Spherical half-designs of high order

Daniel Hughes and Shayne Waldron

Department of Mathematics
University of Auckland
Private Bag 92019, Auckland, New Zealand
e-mail: waldron@math.auckland.ac.nz

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Abstract

We give some explicit examples of putatively optimal spherical half-designs, i.e., ones for which there is numerical evidence that they are of minimal size. These include a 16-point weighted spherical half-design of order 8 for $\mathbb{R}^3$ based on the pentakis dodecahedron. This gives rise to a 32-point weighted spherical 9-design for the sphere.

Key Words: spherical t-designs, spherical half-designs, tight spherical designs, finite tight frames, integration rules, cubature rules, cubature rules for the sphere, pentakis dodecahedron

AMS (MOS) Subject Classifications: primary 05B30, 42C15, 65D30; secondary 94A12.

1 Introduction

Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^d$ and σ be the surface area measure on $\mathbb{S}$, normalised to have $\sigma(\mathbb{S}) = 1$, i.e., to be a probability measure. A "spherical design" is a sequence of points $v_1, \ldots, v_n$ in $\mathbb{S}$ for which the integration (cubature) rule

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \frac{1}{n} \sum_{j=1}^{n} p(v_j), \tag{1.1}$$

holds for all p in some finite dimensional space of polynomials P. The existence of such a spherical design for n sufficiently large was proved in [SZ84]. When P is the polynomials of degree $\leq t$, one has a **spherical t-design**. Suppose that $P = \Pi_k^\circ(\mathbb{R}^d)$, the space of homogeneous polynomials of degree k. If k is even, then (1.1) integrates all the homogeneous polynomials q of even degrees $2m \leq k = 2t$, since taking

$$p(x) = (x_1^2 + \cdots + x_d^2)^{t-m} q(x)$$

in (1.1) gives

$$\int_{\mathbb{S}} q(x)\, d\sigma(x) = \int_{\mathbb{S}} p(x)\, d\sigma(x) = \frac{1}{n} \sum_{j=1}^{n} p(v_j) = \frac{1}{n} \sum_{j=1}^{n} q(v_j).$$

For this reason, the spherical designs which integrate the homogeneous polynomials of degree $2t$ (and hence of degrees $0, 2, \ldots, 2t$) are called **spherical half-designs** of order $2t$. The terms spherical $2t$-design [Sei01] and spherical (t,t)-design [Wal17] are also used. The related spherical designs of harmonic index $2t$ [BOT15] integrate the subspace of harmonic homogeneous polynomials of degree $2t$.

We also observe that if q is homogeneous of odd degree, then $q(-x) = -q(x)$ and its integral over $\mathbb{S}$ is zero. Thus if a spherical design is **centrally symmetric**, i.e., of the form $\{\pm v_j\}$, then it integrates all homogeneous polynomials of odd degree. Therefore

Proposition 1.1 *The following are equivalent:*

(i) $(\pm v_j)$ *is a centrally symmetric spherical* $(2t+1)$*-design of* $2n$ *vectors for* $\mathbb{R}^d$.

(ii) (v_j) *is a spherical half-design of order* $2t$ *of* n *vectors for* $\mathbb{R}^d$.

The analogue of Proposition 1.1 for complex spherical designs is discussed in [RS14] (Lemma 3.4) and [MW19].

There are various equivalent conditions to being a spherical design [DGS77], [BB09]. These include being an integration rule for a subspace of harmonic polynomials, and a variational characterisation. The spherical half-designs (v_j) of order $2t$ are characterised in [Wal17] as the vectors in $\mathbb{R}^d$ which give equality in the inequality

$$\sum_{j=1}^{n} \sum_{k=1}^{n} |\langle v_j, v_k \rangle|^{2t} \geq \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{d(d+2)\cdots(d+2(t-1))} \Big(\sum_{\ell=1}^{n} \|v_\ell\|^{2t} \Big)^2. \tag{1.2}$$

This implies that a spherical half-design (v_j) is **projectively unitarily invariant**, i.e., $(c_j U v_j)$ is also a half-design when $c_j \in \{-1,1\}$ and U is unitary. A spherical t-design has this property if and only if it is centrally symmetric. In view of this (and their definitions), a spherical t-design can be thought of as a set of points that are evenly spaced on the sphere, and a spherical half-design as set of lines (antipodal points) which are evenly spaced on the sphere. Using results from Brouwer degree theory, [BRV13] have shown that the minimum number of points in a spherical t-design (and hence in a spherical half-design of order $2t$) grows like t^{d-1} (with d fixed).

