Technical Report
26 August 2010

A classification of the harmonic frames up to unitary equivalence

Tuan–Yow Chien and Shayne Waldron

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
e–mail: waldron@math.auckland.ac.nz

ABSTRACT

Up to unitary equivalence, there are a *finite* number of tight frames of n vectors for $\mathbb{C}^d$ which can be obtained as the orbit of a single vector under the unitary action of an abelian group G (for nonabelian groups there may be uncountably many). These so called *harmonic frames* (or *geometrically uniform tight frames*) have recently been used in applications including signal processing (where G is the cyclic group).

In an effort to find optimal harmonic frames for such applications, we seek a simple way to describe the unitary equivalence classes of harmonic frames. By using Pontryagin duality, we show that all harmonic frames of n vectors for $\mathbb{C}^d$ can be constructed from d–element subsets of G ($|G| = n$). We then show that in *most*, but *not all* cases, unitary equivalence preserves the group structure, and thus can be described in a simple way. This considerably reduces the complexity of determining whether harmonic frames are unitarily equivalent. We then give extensive examples, and make some steps towards a classification of all harmonic frames obtained from a cyclic group.

Key Words: signal processing, information theory, finite abelian groups, character theory, roots of unity, Pontryagin duality, group characters, group frames, Gramian matrices, tight frames, harmonic frames, geometrically uniform frames, automorphism groups

AMS (MOS) Subject Classifications: primary 11Z05, 20C15, 42C15, 94A12, secondary 11R18, 20F28, 65T60, 94A11, 94A15

1. Introduction

Recently, equal–norm (uniform) finite tight frames of n distinct vectors for $\mathbb{C}^d$ have found diverse applications (cf. [CK07]), including signal analysis, quantum information theory and multivariate orthogonal polynomials. A prominent class of such frames occurs in a number of guises:

- *Geometrically uniform* tight frames – which are the orbit of a single vector under the action of an abelian group of n unitary matrices [BE03].
- *Harmonic frames* – which are obtained as projections of the columns of the Fourier matrix of an abelian group of order n (cf. [GVT98], [CK03] for G cyclic).
- Tight *G–frames* for an abelian group G of order n [H07].

In [VW05] it was shown that these notions are equivalent – we will call such frames *harmonic frames*. Similar constructions have also appeared earlier in other contexts, e.g., as the vertices of polyhedra [H40] and as group codes [S68].

Since there are a *finite* number of abelian groups of order n, and a *finite* number of ways of selecting d rows of the character table of such a group, it follows there are a *finite* number of harmonic frames of n vectors for $\mathbb{C}^d$. The number of harmonic frames given by this construction is $\binom{n}{d} \approx n^d$, $n \to \infty$, times the number of abelian groups of order n. This is only an *upper bound* for the number of harmonic frames, since some of these may be unitarily equivalent. Further, it is reasonable to consider only those with distinct vectors, since those with repeated vectors are simply harmonic frames (for a smaller n) repeated a fixed number of times.

Computations in [HW06] indicate that the number of *unitarily inequivalent* harmonic frames of n *distinct* vectors for $\mathbb{C}^d$ grows like n^{d-1} (for d fixed). It also appears that the majority of these come from the cyclic group – we call these *cyclic harmonic frames* (cf. [K06]). The same harmonic frame may come from several nonisomorphic abelian groups.

It is the purpose of this paper to shed light on precisely *when* and *why* harmonic frames obtained from a character table are unitarily equivalent. The key idea is to use Pontryagin duality to observe that harmonic frames can be constructed by taking d–element subsets of an abelian group G (rather than by taking subsets of characters). Thus determining whether two harmonic frames from the same group are unitarily equivalent becomes a question about the relationship between d–element subsets of the group G. For *most*, but *not all* unitary equivalences there is a simple description in terms of subsets of G. This considerably reduces the complexity of determining which harmonic frames are unitarily equivalent. The exceptional cases are when the unitary equivalence does not preserve the group structure. We give extensive examples, and make some steps towards a classification of all cyclic harmonic frames. Ultimately, a full classification depends on knowing which sums of n–th roots of unity add to zero. This is an active area of research in number theory, e.g., the sum of all *primitive* n–th roots of unity is the *Möbius function* $\mu(n)$.

The rest of this paper is set out as follows. At the end of this section, we give the definitions required. Next we describe two equivalent ways of constructing all harmonic frames from the characters of an abelian group G. In Section 3, we describe the unitary equivalence of harmonic frames in terms of the subsets of G defining them. Then we give a complete description of the harmonic frames for $\mathbb{C}^1$ and $\mathbb{C}^2$. In Section 5, we consider $\mathbb{C}^3$, and the first examples of unitary equivalences which do not preserve the group structure. This is followed by some more general results motivated by these examples.

Basic definitions

A finite sequence of n vectors $(f_j)_{j=1}^n$ for a d–dimensional Hilbert space $\mathcal{H}$ over the field $\mathbb{F} = \mathbb{C}, \mathbb{R}$ is a **tight frame** if it has a *Parseval type expansion*

$$f = \frac{1}{A} \sum_{j=1}^{n} \langle f, f_j \rangle f_j, \qquad \forall f \in \mathcal{H},$$

where $A > 0$. By the polarisation identity, this is equivalent to the more familiar definition

$$A\|f\|^2 = \sum_{j=1}^{n} |\langle f, f_j \rangle|^2, \qquad \forall f \in \mathcal{H}.$$

The **Gramian** of such a sequence $\Phi = (f_j)_{j \in J}$ is the matrix

$$\mathrm{Gram}(\Phi) := [\langle f_k, f_j \rangle]_{j,k \in J}.$$

Tight frames $\Phi = (\phi_j)_{j \in J}$ and $\Psi = (\psi_k)_{k \in K}$ for $\mathcal{H}$ are **unitarily equivalent** if there is a bijection $\sigma : J \to K$, a unitary map U, and a $c > 0$ such that

$$\phi_j = cU\psi_{\sigma j}, \qquad \forall j \in J, \tag{1.1}$$

i.e., up to a reordering and rescaling of the vectors they have the same Gramian matrices

$$\mathrm{Gram}(\Phi) = |c|^2 P_\sigma^* \, \mathrm{Gram}(\Psi) P_\sigma, \tag{1.2}$$

where $P_\sigma : \mathbb{F}^J \to \mathbb{F}^K$ is the $K \times J$ permutation matrix given by $P_\sigma e_j := e_{\sigma j}$. Our counting of harmonic frames will be up to this unitary equivalence, which is an equivalence relation. There are a number of other coarser notions of equivalence in the literature, e.g., where c in (1.1) is replaced by c_j of constant modulus (cf. [F01], [GKK01] and [HP04]).

A tight frame $\Phi = (f_j)$ for $\mathbb{F}^d$ is **real** if all the entries of its Gramian are real. This is equivalent to the existence of a unitary map U with $U\Phi = (Uf_j) \subset \mathbb{R}^d$.

2. Character tables and Pontryagin duality

A (finite) tight frame Φ for $\mathcal{H}$ is **geometrically uniform** [BE03] if its vectors are the orbit of a single (nonzero) vector $v \in \mathcal{H}$ under the action of a finite abelian group G of unitary matrices, i.e., $\Phi = (gv)_{g \in G}$. Necessarily, such a frame has *distinct* vectors. More generally, Φ is a G–**frame** (cf. [H07]) if it has the form $\Phi = (\rho(g)v)_{g \in G}$, where $\rho : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a finite group G, i.e., a group homomorphism into the unitary maps on $\mathcal{H}$ (possibly not injective). There is also the unrelated notion of a generalised frame or g–frame (for short), which generalise fusion frames (cf. [S06]).

