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The Fourier transform of a projective group frame

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A B S T R A C T

Many tight frames of interest are constructed via their Gramian matrix (which determines the frame up to unitary equivalence). Given such a Gramian, it can be determined whether or not the tight frame is projective group frame, i.e., is the projective orbit of some group G (which may not be unique). On the other hand, there is complete description of the projective group frames in terms of the irreducible projective representations of G. Here we consider the inverse problem of taking the Gramian of a projective group frame for a group G, and identifying the cocycle and constructing the frame explicitly as the projective group orbit of a vector v (decomposed in terms of the irreducibles). The key idea is to recognise that the Gramian is a group matrix given by a vector $f \in \mathbb{C}^G$, and to take the Fourier transform of f to obtain the components of v as orthogonal projections.

This requires the development of a theory of group matrices and the Fourier transform for projective representations. Of particular interest, we give a block diagonalisation of (projective) group matrices. This leads to a unique Fourier decomposition of the group matrices, and a further fine-scale decomposition into low rank group matrices.

1. Introduction

Let G be a finite abstract group. A map $\rho : G \to \mathrm{GL}(\mathbb{C}^d)$ is said to be a **projective representation** of G of dimension $d = d_\rho$ with **cocycle** (or **multiplier**) $\alpha : G \times G \to \mathbb{C}$ if

$$\rho(g)\rho(h) = \alpha(g, h)\rho(gh), \qquad \forall g, h \in G. \tag{1.1}$$

Two projective representations ρ, α and $\tilde{\rho}, \tilde{\alpha}$ are **equivalent** if there is a map $c : G \to \mathbb{C}$ with $\tilde{\rho} = cT\rho T^{-1}$. The theory of projective representations follows that of *ordinary* representations (when $\alpha = 1$), and will be

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introduced as needed. The key point for now, is that for certain groups there are representations which are not ordinary representations (the possible α are indexed by the Schur multiplier group).

For a nonzero vector $v \in \mathbb{C}^d$ and a projective unitary representation ρ, α, we define a **projective group frame** or (G, α)**-frame** to be the projective G-orbit of v, i.e., the sequence of vectors

$$(\phi_g) = (gv) = (\rho(g)v)_{g \in G}, \qquad gv := \rho(g)v.$$

By (1.1), this satisfies

$$g\phi_h = \rho(g)\rho(h)v = \alpha(g,h)\phi_{gh}, \tag{1.2}$$

and if the representation is unitary, i.e., $\rho(g)^* = \rho(g)^{-1}$, then

$$\langle \phi_h, \phi_g \rangle = \langle \rho(h)v, \rho(g)v \rangle = \langle \rho(g)^{-1}\rho(h)v, v \rangle = \langle \frac{\rho(g^{-1}h)v}{\alpha(g, g^{-1}h)}, v \rangle. \tag{1.3}$$

These generalise *group frames* (also known as *G-frames*), which are the case when $\alpha = 1$. A sequence of vectors (v_j) in $\mathbb{C}^d$ is a (**normalised**) **tight frame** for $\mathbb{C}^d$ if it satisfies

$$f = \sum_{j=1}^{n} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{C}^d. \tag{1.4}$$

The term *Parseval frame* is also commonly used. Normalised tight G-frames, which are natural generalisations of orthonormal bases, have numerous applications [5], [2], [10] [17], and their structure is now well understood [21].

On the other hand, tight (G, α)-frames are of great interest also [14], [11], [6]. These are sometimes referred to as *projective frame representations* or *group-like systems*. As an example, *SICs* (sets of d^2 equiangular lines in $\mathbb{C}^d$) come as a tight projective group frame for $\mathbb{Z}_d \times \mathbb{Z}_d$ [1], as do *MUBs* (*mutually unbiased bases*) [22], [13], and many real and complex *spherical t-designs* [12], [20].

Until recently, such tight frames have been studied by considering them as G-frames for a larger group for which $\rho(G)$ contains scalar matrices, and by accounting for the scalar multiples of a given vector. One such approach is to consider a *canonical abstract error group* (the enlarged group) with *index group G* [7]. Recent calculations, most notably [3] (which uses the term *twisted* group frame), suggest that they can be viewed as projective group frames, with the theory of group frames following with little extra work (a twist if you like). Our contribution to this burgeoning theory of projective group frames is to study them via their **Gramian** matrix $V^*V \in \mathbb{C}^{G \times G}$, $V = [\phi_g]_{g \in G}$, which has (g, h)-entry given by (1.3). The key points and results are the following:

- The Gramian determines a projective group frame up to unitary equivalence. If the representation ρ is unitary (as must be the case for a tight frame), then the Gramian is a (G, α)-matrix (see Definition 3.1).
- Each (G, α)-matrix is determined by a $\nu \in \mathbb{C}^G$, and they form a C^*-algebra $M_{(G,\alpha)}$.
- The tight (G, α)-frames correspond to the orthogonal projections in $M_{(G,\alpha)}$.
- Each tight (G, α)-frame $(gv)_{g \in G}$ can be decomposed $v = \oplus_j v_j$, $v_j \in V_j$, where the action of ρ on V_j is irreducible.
- Given the Gramian of tight projective group frame for G, i.e., a (G, α)-matrix P which is an orthogonal projection, we find a vector $v = \oplus_j v_j$ and representations $\rho_j \in R$, where R is a complete set of irreducible representations for α, such that the projective group frame $(\sum_j \rho_j(v_j))_{g \in G}$ has Gramian P. In other words, we give a concrete construction of a projective group frame (known only from its Gramian) in terms of the irreducibles involved (Theorem 8.2).

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- The construction above relies on a simultaneous unitary block diagonalisation of the (G,α)-matrices (Theorem 7.2), related to an appropriately defined Fourier transform for projective representations of a finite group. The blocks preserve the spectral structure of the (G,α)-matrix. In particular, the tight (G,α)-frames correspond to the case when all of the blocks are orthogonal projections. This characterisation of the tight (G,α)-frames, gives the usual characterisation (in terms of the decomposition of v into irreducibles), and gives a natural description of the *central* (G,α)*-frames* (as those where the blocks are 0 or I). The decomposition of (G,α)-matrices given by our Fourier transform is of independent interest, e.g., it allows the determinant of a general (G,α)-matrix to be factored. There is also a fine-scale decomposition into low rank (G,α)-matrices, which is not unique.

We now proceed, following the above outline.

2. The Schur multiplier

We first consider the functions $\alpha : G \times G \to \mathbb{C}$ which can be cocycles of a projective representation of G. Given (1.1), multiplying out $\{\rho(x)\rho(y)\}\rho(z) = \rho(x)\{\rho(y)\rho(z)\}$, leads to the following multiplication rule for cocycles

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z). \tag{2.5}$$

Every α satisfying (2.5) does come from a projective representation. Indeed, with $(e_g)_{g\in G}$ being the standard basis vectors for $\mathbb{C}^G$, we can define $\rho : G \to \mathrm{GL}(\mathbb{C}^G)$ by

$$\rho(g)e_h := \alpha(g,h)e_{gh}, \tag{2.6}$$

and use (2.5) to verify that it is such a representation:

$$\begin{aligned}\rho(g)\rho(h)e_k &= \rho(g)\alpha(h,k)e_{hk} = \alpha(h,k)\alpha(g,hk)e_{ghk} \\ &= \alpha(g,h)\alpha(gh,k)e_{ghk} = \alpha(g,h)\rho(gh)e_k.\end{aligned}$$

Henceforth a map $\alpha : G \times G \to \mathbb{C}$ satisfying (2.5) will be called a **cocycle** (more properly a **2-cocycle**) or **multiplier** of G. The set of cocycles is an abelian group under pointwise multiplication, which is commonly denoted by $Z^2(G,\mathbb{C}^\times)$. If projective representations ρ,α and $\tilde\rho,\tilde\alpha$ are **equivalent**, i.e., $\tilde\rho = cT\rho T^{-1}$, where $c : G \to \mathbb{C}$, then

$$\tilde\rho(g)\tilde\rho(h) = c_gT\rho(g)T^{-1}c_hT\rho(h)T^{-1} = c_gc_hT\alpha(g,h)\rho(gh)T^{-1} = \frac{c_gc_h}{c_{gh}}\alpha(g,h)\tilde\rho(gh),$$

so that

$$\tilde\alpha(g,h) = \beta(g,h)\alpha(g,h), \qquad \beta(g,h) := \frac{c_gc_h}{c_{gh}}.$$

The function β above is a cocycle, called a **coboundary** (or 2-coboundary), since

$$\beta(x,y)\beta(xy,z) = \frac{c_xc_y}{c_{xy}}\frac{c_{xy}c_z}{c_{xyz}} = \frac{c_xc_yc_z}{c_{xyz}} = \frac{c_xc_{yz}}{c_{xyz}}\frac{c_yc_z}{c_{yz}} = \beta(x,yz)\beta(y,z).$$

The coboundaries form a subgroup $B^2(G,\mathbb{C}^\times)$ of $Z^2(G,\mathbb{C}^\times)$. The quotient group

$$M(G) = H^2(G,\mathbb{C}^\times) := Z^2(G,\mathbb{C}^\times)/B^2(G,\mathbb{C}^\times)$$

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is called the **Schur multiplier** (or **second homology group** $H_2(G,\mathbb{Z})$ of G). The Schur multiplier is a finite abelian group whose exponent divides the order of G. If G has a nontrivial cyclic Sylow p-subgroup, then p does not divide $|M(G)|$. There do exist finite groups with nontrivial Schur multipliers, and hence projective representations which are not ordinary. The first few cases of nontrivial Schur multipliers are for certain groups of order $4, 8, 9, 12$.