When the vectors $v_1,\ldots,v_n$ in $\mathbb{R}^d$ giving equality in (1.2) are not all of unit norm and not all zero, then one has the weighted integration rule

$$\int_{\mathbb{S}} p\,d\sigma = \frac{1}{\sum_k \|v_k\|^{2t}} \sum_{j=1}^n p(v_j) = \sum_{\substack{j=1\\ v_j\neq 0}}^n w_j p(\frac{v_j}{\|v_j\|}), \quad w_j := \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}}, \quad \forall p\in \Pi_{2t}^\circ(\mathbb{R}^d),$$

and we call (v_j) a **weighted** spherical half-design of order $2t$ with weights (w_j) [KP11].

Since spherical half-designs (v_j) satisfy (1.2), and are determined by the equation

$$\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2t} = \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+2)\cdots(d+2(t-1))}\Big(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Big)^2, \tag{1.3}$$

it is possible to find them numerically, for n sufficiently large [Bra11]. In this way, [HW18] found *putatively optimal* spherical half-designs of order $2t$ for a given t and d, i.e., those with the smallest number of vectors. This was done by using an iterative algorithm that attempts to minimise the difference between the left hand and right hand sides of (1.2) by making appropriate perturbations, starting from an initial guess. A similar search for putatively optimal spherical t-designs for the sphere ($d=3$) was done by Hardin and Sloane [HS96]. Analogous searches for complex (projective) spherical designs include Renes, et al. [RBKSC04] (complex equiangular lines), and Roy and Scott [RS07] (complex weighted t-designs, with predetermined weights).

In this paper, we give some explicit spherical half-designs motivated by the putatively optimal ones found in [HW18]. Since these exhibit a high degree of symmetry, we believe them to be optimal, i.e., to have the minimum number of vectors possible. Before doing this, we give a couple of examples.

Example 1.1 *($t=1$) The spherical half-designs of order 2 are precisely the tight frames for $\mathbb{R}^d$ [Wal03]. A sequence of vectors (v_j) is a* **tight frame** *for $\mathbb{R}^d$ (see [Wal18]) if it satisfies the generalised Parseval identity*

$$x = \frac{1}{A}\sum_{j=1}^n \langle x,v_j\rangle v_j, \qquad \forall x\in\mathbb{R}^d, \qquad (\textit{where } dA = \textstyle\sum_j \|v_j\|^2).$$

Example 1.2 *(tight spherical t-designs) The term “tight” is also used for a spherical t-design which gives equality in the estimate*

$$N(d,t) \ge \begin{cases} \binom{d-1+k}{d-1} + \binom{d-2+k}{d-1}, & t=2k;\\ 2\binom{d-1+k}{d-1}, & t=2k+1, \end{cases} \tag{1.4}$$

of [DGS77] for the minimal number $N(d,t)$ of vectors in a spherical t-design for $\mathbb{R}^d$. A tight spherical $(2t+1)$-design is centrally symmetric. Therefore, if $(\pm v_j)$ is a tight spherical $(2t+1)$-design for $\mathbb{R}^d$, then (v_j) is an optimal spherical half-design of order $2t$.

From the known tight spherical $(2t+1)$-designs for $\mathbb{R}^d$, $d \geq 3$ [BB09], [NV12], we have the following optimal spherical half-designs: an orthonormal basis (Example 1.1) which comes from the cross polytope $(\pm e_j)$, 6 vectors in $\mathbb{R}^3$ (order 4) obtained from 12 vertices of the icosahedron, 28 vectors in $\mathbb{R}^7$ (order 4), 276 vectors in $\mathbb{R}^{23}$ (order 4), 120 vectors in $\mathbb{R}^8$ (order 6), 2300 vectors in $\mathbb{R}^{23}$ (order 6), 98280 vectors in $\mathbb{R}^{24}$ (order 10).