The Gramian matrix of a G–frame $\Phi = (\phi_g)_{g\in G}$ has a special (G–matrix) structure

$$\langle \phi_h, \phi_g\rangle = \langle \rho(h)\phi_1, \rho(g)\phi_1\rangle = \langle \rho(g)^*\rho(h)\phi_1, \phi_1\rangle = \langle \rho(g^{-1}h)\phi_1, \phi_1\rangle.$$

Thus each row and column of the Gramian has the same multiset of entries. We call this *multiset* minus the diagonal entry the **angle multiset** of the G–frame, and denote it

$$\begin{aligned}\mathrm{Ang}(\Phi) : &= \{\langle \phi_g, \phi_1\rangle : g \in G, g \neq 1\} \\ &= \text{multiset of off diagonal entries of any row/column of } \mathrm{Gram}(\Phi).\end{aligned}$$

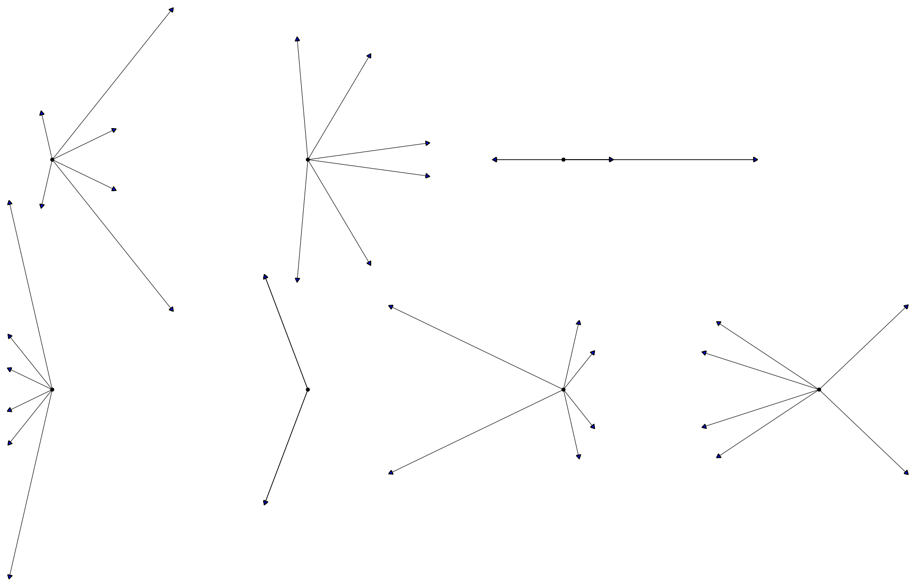

Fig. 1. The angle multisets of the inequivalent harmonic frames of 7 vectors in $\mathbb{C}^3$.

Clearly, unitarily equivalent G–frames have the *same* angle multisets (up to a positive scalar). Unfortunately, this is not enough to characterise them in general.

We now show how, for G abelian, all such G–frames can be constructed from the character table of G. The **character table** (or **Fourier matrix**) of an abelian group G of order n is the $n \times n$ matrix whose rows are the (irreducible) characters of G, i.e., maps $\chi : G \to \mathbb{C} \setminus \{0\}$ satisfying

$$\chi(g + h) = \chi(g)\chi(h), \qquad \forall g, h \in G. \tag{2.1}$$

It is well known (cf. [JL93]) that the set of characters, denoted by $\hat{G}$, forms a group (under pointwise multiplication) which is isomorphic to G, the values of a character are n–th roots of unity, and the characters (rows of the character table) are orthogonal, i.e.,

$$\langle \xi, \eta\rangle := \sum_{g\in G} \xi(g)\overline{\eta(g)} = \begin{cases} 0, & \xi \neq \eta; \\ n, & \xi = \eta \end{cases} \qquad \xi, \eta \in \hat{G}. \tag{2.2}$$

By (2.2), the character table is (a scalar multiple of) a unitary matrix. Since the projection of an orthonormal basis onto a subspace is a tight frame (Naimark's theorem), it follows that an equal–norm tight frame $(v_g)_{g\in G}$ is obtained by taking the columns of the submatrix of the character table given by a selection $\hat{J}\subset\hat{G}$ of rows (characters), i.e.,

$$v_g := (\xi(g))_{\xi\in\hat{J}},$$

This is a G–frame, since by (2.1),

$$v_g = \rho(g)v_1, \qquad \rho(g) := \operatorname{diag}(\xi(g))_{\xi\in\hat{J}}, \quad v_1 := (\xi(1))_{\xi\in\hat{J}}.$$

A frame unitarily equivalent to one given by this construction is called a **harmonic frame**, and a **cyclic harmonic frame** when G can be taken to be the cyclic group $\mathbb{Z}_n$. Cyclic harmonic frames appear in applications as early as [CK03]. It turns out that all G–frames for abelian G are harmonic frames.

Theorem 2.3 ([VW05:Th. 5.4]). *Let Φ be an equal–norm tight frame for $\mathcal{H}$. Then the following are equivalent*
(a) Φ *is a G–frame, where G is an abelian group.*
(b) Φ *is harmonic (obtained from the character table of G).*
For each Φ, G can be taken to be the same in (a) *and* (b)*, but it need not be unique.*

This implies that there is a *finite* number of harmonic frames of n vectors for $\mathbb{C}^d$ (up to unitary equivalence). By way of comparison, there may be *uncountably* many G–frames for G nonabelian (cf. [H07], [VW08]).

In the construction of harmonic frames, one might instead have selected *columns* of the character table, i.e., a subset $J\subset G$ (unitary matrices have orthogonal rows and columns) to obtain an equal–norm tight frame $(w_\xi)_{\xi\in\hat{G}}$, where

$$w_\xi := \xi|_J.$$

Again this is a G–frame, since $\hat{G}$ is isomorphic to G, and

$$w_\xi = \rho(\xi)w_1, \qquad \rho(\xi) := \operatorname{diag}(\xi|_J), \quad w_1 := 1|_J. \tag{2.4}$$

Further, by the *Pontryagin duality map* (canonical group isomorphism)

$$G\to\hat{\hat{G}} : g\mapsto\hat{\hat{g}}, \qquad \hat{\hat{g}}(\chi) := \chi(g), \quad \forall\chi\in\hat{G},\ g\in G.$$

we may write

$$v_g = (\xi(g))_{\xi\in\hat{J}} = (\hat{\hat{g}}(\xi))_{\xi\in\hat{J}} = \hat{\hat{g}}|_{\hat{J}}.$$

Thus $(v_g)_{g\in G}$ is the restriction of the elements of the character group to a subset of the group, and so the construction (2.4) gives all harmonic frames. This construction is the most convenient for us here, as G–frames are determined by subsets J of G rather than (the isomorphic group) $\hat{G}$. We will refer to

$$\Phi = (\xi|_J)_{\xi\in\hat{G}}$$

as the **harmonic frame given by** the subset J of the group G.

A harmonic frame $\Phi=(f_j)$ is said to be **unlifted** if $\sum_j f_j = 0$, otherwise it is **lifted**. The conditions on J for such a harmonic frame to have *distinct* vectors, to be *real*, and to be *lifted* are as follows.

