Technical Report
6 January 2006

On computing all harmonic frames of n vectors in $\mathbb{C}^d$

Shayne Waldron and Nick Hay

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
e–mail: waldron@math.auckland.ac.nz (www.math.auckland.ac.nz/~waldron)

ABSTRACT

There are a finite number of inequivalent isometric frames (equal–norm tight frames) of n vectors for $\mathbb{C}^d$ which are generated from a single vector by applying an abelian group G of symmetries. Each of these so-called *harmonic frames* can be obtained by taking d rows of the character table of G; often in many different ways, which may even include using different abelian groups.

Using an algorithm implemented in the algebra package Magma, we determine which of these are equivalent. The resulting list of all harmonic frames for various choices of n and d is freely available, and it includes many properties of the frames such as: a simple description, which abelian groups generate it, identification of the full group of symmetries, the minimum, average and maximum distance between vectors in the frame, and whether it is real or complex, lifted or unlifted. Additional attributes aimed at specific applications include: a measure of the cross correlation (Grassmannian frames), the number of erasures (robustness to erasures), and the diversity product of the full group of its symmetries (multiple–antenna code design). Some outstanding frames are identified and discussed, and a number of questions are answered by considering the examples on the list.

Key Words: isometric frame, equal–norm tight frame, harmonic frame, equiangluar frame, Grassmannian frame

AMS (MOS) Subject Classifications: primary 42C15, secondary 20C15, 52B15,

1. Introduction

The set $\Phi = \{\phi_j\} \subset \mathbb{C}^d$ of $n \geq d$ nonzero vectors forms an **isometric** (equal–norm tight) **frame** for $\mathbb{C}^d$ if they have equal length and satisfy

$$f = \frac{d}{n} \sum_{j=1}^{n} \langle f, \frac{\phi_j}{\|\phi_j\|} \rangle \frac{\phi_j}{\|\phi_j\|}, \qquad \forall f \in \mathbb{C}^d.$$

Recent applications of such redundant orthogonal type expansions include wavelets, signal processing (cf [CK03]) and orthogonal polynomials of several variables (cf [PW02]).

The unitary images and nonzero scalar multiples of an isometric frame are isometric frames, which we consider to be equivalent. Until recently, the existence of at least one isometric frame for each $n \geq d$ was not widely known. An existence proof based on 'frame potentials' was given by [F01], an inductive construction by [RW02], and there is an explicit construction dating back at least to [GVT98] which (with hindsight) amounts to the following. Consider the character table of the cyclic group of order n

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}, \qquad \omega := e^{2\pi i/n}, \tag{1.1}$$

whose columns form an orthogonal basis of $\mathbb{C}^n$. Split column j into its first d components $\phi_j \in \mathbb{C}^d$, and those remaining ψ_j. Then the orthogonal expansion for $(f, 0)$, $f \in \mathbb{C}^d$ yields

$$\begin{pmatrix} f \\ 0 \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} \langle \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} \rangle \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix} \quad \Longrightarrow \quad f = \frac{1}{n} \sum_{j=1}^{n} \langle f, \phi_j \rangle \phi_j, \quad \forall f \in \mathbb{C}^d,$$

and so $\{\phi_j\}$ forms an isometric frame (since $\|\phi_j\|^2 = d$). The vectors $\{\phi_j\}$ are the columns of the matrix obtained by taking the first d rows of the character table.

In [VW05] it was realised that instead of taking the first d rows of the character table of the cyclic group, one can take *any* d rows of the character table of *any* abelian group of order n. Each of the *finite* number of isometric frames so obtained is generated from a single vector by applying an abelian group of symmetries (such frames will be called harmonic). In this paper we will calculate all harmonic frames (for a given n and d), i.e., precisely which of the above harmonic frames are equivalent. First some definitions.

Definition. *The* **symmetry group** *of an isometric frame* Φ *is the group of unitary maps*

$$\mathrm{Sym}(\Phi) := \{U : U\Phi = \Phi\},$$

i.e., those which permute the vectors of Φ.

For example, the isometric frame $\phi_j := (1, \omega^{j-1}, \omega^{2(j-1)}, \ldots, \omega^{(d-1)(j-1)})$, $j = 1, \ldots, n$ obtained by taking the first $d > 1$ rows of (1.1) has a symmetry given by the diagonal unitary matrix $U := \mathrm{diag}(1, \omega, \ldots, \omega^{d-1})$. This U generates a cyclic group $G \subset \mathrm{Sym}(\Phi)$ of order n which acts transitively on Φ.

Definition. *A* **harmonic frame** *is an isometric frame Φ of n vectors, which is generated by an abelian group $G \subset \mathrm{Sym}(\Phi)$ of order n, i.e., $\Phi = G\phi$, $\forall \phi \in \Phi$. Those harmonic frames where G can be taken to be the cyclic group are called* **cyclic frames**.

The following implies there is a finite number of harmonic frames of n vectors for $\mathbb{C}^d$, and that they can be *all* be obtained by 'taking rows of a character table'.

Theorem ([VW05]). *An isometric frame of n vectors for $\mathbb{C}^d$ is harmonic if and only if it comes from the character table of an abelian group, i.e., is equivalent to one given by the columns of a submatrix obtained by taking d rows of the character table of an abelian group of order n.*

Thus by 'taking rows of a character table' one obtains $\binom{n}{d}$ harmonic frames for each of the $a(n)$ abelian groups of order n. The calculations done by hand in [VW05] gave little indication of how many of these $a(n)\binom{n}{d}$ harmonic frames might be equivalent. It became apparent that a computer algebra calculation would be necessary to determine how many inequivalent frames there are, and which is best for a specific application.

In this paper we give an algorithm to determine how many inequivalent harmonic frames there are, and outline the results from its implementation in the computer algebra system `Magma` (cf [BCP97]). It appears that for d fixed, there are on the order of n^{d-1} inequivalent harmonic frames of n vectors in $\mathbb{C}^d$.