2 Optimal spherical half-designs for $\mathbb{R}^2$

The putatively optimal spherical half-designs for $\mathbb{R}^2$ are given by uniformly spaced lines.

Proposition 2.1 *The $n = t+1$ uniformly spaced lines in $\mathbb{R}^2$ given by the vectors*

$$(v_j) = \Bigl\{ \bigl(\cos \frac{\pi}{n} j, \sin \frac{\pi}{n} j\bigr) : j = 0, \ldots, n-1 \Bigr\}$$

are a spherical half–design of order $2t$.

Proof: We will use the cubature rule that for all bivariate polynomials $p \in \Pi_{n-1}(\mathbb{R}^2)$

$$\int_{\mathbb{S}(\mathbb{R}^2)} p(x,y)\, d\sigma(x,y) = \frac{1}{2\pi} \int_0^{2\pi} p(\cos\theta, \sin\theta)\, d\theta = \frac{1}{n} \sum_{j=1}^{n} p\bigl(\cos \frac{2\pi}{n} j, \sin \frac{2\pi}{n} j\bigr).$$

We now verify that (w_j) is a spherical (t,t)–design for $\mathbb{R}^2$, i.e., (1.3) holds. Using the trigonometric identity $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$, and the cubature rule, we have

$$\begin{aligned}
\sum_j \sum_k |\langle v_j, v_k \rangle|^{2t} &= \sum_j \sum_k \Bigl(\cos j \frac{\pi}{n} \cos k \frac{\pi}{n} + \sin j \frac{\pi}{n} \sin k \frac{\pi}{n} \Bigr)^{2t} \\
&= \sum_j \sum_k \Bigl(\cos(j-k) \frac{\pi}{n} \Bigr)^{2t} = \sum_j \sum_k \Bigl(\frac{\cos \frac{2\pi}{n}(j-k) + 1}{2} \Bigr)^{t} \\
&= n^2 \frac{1}{n} \sum_j \Bigl(\frac{\cos \frac{2\pi}{n} j + 1}{2} \Bigr)^{t} = n^2 \frac{1}{2\pi} \int_0^{2\pi} \Bigl(\frac{\cos\theta + 1}{2} \Bigr)^{t} d\theta.
\end{aligned}$$

The integral above simplifies to

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \Bigl(\frac{\cos\theta + 1}{2} \Bigr)^{t} d\theta &= \frac{1}{\pi} \int_0^{2\pi} \Bigl(\cos \frac{\theta}{2} \Bigr)^{2t} \frac{d\theta}{2} = \frac{1}{\pi} \int_0^{\pi} (\cos x)^{2t}\, dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} (\cos x)^{2t}\, dx = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2t-3}{2t-2} \cdot \frac{2t-1}{2t},
\end{aligned}$$

which gives the result. $\square$

The corresponding spherical $(2t+1)$-design of $2(t+1)$ vectors for the circle is a *tight* spherical design (see Example 1.2), and so this configuration is an optimal spherical half–design, which is unique in the class of *rigid* spherical designs (see [BB09]).

3 Optimal spherical half-designs for $\mathbb{R}^3$ and $\mathbb{R}^5$

The putatively optimal spherical half-designs we present here are *weighted*. There is a corresponding notion for t-designs: a sequence of points $v_1, \ldots, v_n$ in $\mathbb{S} \subset \mathbb{R}^d$ and *weights* $w_1, \ldots, w_n \geq 0$, $w_1 + \cdots + w_n = 1$, is said to be a **weighted spherical t-design** if

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{j=1}^{n} w_j\, p(v_j), \tag{3.5}$$

holds for all polynomials of degree $\leq t$. The correspondence of Proposition 1.1 extends. The following is proved in [KP11] using a different definition of spherical half-designs.