Theorem 2.5. *Let G be an abelian group of order n, and $\Phi = \Phi_J = (\xi|_J)_{\xi\in\hat{G}}$ be the harmonic frame of n vectors for $\mathbb{C}^d$ given by a choice $J \subset G$, where $|J| = d$. Then*
(a) *Φ has distinct vectors if and only if J generates G.*
(b) *Φ is a real frame if and only J is closed under taking inverses.*
(c) *Φ is a lifted frame if and only if the identity is an element of J.*

Proof: (a) Let H be the subgroup generated by J. Then Φ has distinct vectors if and only if the composition of maps $\hat{G} \mapsto \hat{H} \mapsto \mathbb{C}^J : \xi \mapsto \xi|_H \mapsto \xi|_J$ is 1–1. Since each $h \in H$ can be written as a sum of elements in J, and ξ is a character, $\xi(h)$ is determined by $\xi|_J$, and so $\xi|_H \mapsto \xi|_J$ is 1–1. Hence $\xi \mapsto \xi|_J$ is 1–1 if and only if the group homomorphism given by $\hat{G} \mapsto \hat{H} : \xi \mapsto \xi|_H$ is 1–1, i.e., $\hat{G} = \hat{H}$, and so $G = H = \langle J\rangle$.

(b) The frame Φ is real if and only if its multiset of angles is real, i.e.,

$$\sum_{j\in J} \xi(j) = \sum_{j\in J} \hat{\hat{j}}(\xi) \in \mathbb{R}, \quad \forall \xi \in \hat{G} \quad \iff \quad \psi := \sum_{j\in J} \hat{\hat{j}} \in \mathbb{R}^{\hat{G}}.$$

Suppose that J is closed under taking inverses, and $j \in J$. Then either j is its own inverse, so $\xi(j) = \xi(-j) = \overline{\xi(j)} \in \mathbb{R}$, or $j, -j \in J$, so they contribute $\xi(j)+\xi(-j) = \xi(j)+\overline{\xi(j)} \in \mathbb{R}$ to the sum for the angle. Thus we conclude each angle is real. Conversely, suppose the multiset of angles is real, so that $\overline{\psi} = \psi$. Let $\langle \zeta, \chi\rangle$ be the inner product on $\mathbb{C}^{\hat{G}}$ for which the characters of $\hat{G}$ are orthogonal, i.e., $\langle \zeta, \chi\rangle := \frac{1}{|\hat{G}|}\sum_{\xi\in\hat{G}} \zeta(\xi)\overline{\chi(\xi)}$. Then

$$j \in J \quad \iff \quad \langle \psi, \hat{\hat{j}}\rangle = 1 \quad \iff \quad \langle \overline{\psi}, \overline{\hat{\hat{j}}}\rangle = \langle \psi, \hat{\hat{(-j)}}\rangle = 1 \quad \iff \quad -j \in J.$$

(c) By the orthogonality relations for characters, Φ is unlifted if and only if

$$\sum_{\xi\in\hat{G}} \xi|_J = 0 \quad \iff \quad \sum_{\xi\in\hat{G}} \xi(j) = \sum_{\xi\in\hat{G}} \xi(j)\overline{\xi(1)} = 0, \quad \forall j \in J \quad \iff \quad j \neq 1, \quad \forall j \in J.$$

□

Corollary 2.6. *Let G be a finite abelian group, and d^* be the minimum number of generators for G. Then there is a G–frame of distinct vectors for $\mathbb{C}^d$ if and only if $d \geq d^*$.*

Example 2.7. Let G be an *elementary abelian group*, i.e., $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k times), where p is prime. Then G gives harmonic frames of distinct vectors for $\mathbb{C}^d$ only for $d \geq k$ ($d^* = k$ since $0 \neq g \in G$ has order p).

Example 2.8. In $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ all nonzero elements have order 2, and so are equal to their inverse. Thus all harmonic frames given by this group are real. Alternatively, observe this condition on the group element orders implies that all the characters are real.

3. Unitary equivalence and preservation of the group structure

Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G, i.e., isomorphisms $\sigma : G \to G$.

Definition 3.1. *We say G–frames $\Phi = (\phi_g)_{g\in G}$, $\Psi = (\psi_g)_{g\in G}$ are* **unitarily equivalent via an automorphism** *if the map $\sigma : G \to G$ in* (1.1) *can be taken to be in* $\mathrm{Aut}(G)$.

In most, but not all cases (see §5) unitary equivalence of G–frames occurs via an automorphism.

Example 3.2. If G–frames Φ and Ψ are equal, then the set of permutations σ in the unitary equivalences (1.1) between them form a group called the symmetry group of Φ [VW10]. This group, denoted by $\mathrm{Sym}(\Phi)$, contains a subgroup of order $|G|$ consisting of the permutations

$$\sigma : g \mapsto hg, \qquad h \in G,$$

with only the identity being an automorphism of G. From this, it follows that if σ can be chosen to be an automorphism, then there are also choices which are not in $\mathrm{Aut}(\sigma)$.

We now give a simple condition which ensures harmonic frames are unitarily equivalent via an automorphism.

Definition 3.3. *We say subsets J and K of a finite abelian group G are* **multiplicatively equivalent** *if there is an automorphism $\sigma : G \to G$ for which $K = \sigma J$.*

Example 3.4. For $G = \mathbb{Z}_n$, each $\sigma \in \mathrm{Aut}(G)$ has the form $g \mapsto ag$, with $a \in \mathbb{Z}_n^*$ a unit, and hence J and K are multiplicatively equivalent if and only if $K = aJ$ for some $a \in \mathbb{Z}_n^*$.

Multiplicative equivalence is an equivalence relation, with the equivalence classes being the orbits of the natural action of $\mathrm{Aut}(G)$ on the d–element subsets of G.

Theorem 3.5. *Suppose J and K are subsets of a finite abelian group G. Then the following are equivalent*
(a) *The subsets J and K are multiplicatively equivalent.*
(b) *The harmonic frames given by J and K are unitarily equivalent via an automorphism.*

Proof: (a)$\Longrightarrow$(b): Suppose that $K = \sigma J$, where $\sigma \in \mathrm{Aut}(G)$. The natural action of $\mathrm{Aut}(G)$ on $\hat{G}$, which is given by

$$\sigma\chi = \hat{\sigma}\chi := \chi \circ \sigma^{-1}, \qquad \sigma \in \mathrm{Aut}(G), \quad \chi \in \hat{G},$$

induces automorphisms of $\hat{G}$, since

$$\hat{\sigma}(\xi\eta) = (\xi\eta) \circ \sigma^{-1} = (\xi \circ \sigma^{-1})(\eta \circ \sigma^{-1}) = (\hat{\sigma}\xi)(\hat{\sigma}\eta), \qquad \xi, \eta \in \hat{G}.$$

Using $\chi(j) = (\chi \circ \sigma^{-1})(\sigma j) = \hat{\sigma}\chi(\sigma j)$, we calculate

$$\langle \xi|_J, \eta|_J \rangle = \sum_{j\in J} \xi(j)\overline{\eta(j)} = \sum_{j\in J} \hat{\sigma}\xi(\sigma j)\overline{\hat{\sigma}\eta(\sigma j)} = \sum_{k\in K} \hat{\sigma}\xi(k)\overline{\hat{\sigma}\eta(k)} = \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle.$$

Hence, by the condition (1.2), the $\hat{G}$–frames $(\xi|_J)_{\xi\in\hat{G}}$ and $(\xi|_K)_{\xi\in\hat{G}}$ are unitarily equivalent via the automorphism $\hat{\sigma}:\hat{G}\to\hat{G}:\chi\mapsto\chi\circ\sigma^{-1}$.