Every projective representation ρ, α is equivalent to one $\tilde{\rho}, \tilde{\alpha}$, where the $\tilde{\rho}$ is unitary. For a unitary representation, $|\det(\rho(g))| = 1$, and so by taking determinants of (1.1), we have

$$|\alpha(g,h)| = 1, \qquad g,h \in G. \tag{2.7}$$

Consequently, a cocycle satisfying (2.7) is said to be a **unitary** cocycle. For the purpose of defining the Schur multiplier, one can suppose that all of the cocycles are unitary, and satisfy the normalisation condition that $\alpha(1,1) = 1$.

We now list some properties of cocycles that we will often use. Since

$$\rho(1)^2 = \alpha(1,1)\rho(1), \qquad \rho(g)\rho(g^{-1}) = \alpha(g,g^{-1})\rho(1),$$

we have $\rho(1) = \alpha(1,1)I$, and

$$\rho(g)^{-1} = \frac{\rho(g^{-1})}{\alpha(g,g^{-1})\alpha(1,1)}. \tag{2.8}$$

Similarly, $\rho(g)\rho(1) = \alpha(g,1)\rho(g)$ and $\rho(1)\rho(g) = \alpha(1,g)\rho(g)$, give

$$\alpha(1,g) = \alpha(g,1) = \alpha(1,1), \quad \alpha(g,g^{-1}) = \alpha(g^{-1},g), \qquad \forall g \in G. \tag{2.9}$$

3. The C^*-algebra of (G,α)-matrices

Motivated by the formula (1.3) for the Gramian of a (G,α)-frame, we have:

Definition 3.1. We say that $A = [a_{g,h}]_{g,h\in G} \in \mathbb{C}^{G\times G}$ is a (G,α)**-matrix** if

$$a_{g,h} = M_\alpha(\nu)_{g,h} := \frac{\nu(g^{-1}h)}{\alpha(g,g^{-1}h)}, \tag{3.10}$$

for some $\nu : G \to \mathbb{C}$. We denote the set of (G,α)-matrices by $M_{(G,\alpha)}$.

Other variations are discussed in §10, e.g., [3] consider the matrices $M(\nu) = M_{1/\alpha}(\nu)$.

Given a (G,α)-matrix $A \in M_{(G,\alpha)}$, the cocycle α (or part of it) can be recovered via

$$\frac{\alpha(g,h)}{\alpha(1,1)} = \frac{a_{1,h}}{a_{g,gh}} = \frac{\nu(h)}{\alpha(1,h)}\frac{\alpha(g,h)}{\nu(h)}, \qquad \nu(h) \neq 0,$$
$$\frac{\alpha(g,h)}{\alpha(1,1)} = \frac{a_{gh,g}}{a_{h,1}} = \frac{\nu(h^{-1})}{\alpha(gh,h^{-1})}\frac{\alpha(h,h^{-1})}{\nu(h^{-1})}, \qquad \nu(h^{-1}) \neq 0.$$

From now on, we let $n = |G|$, and refer to matrices indexed by the elements of G as $n \times n$ matrices or $G \times G$ matrices.

Lemma 3.1. *The (G,α)-matrices are an n-dimensional subspace of the $n \times n$ matrices which is closed under matrix multiplication, i.e.,*

$$M_\alpha(a\nu+b\mu)=aM_\alpha(\nu)+bM_\alpha(\mu), \qquad a,b\in\mathbb{C},$$
$$M_\alpha(\nu)M_\alpha(\mu)=M_\alpha(\nu *_\alpha \mu), \qquad (\nu *_\alpha \mu)(g):=\sum_{t\in G}\frac{\nu(gt^{-1})\mu(t)}{\alpha(gt^{-1},t)}. \tag{3.11}$$

When α is unitary, i.e., $|\alpha|=1$, then they are also closed under the Hermitian transpose

$$(M_\alpha(\nu))^*=M_\alpha(\nu^{*,\alpha}), \qquad \nu^{*,\alpha}(a):=\overline{\nu(a^{-1})}\alpha(a,a^{-1})\alpha(1,1). \tag{3.12}$$

Proof. Since $\alpha(g,g^{-1}a)\alpha(a,a^{-1}h)=\alpha(g,g^{-1}h)\alpha(g^{-1}a,a^{-1}h)$, we have

$$\begin{aligned}\big(M_\alpha(\nu)M_\alpha(\mu)\big)_{g,h} &= \sum_{a\in G} M_\alpha(\nu)_{g,a}M_\alpha(\mu)_{a,h} = \sum_{a\in G}\frac{\nu(g^{-1}a)}{\alpha(g,g^{-1}a)}\frac{\mu(a^{-1}h)}{\alpha(a,a^{-1}h)}\\ &= \frac{1}{\alpha(g,g^{-1}h)}\sum_{a\in G}\frac{\nu(g^{-1}h(a^{-1}h)^{-1})\mu(a^{-1}h)}{\alpha(g^{-1}h(a^{-1}h)^{-1},a^{-1}h)} = \frac{(\nu *_\alpha \mu)(g^{-1}h)}{\alpha(g,g^{-1}h)}.\end{aligned}$$

Since $\alpha(h,h^{-1}g)\alpha(g,g^{-1}h)=\alpha(h,1)\alpha(h^{-1}g,g^{-1}h)$, we have

$$\begin{aligned}(M_\alpha(\nu)^*)_{g,h} &= \overline{M_\alpha(\nu)_{h,g}} = \overline{\nu(h^{-1}g)}\alpha(h,h^{-1}g)\\ &= \frac{\overline{\nu((g^{-1}h)^{-1})}\alpha(g^{-1}h,(g^{-1}h)^{-1})\alpha(1,1)}{\alpha(g,g^{-1}h)} = M_\alpha(\nu^{*,\alpha})_{g,h}.\end{aligned}$$

We will call $\nu *_\alpha \mu$ the α-**convolution** (of ν and μ). □

Example 3.1. A calculation gives

$$e_g *_\alpha e_h = \frac{e_{gh}}{\alpha(g,h)},$$

so that

$$M_\alpha(e_g)M_\alpha(e_h)=\frac{M_\alpha(e_{gh})}{\alpha(g,h)}.$$

In particular, taking $g=h=1$, gives $I=\alpha(1,1)M_\alpha(e_1)$, so the identity matrix is a (G,α)-matrix, hence (by Cayley–Hamilton) the inverse of a nonsingular (G,α)-matrix is a (G,α)-matrix. Further, for α unitary, the limit formulas

$$A^+=\lim_{\delta\to 0^+}(A^*A+\delta I)^{-1}A^* = \lim_{\delta\to 0^+}A^*(AA^*+\delta I)^{-1}$$

for the pseudoinverse, shows that $M_{(G,\alpha)}$ is closed under the pseudoinverse.

Example 3.2. For α unitary, the matrix $P=M_\alpha(\nu)$ is an orthogonal projection, i.e., satisfies $P^2=P$, $P^*=P$, if and only if $\nu *_\alpha \nu=\nu$ and $\nu^{*,\alpha}=\nu$.

Example 3.3. For ρ (and hence α) unitary, we have $\rho_{jk}^{*,\alpha}=\rho_{kj}$, $1\le j,k\le d_\rho$, i.e.,

$$M_\alpha(\rho_{jk})^*=M_\alpha(\rho_{kj}).$$

This follows from (2.8), by taking the (k,j)-entry of $\rho(g)=\alpha(g,g^{-1})\alpha(1,1)\rho(g^{-1})^*$.

4. The twisted group algebra

We now describe how the basic theory of representations extends to the projective case. Let $\rho : G \to \mathrm{GL}(V)$ be a projective representation of G for a cocycle α on V. This gives a "projective group action" of G on the $\mathbb{C}$-vector space V

$$g \cdot v = gv := \rho(g)v, \qquad \forall v \in V. \tag{4.13}$$

This is not a *group action* in the usual sense, since

$$g \cdot (h \cdot v) = \alpha(g,h)(gh \cdot v).$$

Nevertheless, we will talk about G-invariant subspaces of V, etc. As in the ordinary case, we say that a representation ρ, α on V is **irreducible** if the only G-invariant subspaces of V are 0 and V, i.e., for any nonzero vector v the G-orbit $(g \cdot v)_{g \in G}$ spans V.

Let $\mathbb{C}G$ denote the set of formal $\mathbb{C}$-linear combinations of the elements of G. This becomes are ring, which we denote by $(\mathbb{C}G)_\alpha$, under the multiplication

$$g \cdot_\alpha h := \alpha(g,h)gh,$$

extended linearly. We note that the cocycle multiplication rule (2.5) is equivalent to the associativity of the multiplication, since

$$(g \cdot_\alpha h) \cdot_\alpha k = \alpha(g,h)(gh \cdot_\alpha k) = \alpha(g,h)\alpha(gh,k)ghk$$
$$g \cdot_\alpha (h \cdot_\alpha k) = g \cdot_\alpha (\alpha(h,k)hk) = \alpha(h,k)\alpha(g,hk)ghk.$$

Moreover, $(\mathbb{C}G)_\alpha$ is an algebra, which generalises the group algebra $\mathbb{C}G$ (the case $\alpha = 1$), and is called the **α-twisted group algebra** (over $\mathbb{C}$). The vector space V becomes an $(\mathbb{C}G)_\alpha$-module under the operation $(\mathbb{C}G)_\alpha \times V \to V$ given by extending (4.13) linearly. Conversely, if V is a $(\mathbb{C}G)_\alpha$-module, then

$$\rho(g)v := g \cdot v, \qquad g \in G, \; v \in V,$$

defines a projective representation $\rho : G \to \mathrm{GL}(V)$ for α. In the language of modules:

The G-invariant subspaces V are precisely the $(\mathbb{C}G)_\alpha$-submodules of V.