The Magma package was chosen as it is particularly well suited to the task of computing and then identifying the full group of symmetries of each harmonic frame.

In the next section we outline our algorithm which produces a list of all harmonic frames Φ of n vectors in $\mathbb{C}^d$ by deciding which of the $a(n)\binom{n}{d}$ harmonic frames described above are equivalent to each other.

This list, a freely available text file, gives all inequivalent harmonic frames for various $n \geq d$, and many properties of them including: a simple description, which abelian groups generate it, identification of the full group of symmetries, the minimum, average and maximum distance between vectors in the frame, and whether it is real or complex, lifted or unlifted. Additional attributes aimed at specific applications include: a measure of the cross correlation (Grassmannian frames), the number of erasures (robustness to erasures), and the diversity product of the full group of its symmetries (multiple–antenna design).

The remaining sections detail and interpret the results.

2. The algorithm for computing all harmonic frames

Recall that two isometric frames $(\phi_j)_{j=1}^n$ and $(\psi_j)_{j=1}^n$ for $\mathbb{C}^d$ are equivalent if there exists a unitary matrix U, a scalar c, and a permutation π of the indices for which

$$\psi_{\pi j} = cU\phi_j, \qquad \forall j.$$

Hence in the *worst case* to determine whether or not two of the $a(n)\binom{n}{d}$ harmonic frames of n vectors in $\mathbb{C}^d$ obtained from character tables are equivalent or not, one must consider the

$n(n-1)\cdots(n-d+1)$ possible unitary matrices U above. *Fortunately*, it turns out that in many cases there are much easier tests to check equivalence. Our algorithm applies three of these "easy" tests as required, then if all three are inconclusive does the computationally "expensive" check for the existence of a U. In pseudo-code our proposed algorithm is:

```
H:={}                      % let H be the set of inequivalent frames found so far
for j = 1 to a(n)          % count over all abelian groups of order n
 for Theta in F[j]         % count over the frames given by the j-th abelian group
   if Theta is not equivalent to a frame in H by test 1
     then add Theta to H      % easy test for inequivalence
     else
       if Theta is equivalent to a frame in H by tests 2 and 3
         then discard Theta   % easy tests for equivalence
         else
           if Theta is equivalent a frame in H by test 4
             then discard Theta     % expensive, but conclusive test for
             else add Theta to H    % for equivalence or inequivalence
 end for
end for
```

A complexity analysis of this algorithm is complicated by the fact that a priori, one does not know how often the computationally expensive test for equivalence must be used. However, our results indicate this test is infrequently required, and it is in this sense we claim that our algorithm is "efficient". We now give a precise description of our tests.

To illustrate the tests we consider the case $d=2$, $n=4$. Here there are two abelian groups to consider: the cyclic group C_4 and $C_2\times C_2$, which have character tables

$$\begin{bmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{bmatrix}, \qquad \begin{bmatrix}1&1\\1&-1\end{bmatrix}\otimes\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}. \tag{2.1}$$

For each of these, there are 6 possible pairs of rows we can take, giving a total of 12 possible harmonic frames $\{\phi_j\}$ to be examined. Some have repeated vectors, e.g., taking rows $\{1,3\}$ of the first character table gives

$$\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\},$$

and so are not harmonic frames of n vectors. For those choices which do give harmonic frames, i.e., the set `F[j]` above, the tests for equivalence are as follows.

Test 1: (for inequivalence) Since unitary maps preserve angles, the multiset of inner products $\{\langle\phi_1,\phi_j\rangle\}_{j=1}^n$ is the same for all equivalent choices. For example, taking rows $\{1,2\}$ and $\{2,3\}$ of the first character table gives

$$\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\i\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\-i\end{bmatrix}\}, \qquad \{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}i\\-1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}-i\\-1\end{bmatrix}\}$$

which are inequivalent harmonic frames since their inner products $\{2, 1+i, 0, 1-i\}$ and $\{2, i-1, 0, -1-i\}$ are different. We refer to this test as **comparison of inner products**.

Test 2: (for equivalence) Some choices give the same frame after reordering, e.g., taking rows $\{1,2\}$ and $\{1,4\}$ of the first character table gives

$$\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\i\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\-i\end{bmatrix}\}, \qquad \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-i\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\i\end{bmatrix}\}.$$

We refer to this test as **reordering**.

Test 3: (for equivalence) Some choices give the same frame after permuting the coordinates (and reordering), e.g., taking rows $\{2,3\}$ and $\{3,4\}$ of the first character table gives

$$\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}i\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-i\\-1\end{bmatrix}\}, \qquad \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\-i\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\i\end{bmatrix}\}.$$

We refer to this test as **permuting the coordinates**.

Finally, should these three (simple) tests fail to determine whether harmonic frames Φ and Ψ are equivalent, then it is necessary to try and find a unitary U with $\Psi = U\Phi$.

Test 4: For Φ and Ψ we determine equivalence by **checking for the existence** of a unitary U with $\Psi = U\Phi$. This expensive calculation is simplified as follows. Choose a basis from Φ, say wlog $\phi_1, \ldots, \phi_d$. Then U is determined by the linearly independent images $\psi_j = U\phi_j \in \Psi$, $1 \leq j \leq d$, which must satisfy $\langle \psi_j, \psi_k \rangle = \langle \phi_j, \phi_k \rangle$, $1 \leq j, k \leq d$. For all such U (which are unitary by construction) we then check whether $U\Phi = \Psi$. For example, taking rows $\{2,4\}$ of first character table and $\{2,3\}$ of the second gives

$$\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}i\\-i\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}-i\\i\end{bmatrix}\}, \qquad \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\}$$

which are equivalent, but can not be recognised as such using the first three tests. In such (fortunately) rare cases we keep copies of harmonic frames which could not be resolved using the first three tests (equivalent or not) for use in any subsequent comparisons.