(b)$\Longrightarrow$(a): Suppose the harmonic frames given by $J,K\subset G$ are unitarily equivalent via an isomorphism $\hat{\sigma}:\hat{G}\to\hat{G}$, i.e.,

$$\langle\xi|_J,\eta|_J\rangle=\langle\hat{\sigma}\xi|_K,\hat{\sigma}\eta|_K\rangle,\qquad\forall\xi,\eta\in\hat{G}.$$

Taking $\eta=1$, the trivial character, above, gives

$$\sum_{j\in J}\xi(j)=\sum_{k\in K}(\hat{\sigma}\xi)(k),\qquad\forall\xi\in\hat{G}.\tag{3.6}$$

We now seek to define an automorphism $\sigma=\tau^{-1}:G\to G$ satisfying

$$(\hat{\sigma}\chi)(g)=(\chi\circ\sigma^{-1})(g),\qquad\forall\chi\in\hat{G},\quad\forall g\in G.$$

Since $\hat{\sigma}:\hat{G}\to\hat{G}$ is an automorphism, $\chi\mapsto\hat{\sigma}\chi(g)$ belongs to $\hat{\hat{G}}$, and so we can use Pontryagin duality to define τg by

$$(\tau g)\hat{}\,\hat{}(\chi):=\hat{\sigma}\chi(g),\qquad\forall\chi\in\hat{G}.$$

This map $\tau:G\to G$ is a bijection, since

$$\begin{aligned}\tau g=\tau h\quad&\Longleftrightarrow\quad\hat{\sigma}\chi(g)=\hat{\sigma}\chi(h),\quad\forall\chi\in\hat{G}\quad\Longleftrightarrow\quad\hat{\hat{g}}(\hat{\sigma}\chi)=\hat{\hat{h}}(\hat{\sigma}\chi),\quad\forall\chi\in\hat{G}\\&\Longleftrightarrow\quad\hat{\hat{g}}(\xi)=\hat{\hat{h}}(\xi),\quad\forall\xi\in\hat{G}\quad\Longleftrightarrow\quad\hat{\hat{g}}=\hat{\hat{h}}\quad\Longleftrightarrow\quad g=h,\end{aligned}$$

and it is a homomorphism since

$$\begin{aligned}\hat{\sigma}\xi\in\hat{G}\quad&\Longrightarrow\quad(\hat{\sigma}\xi)(g+h)=(\hat{\sigma}\xi)(g)(\hat{\sigma}\xi)(h),\quad\forall\xi\in\hat{G}\\&\Longleftrightarrow\quad(\tau(g+h))\hat{}\,\hat{}(\xi)=(\tau g)\hat{}\,\hat{}(\xi)(\tau h)\hat{}\,\hat{}(\xi),\quad\forall\xi\in\hat{G}\\&\Longleftrightarrow\quad(\tau(g+h))\hat{}\,\hat{}=(\tau g)\hat{}\,\hat{}(\tau h)\hat{}\,\hat{}\quad\Longleftrightarrow\quad\tau(g+h)=\tau g+\tau h\end{aligned}$$

(where we write the group operation in $\hat{\hat{G}}$ as $\cdot$). Thus $\sigma:=\tau^{-1}\in\mathrm{Aut}(G)$, which satisfies

$$(\hat{\sigma}\xi)(k)=(\sigma^{-1}k)\hat{}\,\hat{}(\xi),$$

Hence, by Pontryagin duality, (3.6) gives

$$\sum_{j\in J}\hat{\hat{j}}(\xi)=\sum_{k\in K}(\sigma^{-1}k)\hat{}\,\hat{}(\xi),\quad\forall\xi\in\hat{G}\quad\Longrightarrow\quad\sum_{j\in J}\hat{\hat{j}}=\sum_{k\in K}(\sigma^{-1}k)\hat{}\,\hat{}.$$

Since characters of a finite abelian group are linearly independent, we conclude

$$\{\hat{\hat{j}}:j\in J\}=\{(\sigma^{-1}k)\hat{}\,\hat{}:k\in K\}\quad\Longrightarrow\quad\{j:j\in J\}=\{\sigma^{-1}k:k\in K\}\quad\Longrightarrow\quad K=\sigma J,$$

i.e., J and K are multiplicatively equivalent subsets of G. $\square$

The number of multiplicative equivalence classes of d–element subsets of a group G which generate G is essentially Hall's *Eulerian function* $\Phi_d(G)$, which counts the ordered d–element generating subsets of G [H36].

A simple instance where multiplicative inequivalence of subsets implies the unitary inequivalence of the harmonic frames they give is when their angle multisets differ (see §6). These observations, together with Theorem 3.5, considerably reduce the calculations required to determine whether harmonic frames are unitarily equivalent (cf. [HW06]).

Example 3.7. *Four vectors in* $\mathbb{C}^2$. First consider $G = \mathbb{Z}_4$. The automorphism group has order 2, generated by $\sigma : g \mapsto 3g$ ($\mathbb{Z}_4^* = \{1, 3\}$). Thus the multiplicative equivalence classes of 2–element subsets of G, which are the orbits under the action of $\mathrm{Aut}(G)$, are

$$\{\{0,1\},\{0,3\}\}, \qquad \{\{1,2\},\{2,3\}\}, \qquad \{\{1,3\}\}, \qquad \{\{0,2\}\}.$$

The first three give cyclic harmonic frames with distinct vectors (since 1 generates G), while the last does not. None are unitarily equivalent, since the (respective) angle multisets are

$$\{-i+1,0,i+1\}, \qquad \{0,-i-1,i-1\}, \qquad \{0,0,-2\}, \qquad \{0,0,2\}.$$

Now consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is generated by any two of its three elements $\{a, b, a+b\}$ of order 2. The automorphism group has order 6, with an automorphism corresponding to each permutation of $\{a, b, a+b\}$. Thus the multiplicative equivalence classes are

$$\{\{a,b\},\{a,a+b\},\{b,a+b\}\}, \qquad \{\{0,a\},\{0,b\},\{0,a+b\}\}.$$

Only the first gives a harmonic frame with distinct vectors. This real frame has angles $\{0, 0, -2\}$, and is unitarily equivalent to the cyclic harmonic frame with these angles.

Example 3.8. *Seven vectors in* $\mathbb{C}^3$. For $G = \mathbb{Z}_7$, the seven multiplicative equivalence classes have representatives

$$\{1,2,6\},\ \{1,2,3\},\ \{0,1,2\},\ \{0,1,3\},\ \{1,2,5\} \quad \text{(size 6)}$$

$$\{0,1,6\} \quad \text{(size 3)} \qquad \{1,2,4\} \quad \text{(size 2)}.$$

Each gives a cyclic harmonic frame of distinct vectors (as nonzero elements generate G). None of these are unitarily equivalent since their angle multisets are different (see Fig. 1).

A finite abelian group G can be written as a direct sum of p–groups

$$G_p = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_m}}$$

where p are the prime divisors of $|G|$. The automorphism group of G_p has order

$$|\mathrm{Aut}(G_p)| = \prod_{k=1}^{m}(p^{d_k} - p^{k-1}) \prod_{j=1}^{m}(p^{e_j})^{m-d_j} \prod_{i=1}^{m}(p^{e_i-1})^{m-c_i+1}, \tag{3.9}$$

where

$$c_k := \min\{r : e_r = e_k\} \leq k, \qquad d_k := \max\{r : e_r = e_k\} \geq k,$$

and so the order of $\mathrm{Aut}(G)$ is the product of these orders (cf. [HR07]). In effect, the less cyclic an abelian group is, the larger its automorphism group becomes. This gives a heuristic explanation for the observation of [HW06] that most harmonic frames are cyclic, with increasingly fewer as G becomes less cyclic, via the following mechanisms:

- As G becomes less cyclic, $|\mathrm{Aut}(G)|$ becomes larger, and so the number of multiplicative equivalence classes becomes smaller.
- As G becomes less cyclic, the orders of its elements become smaller, so $J \subset G$ is less likely to generate G, and hence give a harmonic frame with distinct vectors.

4. Cyclic frames in $\mathbb{C}^1$ and $\mathbb{C}^2$

There is just one harmonic frame of n distinct vectors for $\mathbb{C}^1$.