Theorem 3.1 *Let (v_j) be a sequence of n vectors in $\mathbb{R}^d$, and*

$$w_j = w_j^{(t)} := \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}}$$

be its weights as spherical half-design of order $2t$. Then following are equivalent:

(i) $(\pm v_j/\|v_j\|)$, $(w_j/2)$ *is a weighted spherical $(2t+1)$-design of $2n$ vectors for $\mathbb{R}^d$.*

(ii) (v_j) *is a weighted spherical half-design of order $2t$ of n vectors for $\mathbb{R}^d$.*

Proof: First suppose that $(\pm v_j/\|v_j\|)$, $(w_j/2)$ is a weighted spherical $(2t+1)$-design of $2n$ vectors for $\mathbb{R}^d$ (the weight for $\pm v_j/\|v_j\|$ is $w_j/2$). Then

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{j=1}^{n} \frac{w_j^{(t)}}{2} \Big\{ p\Big(\frac{v_j}{\|v_j\|}\Big) + p\Big(-\frac{v_j}{\|v_j\|}\Big) \Big\} = \sum_{j=1}^{n} w_j^{(t)}\, p\Big(\frac{v_j}{\|v_j\|}\Big), \qquad \forall p \in \Pi_{2t}^{\circ}(\mathbb{R}^d),$$

so that (v_j) is a a weighted spherical half-design of order $2t$.

Now suppose that (v_j) is a a weighted spherical half-design of order $2t$. Then the integration rule with points $(\pm v_j/\|v_j\|)$ and weights $(w_j/2)$ integrates $\Pi_{2t}^{\circ}(\mathbb{R}^d)$ (by the above calculation). It also integrates the homogeneous polynomials of odd order (since $p(x) + p(-x) = 0$ when p is odd), and the constants (since the weights add to 1). It therefore only remains to show that this rule integrates $\Pi_{2r}^{\circ}(\mathbb{R}^d)$, $1 \leq r < t$. A direct calculation (see [Wal17]) of the condition (1.3) shows that $(\|v_j\|^{t/r-1} v_j)$ is a spherical half-design of order $2r$, $1 \leq r \leq t$. Thus, we have the integration rule

$$\int_{\mathbb{S}} p(x)\, d\sigma(x) = \sum_{j=1}^{n} w_j^{(r)}\, p\Big(\frac{v_j}{\|v_j\|}\Big), \qquad \forall p \in \Pi_{2r}^{\circ}(\mathbb{R}^d),$$

where

$$w_j^{(r)} = \frac{\| \|v_j\|^{t/r-1} v_j \|^{2r}}{\sum_k \| \|v_k\|^{t/r-1} v_k \|^{2r}} = \frac{\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}} = w_j^{(t)}, \qquad 1 \le r \le t.$$

Therefore, for $p \in \Pi_{2r}^\circ(\mathbb{R}^d)$, we have

$$\sum_{j=1}^n \frac{w_j^{(t)}}{2} \Big\{ p\big(\frac{v_j}{\|v_j\|}\big) + p\big(-\frac{v_j}{\|v_j\|}\big) \Big\} = \sum_{j=1}^n w_j^{(r)}\, p\big(\frac{v_j}{\|v_j\|}\big) = \int_{\mathbb{S}} p(x)\, d\sigma(x),$$

as desired. □

This gives a 1-1 correspondence, where the weighted spherical half-designs are given up to multiplication of its vectors by ± 1 and fixed nonzero scalar. In particular, if $(\pm u_j)$, (w_j) is a weighted spherical $(2t+1)$-design, then $v_j := w_j^{1/(2t)} u_j$ gives the corresponding spherical half-design of order $2t$. We note that the minimal number of vectors in a *weighted* spherical half-design of order $2t$ is an increasing function of t (as it is for weighted spherical t-designs). We also observe that a (weighted) spherical $2t$-design is a (weighted) spherical half-design of order $2t$.

When describing our weighted spherical designs, we will use the normalised weights

$$\hat{w}_j := n w_j = \frac{n\|v_j\|^{2t}}{\sum_k \|v_k\|^{2t}}, \qquad 1 \le j \le n,$$

which are all 1 for an unweighted spherical design.

We now summarise our new constructions, with details and explanation to follow.