3. The Magma implementation and results

Here we give details of the implementation of our algorithm in the computer algebra system `Magma` (cf [BCP97]), which is ideally suited to computing with groups. Magma has a unique identification for each finite group of order $n \leq 1000$ of the form `<n,k>`. A sample of the results is given in the appendix, and the full results are available at the website

`www.math.auckland.ac.nz/~waldron/Harmonic-frames`

First we need to count over *all* abelian groups of order n. Each can be written as a product of cyclic groups $C_{t_1} \times \cdots \times C_{t_k}$, where $(t_1, \ldots, t_k)$ are the torsion invariants, which satisfy $t_1 t_2 \cdots t_k = n$ and t_j divides t_{j+1}, and uniquely determine the group. These we order so that the cyclic frames appear first on our list. The character table of $C_{t_1} \times \cdots \times C_{t_k}$ is the Kronecker product of the character tables of the cyclic groups C_{t_j}, see, e.g., (2.1).

Again, we illustrate using the example $d = 2$, $n = 4$ (all harmonic frames of four vectors in $\mathbb{C}^2$). We can take $d = 2$ rows from each of the two character tables (2.1) in six ways. This we do in the lexicographic order $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$. The `Magma` output from our implementation of the algorithm is the following:

```
St 4 2 (0.031) Ex 12 Fl 8 EV 3 EP 1 FS 1 FC 3
Gp 1 0.011 *-e**p
Gp 2 0.02 ---Fee
Fr 1 C L 1 1.0000 1.1380 1.4142 2 0.000 0.707 1 <4, 1> {1} 0.00 {1,2} 0.0
Fr 2 C U 1 1.4142 1.6261 1.7320 2 0.000 0.707 1 <4, 1> {1} 0.00 {2,3} 0.0
Fr 3 R U 2 1.4142 1.6094 2.0000 1 0.000 1.000 2 <8, 3> {1,2} 0.00 {2,4} 0.0
```

The line “ `St 4 2 (0.031) Ex 12` ... `FC 3` ” indicates the start of the list of harmonic frames of `4` vectors in $\mathbb{C}^2$, a calculation which took `0.031` seconds, involved the examination of `12` possibilities (`Ex`), ultimately yielding `3` inequivalent harmonic frames (`FC`).

The line “ `Gp 1 0.011 *-e**p` ” indicates the first group C_4 has the label `<4,1>`, that it took `0.11` seconds to determine which of the six subsets of 2 rows gave inequivalent harmonic frames, with `*-e**p` being a summary of how this decision was made.

The first `*` indicates taking rows $\{1,2\}$ gave a harmonic frame which was not equivalent to any found so far by *comparison of inner products*, and so was added to the list. The `-` indicates taking rows $\{1,3\}$ did not yield a harmonic frame as the vectors obtained were not distinct. The `e` indicates taking rows $\{1,4\}$ gave a frame which was equal to one already on the list (after *reordering*). The `p` indicates the frame from rows $\{3,4\}$ was equal to one on the list after *permuting the coordinates* (and reordering).

Searching the second character table “ `Gp 2 0.02 ---Fee` ” for the group `<4,2>` turned up no extra frames. The `F` indicates the frame from rows $\{2,3\}$ was equivalent to one already on the list (and the *reordering* and *permutation of coordinate* tests failed). In these rare cases the other ‘version’ of the frame is kept for future comparison purposes. A `@` is recorded when a new frame is found which has the same multiset of inner products as one on the list. In both the `F` and `@` cases, where the simple tests for equivalence fail, we must use the *check for the existence* of a unitary U.

The end of the first line “ ... `Ex 12 Fl 8 EV 3 EP 1 FS 1 FC 3` ” summarises that `12` frames were examined, `8` consisted of distinct vectors (`Fl`), of which `3` were discarded because they had equal vectors after *reordering* (`EV`), `1` because it was equal after a *permutation of coordinates* (`EP`), and `1` because it was equivalent via some ‘frame symmetry’ U to one on the list (`FS`), for a total frame count of `3` inequivalent harmonic frames (`FC`).

The frames and their attributes

Each of the frames obtained is numbered and is recorded on a line:

```
Fr 1 C L 1 1.0000 1.1380 1.4142 2 0.000 0.707 1 <4, 1> {1} 0.00 {1,2} 0.0
```

" Fr 1 C L 1 ... " indicates Frame 1, which is Complex (as opposed to Real), Lifted (as opposed to Unlifted), and occurs in only 1 way as rows of a character table up to reordering and a permutation of coordinates. We recall from [VW05] that:

Definition. *An isometric frame* $\Phi \subset \mathbb{C}^d$ *is said to be* **real** *if there is a unitary map* U *with* $U\Phi \subset \mathbb{R}^d$, *otherwise it is said to be* **complex**.

To determine whether a harmonic frame Φ is real we use the test that it is if and only if *all* the inner products $\{\langle \phi_1, \phi_j \rangle\}_{j=1}^n$ are real.

Definition. *A harmonic frame* Φ *is said to be* **unlifted** *if the sum of its vectors is* 0, *otherwise it is* **lifted**.

A harmonic frame Φ for $\mathbb{C}^{d+1}$ is lifted if and only if it is equivalent to one obtained by taking the row of 1's and d other rows of the character table of G. The frame for $\mathbb{C}^d$ obtained by taking just the d rows is unlifted, and can be 'lifted' by adding an extra constant component to each vector to obtain Φ. This gives a 1–1 correspondence between the unlifted frames for $\mathbb{C}^d$ and the lifted frames for $\mathbb{C}^{d+1}$, with the lifting process preserving symmetries (but adding no additional ones).