Proposition 4.1. *There is a unique harmonic frame of n distinct vectors for $\mathbb{C}^1$, namely the cyclic harmonic frame given by the n–th roots of unity.*

Proof: Use Theorems 2.5 and 3.5. If g generates an abelian group G of order n, then G must be $\mathbb{Z}_n$. If g_1,g_2 generate $\mathbb{Z}_n$, then $\{g_1\}$,$\{g_2\}$ are multiplicatively equivalent (as $g_1 \mapsto g_2$ gives an automorphism of G), and so give unitarily equivalent frames. $\square$

From this, we deduce there is a unique lifted harmonic frame of n vectors for $\mathbb{C}^2$, namely the cyclic harmonic frame given by the subset $J = \{0, g\}$, where $\mathbb{Z}_n = \langle g \rangle$.

The angle multiset of the cyclic harmonic frame for $\mathbb{C}^2$ given by $\{j_1, j_2\} \subset \mathbb{Z}_n$ is

$$\{\omega^{aj_1} + \omega^{aj_2} : a \in \mathbb{Z}_n, a \neq 0\}, \qquad \omega := e^{\frac{2\pi i}{n}}.$$

We now show that if 2–element subsets of $\mathbb{Z}_n$ are multiplicatively inequivalent, then the angle multisets of the harmonic frames that they give are not equal, and hence give unitarily inequivalent cyclic harmonic frames. To find an angle in one but not the other, we need to understand which sums of n–th roots of unity are zero.

Lemma 4.2. *Suppose that z_1, z_2, w_1, w_2 are unit modulus complex numbers. Then*

$$z_1 + z_2 = w_1 + w_2 \neq 0 \quad \Longrightarrow \quad \{z_1, z_2\} = \{w_1, w_2\}.$$

Lemma 4.3. *Let $\omega = e^{\frac{2\pi i}{n}}$. If $\omega^{j_1} + \omega^{j_2} = 0$, $j_1, j_2 \in \mathbb{Z}_n$, then n is even, and*

$$\omega^{aj_1} + \omega^{aj_2} = \begin{cases} 0, & a \text{ odd;} \\ 2\omega^{aj_1}, & a \text{ even.} \end{cases}$$

Recall the cyclic group $\mathbb{Z}_n$ has a unique cyclic subgroup of each order dividing n, and no other subgroups. Thus, if $j_1, j_2 \in \mathbb{Z}_n$ have the same order, then they generate the same subgroup, i.e.,

$$\mathrm{ord}(j_1) = \mathrm{ord}(j_2) \quad \Longleftrightarrow \quad \langle j_1 \rangle = \langle j_2 \rangle.$$

We will also repeatedly use the facts

$$\text{ord}(aj) \le \text{ord}(j), \quad \forall a \in \mathbb{Z},\ j \in \mathbb{Z}_n, \qquad \text{ord}(b) = n \Longleftrightarrow b \in \mathbb{Z}_n^*. \tag{4.4}$$

Theorem 4.5. *Cyclic frames of n distinct vectors for $\mathbb{C}^2$ are unitarily equivalent if and only if the subsets of $\mathbb{Z}_n$ that give them are multiplicatively equivalent.*

Proof: Suppose the subsets $\{j_1, j_2\}$ and $\{k_1, k_2\}$ of $\mathbb{Z}_n$ are not multiplicatively equivalent, and give harmonic frames of distinct vectors, i.e., $\langle j_1, j_2\rangle = \langle k_1, k_2\rangle = \mathbb{Z}_n$. We will show that the cyclic harmonic frames they give have different angle multisets, and so are not unitarily equivalent. Since multiplicatively equivalent subsets give the same angle multisets, it suffices to consider the following cases.

Case (a): $\omega^{j_1} + \omega^{j_2} \neq 0$. By Lemma 4.2, if this angle is appears in the second frame as $\omega^{bk_1} + \omega^{bk_2}$, $b \in \mathbb{Z}_n$, then $\{j_1, j_2\} = \{bk_1, bk_2\}$. Since the frames are not multiplicatively equivalent, we must have $b \notin \mathbb{Z}_n^*$, and hence $\langle b\rangle \neq \mathbb{Z}_n$. But this implies $\langle j_1, j_2\rangle = \langle bk_1, bk_2\rangle \subset \langle b\rangle \neq \mathbb{Z}_n$, and so $\omega^{j_1} + \omega^{j_2}$ cannot be an angle in the second frame.

Case (b): $\omega^{aj_1} + \omega^{aj_2} = \omega^{bk_1} + \omega^{bk_2} = 0$, $\forall a, b \in \mathbb{Z}_n^*$. Suppose first that there is a unit in each of the subsets. Then by going to multiplicatively equivalent subsets, we may assume that $j_1 = k_1 = 1$, and thus obtain $\omega + \omega^{j_2} = 0 = \omega + \omega^{k_2}$, which gives $j_2 = k_2$, and so the two subsets are equal. Thus we may assume that $j_1, j_2 \notin \mathbb{Z}_n^*$. By Lemma 4.3, n is even, and the nonzero angles of the first frame are $\{2\omega^{2kj_1} : 1 \le k \le \frac{n}{2}\} = \{2\omega^{2kj_2} : 1 \le k \le \frac{n}{2}\}$, and we conclude $\langle 2j_1\rangle = \langle 2j_2\rangle$. Since j_1, j_2 are not units, they cannot have the same order (and generate $\mathbb{Z}_n$), and so we can assume that $\text{ord}(j_1) < \text{ord}(j_2)$. The group $\langle 2j_1\rangle$ is either equal to $\langle j_1\rangle$, or has half its order, and similarly for $\langle j_2\rangle$. Thus the only way to have $\langle 2j_1\rangle = \langle 2j_2\rangle$ is for $\langle j_1\rangle = \langle 2j_1\rangle$, in which case $j_1 \in \langle 2j_2\rangle \subset \langle j_2\rangle$, and $\langle j_1, j_2\rangle = \langle j_2\rangle \neq \mathbb{Z}_n$. We conclude that case (b) can never occur. □

A careful reading of the proof shows that if $\omega^{j_1} + \omega^{j_2} \neq 0$, then

$$\{\omega^{aj_1} + \omega^{aj_2} \neq 0 : a \in \mathbb{Z}_n^*\} \tag{4.6}$$

is a set of nonzero angles, which is unique to frame given by $\{j_1, j_2\}$ (or any multiplicatively equivalent subset), and that for n even, there is a unique frame in which the angles given by (4.6) are all zero, namely that given by $\{1, 1 + \frac{n}{2}\}$.

Not all harmonic frames for $\mathbb{C}^2$ are cyclic. We now give a detailed description of the first example: a complex frame of $n = 8$ vectors obtained from $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. This also serves to illustrate the angle set (4.6).

Example 4.7. *A noncyclic harmonic frame in $\mathbb{C}^2$.* There a seven unitarily inequivalent cyclic harmonic frames of $n = 8$ distinct vectors for $\mathbb{C}^2$. We now list them, giving a representative of the multiplicative equivalence class they correspond to, followed by the 4 angles given by (4.6) – note these are unique, and then the remaining 3 angles.