Since $(\mathbb{C}G)_\alpha$ is semisimple, it follows that if ρ, α is a projective representation on V, then V decomposes as direct sum $V = \oplus_j V_j$ of irreducible $(\mathbb{C}G)_\alpha$-submodules, i.e., each $\rho|_{V_j}$ is an irreducible projective representation for the cocycle α. When ρ is unitary, the decomposition of V can be taken to be an orthogonal direct sum. There are finitely many such irreducible representations of G up to equivalence, which we now describe.

The representation $\rho : G \to \mathrm{GL}(\mathbb{C}^G)$ for α given by (2.6) generalises the (left) regular (ordinary) representation. It can also be thought of as a representation on $\mathbb{C}G$, via the identification of $\mathbb{C}^G$ with $\mathbb{C}G$, i.e., $\rho(g)h := \alpha(g,h)gh$. We will call it the **regular α-representation for G**. As in the ordinary case, the regular α-representation decomposes as a direct sum of irreducible representations (for the fixed cocycle α), with each irreducible occuring with multiplicity given by its dimension. In particular, if $R = R_\alpha$ is a **complete set of irreducible representations** i.e., each representation occurs exactly once in R up to equivalence, then we have

$$n = |G| = \dim(\mathbb{C}^G) = \sum_{\rho \in R} d_\rho^2. \tag{4.14}$$

We write $[\rho]$ for the equivalence class of ρ (where α is fixed) and $\rho \approx \xi$ for the equivalence, e.g., $\chi_\rho = \chi_{[\rho]}$ means that χ_ρ depends only on ρ up to equivalence. Schur's lemma extends (for $(\mathbb{C}G)_\alpha$-homomorphisms between irreducibles), and from it (see [4]), one has

Theorem 4.1. *Fix α. If ρ and ξ are irreducible projective representations of G on V_1 and V_2, then*

$$\frac{1}{|G|}\sum_{g\in G}\xi(g)^{-1}L\rho(g) = \frac{1}{|G|}\sum_{g\in G}\frac{\xi(g^{-1})L\rho(g)}{\alpha(g,g^{-1})\alpha(1,1)} = \frac{\mathrm{trace}(L)}{\dim(V_1)}\begin{cases}0, & \rho\not\approx\xi;\\ I, & \rho=\xi,\end{cases}$$

for all linear maps $L: V_1 \to V_2$. In particular, if ρ and ξ map to matrices, then

$$\sum_{g\in G}\frac{\xi_{\ell m}(g^{-1})\rho_{jk}(g)}{\alpha(g,g^{-1})\alpha(1,1)} = \frac{|G|}{d_\rho}\begin{cases}0, & \rho\not\approx\xi;\\ \delta_{j\ell}\delta_{km}, & \rho=\xi.\end{cases} \tag{4.15}$$

When ρ is unitary, by (2.8) we may write (4.15) as

$$\langle\rho_{jk},\xi_{m\ell}\rangle = \frac{|G|}{d_\rho}\begin{cases}0, & \rho\not\approx\xi;\\ \delta_{jm}\delta_{k\ell}, & \rho=\xi,\end{cases} \tag{4.16}$$

which we will refer to the *orthogonality of coordinates* (of irreducible representations). This leads to a character theory and Fourier transform for projective representations.

5. Character theory for projective representations

Invariants of the equivalence class

$$[\rho] = \{T\rho T^{-1} : T \text{ is invertible}\}$$

of a projective representation $\rho: G \to \mathrm{GL}(\mathbb{C}^{d_\rho})$ include its α-**character** $\chi_\rho = \chi_{[\rho]} \in \mathbb{C}^G$

$$\chi_\rho(g) := \mathrm{trace}(\rho(g)) = \textstyle\sum_{j=1}^{d_\rho}\rho_{jj}(g), \qquad g\in G,$$

and the subspace of its coordinates

$$U_{\rho,\alpha} = U_{[\rho],\alpha} := \mathrm{span}\{\rho_{jk} : 1 \le j,k \le d_\rho\}.$$

It follows from (4.14) and (4.15) that we have the orthogonal decomposition

$$\mathbb{C}^G = \bigoplus_{\rho\in R} U_{\rho,\alpha}. \tag{5.17}$$

Since $\chi_\rho \in U_{\rho,\alpha}$, the α-characters of inequivalent irreducibles are orthogonal. From

$$\rho(1) = \alpha(1,1)I, \qquad \rho(hgh^{-1}) = \frac{\alpha(hgh^{-1},h)}{\alpha(h,g)}\rho(h)\rho(g)\rho(h)^{-1},$$

it follows that the α-characters satisfy

$$\chi_\rho(1) = \alpha(1,1)d_\rho, \qquad \chi_\rho(hgh^{-1}) = \frac{\alpha(hgh^{-1},h)}{\alpha(h,g)}\chi_\rho(g). \tag{5.18}$$

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Furthermore, if ρ is unitary, then (2.8) gives

$$\chi_\rho(g^{-1}) = \alpha(g,g^{-1})\alpha(1,1)\overline{\chi_\rho(g)}. \tag{5.19}$$

If $\rho : G \to \mathrm{GL}(V)$ is a representation on V, and $V = \oplus_j V_j$ is a decomposition into irreducibles, then

$$\chi_\rho = \sum_j \chi_{\rho|_{V_j}} = \sum_{\xi\in R} m_\xi \chi_\xi, \tag{5.20}$$

where m_ξ is the multiplicity of the irreducible ξ. Since the α-characters of inequivalent irreducibles are orthogonal, we may determine m_ξ from χ_ρ via

$$\langle \chi_\rho, \chi_\xi\rangle = m_\xi \langle \chi_\xi, \chi_\xi\rangle = m_\xi \tfrac{|G|}{d_\xi} d_\xi = m_\xi |G|.$$

Applying this to the regular α-represention (2.6) gives $m_\xi = d_\xi$, and hence (4.14).

Motivated by (5.18), a function $f : G \to \mathbb{C}$ is called an α-**class function** if

$$f(hgh^{-1}) = \frac{\alpha(hgh^{-1},h)}{\alpha(h,g)} f(g), \qquad \forall g,h \in G,$$

and $g \in G$ is a α-**element** of G if

$$\frac{\alpha(hgh^{-1},h)}{\alpha(h,g)} = 1, \quad \forall h \in C_G(g) \quad \iff \quad \alpha(g,h) = \alpha(h,g), \quad \forall h \in \mathrm{C}_G(g).$$

It can be shown that the subspace of α-class functions (which contains the α-characters) has an orthogonal basis given by the irreducible α-characters, and its dimension, i.e., the number of irreducible projective representations (up to equivalence), equals the number of conjugacy classes of G which contain a α-element (see [4] for details).

6. Fourier analysis

Let $R = R_\alpha$ be a complete set of inequivalent irreducible projective representations of a finite group G for a given cocycle α. We define the α-**Fourier transform** F_α of $f : G \to \mathbb{C}$ at $\rho \in R$ to be the linear map given by

$$F_\alpha(f)(\rho) = (F_\alpha f)_\rho := \sum_{a\in G} \frac{f(a)\rho(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)}. \tag{6.21}$$

When ρ is unitary, i.e., $\rho(a)^* = \rho(a)^{-1}$, this simplifies to

$$(F_\alpha f)_\rho = \sum_{a\in G} f(a)\rho(a)^{-1} = \sum_{a\in G} f(a)\rho(a)^*. \tag{6.22}$$

We observe that the spectral structure of $(F_\alpha f)_\rho$ depends only on ρ up to equivalence, since

$$(F_\alpha f)_{T\rho T^{-1}} = T(F_\alpha f)_\rho T^{-1}. \tag{6.23}$$

To be able to compare with some presentations, we also define the following variant

$$(\mathcal{F}_\alpha f)_\rho := \sum_{a\in G} \frac{f(a)\rho(a)}{\alpha(a,a^{-1})\alpha(1,1)} = (F_\alpha \tilde{f})_\rho, \quad \tilde{f}(a) := f(a^{-1}). \tag{6.24}$$

Example 6.1. By (2.8), the Fourier transforms of the standard basis vectors are

$$(F_\alpha e_g)_\rho = \frac{\rho(g^{-1})}{\alpha(g,g^{-1})\alpha(1,1)} = \rho(g)^{-1}, \qquad (\mathcal{F}_\alpha e_g)_\rho = \frac{\rho(g)}{\alpha(g,g^{-1})\alpha(1,1)} = \rho(g^{-1})^{-1}.$$

We note that $f \mapsto ((F_\alpha f)_\rho)_{\rho\in R}$ and $f \mapsto ((\mathcal{F}_\alpha f)_\rho)_{\rho\in R}$ are linear maps $\mathbb{C}^G \to \oplus_\rho M_{d_\rho}(\mathbb{C})$ between spaces of dimension $n = |G| = \sum_\rho d_\rho^2$. The corresponding **inverse α-Fourier transforms** at $A = (A_\rho)_{\rho\in R}$ are given by

$$(F_\alpha^{-1}A)(a) := \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}(A_\rho \rho(a)), \qquad (\mathcal{F}_\alpha^{-1}A)(a) := (F_\alpha^{-1}A)(a^{-1}). \tag{6.25}$$

These are inverses of each other, and we can extend the other basic results of Fourier analysis (for when the multiplier is $\alpha = 1$), as follows.