The entries " 1.0000 1.1380 1.4142 2 0.0000 0.7071 " are the *minimum*, *average* and *maximum* distance between the normalised vectors of the frame

$$0 < \min_{j\neq k} \|\hat\phi_j - \hat\phi_k\| \le \operatorname*{ave}_{j\neq k} \|\hat\phi_j - \hat\phi_k\| \le \max_{j\neq k} \|\hat\phi_j - \hat\phi_k\| \le 2, \qquad \hat\phi_j := \frac{\phi_j}{\|\phi_j\|} = \frac{\phi_j}{\sqrt{d}},$$

the number of **erasures**, i.e., the number of vectors that can be removed from Φ so those remaining still span (which is $\le n-d$) and the measures of *cross correlation* (discussed later)

$$0 \le \min_{j\neq k} |\langle \phi_j, \phi_k \rangle| \le \max_{j\neq k} |\langle \phi_j, \phi_k \rangle| \le 1.$$

The entries " 1 <4, 1> " for frame 1 indicate that its *full symmetry group* $\mathrm{Sym}(\Phi)$ is <4, 1> which has a subgroup consisting of scalar multiplication by the 1–th roots of unity.

The end of the line for frame 1 " ... {1} 0.00 {1,2} 0.0 " records that it can be constructed by taking rows {1,2} of the character table of the group <4,1>, that its *diversity product* (discussed later) is 0.00, and calculating its various properties took 0.0 seconds. Equivalent harmonic frames may come from more than one abelian group, e.g., " {1,2} 0.00 {2,4} 0.0 " indicates that frame 3 can be obtained from the groups <4,1> and <4,2> by taking rows {2,4} from the character table of the first of these.

In summary, a quick glance at the list shows there are three inequivalent harmonic frames of four vectors in $\mathbb{C}^2$; the first two are complex (one lifted, one unlifted) with symmetry groups of order 4, and the third is real (unlifted) with symmetry group of order 8 (and comes from two abelian groups).

4. The number of harmonic frames and their erasures

In [GVT98], it was shown (apparently for the first time) that there was an isometric frame of n vectors in $\mathbb{C}^d$ for every $n \geq d$ by explicitly giving *one* harmonic frame. Our computations indicate there are *many* such harmonic frames (on the order of n^{d-1}).

There is a unique harmonic frame of n vectors in $\mathbb{C}$ given by the n–th roots of unity. For $n \neq 1$ this is unlifted, and so can be lifted to $\mathbb{C}^2$. In $\mathbb{C}^2$ there are other (unlifted) inequivalent harmonic frames, and we plot the total number of them verses n.

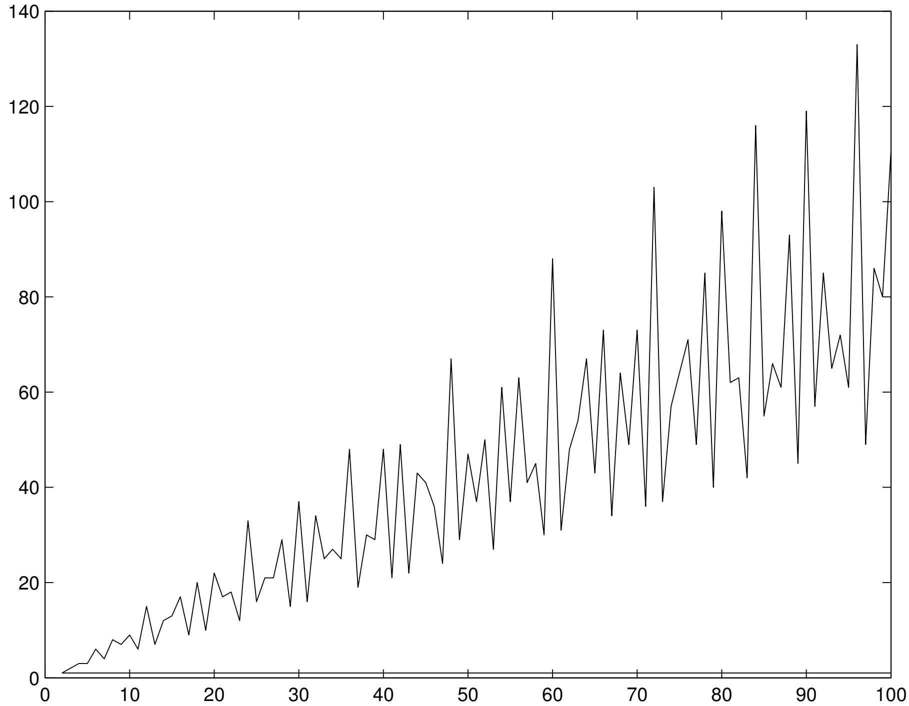
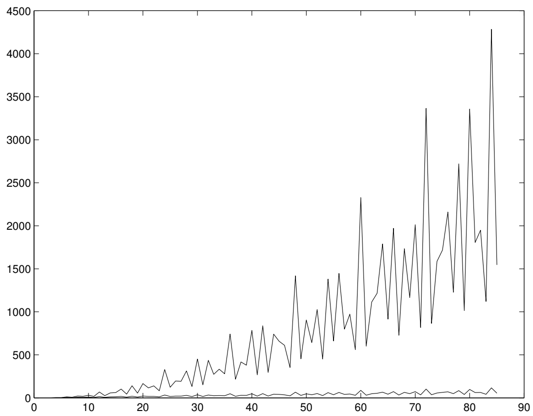


Fig. 1. The number of inequivalent harmonic frames of n vectors in $\mathbb{C}^2$.

We observe that the number of harmonic frames in $\mathbb{C}^2$ appears to grow like n, and also be influenced by the number of prime factors of n (more factors giving more frames). A similar pattern appears in $\mathbb{C}^d$ with the number growing like n^{d-1}.