$\{0,1\}$	$1+\omega, 1+\omega^3, 1+\omega^5, 1+\omega^7$	$1+\omega^2, 1+\omega^4=0, 1+\omega^6$
$\{1,2\}$	$\omega+\omega^2, \omega^3+\omega^6, \omega^5+\omega^2, \omega^7+\omega^6$	$\omega^2+\omega^4, \omega^4+1=0, \omega^6+\omega^4$
$\{1,3\}$	$\omega+\omega^3, \omega^5+\omega^7$ (twice)	$\omega^2+\omega^6=0, \omega^4+\omega^4, \omega^6+\omega^2=0$
$\{1,4\}$	$\omega+\omega^4, \omega^3+\omega^4, \omega^5+\omega^4, \omega^7+\omega^4$	$\omega^2+1, \omega^4+1=0, \omega^6+1$
$\{1,5\}$	$\omega+\omega^5=\omega^3+\omega^7=0$ (twice)	$\omega^2+\omega^2, \omega^4+\omega^4, \omega^6+\omega^6$
$\{1,6\}$	$\omega+\omega^6, \omega^3+\omega^2, \omega^5+\omega^6, \omega^7+\omega^2$	$\omega^2+\omega^4, \omega^4+1=0, \omega^6+\omega^4$
$\{1,7\}$	$\omega+\omega^7, \omega^3+\omega^5$ (twice)	$\omega^2+\omega^6=0, \omega^4+\omega^4, \omega^6+\omega^2$

There are two harmonic frames of distinct vectors given by the group $G=\mathbb{Z}_4\times\mathbb{Z}_2$. Here is a representative subset giving them, followed by the angle multiset.

$\{(0,1),(1,0)\}$	$0,0,1+\omega^2,1+\omega^6,\omega^2+\omega^4,\omega^4+\omega^4,\omega^6+\omega^4$
$\{(1,0),(1,1)\}$	$0,0,0,0,\omega^2+\omega^2,\omega^4+\omega^4,\omega^6+\omega^6$

The last of these has the same angles as the cyclic harmonic frame given by $\{1,5\}$, and it is easy to check that it is unitarily equivalent to it. The angle multiset of the first is not shared by any cyclic harmonic frame, and so is an example of a noncyclic harmonic frame. This noncyclic harmonic frame $(\xi|_J)_{\xi\in\hat{G}}$ for $J=\{(0,1),(1,0)\}$ is

$$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}i\\1\end{bmatrix},\begin{bmatrix}i\\-1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}-i\\1\end{bmatrix},\begin{bmatrix}-i\\-1\end{bmatrix}\right\}.$$

Here is a list of the numbers noncyclic harmonic frames determined by our calculations.

n	non	cyc
4	0	3
8	1	7
9	1	6
12	2	13
16	4	13
18	2	18
20	3	19
24	6	27
25	1	15
27	3	18
28	4	25
32	9	25

n	non	cyc
4	0	3
8	5	16
9	3	15
12	11	57
16	28	74
18	19	121
20	29	137
24	89	241
25	8	115
27	33	159
28	57	255
32	158	278

n	non	cyc
4	0	1
8	8	21
9	5	23
12	30	141
16	139	228
18	80	494
20	154	622
24	604	1349
25	37	636
27	202	973
28	443	1697
32	1379	2152

Table 1. The numbers of inequivalent *non*cyclic, *cyc*lic harmonic frames of $n\leq 35$ distinct vectors for $\mathbb{C}^d$, $d=2,3,4$ when a nonabelian group of order n exists.

5. Unitary equivalence without preserving the group structure

Theorem 4.5 implies that unitary and multiplicative equivalence are the same for cyclic harmonic frames for $\mathbb{C}^3$, except if both frames are unlifted. In this case, there are subsets of $\mathbb{Z}_n$ which are multiplicatively inequivalent, and do give unitarily equivalent frames.

Example 5.1. ($n = 8$, $d = 3$). For $\mathbb{Z}_8$ there are 17 multiplicative equivalence classes of 3–element subsets which generate it. Only two of these give frames with the same angles, namely

$$\big\{\{1,2,5\},\{3,6,7\}\big\}, \qquad \big\{\{1,5,6\},\{2,3,7\}\big\}.$$

The common angle multiset is

$$\{-1, i, i, -i, -i, -2i-1, 2i-1\} \qquad (\omega^2 = i,\, \omega^4 = -1,\, \omega^6 = -i,\, \omega := e^{\frac{2\pi i}{8}}).$$

Notice here that in many of the angles $\omega^{aj_1} + \omega^{aj_2} + \omega^{aj_2}$, $a \neq 0$ there is cancellation, as outlined in Lemma 4.3. This explains why the angles multisets for multiplicatively inequivalent subsets can be the same. These frames are unitarily equivalent (to be proved next), but not via an automorphism.

Definition 5.2. *Let p be prime with $p^2 \mid n$, and define*

$$B_{p,n} := \mathbb{Z}_n^*\{b : 1 \le b < n,\, p^2 b \text{ divides } n \text{ in } \mathbb{Z}\} \subset \mathbb{Z}_n.$$

We observe that the subsets $B_{n,p}$ and $\frac{n}{p}\mathbb{Z}_n$ of $\mathbb{Z}_n$ are invariant under multiplication by units (by construction, and since $m\mathbb{Z}_n = \mathbb{Z}_n$, $m \in \mathbb{Z}_n^*$).

Lemma 5.3. *Let p be prime, $d = p + 1$, $n \ge d$ with $p^2 \mid n$, and*

$$A := \frac{n}{p}\mathbb{Z}_n + a = \Big\{a, \frac{n}{p} + a, \frac{2n}{p} + a, \ldots, (p-1)\frac{n}{p} + a\Big\}, \quad a \in \mathbb{Z}_n.$$

Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets $J, K \subset G = \mathbb{Z}_n$, defined by

$$J := A \cup \{b\}, \quad K := A \cup \Big\{b + r\frac{n}{p}\Big\}, \qquad b \in B_{p,n}, \quad b \notin A$$

are unitarily equivalent.

Proof: Since multiplicative equivalence of subsets implies the unitary equivalence of the frames they give (Theorem 3.5), we can multiply these subsets by some unit $m \in \mathbb{Z}_n^*$. This gives subsets of the same form since $mA = \frac{n}{p}\mathbb{Z}_n + ma$ and $mB_{p,n} = B_{p,n}$. Hence, by the definition of $B_{p,n}$, we can suppose without loss of generality that $p^2 b \mid n$ (in $\mathbb{Z}$).

We now show the harmonic frames $(\xi|_J)_{\xi\in\hat{G}}$ and $(\xi|_K)_{\xi\in\hat{G}}$ are unitarily equivalent. Let

$$\omega := e^{\frac{2\pi i}{n}}, \qquad \zeta = \omega^{\frac{n}{p}} = e^{\frac{2\pi i}{p}} \text{ (p–th root of unity)}, \qquad \chi(\ell) := \omega^{\ell}.$$

Then χ is a generator of $\hat{G}$, so $\xi, \eta \in \hat{G}$ can be written $\xi = \chi^j$, $\eta = \chi^k$, and we compute

$$\begin{aligned}\langle \xi|_J, \eta|_J \rangle &= \sum_{\alpha \in A} \xi(\alpha)\overline{\eta(\alpha)} + \xi(b)\overline{\eta(b)} \\ &= \omega^{aj}\overline{\omega^{ak}} + \ldots + \omega^{(a+(p-1)\frac{n}{p})j}\overline{\omega^{(a+(p-1)\frac{n}{p})k}} + \omega^{bj}\overline{\omega^{bk}} \\ &= \omega^{a(j-k)}\{1 + \zeta^{j-k} + \ldots + \zeta^{(p-1)(j-k)}\} + \omega^{b(j-k)},\end{aligned} \tag{5.4}$$

and similarly

$$\langle \xi|_K, \eta|_K \rangle = \omega^{a(j-k)}\{1 + \zeta^{j-k} + \ldots + \zeta^{(p-1)(j-k)}\} + \omega^{(b+r\frac{n}{p})(j-k)}. \tag{5.5}$$

Since $p^2 b \mid n$, we can define a permutation σ of $\mathbb{Z}_n$ by

$$\sigma j := j - r\frac{n}{pb}j^*, \qquad j^* := j \mod p,$$

and an associated permutation $\hat{\sigma}$ of $\hat{G}$ by

$$\hat{\sigma}(\chi^j) := \chi^{\sigma j}.$$

This σ is clearly a well defined map $G \to G$, and it is 1–1

$$\begin{aligned}\sigma j = \sigma k \quad &\Longrightarrow \quad j - r\frac{n}{pb}j^* = k - r\frac{n}{pb}k^* \quad \Longrightarrow \quad j \equiv k \mod p \quad (\text{since } p \text{ divides } r\tfrac{n}{pb}) \\ &\Longrightarrow \quad j^* = k^* \quad \Longrightarrow \quad j = k.\end{aligned}$$

We now show that $\hat{\sigma}$ gives a unitary equivalence, i.e., $\langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle = \langle \xi|_J, \eta|_J \rangle$, $\forall \xi, \eta \in \hat{G}$.