Theorem 3.2 *There exists*

(i) *a weighted spherical half-design of* 16 *vectors for* $\mathbb{R}^3$ *of order* 8 *(Example 3.1).*

(ii) *a weighted spherical half-design of* 16 *vectors for* $\mathbb{R}^5$ *of order* 4 *(Example 3.3).*

and correspondingly (by Theorem 3.1)

(iii) *a weighted spherical* 9*-design of* 32 *vectors for* $\mathbb{R}^3$.

(iv) *a weighted spherical* 5*-design of* 32 *vectors for* $\mathbb{R}^5$.

The normalised weights $\hat{w}_j$ *for all of these designs are* $\frac{20}{21} \approx 0.9523$, $\frac{36}{35} \approx 1.0286$.

In particular, we observe that tight t-designs for $\mathbb{R}^n$ can exist only for $t \le 5$, $t = 7$ or $t = 11$, and for $n = (2m+1)^2 - 2$ [BD80], [BMV04], so that there is no tight spherical 5-design of 30 points for $\mathbb{R}^5$, and no tight spherical 9-design of 30 points for $\mathbb{R}^3$. This suggests the weighted spherical 5-design and 9-design of 32 points are indeed optimal.

Let U be unitary. Since the unitary image (Uv_j) of a spherical half-design (v_j) for $\mathbb{R}^d$ is also a spherical half-design, one cannot recognise an exact form for spherical half-design from the individual coordinates of the vectors of a numerically generated one. Instead, one must consider the Gramian matrix $(\langle v_j, v_k \rangle)$ of the design, which determines it up to the above unitary equivalence.

In the following example, we first found an exact form for the Gramian, and then recognised the vectors to be the vertices of a pentakis dodecahedron (a Catalan solid). A **pentakis dodecahedron** (or **kisdodecahedron**) is a dodecahedron with a pentagonal pyramid covering each face, i.e., the Kleetope of the dodecahedron.

Example 3.1 *There is a weighted spherical half-design (v_j) of 16 vectors for $\mathbb{R}^3$ of order 8, which is given by the lines through the antipodal vertices of the pentakis dodecahedron (take one of the two vertices) as follows (the six vertices/lines of the icosahedron are the first columns)*

$$[v_j] := \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & \tau & 0 & -1 & \tau & 1 & 1 & 1 & 1 & 0 & 0 & \frac{1}{\tau} & \frac{1}{\tau} & \tau & -\tau \\ \tau & 0 & 1 & \tau & 0 & -1 & 1 & 1 & -1 & -1 & \frac{1}{\tau} & \frac{1}{\tau} & \tau & -\tau & 0 & 0 \\ 1 & \tau & 0 & -1 & \tau & 0 & 1 & -1 & 1 & -1 & \tau & -\tau & 0 & 0 & \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \alpha\Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$$

$$\tau := \frac{1+\sqrt{5}}{2} \text{ (the golden ratio)}, \quad \alpha := \sqrt{\frac{3}{1+\tau^2}}, \quad \Lambda_1 := \left(\tfrac{20}{21}\right)^{\frac{1}{8}} I_6, \quad \Lambda_2 := \left(\tfrac{36}{35}\right)^{\frac{1}{8}} I_{10}.$$

It is easy to verify that (1.3) holds. These vectors have lengths $\|v_j\| = \left(\frac{20}{21}\right)^{\frac{1}{8}}, \left(\frac{36}{35}\right)^{\frac{1}{8}}$ (respectively), and the corresponding normalised weights are

$$\frac{16\left(\frac{20}{21}\right)}{6\left(\frac{20}{21}\right)+10\left(\frac{36}{35}\right)} = \frac{20}{21} \approx 0.9523, \qquad \frac{16\left(\frac{36}{35}\right)}{6\left(\frac{20}{21}\right)+10\left(\frac{36}{35}\right)} = \frac{36}{35} \approx 1.0286. \tag{3.6}$$

By Theorem 3.1, this gives a weighted spherical 9-design of 32 points for $\mathbb{R}^3$. By way of comparison, Hardin and Sloan [HS96] give numerical evidence for a spherical 8-design of $n = 36, 40, 42, \geq 44$ points, and of a spherical 9-design of $n = 48, 50, 52, \geq 54$ points for $\mathbb{R}^3$ (Womersley [Wom17] suggests $n = 50$).