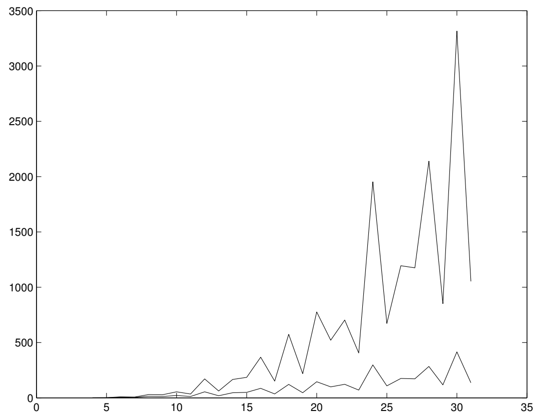
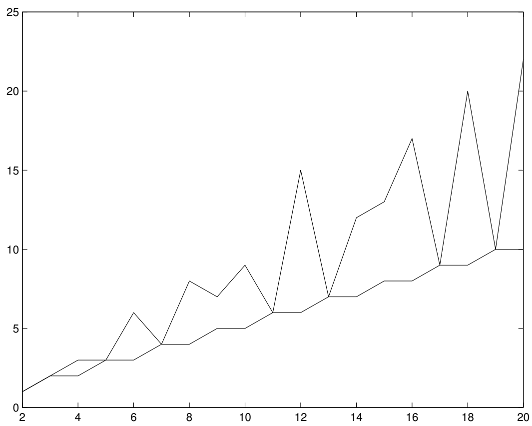


Fig. 2. The number of inequivalent harmonic frames of n vectors in $\mathbb{C}^3$ and $\mathbb{C}^4$. The lower graph shows how many of them are lifted.

Given that the existence of a general isometric frames was only recently established, we think this list giving a *choice* of many harmonic frames is a big step forward. Of course, it remains to decide which frame is the best for a particular application, and to understand their geometry better. In [MW04] estimates for the number of harmonic frames are given.

An isometric frame Φ of n vectors for $\mathbb{C}^d$ is **robust to k erasures** if removing any k vectors leaves a spanning set. Clearly, $k \leq n-d$ and k of the coefficients $\{\langle f, \phi\rangle : \phi \in \Phi\}$ can be lost with it still being possible to reconstruct $f \in \mathbb{C}^d$ from those remaining. [GKK01] proposed the use of frames robust to high numbers of erasures for the transmission of data subject to losses. They showed that the harmonic frame obtained by taking the first d rows of the character table of the cyclic group of order n is robust to the maximum possible $n-d$ erasures. Not all harmonic frames are robust to $n-d$ erasures, e.g., 4 equally spaced vectors in $\mathbb{R}^2$ are robust to only 1 erasure, since removing two antipodal vectors does not leave a spanning set. Below we indicate graphically the number of harmonic frames of n vectors and how robust to erasures they are.



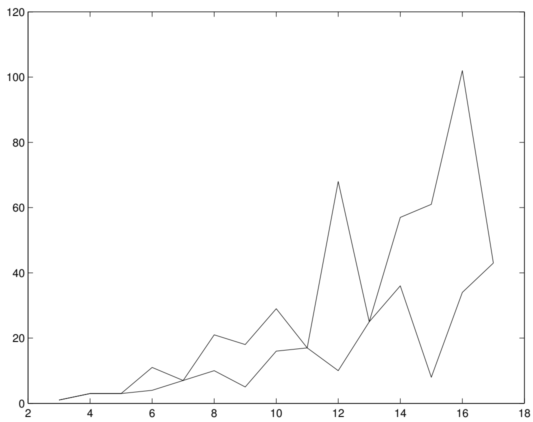
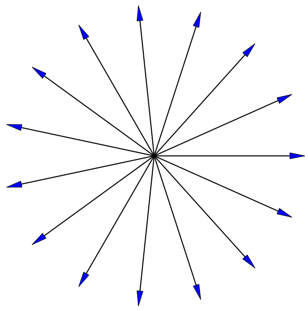


Fig. 3. The number of harmonic frames of n vectors in $\mathbb{C}^2$ and $\mathbb{C}^3$, and those with $n-d$ erasures (lower graph).

Though not all harmonic frames of n vectors are robust to $n-d$ erasures, unless n is prime (easily proved), there are many which are. Amongst this reasonably large class one might seek a harmonic frame robust to $n-d$ erasures which has additional desirable properties for accurate data transmission, e.g., a large minimal spacing between the vectors.

5. Equally spaced and Equiangular frames

The n equally spaced unit vectors in $\mathbb{R}^2$ form a harmonic frame (generated by a rotation through $2\pi/n$).



This motivating example led a number of people to investigate isometric frames for their possible connection with (the big problem) of finding equally spaced points on a sphere.

The only harmonic frames for $\mathbb{C}^d$ with $\langle \phi_j, \phi_k \rangle$ (and hence $\|\phi_j - \phi_k\|$) constant for $j \neq k$ are the orthonormal bases (the unique harmonic frame for $n = d$) and the vertices of the regular d–simplex in $\mathbb{R}^d$ (the unique unlifted harmonic frame for $n = d + 1$). One might slightly weaken this condition, as some have done, in an attempt to find additional nicely spaced frames.

First we look for harmonic frames with $\|\phi_j - \phi_k\|$ (equivalently $\Re\langle \phi_j, \phi_k \rangle$) constant, which we naively call **equispaced**. They exist! Take the second and third rows of the character table for the cyclic group of order 5 to obtain

$$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^4 \end{bmatrix}, \begin{bmatrix} \omega^3 \\ \omega \end{bmatrix}, \begin{bmatrix} \omega^4 \\ \omega^3 \end{bmatrix}\}, \qquad \omega := e^{2\pi i/5},$$

which is an unlifted complex harmonic frame of five vectors with $\|\phi_j - \phi_k\| = \sqrt{5}$, $j \neq k$. This belongs to the *family* of $n = 2d + 1$ equispaced vectors $\Phi = (\phi_j)_{j=1}^{2d+1}$ for $\mathbb{C}^d$ given by $\phi_j := (\omega^j, \omega^{2j}, \ldots, \omega^{dj})$, with $\omega := e^{2\pi i/(2d+1)}$. We calculate

$$\|\phi_j - \phi_k\|^2 = \sum_{l=1}^{d} |\omega^{jl} - \omega^{kl}|^2 = 2d - \sum_{m=1}^{2d+1} \omega^m = 2d + 1, \qquad j \neq k,$$

and so $\|\hat{\phi}_j - \hat{\phi}_k\| = \sqrt{2 + 1/d}$, $j \neq k$. Similarly, our computations also turned up a family of $n = 2d - 1$ equispaced vectors $\Psi = (\psi_j)_{j=1}^{2d-1}$ given by $\psi_j := (1, \omega^j, \ldots, \omega^{(d-1)j})$, with $\omega := e^{2\pi i/(2d-1)}$. For this $\|\hat{\psi}_j - \hat{\psi}_k\| = \sqrt{2 - 1/d}$, $j \neq k$.