If $j - k \equiv 0 \mod p$, then $\sigma j - \sigma k = j - k$, so that (5.4) and (5.5) give

$$\langle \xi|_J, \eta|_J \rangle = p\omega^{a(j-k)} + \omega^{b(j-k)}, \qquad \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle = p\omega^{a(j-k)} + \omega^{(b+r\frac{n}{p})(j-k)},$$

which are equal, since $\omega^{r\frac{n}{p}(j-k)} = \zeta^{r(j-k)} = \zeta^0 = 1$.

Now consider $j - k \not\equiv 0 \mod p$. Since $\zeta^{j-k} \neq 1$ is a primitive p–th root of unity, the sums of p–th roots of unity in (5.4) and (5.5) vanish, and we obtain

$$\langle \xi|_J, \eta|_J \rangle = \omega^{b(j-k)}, \qquad \langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle = \omega^{(b+r\frac{n}{p})(\sigma j - \sigma k)} = \omega^{(b+r\frac{n}{p})(j-k-r\frac{n}{pb}(j^*-k^*))} =: \omega^c.$$

Since $bp^2 \mid n$, and $j^* - k^* = j - k + p\ell$, $\ell \in \mathbb{Z}$, we have

$$\begin{aligned}c &= (b + r\frac{n}{p})(j - k - r\frac{n}{pb}(j^* - k^*)) \equiv b(j - k - r\frac{n}{pb}(j^* - k^*)) + r\frac{n}{p}(j - k) \\ &\equiv b(j - k - r\frac{n}{pb}(j - k + p\ell)) + r\frac{n}{p}(j - k) \equiv (j - k)\{b - r\frac{n}{p} + r\frac{n}{p}\} \equiv b(j - k) \mod n.\end{aligned}$$

Hence $\langle \hat{\sigma}\xi|_K, \hat{\sigma}\eta|_K \rangle = \omega^c = \omega^{b(j-k)} = \langle \xi|_J, \eta|_J \rangle$. $\square$

Lemma 5.6. *Let p be prime, $d = p+1$, $n \geq d$ with $p^2 \mid n$, $A = \frac{n}{p}\mathbb{Z}_n + a$, where $a \in \{1, \ldots, p-1\}$, and $b \in \mathbb{Z}$ with $p \mid b$. Then the cyclic harmonic frames for $\mathbb{C}^d$ given by the subsets*

$$J_r := A \cup \{b + r\frac{n}{p}\}, \quad r \in \{0, 1, \ldots, p-1\}$$

are not multiplicatively equivalent.

Proof: Suppose, by way of contradiction, that the subsets J_{r_1} and J_{r_2}, $r_1 \neq r_2$ are multiplicatively equivalent. i.e., $mJ_{r_1} = J_{r_2}$, $m \in \mathbb{Z}_n^*$. Since $mA = \frac{n}{p}\mathbb{Z}_n + ma$, this implies

$$ma = a + s\frac{n}{p}, \qquad m(b + r_1\frac{n}{p}) = b + r_2\frac{n}{p}.$$

Thus

$$(a + s\frac{n}{p})(b + r_1\frac{n}{p}) = ma(b + r_1\frac{n}{p}) = a(m(b + r_1\frac{n}{p})) = a(b + r_2\frac{n}{p})$$

which gives

$$s\frac{b}{p}n = a(r_2 - r_1)\frac{n}{p} - s\frac{n}{p^2}r_1 n \quad \Longrightarrow \quad 0 \equiv a(r_2 - r_1)\frac{n}{p} \not\equiv 0 \mod n,$$

a contradiction. Therefore J_{r_1} and J_{r_2} are not multiplicatively equivalent. □

We can now prove a very general form of Example 5.1.

Theorem 5.7. *Suppose $p^3 \mid n$, where p is prime. Then the p subsets of $\mathbb{Z}_n$*

$$J_r := \{1, \frac{n}{p} + 1, \ldots (p-1)\frac{n}{p} + 1\} \cup \{p + r\frac{n}{p}\}, \quad r \in \{0, 1, \ldots, p-1\}$$

give cyclic harmonic frames of n distinct vectors for $\mathbb{C}^{p+1}$ which are unitarily equivalent, but not via an automorphism.

Proof: Let $a = 1$, $b = p$ in Lemmas 5.3 and 5.6. Since $p^2 b = p^3 \mid n$, $b \in B_{pn}$, and Lemma 5.3 implies the subsets J_r give unitarily equivalent frames. Since $p \mid b$, Lemma 5.6 implies these subsets are not multiplicatively equivalent, and so the cyclic harmonic frames they give are not unitarily equivalent via an automorphism (Theorem 3.5). Finally, since $1 \in J_r$ generates $\mathbb{Z}_n$, the frame given by J_r has distinct vectors (Theorem 2.5). □

For $p = 2$, $n = 8$, $J_0 = \{1, 5\} \cup \{2\}$, $J_1 = \{1, 5\} \cup \{6\}$, and we have Example 5.1. Our computations suggest that for $\mathbb{C}^3$ the only cases where multiplicatively inequivalent subsets give unitarily equivalent cyclic harmonic frames are those of Theorem 5.7. For $\mathbb{C}^4$ there are examples not covered by Theorem 5.7. We now give indicative examples (see [C10] for further detail).

Example 5.8. ($n = 8$, $d = 4$) We can 'lift' the Example 5.1, i.e., add 0 to each subset to obtain

$$\{\{0, 1, 2, 5\}, \{0, 3, 6, 7\}\}, \qquad \{\{0, 1, 5, 6\}, \{0, 2, 3, 7\}\}.$$

These are still multiplicative equivalence classes, since $m0 = 0$, $m \in \mathbb{Z}_n$, and by the same reasoning are not multiplicatively equivalent. They still give the same angles, since the angle $\theta = \omega^{aj_1} + \omega^{aj_2} + \omega^{aj_3}$ transforms to $\omega^0 + \omega^{aj_1} + \omega^{aj_2} + \omega^{aj_3} = 1 + \theta$, and they are unitarily equivalent since

$$\langle \xi|_{J\cup\{0\}}, \eta|_{J\cup\{0\}} \rangle = \sum_{j\in J} \xi(j)\overline{\eta(j)} + \xi(0)\overline{\eta(0)} = \langle \xi|_J, \eta|_J \rangle + 1.$$

Example 5.9. ($n = 9$, $d = 4$) For $\mathbb{Z}_9$, the following multiplicative equivalence classes of 4–element subsets give cyclic frames with the same angles

$$\big\{\{1,4,6,7\},\{2,3,5,8\}\big\}, \qquad \big\{\{1,3,4,7\},\{2,5,6,8\}\big\}.$$

The common angle multiset is

$$\{\omega^3,\omega^3,\omega^3,\omega^6,\omega^6,\omega^6,1+3\omega^3,1+3\omega^6\}, \qquad \omega := e^{\frac{2\pi i}{9}}.$$

By a similar argument to that of Theorem 5.7, it can be verified that the frames these give are unitarily equivalent (but not via an automorphism). Here the permutation σ is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 1 & 3 & 2 & 4 & 6 & 5 & 7 & 0 & 8 \end{pmatrix} \qquad \text{(for } \{1,4,6,7\} \text{ and } \{1,3,4,7\}\text{)}.$$

The *exceptional* cases given in this section, of multiplicatively inequivalent subsets which give unitarily equivalent harmonic frames, hinge on certain sums of n–th roots of unity vanishing. This question, the vanishing of sums of n–th roots, is an active area of number theory research (cf. [M65], [CJ76], [LL00]). Clearly, a complete classification of all cyclic harmonic frames using the techniques outlined here is intimately related to this as yet unresolved question. More details are given in the thesis [C10].