Equiangular lines have long been studied in relation to spherical designs. The unit vectors (v_j) in $\mathbb{R}^d$ (or the lines that they give) are said to be **equiangular** if they have equal cross-correlation, i.e.,

$$|\langle v_j, v_k\rangle| = \alpha, \quad j \neq k, \qquad \text{for some } \textit{angle } \alpha > 0.$$

Example 3.2 *(Maximal lines) The number n of equiangular lines in $\mathbb{R}^d$ satisfies the absolute (or Gerzon) bound $n \leq \frac{1}{2}d(d+1)$. When this bound is attained, the set of lines has angle $\frac{1}{\sqrt{d+2}}$, and hence is a spherical half-design of order 4, by checking (1.3), i.e.,*

$$n \cdot 1 + (n^2 - n)\Big(\frac{1}{\sqrt{d+2}}\Big)^4 = \frac{3}{4}\frac{d(d+1)^2}{d+2} = \frac{1 \cdot 3}{d(d+2)} n^2.$$

Such lines can exist only when $d = 2, 3$ or $d+2$ is the square of an odd integer. On the other hand, the spherical 5-design of $2n = d(d+1)$ vectors these lines give is tight, since (1.4) holds as

$$N(d,5) = 2\binom{d-1+2}{d-1} = d(d+1) = 2n.$$

Therefore a set of $n = \frac{1}{2}d(d+1)$ equiangular lines in $\mathbb{R}^d$ gives an optimal spherical half-design of order 4. These are known to exist for $d = 2, 3, 7, 23$ (there are just a few cases of tight spherical t-designs known, see Example 1.2).

In our final example, various subsets of equiangular lines were recognised from the Gramian, which ultimately led to the presentation we now give.

Example 3.3 *There is a weighted spherical half-design* (v_j) *of* 16 *vectors for* $\mathbb{R}^5$ *of order* 4*. This consists of* 6 *equiangular lines in* $\mathbb{R}^5$ *at an angle of* $\frac{1}{5}$ *(the vertices of a simplex) given by vectors of length* $(\frac{20}{21})^{\frac{1}{4}}$*, and* 10 *equiangular lines in* $\mathbb{R}^5$ *at an angle of* $\frac{1}{3}$ *given by vectors of length* $(\frac{36}{35})^{\frac{1}{4}}$*, where the angle between lines from different families is* $\frac{1}{\sqrt{5}}$*. A direct calculation shows that (1.3) holds for* $t = 2$*, i.e.,*

$$\Big(\frac{20}{21}\Big)^{\frac{1}{2}4}\big(30(\tfrac{1}{5})^4+6\big)+\Big(\frac{36}{35}\Big)^{\frac{1}{2}4}\big(90(\tfrac{1}{3})^4+10\big)+\Big(\frac{20}{21}\frac{36}{35}\Big)^{\frac{1}{4}4}\big(120(\tfrac{1}{\sqrt{5}})^4\big)=\frac{3}{35}\Big(6\frac{20}{21}+10\frac{36}{35}\Big)^2.$$

The Gramian can be presented as the (rank 5*) block matrix (the six lines first)*

$$\begin{pmatrix}\Lambda_1 & \\ & \Lambda_2\end{pmatrix}\begin{pmatrix}\frac{1}{2}BB^T & B\\ B^T & \frac{5}{6}B^TB\end{pmatrix}\begin{pmatrix}\Lambda_1 & \\ & \Lambda_2\end{pmatrix}, \qquad \Lambda_1 := \big(\tfrac{20}{21}\big)^{\frac{1}{4}}I_6, \quad \Lambda_2 := \big(\tfrac{36}{35}\big)^{\frac{1}{4}}I_{10},$$

where

$$B = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1\\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1\\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix}. \tag{3.7}$$