Notice that the separation between points in Φ is larger than that for those in Ψ (which has *fewer* points), and so equispacing is perhaps not the most useful notion of ‘equal spacing’. The difficulty here is that though an equal distance from each other, the points are clumped in a particular cone. This phenomenon can best be seen for the standard basis vectors in $\mathbb{C}^3$; which though equally spaced, all lie in the first octant. Indeed for d large the separation in both Φ and Ψ approaches $\sqrt{2}$ (that of an orthonormal basis).

Heath and Strohmer [HS04] call an isometric frame **equiangular** when $|\langle \phi_j, \phi_k \rangle|$ is constant for $j \neq k$, and show they can only exist if $n \leq d(d+1)/2$ (real frames) or $n \leq d^2$ (complex frames). There exist harmonic frames which are equiangular, e.g., there are two inequivalent complex ones of 7 vectors for $\mathbb{C}^3$ given by

$$\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \\ \omega^j \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^4 \\ \omega^{2j} \end{bmatrix}, \begin{bmatrix} \omega^3 \\ \omega^6 \\ \omega^{3j} \end{bmatrix}, \begin{bmatrix} \omega^4 \\ \omega \\ \omega^{4j} \end{bmatrix}, \begin{bmatrix} \omega^5 \\ \omega^3 \\ \omega^{5j} \end{bmatrix}, \begin{bmatrix} \omega^6 \\ \omega^5 \\ \omega^{6j} \end{bmatrix}\}, \qquad \omega := e^{2\pi i/7},$$

where $j = 5$ or $j = 6$. These belong to the family of equiangular frames for $d = p^l + 1$ (p prime, $l \in \mathbb{N}$) and $n = d^2 - d + 1$ introduced in [K99]. Our computations yield (often several) equiangular complex harmonic frames of $n = 2d + 1$ (d odd) and $n = 2d - 1$ (d even) vectors for $\mathbb{C}^d$ for d up to 6 (as far as we have gone). For example, there are four inequivalent equiangular harmonic frames of 11 vectors in $\mathbb{C}^5$. Given the known examples of nontrivial equiangular isometric frames, and the obvious conjecture here, we feel these special harmonic frames are worthy of further investigation.

There is considerable interest in the conjecture (for which there is strong numerical evidence) that: *there exists an equiangular frame of $n = d^2$ vectors for $\mathbb{C}^d$*. Unfortunately, there are *no* equiangular harmonic frames of $n = d^2$ vectors on our list!

6. Symmetries of harmonic frames

By definition, the symmetry group of a harmonic frame of n vectors contains an abelian group of order n. Our algorithm for finding all harmonic frames for a given d, n was implemented in `Magma`, so, once it was established there were many such frames, we could easily enquire which – if any – had additional symmetries, i.e., $|\operatorname{Sym}(\Phi)| > n$.

As an example, there are 18 harmonic frames of 9 vectors for $\mathbb{C}^3$. Of these 13 have no additional symmetries, 4 have a symmetry group of order 18, and 1 has symmetry group of order 162 (the group `<162,10>`)! So how can one make sense of such results? For a start, the group of order 162 can be explained as follows.

Theorem 6.1. *There is a harmonic frame of $n = dm$ vectors for $\mathbb{C}^d$ with symmetry group of order $d!m^d$. This can be realised as the standard basis $\{e_1, \ldots, e_d\}$ tensored with the m–th roots of unity, i.e., $\Phi := \{\mu^j e_k\}_{j=1,k=1}^{m\ \ d}$, $\mu := e^{2\pi i/m}$. Φ has symmetries given by permutation of coordinates and multiplication of a coordinate by an m–th root of unity.*

Proof: Take rows $2, m + 2, 2m + 2, \ldots, (d - 1)m + 2$ of the character table of the cyclic group of order n, express the resulting harmonic frame in terms of μ and $\nu := e^{2\pi i/d}$, and use the fact that vectors $(1, \nu^j, \nu^{2j}, \ldots, \nu^{(d-1)j})$, $j = 0, \ldots d - 1$ are orthogonal. $\square$

For $d = 3$, $n = 9$ ($m = 3$), we get the harmonic frame with symmetry group of $3!3^3 = 162$. This frame has the largest minimum distance between the normalised vectors $\sqrt{2} \approx 1.4142$, as one might hope. Interestingly, there are also two inequivalent harmonic frames (of 9 vectors in $\mathbb{C}^3$) which share this minimum (and the average and maximum too). One of these has symmetry group of order 9, the other 18.

Now consider the symmetry groups of harmonic frames of n vectors for $\mathbb{C}^2$. The harmonic frame of $n > 2$ equally spaced vectors in $\mathbb{R}^2$ has symmetries given by the dihedral group which has order $2n$. While for n even, Theorem 6.1 gives the harmonic frame

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ -\omega \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^3 \\ -\omega^3 \end{bmatrix}, \begin{bmatrix} \omega^4 \\ \omega^4 \end{bmatrix}, \ldots, \begin{bmatrix} \omega^{n-2} \\ \omega^{n-2} \end{bmatrix}, \begin{bmatrix} \omega^{n-1} \\ -\omega^{n-1} \end{bmatrix} \right\}, \qquad \omega := e^{2\pi i/n}$$

which has symmetry group of order $n^2/2$. A cursory glance at the list indicates these orders account for largest symmetry groups in each case, and for n odd but not prime there can be other harmonic frames with symmetry group of order $2n$ (apparently with nonisomorphic symmetry groups). For example, the complex harmonic frame of 9 vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} \omega \\ \omega \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega^2 \end{bmatrix} \right\}, \qquad \omega := e^{2\pi i/3} \tag{6.2}$$

has symmetry group of order 18 (multiplication of each coordinate by a third root of unity and permutation of the coordinates). For n prime we can show the following (cf [MW04]).