6. A family of cyclic harmonic frames

We now describe a family of cyclic harmonic frames for which unitary equivalence is the same as multiplicative equivalence of the subsets which give them. This is essentially a general form of the argument of case (a) in the proof of Theorem 4.5.

Let θ be the angle map on d–element subsets of $\mathbb{Z}_n$ given by

$$\theta(J) := \sum_{j\in J} \omega^j, \qquad \omega = e^{\frac{2\pi i}{n}}.$$

Proposition 6.1. *Let $\mathcal{C}_d$ be the collection of d–element subsets of $\mathbb{Z}_n$ given by*

$$\mathcal{C}_d := \{J : \theta^{-1}(\theta(J)) = J\}.$$

If $J \in \mathcal{C}_d$, then J and K give unitarily equivalent cyclic harmonic frames of distinct vectors if and only if they are multiplicatively equivalent subsets.

Proof: ($\Longrightarrow$) Suppose, by way of contradiction, that J and K are not multiplicatively equivalent. Then the angle $\theta(J) = \sum_{j\in J} \omega^j$ in the frame given by J is in the frame given by K if and only if

$$\sum_{j\in J} \omega^j = \sum_{k\in K} \omega^{bk} \quad \Longrightarrow \quad J = bK \quad (\text{since } \theta \text{ is 1–1 on } \mathcal{C}_d),$$

where $b \notin \mathbb{Z}_n^*$ (since the frames are not multiplicatively equivalent). Since the frame given by J has distinct vectors, $\mathbb{Z}_n = \langle J \rangle$, and we have

$$\mathbb{Z}_n = \langle J \rangle = \langle bK \rangle \subset \langle b \rangle \neq \mathbb{Z}_n,$$

a contradiction.

($\Longleftarrow$) By Theorem 3.5. □

The subsets in $\mathcal{C}_d$ are the analogue of the subsets of a *normal basis* for a cyctomic field.

Example 6.2. ($d = 2$, n odd). Here $\mathcal{C}_2$ is all 2–element subsets of $\mathbb{Z}_n$, as in the case (a) in the proof of Theorem 4.5.

Example 6.3. ($n = p$ a prime). Here the p–th roots of unity are linearly independent over $\mathbb{Q}$, so they form a normal basis, and thus $\mathcal{C}_d$ is all d–element subsets of $\mathbb{Z}_p$. Moreover, unitarily inequivalent frames share no angles. Thus the number of unitarily equivalent harmonic frames of p vectors in $\mathbb{C}^d$ is the number of orbits of the d–element subsets of $\mathbb{Z}_p$ under the (multiplicative) action of $\mathbb{Z}_p^*$. A formula in terms of the Euler φ function is given [MW10], a recursive formula in [H10] (cf. [H36], [S00]).

This example can be further generalised as follows:

Theorem 6.4 ([C10]). *Let n be square free, i.e., be a product of distinct primes. Then d–element subsets of $\mathbb{Z}_n^*$ give unitarly equivalent harmonic frames if and only if they are multiplicatively equivalent.*

References

[BE03] H. Bölcskei and Y. C. Eldar, Geometrically uniform frames, *IEEE Trans. Inform. Theory* **49 no. 4** (2003), 993–1006.

[CK03] P. G. Casazza and J. Kovačević, Equal-norm tight frames with erasures, *Advances in Comp. Math.* **18** (2003), 387–430.

[CK07] A. Chebira and J. Kovačević, Life Beyond Bases: The Advent of Frames (Part I), *IEEE Signal Processing Mag.* **24** (86–104), 2007.

[C10] T. Chien, Masters thesis, University of Auckland, 2010.

[CJ76] J. H. Conway and A. J. Jones, Trigonometric Diophantine equations (On vanishing sums of roots of unity), *Acta Arith.* **30** (1976), 229–240.

[S00] M du Sautoy, Counting p-groups and nilpotent groups, *Inst. Hautes Études Sci. Publ. Math.* **92** (2000), 63–112.

[F01] M. Fickus, Finite normalized tight frames and spherical equidistribution, Dissertation (University of Maryland), 2001.

[GVT98] V. K. Goyal, M. Vetterli, and N. T. Thao, Quantized overcomplete expansions in $\mathbb{R}^N$: analysis, synthesis, and algorithms, *IEEE Trans. Inform. Theory* **44(1)** (1998), 16–31.

[H40] H. Hadwiger, Über ausgezeichnete Vektorsterne und reguläre Polytope, *Comment. Math. Helv.* **13** (1940), 90–107.

[H36] P. Hall, The Eulerian functions of a group, *Q. J. Math.* **7** (1936), 134–151.

[H07] D. Han, Classification of Finite Group–Frames and Super–Frames, *Canad. Math. Bull.* **50** (2007), 85–96.

[H10] M Hirn, The number of harmonic frames of prime order, *Linear Algebra Appl.* **432** (2010), 1105–1125.

[HP04] R. B. Holmes and V. I. Paulsen, Optimal frames for erasures, *Linear Algebra Appl.* **377** (2004), 31–51.

[HW06] N. Hay and S. Waldron, On computing all harmonic frames of n vectors in $\mathbb{C}^d$, *Appl. Comput. Harmonic Anal.* **21** (2006), 168-181.

[HR07] C. J. Hillar and D. L. Rhea, Automorphisms of finite abelian groups, *Amer. Math. Monthly* **114** (2007), 917–923.

[JL93] G. James and M. Liebeck, "Representations and Characters of Groups", Cambridge University Press, Cambridge, 1993.

[K06] D. Kalra, Complex equiangular cyclic frames and erasures, *Linear Algebra Appl.* **419** (2006), 373–399.

[LL00] T. Y. Lam and K. H. Leung, On vanishing sums of roots of unity, *J. Algebra* **224** (2000), 91–109.

[GKK01] V. K. Goyal, J. Kovačević, and J. A. Kelner, Quantized Frame Expansions with Erasures, *Appl. Comput. Harmonic Anal.* **10** (2001), 203–233.

[M65] H. B. Mann, On linear relations between roots of unity, *Mathematika* **12** (1965), 107–117.

[MW10] S. Marshall and S. Waldron, On the number of harmonic frames, Preprint, 2010.

[S68] D. Slepian, Group codes for the Gaussian channel, *Bell System Tech. J.* **47** (1968), 575–602.

[S06] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.* **322** (2006), 437–452.

[VW05] R. Vale and S. Waldron, Tight frames and their symmetries, *Constr. Approx.* **21** (2005), 83–112.

[VW08] R. Vale and S. Waldron, Tight frames generated by finite nonabelian groups, *Numer. Algorithms* **48** (2008), 11–28.

[VW10] R. Vale and S. Waldron, The symmetry group of a finite frame, *Linear Algebra Appl.* **433** (2010), 248-262.