The weights for this design are $\frac{20}{21} \approx 0.9523$*,* $\frac{36}{35} \approx 1.0286$*, the same as in Example 3.1. This design gives a 32-point weighted spherical* 5*-design for* $\mathbb{R}^5$*. By way of comparison, a tight spherical* 5*-design for* $\mathbb{R}^5$ *(which does not exist) would have* $N(5,5) = 30$ *points.*

The 6×10 matrix B of (3.7) is very interesting, since its columns and its rows give equiangular lines in $\mathbb{R}^5$, i.e.,

$$A = \tfrac{1}{2}BB^T = \begin{pmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5}\\ -\frac{1}{5} & 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5}\\ -\frac{1}{5} & -\frac{1}{5} & 1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5}\\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 1 & -\frac{1}{5} & -\frac{1}{5}\\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 1 & -\frac{1}{5}\\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 1 \end{pmatrix},$$

$$C = \tfrac{5}{6}B^TB = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}.$$

The presentation of these lines also seems to be new, as they do not appear in Janet Tremain's list of concrete constructions of equiangular lines [Tre08].

The weighted 16-point design for $\mathbb{R}^3$ and $\mathbb{R}^5$ share the following properties

- They have the same weights $\frac{20}{21} \approx 0.9523$, $\frac{36}{35} \approx 1.0286$ (which are rationals).
- For the 32-point weighted spherical t-designs that they correspond to, the number of points in an unweighted tight spherical design is $30 = N(3,9) = N(5,5)$.
- Both are the orbit of two vectors, under the projective symmetry group.

This seems to be a curious coincidence, since the designs are of different orders and are in different dimensions. Moreover, the projective symmetry groups (see [CW18]) of the designs are different: for the design for $\mathbb{R}^3$ it is A_5 (the symmetries of the dodecahedron factored by $\langle -I \rangle$), and for the design for $\mathbb{R}^5$ it is S_6.

Motivated by the fact that the projective symmetry group of the design for $\mathbb{R}^5$ of Example 3.3 is (isomorphic to) S_6, we can give the following neat presentation of it:

$$V = [v_1, \ldots, v_{16}] = [\alpha BB^T, \beta B] \in \mathbb{R}^{6\times 16}, \qquad \frac{12}{5}\alpha^2 = \frac{1}{2}\sqrt{\frac{20}{21}}, \quad \beta^2 = \frac{5}{6}\sqrt{\frac{36}{35}},$$

where B is given by (3.7), since the Gramian of this V is

$$V^T V = \begin{pmatrix} \alpha BB^T \\ \beta B^T \end{pmatrix} \begin{pmatrix} \alpha BB^T & \beta B \end{pmatrix} = \begin{pmatrix} \alpha^2 (BB^T)^2 & \alpha\beta BB^T B \\ \alpha\beta B^T BB^T & \beta^2 B^T B \end{pmatrix} = \begin{pmatrix} \frac{12}{5}\alpha^2 BB^T & \frac{12}{5}\alpha\beta B \\ \frac{12}{5}\alpha\beta B^T & \beta^2 B^T B \end{pmatrix}.$$

The vectors (v_j) are in the 5-dimensional subspace $\{x \in \mathbb{R}^6 : x_1 + \cdots + x_6 = 0\}$ of $\mathbb{R}^6$, and S_6 acts on them by permutation of the coordinates (the first six vectors are the orthogonal projections of the standard basis vectors onto the subspace). This action on the subspace is irreducible, indeed it is the complex reflection group $G(1,1,6)$ in the first infinite family of the Shephard-Todd classification of complex reflection groups [LT09].

4 Conclusion

We gave explicit examples of putatively optimal weighted spherical half-designs, which are the orbit of *two* vectors (Examples 3.1 and 3.3) of close to equal norm. This suggests that the *weighted* spherical designs with a high degree of symmetry and a small number of vectors are natural in some cases. The study of such spherical t-designs is still in its infancy (see [SW04], [BG12], [Wom17], [ZC18]).

We also clarified the very close relationship between (centrally symmetric) weighted spherical $(2t+1)$-designs and weighted spherical half-designs of order $2t$ (Theorem 3.1). In this regard, it would be interesting to know if there are any optimal (weighted) spherical $(2t+1)$-designs which are not centrally symmetric for t large.

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