S. Waldron, Tight frames generated by finite nonabelian groups, Numer. Algorithms 48 (2008) 11–28.

[30] R. Vale, S. Waldron, The symmetry group of a finite frame, Linear Algebra Appl. 433 (2010) 248–262.

[31] S. Waldron, Continuous and discrete tight frames of orthogonal polynomials for a radially symmetric weight, Constr. Approx. 30 (2009) 33–52.

[32] S. Waldron, An Introduction to Finite Tight Frames, Birkhäuser, Boston, 2014.