Theorem 6.3. *For $n = p > 2$ prime, there are $\frac{p+1}{2}$ harmonic frames (all cyclic) for $\mathbb{C}^2$*

1 *real unlifted: the n equally spaced vectors in $\mathbb{R}^2$*

1 *complex lifted:* $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \omega^{p-1} \end{bmatrix} \right\}$

$\frac{p-3}{2}$ *complex unlifted:* $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^j \end{bmatrix}, \ldots, \begin{bmatrix} \omega^{p-1} \\ \omega^{j(p-1)} \end{bmatrix} \right\}$, $j \in J$

where $\omega := e^{2\pi i/p}$, and $J \subset \{2, \ldots, p-2\}$ is any set of $\frac{p-3}{2}$ indices with

$$j_1 j_2 \neq 1 \bmod p, \quad \forall j_1, j_2 \in J.$$

The real frame has $|\operatorname{Sym}(\Phi)| = 2n$, and the complex frames $|\operatorname{Sym}(\Phi)| = n$.

Example. The cyclic frames of n vectors in $\mathbb{C}^2$ with symmetry groups larger than n can be decomposed as follows. Let $k \geq 3$ be a divisor of n, i.e., $n = mk$. Then the tensor product of the m–th roots of unity with the k equally spaced vectors in $\mathbb{R}^2$ (with $m = 2$ allowed only when k is odd) gives a harmonic frame $\Phi = \{\phi_{rs}\}$

$$\phi_{rs} := \omega^r \begin{bmatrix} \cos \frac{2\pi s}{k} \\ \sin \frac{2\pi s}{k} \end{bmatrix}, \qquad 1 \leq r \leq m, \ 1 \leq s \leq k, \quad \omega := e^{2\pi i/m}.$$

This is cyclic with symmetry group $C_m \times D_k$ which has order order $2n$. For n even, $n = 2m$, the tensor product of the m–th roots of unity with an orthonormal basis is a cyclic frame with symmetry group of order $2m^2 = n^2/2$.

The maximum order of the symmetry groups of harmonic frames for $\mathbb{C}^3$, $\mathbb{C}^4$, ... is more complicated, with larger symmetry groups appearing when n is composite.

There exist harmonic frames, which cannot be decomposed as tensor products of other harmonic frames, with large symmetry groups, e.g, the real harmonic frame of 14 vectors for $\mathbb{C}^5$ given by $\phi_j := (\omega, \bar{\omega}^j, (-1)^j, \omega^{6j}, \bar{\omega}^{6j})$, $j = 1, \ldots, 14$, where $\omega := e^{2\pi i/14}$, which has a symmetry group of order $392 = 12 \times 28$. Of the 336 harmonic frames of 14 vectors for $\mathbb{C}^5$ only two have a symmetry group of order 392 (the other orders are $98, 42, 28$ and 14, and the number of frames with symmetry group of this order is $4, 2, 7$ and 321, respectively). This harmonic frame (ϕ_j) is arguably the *most* symmetric as it is unlifted, whereas the other with symmetry group of order 392 (which is also real) is lifted.

It was observed in [F01] that the vertices of the **platonic solids** form isometric frames for $\mathbb{C}^3$ (with a high degree of symmetry). By consulting our list we can check whether these are harmonic, i.e., are generated by an abelian subgroup of the full symmetry group. This will be so if there is a real harmonic frame with the same number of vectors and symmetry group that of the platonic solid. The *tetrahedron*, *cube* and *octahedron* are harmonic frames, and are the most symmetric for that number of vectors. The *icosahedron* (12 vertices) and *dodecahedron* (20 vertices) are not harmonic frames, as no harmonic frame with the same number of vectors shares their symmetry group (which has order 120). However, there is a complex unlifted harmonic frame of 12 vectors with a symmetry group of order 384, and complex lifted and unlifted frames of 20 vectors with a symmetry group of order 200.

Finite groups $\mathcal{V}$ of $d \times d$ unitary matrices (*unitary space–time* signals) with large **diversity product**

$$0 \leq \zeta := \frac{1}{2} \min_{\substack{U,V \in \mathcal{V} \\ U \neq V}} |\det(U - V)|^{\frac{1}{d}} = \frac{1}{2} \min_{0 \neq V \in \mathcal{V}} |\det(I - V)|^{\frac{1}{d}} \leq 1$$

have been proposed for use in multiple–antenna code design (which we don't pretend to fully understand), see, e.g., [HM00] and [SHHS01]. For many choices of d, n there are harmonic frames with $\mathcal{V} = \mathrm{Sym}(\Phi)$ having a diversity product in the range 0.4 to 0.6, which we are led to believe is acceptable for applications.

7. Some other interesting harmonic frames

It has been known (at least) since [Z01] that there exists at least one real harmonic frame of $n \geq d$ vectors for $\mathbb{C}^d$. Most harmonic frames appear to be complex, but there do exist inequivalent real harmonic frames.