Theorem 6.1. *The α-Fourier transform and inverse α-Fourier transform are inverses of each other, and*

$$F_\alpha(\nu)F_\alpha(\mu) = F_\alpha(\mu *_\alpha \nu), \qquad \mathcal{F}_\alpha(\nu)\mathcal{F}_\alpha(\mu) = \mathcal{F}_\alpha(\nu *_{\tilde\alpha} \mu), \tag{6.26}$$

where $\tilde\alpha(a,b) := \alpha(b^{-1},a^{-1})$. There is the Plancherel formula

$$\sum_{a\in G} \frac{\nu(a)\mu(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)} = \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\big((F_\alpha\nu)_\rho (F_\alpha\mu)_\rho\big), \tag{6.27}$$

and when each $\rho \in R$ is unitary

$$\langle \nu,\mu\rangle := \sum_{a\in G} \nu(a)\overline{\mu(a)} = \frac{1}{|G|}\sum_\rho d_\rho \langle (F_\alpha\nu)_\rho, (F_\alpha\mu)_\rho\rangle, \tag{6.28}$$

where $\langle A,B\rangle := \operatorname{trace}(AB^*)$ is the Frobenius inner product on matrices.

Proof. Since F_α, F_α^{-1} and $\mathcal{F}_\alpha, \mathcal{F}_\alpha^{-1}$ are linear maps between n-dimensional spaces, to prove they are inverses, it suffices to prove that $F_\alpha^{-1}F_\alpha = I$ and $\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha = I$. Moreover, it suffices to prove the just first, since if it holds, then (6.24) and (6.25) imply

$$\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha f(a) = F_\alpha^{-1}(\mathcal{F}_\alpha f)(a^{-1}) = F_\alpha^{-1}(F_\alpha \tilde f)(a^{-1}) = \tilde f(a^{-1}) = f(a).$$

We will show that $F_\alpha^{-1}F_\alpha = I$ on the standard basis $(e_g)_{g\in G}$. First recall the character of the regular α-representation is $\sum_\rho d_\rho\chi_\rho = \alpha(1,1)|G|e_1$. Therefore, we calculate

$$\begin{aligned}(F_\alpha^{-1}F_\alpha e_g)(h) &= \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\Big(\frac{\rho(g^{-1})\rho(h)}{\alpha(g,g^{-1})\alpha(1,1)}\Big) = \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\Big(\frac{\alpha(g^{-1},h)\rho(g^{-1}h)}{\alpha(g,g^{-1})\alpha(1,1)}\Big)\\ &= \frac{1}{|G|}\sum_\rho d_\rho \frac{\chi_\rho(g^{-1}h)}{\alpha(g,g^{-1}h)} = \frac{1}{|G|}M_\alpha\big(\textstyle\sum_\rho d_\rho\chi_\rho\big)_{g,h} = M_\alpha\big(\alpha(1,1)e_1\big)_{g,h} = I_{g,h} = e_g(h).\end{aligned}$$

The inversion formula can be expanded

$$(F_\alpha^{-1}F_\alpha f)(g) = \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\Big(\sum_{a\in G}\frac{f(a)\rho(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)}\rho(g)\Big). \tag{6.29}$$

Since both sides of the Plancherel formula (6.27) are linear in ν and μ, it suffices (by linearity) to prove it for $\nu = e_g$ and $\mu = e_h$. Using (6.29), we have

$$\begin{aligned}\frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\big((F_\alpha e_g)_\rho (F_\alpha e_h)_\rho\big) &= \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\Big(\sum_{a\in G}\frac{e_g(a)\rho(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)}\frac{\rho(h^{-1})}{\alpha(h,h^{-1})\alpha(1,1)}\Big)\\ &= \frac{e_g(h^{-1})}{\alpha(h,h^{-1})\alpha(1,1)} = \sum_{a\in G}\frac{e_g(a)e_h(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)}.\end{aligned}$$

Since both sides of (6.28) are linear in ν and conjugate linear in μ, it again suffices to consider $\nu = e_g$ and $\mu = e_h$. Since ρ is unitary, $(F_\alpha e_h)_\rho^* = (\rho(h)^{-1})^* = \rho(h)$, and so

$$\begin{aligned}\frac{1}{|G|}\sum_\rho d_\rho \langle (F_\alpha e_g)_\rho, (F_\alpha e_h)_\rho)\rangle &= \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\Big(\sum_{a\in G}\frac{e_g(a)\rho(a^{-1})}{\alpha(a,a^{-1})\alpha(1,1)}\rho(h)\Big)\\ &= e_g(h) = \sum_{a\in G} e_g(a)\overline{e_h(a)}.\end{aligned}$$

We now prove the convolution formulas. On one hand, we have

$$\big(F_\alpha(\nu *_\alpha \mu)\big)_\rho = \sum_{a\in G}\sum_{t\in G}\frac{\nu(at^{-1})\mu(t)}{\alpha(at^{-1},t)}\rho(a)^*.$$

Since $\rho(a)^*\rho(b)^* = (\rho(b)\rho(a))^* = (\alpha(b,a)\rho(ba))^* = \frac{\rho(ba)^*}{\alpha(b,a)}$, and we obtain

$$\begin{aligned}(F_\alpha\nu)_\rho(F_\alpha\mu)_\rho &= \sum_{a\in G}\nu(a)\rho(a)^*\sum_{b\in G}\mu(b)\rho(b)^* = \sum_{a\in G}\sum_{b\in G}\frac{\nu(a)\mu(b)}{\alpha(b,a)}\rho(ba)^*\\ &= \sum_{c\in G}\sum_{t\in G}\frac{\nu(t)\mu(ct^{-1})}{\alpha(ct^{-1},t)}\rho(c)^* = \big(F_\alpha(\mu *_\alpha \nu)\big)_\rho,\end{aligned}$$

which gives the first convolution formula. From this, we have

$$\mathcal{F}_\alpha(\tilde\nu)\mathcal{F}_\alpha(\tilde\mu) = \mathcal{F}_\alpha(\widetilde{\mu *_\alpha \nu}),$$

and a calculation gives

$$\begin{aligned}\widetilde{\mu *_\alpha \nu}(g) = (\mu *_\alpha \nu)(g^{-1}) &:= \sum_{t\in G}\frac{\mu(g^{-1}t^{-1})\nu(t)}{\alpha(g^{-1}t^{-1},t)} = \sum_{t\in G}\frac{\tilde\mu(tg)\tilde\nu(t^{-1})}{\tilde\alpha(t,g^{-1}t^{-1})} = \sum_{s\in G}\frac{\tilde\nu(gs^{-1})\tilde\mu(s)}{\tilde\alpha(sg^{-1},s^{-1})}\\ &= (\tilde\nu *_{\tilde\alpha} \tilde\mu)(g),\end{aligned}$$

which gives the second convolution formula. □

To the best of our knowledge, (6.21) is the first time that a Fourier transform has been defined for projective representations. For ordinary representations, the Fourier transform is well studied, see, e.g., [9], [16].

Example 6.2. The condition for $P = M_\alpha(\nu)$ to satisfy $P^2 = P$ is $\nu *_\alpha \nu = \nu$, which transforms to

$$F_\alpha(\nu *_\alpha \nu) = F_\alpha(\nu)^2 = F_\alpha(\nu).$$

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i.e., each $(F_\alpha \nu)_\rho$ satisfies this condition.

Lemma 6.1. *If $\rho : G \to \mathrm{GL}(\mathbb{C}^{d_\rho})$ is a unitary irreducible projective representation, then*

$$(F_\alpha \rho_{rs})_\xi = \frac{|G|}{d_\rho} \begin{cases} 0, & \xi \not\approx \rho; \\ e_s^* e_r, & \xi = \rho. \end{cases}$$

Proof. In view of (6.23), it suffices to prove this when all the $\xi \in R$ are unitary. Here, the orthogonality of coordinates (4.16) gives

$$(F_\alpha \rho_{rs})_\xi = \sum_{a \in G} \rho_{rs}(a)\xi(a)^* = \sum_{a \in G} \rho_{rs}(a)(\overline{\xi_{kj}(a)})_{j,k=1}^{d_\xi} = (\langle \rho_{rs}, \xi_{kj} \rangle)_{j,k=1}^{d_\xi} = \begin{cases} 0, & \xi \not\approx \rho; \\ \frac{|G|}{d_\rho} e_s^* e_r, & \xi = \rho. \end{cases} \qquad \square$$

Example 6.3. As examples, we have (for ρ unitary or not) that the α-character satisfies

$$(F_\alpha \chi_\rho)_\xi = \frac{|G|}{d_\rho} \begin{cases} 0, & \xi \not\approx \rho; \\ I, & \xi = \rho, \end{cases}$$

and if

$$f(g) = \mathrm{trace}(\rho(g)A) = \langle \rho(g), A^* \rangle = \sum_{j,k} \rho_{jk}(g) a_{kj},$$

then

$$(F_\alpha f)_\xi = \frac{|G|}{d_\rho} \begin{cases} 0, & \xi \not\approx \rho; \\ A, & \xi = \rho. \end{cases} \tag{6.30}$$

In particular, for $f(g) = \langle \rho(g)v, v \rangle = \mathrm{trace}(\rho(g)vv^*)$, i.e., $A = vv^*$ above, we have

$$(F_\alpha f)_\xi = \frac{|G|}{d_\rho} \begin{cases} 0, & \xi \not\approx \rho; \\ vv^*, & \xi = \rho. \end{cases} \tag{6.31}$$

Lemma 6.2. *For α unitary, a (G,α)-matrix $M_\alpha(\nu)$ is Hermitian if and only if*

$$F_\alpha(\nu)_\rho^* = F_\alpha(\nu)_\rho, \qquad \rho \in R.$$

Proof. We have $M_\alpha(\nu)^* = M_\alpha(\nu)$, i.e., $\nu^{*,\alpha} = \nu$, if and only if $F_\alpha(\nu^{*,\alpha})_\rho = F_\alpha(\nu)_\rho$, for all ρ. Using (2.8), we have

$$\begin{aligned} F_\alpha(\nu^{*,\alpha})_\rho &= \sum_{a \in G} \overline{\nu(a^{-1})} \alpha(a, a^{-1}) \alpha(1,1) \rho(a)^* = \Big(\sum_{a \in G} \nu(a^{-1}) \frac{\rho(a)}{\alpha(a, a^{-1})\alpha(1,1)} \Big)^* \\ &= \Big(\sum_{a \in G} \nu(a^{-1}) \rho(a^{-1})^* \Big)^* = F_\alpha(\nu)_\rho^*, \end{aligned}$$

which gives the result. $\square$

Example 6.2 and Lemmas 6.1 and 6.2 are sufficient to obtain the characterisation of tight (G,α)-frames given in §8. Before doing this, we give more precise results about the spectral structure of (G,α)-matrices, which are both enlightening and useful for other applications.

Definition 6.1. If $(F_\alpha \nu)_\xi = 0$, for each $\xi \not\approx \rho$, then we say that $\nu \in \mathbb{C}^G$ is a **ρ-function** and $M_\alpha(\nu)$ is a **ρ-matrix**.

The vector spaces of ρ-functions and ρ-matrices depend only ρ up to equivalence, and have dimension d_ρ^2. It follows from the convolution formula (6.26) for F_α that

- The ρ-functions are closed under the $*_\alpha$ convolution.
- The product of ρ-matrices is a ρ-matrix.

Lemma 6.3. *The following are equivalent*

1. $M_\alpha(\nu)$ *is a ρ-matrix.*
2. ν *is a ρ-function.*
3. $\nu \in \operatorname{span}\{\rho_{jk} : 1 \le j, k \le d_\rho\}$.
4. $\nu \perp \xi_{jk}$, $1 \le j, k \le d_\xi$, $\forall \xi \in R$, $\xi \ne \rho$.

Proof. Let $B = (F_\alpha \nu)_\rho$. Then ν is a ρ-function if and only if $(F_\alpha \nu)_\xi = 0$, $\xi \not\approx \rho$, i.e.,

$$\nu(g) = \frac{1}{|G|} \sum_\xi d_\xi \operatorname{trace}((F_\alpha \nu)_\xi \xi(g)) = \frac{d_\rho}{|G|} \operatorname{trace}(B\rho(g)) = \frac{d_\rho}{|G|} \sum_{j,k} b_{kj} \rho_{jk}(g).$$

This and the fact and the orthogonality of coordinates (4.16) gives the result. □

From $F_\alpha^{-1} F_\alpha = I$ and (5.17), we have that each $f \in \mathbb{C}^G$ can be uniquely decomposed as a sum of orthogonal ρ-functions

$$f = \sum_\rho f_\rho, \qquad f_\rho = f_{[\rho]} := \frac{d_\rho}{|G|} \operatorname{trace}((F_\alpha f)_\rho \rho), \tag{6.32}$$

which we will call **F_α-Fourier decomposition** of $f \in \mathbb{C}^G$ into orthogonal ρ-functions. We will also call

$$M_\alpha(f) = \sum_\rho M_\alpha(f_\rho), \tag{6.33}$$

the **F_α-Fourier decomposition** of $M_\alpha(f)$ into ρ-matrices. The following lemma shows that

$$M_\alpha(f_\rho) M_\alpha(f_\xi) = 0, \quad \rho \not\approx \xi. \tag{6.34}$$

Lemma 6.4. *For $\rho, \xi \in R$, we have*

$$\rho_{jk} *_\alpha \xi_{rs} = \sum_\ell \langle \xi_{rs}, \rho_{k\ell} \rangle \, \rho_{j\ell} = \frac{|G|}{d_\rho} \begin{cases} \rho_{js}, & \xi = \rho, \ r = k; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We have

$$(\rho_{jk} *_\alpha \xi_{rs})(g) = \sum_{t \in G} \frac{\rho_{jk}(gt^{-1}) \xi_{rs}(t)}{\alpha(gt^{-1}, t)}.$$

Since

$$\rho(gt^{-1}) = \frac{\rho(g)\rho(t^{-1})}{\alpha(g, t^{-1})} = \frac{\rho(g)\alpha(t, t^{-1})\alpha(1,1)\rho(t)^*}{\alpha(g, t^{-1})},$$

and $\alpha(g,t^{-1})\alpha(gt^{-1},t)=\alpha(g,t^{-1}t)\alpha(t^{-1},t)=\alpha(1,1)\alpha(t,t^{-1})$, we have

$$\begin{aligned}(\rho_{jk}*_\alpha\xi_{rs})(g)&=\sum_{t\in G}\frac{\alpha(t,t^{-1})\alpha(1,1)}{\alpha(g,t^{-1})\alpha(gt^{-1},t)}(\rho(g)\rho(t)^*)_{jk}\xi_{rs}(t)\\&=\sum_{t\in G}\sum_\ell\rho_{j\ell}(g)\overline{\rho_{k\ell}(t)}\xi_{rs}(t)=\sum_\ell\langle\xi_{rs},\rho_{k\ell}\rangle\,\rho_{j\ell}(g),\end{aligned}$$

and the orthogonality completes the result. □

Further, if α is unitary, then $f_\rho^{*,\alpha}$ is also a ρ-function, since Example 3.3 gives

$$f_\rho=\sum_{j,k}a_{jk}\rho_{jk}\qquad\Longrightarrow\qquad f_\rho^{*,\alpha}=\sum_{j,k}\overline{a_{jk}}\rho_{jk}^{*,\alpha}=\sum_{j,k}\overline{a_{jk}}\rho_{kj},$$

so the ρ-matrices are closed under the Hermitian transpose, and from (6.34) we obtain

$$\langle M_\alpha(f_\rho),M_\alpha(f_\xi)\rangle=\operatorname{trace}(M_\alpha(f_\rho)M_\alpha(f_\xi^{*,\alpha}))=0,\qquad\rho\not\approx\xi.\tag{6.35}$$

Example 6.4. From (6.33) (6.34) and (6.35), we have

$$M_\alpha(\nu)M_\alpha(\mu)=\sum_\rho M_\alpha(\nu_\rho)M_\alpha(\mu_\rho),\qquad M_\alpha(f)^k=\sum_\rho M_\alpha(f_\rho)^k,$$
$$\langle M_\alpha(\nu),M_\alpha(\mu)\rangle=|G|\langle\nu,\mu\rangle=|G|\sum_\rho\langle\nu_\rho,\mu_\rho\rangle=\sum_\rho\langle M_\alpha(\nu_\rho),M_\alpha(\mu_\rho)\rangle.$$

7. The spectral structure of the (G,α)-matrices

The circulant matrices, i.e., the (G,α)-matrices for G a cyclic group and $\alpha=1$, are all simultaneously unitarily diagonalisable by the Fourier matrix, i.e., the characters (representations) of G are the eigenvectors of the circulant matrices. We now investigate to what extent this result extends to general (G,α)-matrices.

We recall from §5 the orthogonal decomposition of $\mathbb{C}^G$ into ρ-functions

$$\mathbb{C}^G=\bigoplus_{\rho\in R}U_{\rho,\alpha},\qquad U_{\rho,\alpha}:=\operatorname{span}\{\rho_{jk}:1\le j,k\le d_\rho\}.$$

For ordinary representations, i.e., $\alpha=1$, we now present the standard diagonalisation result (see [9], [15]), which shows that the $U_{\rho,\alpha}$ are invariant subspaces of the (G,α)-matrices.

Suppose henceforth that each $\rho\in R$ is unitary, and let E be the unitary matrix

$$E=E_R:=[E_{\rho,k}:\rho\in R,1\le k\le d_\rho],\qquad E_{\rho,k}:=\sqrt{\tfrac{d_\rho}{|G|}}[\rho_{k1},\rho_{k2},\dots,\rho_{kd_\xi}].$$

Theorem 7.1. *For ordinary representations, the matrix $E^*M_\alpha(\nu)E$ is block diagonal, with diagonal blocks $(A_\rho:\rho\in R,1\le k\le d_\rho)$, where $A_\rho:=(\mathcal{F}_\alpha\nu)_\rho$.*

Proof. By way of motivation, if we write $E=[\xi_1,\xi_2,\dots]$ and $M_\alpha(v)E=E\Lambda$, then

$$M_\alpha(v)_{g,h}=(E\Lambda E^*)_{g,h}=\sum_s\sum_t E_{g,s}\Lambda_{st}(E^*)_{t,h}=\sum_s\sum_t\xi_s(g)\Lambda_{st}\overline{\xi_t(h)}.$$

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By Fourier inversion, we calculate

$$M_\alpha(\nu)_{g,h} = \nu(g^{-1}h) = (\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha\nu)(g^{-1}h) = (F_\alpha^{-1}\mathcal{F}_\alpha\nu)(h^{-1}g) = \frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\bigl(A_\rho\rho(h^{-1}g)\bigr).$$