For example, in $\mathbb{C}^3$ there is exactly one lifted real frame, namely the lift of $E_n := \{(\cos(2\pi j/n), \sin(2\pi j/n))\}_{j=1}^n$ the n equally spaced vectors in $\mathbb{R}^2$, i.e.,

$$E_n \dot{+} \{1\} := \{ \begin{bmatrix} \cos \frac{2\pi j}{n} \\ \sin \frac{2\pi j}{n} \\ \frac{1}{\sqrt{2}} \end{bmatrix} : j = 1, \ldots, n \},$$

In addition, for n even there are the following unlifted real harmonic frames

$$E_{n/2} \hat{+} \{-1, 1\} := \{ \begin{bmatrix} \cos \frac{\pi j}{n} \\ \sin \frac{\pi j}{n} \\ \frac{(-1)^k}{\sqrt{2}} \end{bmatrix} : j = 1, \ldots, \frac{n}{2}, k = 0, 1 \},$$

and

$$(E_{n/2} \hat{+} \{1\}) \otimes \{-1, 1\} := \{ (-1)^k \begin{bmatrix} \cos \frac{\pi j}{n} \\ \sin \frac{\pi j}{n} \\ \frac{1}{\sqrt{2}} \end{bmatrix} : j = 1, \ldots, \frac{n}{2}, k = 0, 1 \}.$$

All of these are cyclic, except for $(E_{n/2} \hat{+} \{1\}) \otimes \{-1, 1\}$ when n is multiple of 4.

Consider the inequivalent harmonic frames of 3 vectors in $\mathbb{C}^2$ (one real, one complex)

$$\Phi := \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \}, \qquad \Psi := \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} \}, \quad \omega := e^{2\pi i/3}.$$

Both are equiangular with $|\langle \hat{\phi}_j, \hat{\phi}_k \rangle| = |\langle \hat{\psi}_j, \hat{\psi}_k \rangle| = 1/2$, $j \neq k$, and equispaced with

$$\|\hat{\phi}_j - \hat{\phi}_k\| = \sqrt{3} \approx 1.7320, \quad \|\hat{\psi}_j - \hat{\psi}_k\| = \sqrt{3}/\sqrt{2} \approx 1.2247, \quad j \neq k.$$

The real frame has $|\operatorname{Sym}(\Phi)| = 6$ and the complex one $|\operatorname{Sym}(\Psi)| = 3$. This generalises:

Theorem 7.1 (Harmonic frames with $n = d + 1$). *Suppose $d \geq 2$, and let k be the number of positive divisors of $d+1$. Then there are exactly k inequivalent harmonic frames of $n = d + 1$ vectors in $\mathbb{C}^d$. All are cyclic and equiangular, and all but the vertices of the regular d–simplex are lifted.*

Proof: Each of these harmonic frames is obtained by ruling out *one* row of the character table. All possible rows can be obtained as a row of the character table of the cyclic group, and the multiset of entries of the row completely determines the inner products of the vectors in the frame (as the columns of the character table are orthonormal). Hence the number of inequivalent harmonic frames is k, the number of different multisets obtainable by taking a row. $\square$

Example. There are 3 harmonic frames of 4 vectors for $\mathbb{C}^3$, corresponding to ruling out the rows giving multisets $\{1, 1, 1, 1\}, \{1, -1, 1, -1\}, \{1, i, -1, -i\}$. The two frames given by ruling out the real rows are real harmonic frames. For $d + 1$ a prime there are just two harmonic frames of $d + 1$ vectors for $\mathbb{C}^d$.

8. Concluding remarks

We hope that the many examples of harmonic frames given here and on the list at

`www.math.auckland.ac.nz/~waldron/Harmonic-frames`

will be used in applications similar to those suggested. The text listing has the advantage that one can immediately read the output – even if Magma is unavailable. There is a toolbox called `explore` for analysing the list frames for those who have access to Magma.

9. Appendix

Harmonic frames in $\mathbb{C}^2$

Harmonic frames in $\mathbb{C}^3$

Harmonic frames in $\mathbb{C}^4$

References

[BCP97] W. Bosma, J. Cannon, and C. Playoust, The algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[CK03] P. Casazza and J. Kovačević, Equal-norm tight frames with erasures, *Adv. Comput. Math.* **18 no. 2–4** (2003), 387–430.

[F01] M. Fickus, "Finite normalised tight frames and spherical equidistribution", dissertation, University of Maryland, 2001.

[GKK01] V. K. Goyal, J. Kovačević, and J. A. Kelner, Quantized Frame Expansions with Erasures, *Appl. Comput. Harmonic Anal.* **10** (2001), 203–233.

[GVT98] V. K. Goyal, M. Vetterli, and N. T. Thao, Quantized overcomplete expansions in $\mathbb{R}^N$: analysis, synthesis, and algorithms, *IEEE Trans. Inform. Theory* **44(1)** (1998), 16–31.

[HS04] T. Strohmer and R. W. Heath, Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmonic Anal.* **14 no. 3** (2003), 257–275.

[HM00] B. M. Hochwald and T. L. Marzetta, Unitary space-time modulation for multiple-antenna communications in Rayleigh flat fading, *IEEE Trans. Inform. Theory* **46 no. 2** (2000), 543–564.

[K99] H. König, Cubature formulas on spheres, *in* "Advances in multivariate approximation: Witten-Bommerholz, 1998" (W. Haussmann, K. Jetter, M. Reimer Eds.), pp. 201–211, Wiley-VCH, Berlin, 1999.

[MW04] S. Marshall and S. Waldron, Enumeration of harmonic frames, Preprint, 2004.

[PW02] I. Peng and S. Waldron, Signed frames and Hadamard products of Gram matrices, *Linear Algebra Appl.* **347 no. 1–3** (2002), 131–157.

[SHHS01] A. Shokrollahi, B. Hassibi, B. M. Hochwald, and W. Sweldens, Representation theory for high-rate multiple-antenna code design, *IEEE Trans. Inform. Theory* **47 no. 6** (2001), 2335–2367.

[RW02] R. Reams and S. Waldron, Isometric tight frames, *Electron. J. Linear Algebra* **9** (2002), 122–128.

[VW05] R. Vale and S. Waldron, Tight frames and their symmetries, Constr. Approx., 21 no. 1.

[Z01] G. Zimmermann, Normalized tight frames in finite dimensions , *in* "Recent Progress in Multivariate Approximation, ISNM 137" (W. Haussmann and K. Jetter Eds.), pp. 249–252, Birkhäuser, Basel, 2001.