Since the representations are ordinary, i.e., $\alpha = 1$, $\operatorname{trace}\bigl(A_\rho\rho(h^{-1}g)\bigr) = \operatorname{trace}\bigl(\rho(g)A_\rho\rho(h)^*\bigr)$. Hence, by writing $A_\rho = [a_{jk}^\rho]_{j,k=1}^{d_\rho}$, we obtain

$$M_\alpha(\nu)_{g,h} = \frac{1}{|G|}\sum_\rho d_\rho \sum_j (\rho(g)A_\rho\rho(h)^*)_{jj} = \sum_\rho\sum_j\sum_s\sum_t \frac{d_\rho}{|G|}\rho_{js}(g)a_{st}^\rho\overline{\rho_{jt}}(h),$$

which gives the result. □

From the above, it follows that the orthogonal subspaces

$$U_{\rho,\alpha,j} := \operatorname{span}\{\rho_{jk} : 1 \le k \le d_\rho\}, \qquad \rho \in R, \quad 1 \le j \le d_\rho,$$

are invariant subspaces of the (G,α)-matrices when $\alpha = 1$. These subspaces do not give a unique orthogonal decomposition of $U_{\rho,\alpha}$ into invariant subspaces of the (G,α)-matrices, since, for any unitary T, one has

$$U_{\rho,\alpha} = \bigoplus_{j=1}^{d_\rho} U_{T\rho T^{-1},\alpha,j}. \tag{7.36}$$

It is natural to suppose that the $U_{\rho,\alpha,j}$ are invariant subspaces of the (G,α)-matrices for the projective case also, and to adapt the argument above to prove it. Here

$$M_\alpha(\nu)_{g,h} = \sum_\rho\sum_j\sum_s\sum_t \frac{\alpha(h,h^{-1}g)}{\alpha(g,g^{-1}h)}\frac{d_\rho}{|G|}a_{st}^\rho\rho_{js}(g)\overline{\rho_{jt}}(h),$$

and so the remainder of the argument breaks down.

To understand the invariant subspaces of the (G,α)-matrices, we first consider the range of the (G,α)-matrices $M_\alpha(\rho_{jk})$.

Lemma 7.1. *Let $\rho,\xi \in R$. Then*

$$M_\alpha(\rho_{jk})v = \sum_{\ell=1}^{d_\rho}\sum_{h\in G}\rho_{\ell k}(h)v_h\,\overline{\rho_{\ell j}}, \qquad v \in \mathbb{C}^G,$$

and, in particular,

$$M_\alpha(\rho_{jk})\overline{\xi_{st}} = \frac{|G|}{d_\rho}\begin{cases}\overline{\rho_{sj}}, & \xi = \rho,\ t = k;\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Since $\rho(g^{-1}h) = \alpha(g,g^{-1}h)\rho(g)^*\rho(h)$, we have

$$\begin{aligned}(M_\alpha(\rho_{jk})v)_g &= \sum_h M_\alpha(\rho_{jk})_{g,h}v_h = \sum_h \frac{\rho_{jk}(g^{-1}h)}{\alpha(g,g^{-1}h)}v_h = \sum_h(\rho(g)^*\rho(h))_{jk}v_h\\ &= \sum_h\sum_\ell(\rho(g)^*)_{j\ell}\rho(h)_{\ell k}v_h = \sum_h\sum_\ell\overline{\rho_{\ell j}(g)}\rho_{\ell k}(h)v_h.\end{aligned}$$

Hence

$$M_\alpha(\rho_{jk})\overline{\xi_{st}} = \sum_\ell \langle \rho_{\ell k}, \xi_{st}\rangle\, \overline{\rho_{\ell j}} = \begin{cases} \frac{|G|}{d_\rho}\overline{\rho_{sj}}, & \xi=\rho,\ t=k; \\ 0, & \text{otherwise}, \end{cases}$$

since the entries of ρ and ξ are orthogonal. □

We therefore conclude that the orthogonal subspaces

$$V_{\rho,\alpha,j} := \operatorname{span}\{\overline{\rho_{jk}} : 1 \le k \le d_\rho\} = \overline{U_{\rho,\alpha,j}}, \qquad \rho \in R, \quad 1 \le j \le d_\rho,$$

are invariant subspaces of the (G,α)-matrices.

This gives the desired simultaneous unitary (block) diagonalisation of projective group matrices:

Theorem 7.2. *For projective representations, the matrix $\overline{E}^* M_\alpha(\nu)\overline{E}$ is block diagonal, with diagonal blocks $(B_\rho^T : \rho \in R, 1 \le k \le d_\rho)$, where $B_\rho := (F_\alpha \nu)_\rho$.*

Proof. Let $(F_\alpha\nu)_\rho = B_\rho = [b^\rho_{jk}]_{j,k=1}^{d_\rho}$. Then

$$M_\alpha(\nu)_{g,h} = \frac{\nu(g^{-1}h)}{\alpha(g,g^{-1}h)} = \frac{(F_\alpha^{-1}F_\alpha\nu)(g^{-1}h)}{\alpha(g,g^{-1}h)} = \frac{1}{\alpha(g,g^{-1}h)}\frac{1}{|G|}\sum_\rho d_\rho \operatorname{trace}\bigl(B_\rho\rho(g^{-1}h)\bigr).$$

Since $\rho(g^{-1}h) = \alpha(g,g^{-1}h)\rho(g)^*\rho(h)$, we obtain

$$M_\alpha(\nu)_{g,h} = \sum_\rho \frac{d_\rho}{|G|}\operatorname{trace}\bigl(\rho(h)B_\rho\rho(g)^*\bigr) = \sum_\rho \frac{d_\rho}{|G|}\sum_j\sum_s\sum_t \rho_{js}(h) b^\rho_{st}\overline{\rho_{jt}(g)}.$$

Thus, we calculate

$$\begin{aligned}(\overline{E_{\xi,k_\xi}}^* M_\alpha(\nu)\overline{E_{\eta,k_\eta}})_{\ell m} &= \sum_g\sum_h \frac{\sqrt{d_\xi}}{\sqrt{|G|}}\xi_{k_\xi\ell}(g)\sum_\rho\frac{d_\rho}{|G|}\sum_{j,s,t}\rho_{js}(h)b^\rho_{st}\overline{\rho_{jt}(g)}\frac{\sqrt{d_\eta}}{\sqrt{|G|}}\overline{\eta_{k_\eta m}}(h) \\ &= \frac{\sqrt{d_\xi d_\eta}}{|G|}\sum_\rho\frac{d_\rho}{|G|}\sum_{j,s,t}\langle \xi_{k_\xi\ell},\rho_{jt}\rangle\langle\rho_{js},\eta_{k_\eta m}\rangle b^\rho_{st} = \begin{cases} b^\xi_{ml}, & \xi=\eta; \\ 0, & \text{otherwise}, \end{cases}\end{aligned}$$

which gives the result. □

Example 7.1. For ordinary representations, $\overline{\rho}(g)\overline{\rho}(h) = \overline{\rho(g)\rho(h)} = \overline{\rho(gh)} = \overline{\rho}(gh)$, so that $\{\overline{\rho}\}_{\rho\in R}$ is another complete set of ordinary representations for G, with $E_{\overline{\rho},k} = \overline{E_{\rho,k}}$. Hence from Theorem 7.1, we have

$$\overline{E}^* M_\alpha(\nu)\overline{E} = \operatorname{diag}(A_{\overline{\rho}} : \rho \in R, 1 \le k \le d_\rho),$$

where $A_{\overline{\rho}} = \mathcal{F}_\alpha(\nu)_{\overline{\rho}} = F_\alpha(\nu)_\rho^T = B_\rho^T$ (which is Theorem 7.2).

Example 7.2. The determinant of a (G,α)-matrix factors

$$\det(M_\alpha(\nu)) = \det(\overline{E}^* M_\alpha(\nu)\overline{E}) = \prod_\rho \det(B_\rho)^{d_\rho} = \prod_{\rho\in R}\det((F_\alpha\nu)_\rho)^{d_\rho}.$$

This "factorisation of the group determinant" was one of the motivations which lead to the development of representation theory (see [15]).

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Example 7.3. The blocks of the diagonal form of a (G,α)-matrix $M_\alpha(f)$ are unique up to similarity, and in particular, the Jordan canonical form is given by the block diagonal matrix with d_ρ diagonal blocks given by the Jordan canonical form of $(F_\alpha f)_\rho$.

For ordinary representations, invariant subspaces of the (G,α)-matrices are

$$\{U_{\rho,\alpha,j}\}_{\rho\in R,1\le j\le d_\rho} = \{V_{\rho,\alpha,j}\}_{\rho\in R,1\le j\le d_\rho}.$$

For projective representations, the $\{V_{\rho,\alpha,j}\}$ are invariant subspaces (with a neat block diagonal form). It is no longer the case that even $\{V_{\rho,\alpha}\}_{\rho\in R} = \{U_{\rho,\alpha}\}_{\rho\in R}$, see (9.42). Nevertheless, in the specific cases considered so far (see §9.2), it appears that there is a block diagonal form for the $\{\mathcal{U}_{\rho,\alpha}\}_{\rho\in R}$ which is more complicated (the blocks are no longer ordered by ρ, and blocks similar to the same $(F_\alpha\nu)_\rho$ are not all identically equal).

From the orthogonal decomposition

$$\mathbb{C}^G = \bigoplus_{\rho\in R}\bigoplus_{j=1}^{d_\rho} U_{\rho,\alpha,j}, \tag{7.37}$$

we obtain a **fine scale** F_α-Fourier decomposition

$$M_\alpha(f) = \sum_\rho\sum_{j=1}^{d_\rho} M_\alpha(f_{\rho,j}), \qquad f_{\rho,j} = \frac{d_\rho}{|G|}\operatorname{trace}((F_\alpha f)_\rho e_j e_j^*\rho) \in U_{\rho,\alpha,j}.$$

We observe from (7.36), or the formula $f_{\rho,j}(g) = \langle\rho(g)(F_\alpha f)_\rho e_j, e_j\rangle$, that there is not a unique *fine scale* F_α-Fourier decomposition.

Let $M_{\rho,\alpha,j}$ be the d_ρ-dimensional vector space of (G,α)-matrices of the form $M_\alpha(f_{\rho,j})$. It follows from Lemma 6.4 (or the block diagonal form) that $M_{\rho,\alpha,j}$ is closed under multiplication, indeed $M_{\rho,\alpha,j}M_{(G,\alpha)} \subset M_{\rho,\alpha,j}$, and hence is an algebra. Moreover, from Lemma 6.4, we have $M_\alpha(f_{\rho,j})M_\alpha(f_{\xi,k}) = 0$, $(\rho,j)\ne(\xi,k)$, and consequently

$$M_\alpha(\nu)M_\alpha(\mu) = \sum_{\rho\in R}\sum_{j=1}^{d_\rho} M_\alpha(\nu_{\rho,j})M_\alpha(\mu_{\rho,j}).$$

We now show the Fourier decompositions of $M_\alpha(f)$ are into *low rank* (G,α)-matrices.

Proposition 7.1. *The rank of a (G,α)-matrix satisfies*

$$\operatorname{rank}(M_\alpha(f)) = \sum_\rho \operatorname{rank}(M_\alpha(f_\rho)) = \sum_\rho d_\rho \operatorname{rank}((F_\alpha f)_\rho), \tag{7.38}$$

$$\operatorname{rank}(M_\alpha(f)) \le \sum_\rho\sum_j \operatorname{rank}(M_\alpha(f_{\rho,j})), \qquad \operatorname{rank}(M_\alpha(f_{\rho,j})) \in \{0, d_\rho\}. \tag{7.39}$$

In particular, $M_\alpha(f)$ is invertible if and only if $f_{\rho,j}\ne 0$, $\forall\rho,j$.

Proof. The block diagonal matrix $\overline{E}^* M_\alpha(f)\overline{E}$ has diagonal blocks $(F_\alpha f)_\rho^T = (F_\alpha f_\rho)_\rho^T$, each repeated d_ρ times, and so we have

$$\operatorname{rank}(M_\alpha(f)) = \sum_\rho d_\rho \operatorname{rank}((F_\alpha f)_\rho) = \sum_\rho d_\rho \operatorname{rank}((F_\alpha f_\rho)_\rho).$$

The block diagonal form of $M_\alpha(f_\rho)$ has just d_ρ (possibly) nonzero blocks $F_\alpha(f_\rho)_\rho^T$, and so the rank of $M_\alpha(f_\rho)$ is $d_\rho \operatorname{rank}((F_\alpha f_\rho)_\rho)$. The block diagonal form of $M_\alpha(f_{\rho,j})$ has d_ρ (possibly) nonzero blocks

$$(F_\alpha f_{\rho,j})_\rho^T = ((F_\alpha f)_\rho e_j e_j^*)^T = e_j e_j^* (F_\alpha f_\rho)_\rho^T,$$

which either have rank one (since $\operatorname{rank}(e_j e_j^*) = 1$), or are rank zero (when $f_{\rho,j} = 0$). □

Example 7.4. Proposition 7.1 gives a restriction on the possible rank of a (G, α)-matrix, e.g., for $G = D_{4m}$ (the dihedral group of order $4m$) and α the nontrivial cocycle, all irreducibles have $d_\rho = 2$ (see §9.2), and so a (G, α)-matrix must be of even rank.

There is some interest in subspaces (in our case subalgebras) of matrices with a restriction on their rank [8].

8. The characterisation of tight (G, α)-frames

We now give the main application of our results: a simple characterisation and explicit construction of all tight (G, α)-frames. Let R be a complete set of irreducible unitary representations for a (unitary) cocycle α. With very little effort we have:

Lemma 8.1. *Let $P = M_\alpha(\nu)$ be a (G, α)-matrix. Then the following are equivalent*

1. *P is the Gramian of a normalised tight (G, α)-frame.*
2. *P is an orthogonal projection.*
3. *Each $F_\alpha(\nu)_\rho$, $\rho \in R$, is an orthogonal projection.*

Proof. Since a sequence of vectors is a normalised tight frame if and only if its Gramian is an orthogonal projection, we have the equivalence of the first two. The condition that P be an orthogonal projection, i.e., $P^2 = P$ and $P^* = P$, is equivalent to the last condition by Example 6.2 and Lemma 6.2. □

We will refer to $(F_\alpha(\nu)_\rho)_{\rho \in R}$ as the **Fourier coefficients** of $P = M_\alpha(\nu)$, or of any (G, α)-frame with Gramian $M_\alpha(\nu)$.

Example 8.1. If G is abelian and $\alpha = 1$, then all the irreducibles have dimension 1, and so the Fourier coefficients of a normalised tight G-frame must be 0 or 1. Thus there are a finite number of such G-frames for G abelian, the so called **harmonic frames**.

Now we suppose that $\rho : G \to V$ is a unitary action on V (for a cocycle α). We now answer the question: when is $(\rho(g)v)_{g \in G}$ a normalised tight (G, α)-frame for V? The result below is the main structure theorem for tight (G, α)-frames, and was first given in [18] (Theorem 6.18) for the ordinary case, and in [3] (Theorem 2.11) for the projective case. The proof given here uses the Fourier coefficients of the frame, rather than asserting (1.4) for each irreducible, and gives insight into the result.

Theorem 8.1. *Let $\rho : G \to V$ be a unitary action on V for a cocycle α, and $V = \oplus V_j$ be an orthogonal direct sum of irreducible $(\mathbb{C}G)_\alpha$-modules. If $v = \sum_j v_j$, $v_j \in V_j$, then $(gv)_{g \in G} = (\rho(g)v)_{g \in G}$ is a normalised tight (G, α)-frame for V if and only if*

$$\|v_j\|^2 = \frac{\dim(V_j)}{|G|}, \quad \forall j,$$

and in the case V_j is $(\mathbb{C}G)_\alpha$-isomorphic to V_k, $j \neq k$, via $\sigma : V_j \to V_k$ that

$$\langle \sigma v_j, v_k \rangle = 0.$$

Proof. By Lemma 8.1, $\Phi := (gv)_{g\in G}$ is a normalised tight frame for V if and only if its Fourier coefficients are orthogonal projections, and the sum of the ranks of these projections is $\dim(V)$. We now calculate the Fourier coefficients $(c_\xi)_{\xi\in R}$.

Let $\rho_j : V_j \to V_j$ be the irreducible representation given by $\rho_j(g) := \rho(g)|_{V_j}$, and let $\sigma_j : V_j \to V_\xi$ be a unitary $(\mathbb{C}G)_\alpha$-isomorphism to the $\xi : G \to \mathrm{GL}(V_\xi)$ in R with $\rho_j \approx \xi$. We note that $\xi = \sigma_j \rho_j \sigma_j^{-1}$. By the orthogonality of the V_j, the Gramian of $(gv)_{g\in G}$ is the sum of the Gramians $M_\alpha(f_j)$ of the (G, α)-frames $\Phi_j := (\rho_j(g)v_j)_{g\in G}$ for V_j, where

$$\begin{aligned} f_j(g) &:= \langle \rho_j(g)v_j, v_j \rangle = \mathrm{trace}(\rho_j(g)v_j v_j^*) = \mathrm{trace}(\sigma_j \rho_j(g)\sigma_j^{-1}\sigma_j v_j(\sigma_j v_j)^*) \\ &= \mathrm{trace}(\xi(g)\sigma_j v_j(\sigma_j v_j)^*). \end{aligned}$$

It follows from (6.31) that the Fourier coefficients of Φ_j (given by the ξ-function f_j) are

$$(F_\alpha f_j)_\eta = \begin{cases} 0, & \eta \not\approx \xi; \\ w_j w_j^*, & \eta = \xi, \end{cases} \qquad w_j := \sqrt{\frac{|G|}{d_\xi}}\sigma_j v_j \quad (\rho_j \approx \xi),$$

and hence the Fourier coefficients of Φ are

$$c_\xi := \sum_{j:\rho_j\approx\xi} w_j w_j^*.$$

If Φ is a normalised tight frame for V, then each Φ_j must be one for V_j (since orthogonal projections map normalised tight frames to normalised tight frames), i.e., $w_j w_j^*$ is a rank one orthogonal projection, which gives

$$\|w_j\|^2 = \frac{|G|}{d_\xi}\|\sigma_j v_j\|^2 = \frac{|G|}{d_\xi}\|v_j\|^2 = 1 \quad\iff\quad \|v_j\|^2 = \frac{\dim(V_j)}{|G|}.$$

Finally, for c_ξ to be an orthogonal projection, we need $w_j \perp w_k$, $j \neq k$, $\rho_j, \rho_k \approx \xi$, i.e.,

$$w_j \perp w_k \quad\iff\quad \sigma_j v_j \perp \sigma_k v_k \quad\iff\quad \sigma_k^{-1}\sigma_j v_j \perp v_k \quad\iff\quad \sigma v_j \perp v_k,$$

where, by Schur's lemma, we can replace the $(\mathbb{C}G)_\alpha$-isomorphism $\sigma_k^{-1}\sigma_j : V_j \to V_k$ above by any other one σ. □

There a natural description of the various classes of G-frames in [21] (and their generalisation to (G, α)-frames) in terms of the Fourier coefficients, e.g.,

- *Irreducible (G, α)-frames*: There is only one nonzero Fourier coefficient, which is a rank one orthogonal projection (up to a scalar multiple).
- *Homogeneous (G, α)-frames*: There is only one nonzero Fourier coefficient.
  Equivalently, the Gramian is a ρ-matrix for some irreducible $\rho \in R$.
- *Central (G, α)-frames*: All the Fourier coefficients are 0 or a scalar multiple of I.

Example 8.2. A (G, α)-frame with Gramian $M_\alpha(f)$ is **central** if f is a α-class function. Since the irreducible α-characters are a basis for the α-class functions, f must have the form

$$f(g) = \sum_\rho a_\rho \chi_\rho(g) = \sum_\rho a_\rho \operatorname{trace}(\rho(g)I),$$

and so, by (6.30), the Fourier coefficients of the frame are $(F_\alpha f)_\rho = \frac{|G|}{d_\rho} a_\rho I$. For these to be orthogonal projections, we must have

$$a_\rho = \frac{d_\rho}{|G|} = \frac{\chi_\rho(1)}{\alpha(1,1)|G|},$$

which, by taking $a_\rho = 0$ or the above value, gives the characterisation of [19] and [3] for normalised tight central (G, α)-frames in terms of their Gramian.

Theorem 8.2 *(Construction). Let $M_\alpha(f)$ be an orthogonal projection, i.e., the Gramian of some normalised tight (G, α)-frame. Write its Fourier coefficients (which are rank m_ξ orthogonal projections) as*

$$(F_\alpha f)_\xi = \sum_{j=1}^{m_\xi} w_{\xi,j} w_{\xi,j}^*, \qquad w_\xi = (w_{\xi,1}, \dots, w_{\xi,m_\xi}) \in (\mathbb{C}^{d_\xi})^{m_\xi},$$

where $\langle w_{\xi,j}, w_{\xi,k} \rangle = \delta_{jk}$. Let

$$v := \Big(\sqrt{\tfrac{d_\xi}{|G|}} w_\xi\Big)_{\xi \in R} \in V := \bigoplus_{\xi \in R} (\mathbb{C}^{d_\xi})^{m_\xi}. \tag{8.40}$$

Then $(\rho(g)v)_{g \in G}$ is a normalised tight (G, α)-frame for V with Gramian $M_\alpha(f)$, where the unitary action $\rho : G \to \mathrm{GL}(V)$ is given by

$$\rho(g)\big((v_{\xi,1}, \dots, v_{\xi,m_\xi})_{\xi \in R}\big) := (\xi(g) v_{\xi,1}, \dots, \xi(g) v_{\xi,m_\xi})_{\xi \in R}.$$

Proof. Let $M_\alpha(\nu)$ be the Gramian of $(\rho(g)v)_{g \in G}$, i.e.,

$$\nu(g) := \langle \rho(g)v, v \rangle = \sum_{\xi \in R} \sum_{j=1}^{m_\xi} \langle \xi(g) w_{\xi,j}, w_{\xi,j} \rangle.$$

Then by the orthogonality of coordinates (4.16), and (6.31), we calculate

$$\begin{aligned}(F_\alpha \nu)_\eta &= \sum_{a \in G} \sum_{\xi \in R} \frac{d_\xi}{|G|} \sum_{j=1}^{m_\xi} \langle \xi(a) w_{\xi,j}, w_{\xi,j} \rangle \eta(a)^* = \sum_{a \in G} \frac{d_\xi}{|G|} \sum_{j=1}^{m_\eta} \langle \eta(a) w_{\eta,j}, w_{\eta,j} \rangle \eta(a)^* \\ &= \sum_{j=1}^{m_\eta} \sum_{a \in G} \operatorname{trace}\Big(\eta(a) \frac{d_\xi}{|G|} w_{\eta,j} w_{\eta,j}^*\Big) \eta(a)^* = \sum_{j=1}^{m_\eta} w_{\eta,j} w_{\eta,j}^* = (F_\alpha f)_\eta,\end{aligned}$$

as claimed. □

Example 8.3. Consider the normalised tight central (G, α)-frame with Gramian

$$M_\alpha\Big(\sum_{\xi \in S} \frac{d_\xi}{|G|} \chi_\xi\Big), \qquad d_\xi = \frac{\chi_\xi(1)}{\alpha(1,1)},$$

where $S \subset R$ (see Example 8.2). Its nonzero Fourier coefficients are I for $\xi \in S$. By writing these as $I = \sum_j e_j e_j^*$, we can realise this frame as $(\phi_g)_{g \in G}$, where

$$\phi_g := \Big(\sqrt{\tfrac{d_\xi}{|G|}} \xi_{jk}(g)\Big)_{\xi \in S, 1 \le j,k \le d_\rho}.$$

This can be written compactly as $(\sqrt{\frac{d_\xi}{|G|}} \xi)_{\xi \in S} \in \bigoplus_{\xi \in S} \mathbb{C}^{d_\xi \times d_\xi}$, with the Frobenius inner product on $\mathbb{C}^{d_\xi \times d_\xi}$.

A square matrix is the Gramian of a frame (spanning sequence for a vector space) if and only if it is positive semidefinite. Thus a (G, α)-matrix $M_\alpha(f)$ is the Gramian of a (G, α)-frame if and only if its Fourier coefficients are positive semidefinite. Each such Fourier coefficient can be unitarily diagonalised, giving $(F_\alpha f)_\xi = \sum_j \lambda_j w_{\xi,j} w_{\xi,j}^*$, where the $w_{\xi,j}$ are orthonormal and $\lambda_j > 0$. The frame can be realised as in Theorem 8.2, where w_ξ in (8.40) is replaced by $(\sqrt{\lambda_j} w_{\xi,j})_{1 \le j \le m_\xi}$, and $m_\xi = \mathrm{rank}((F_\alpha f)_\xi)$.

9. Examples

We now give some examples of (G, α)-matrices, their block diagonalisations (factorisation of the determinant), and Fourier decompositions.

9.1. The Klein four-group

The first group with a nontrivial Schur multiplier is the Klein four-group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. We order this $1, a, b, ab = (0,0), (1,0), (0,1), (1,1)$, and write $\nu(j,k) = \nu_{jk}$.

For $\alpha = 1$, there are four one-dimensional representations, giving

$$M_1(\nu) = \begin{pmatrix} \nu_{00} & \nu_{10} & \nu_{01} & \nu_{11} \\ \nu_{10} & \nu_{00} & \nu_{11} & \nu_{01} \\ \nu_{01} & \nu_{11} & \nu_{00} & \nu_{10} \\ \nu_{11} & \nu_{01} & \nu_{10} & \nu_{00} \end{pmatrix}, \qquad E = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

with $\overline{E}^* M_1(\nu) \overline{E}$ being diagonal, and

$$\det(M_1(\nu)) = (\nu_{00} + \nu_{10} + \nu_{01} + \nu_{11})(\nu_{00} - \nu_{10} + \nu_{01} - \nu_{11})(\nu_{00} + \nu_{10} - \nu_{01} - \nu_{11})(\nu_{00} - \nu_{10} - \nu_{01} + \nu_{11}).$$

For the nontrivial multiplier, we have (G, α)-matrices

$$M_\alpha(\nu) = \begin{pmatrix} \nu_{00} & \nu_{10} & \nu_{01} & \nu_{11} \\ \nu_{10} & \nu_{00} & \nu_{11} & \nu_{01} \\ \nu_{01} & -\nu_{11} & \nu_{00} & -\nu_{10} \\ -\nu_{11} & \nu_{01} & -\nu_{10} & \nu_{00} \end{pmatrix}, \quad \alpha := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

and a single two-dimensional projective representation ρ for α. This representation, and a $\tilde{\rho}$ equivalent to it, are given by

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(ab) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\tilde{\rho}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\rho}(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\rho}(ab) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

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where $\tilde{\rho} = T\rho T^{-1}$, $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. For these, we have

$$E_\rho = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \qquad E_{\tilde{\rho}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

This shows that invariant subspace orthogonal decompositions

$$U_{\rho,\alpha} = \bigoplus_{j=1}^{d_\rho} U_{\rho,\alpha,j}, \qquad V_{\rho,\alpha} = \bigoplus_{j=1}^{d_\rho} V_{\rho,\alpha,j}$$

are not unique.

Here

$$\overline{E_\rho}^* M_\alpha(\nu)\overline{E_\rho} = \begin{pmatrix} C_\rho & 0 \\ 0 & C_\rho \end{pmatrix}, \qquad C_\rho = (F_\alpha \nu)_\rho^T = \begin{pmatrix} \nu_{00}+\nu_{01} & \nu_{10}-\nu_{11} \\ \nu_{10}+\nu_{11} & \nu_{00}-\nu_{01} \end{pmatrix}$$
$$\overline{E_{\tilde{\rho}}}^* M_\alpha(\nu)\overline{E_{\tilde{\rho}}} = \begin{pmatrix} C_{\tilde{\rho}} & 0 \\ 0 & C_{\tilde{\rho}} \end{pmatrix}, \qquad C_{\tilde{\rho}} = (F_\alpha \nu)_{\tilde{\rho}}^T = \begin{pmatrix} \nu_{00}+\nu_{10} & \nu_{01}+\nu_{11} \\ \nu_{01}-\nu_{11} & \nu_{00}-\nu_{10} \end{pmatrix},$$

and

$$\det(M_\alpha(\nu)) = (\nu_{00}^2 + \nu_{11}^2 - \nu_{10}^2 - \nu_{01}^2)^2.$$

We have a fine-scale Fourier decomposition of $M_\alpha(\nu)$ into (G,α)-matrices

$$M_\alpha(\nu) = M_\alpha(\nu_{\rho,1}) + M_\alpha(\nu_{\rho,2}),$$

where

$$\nu_{\rho,1} = \frac{1}{2}(\nu_{00}+\nu_{01}, \nu_{10}-\nu_{11}, \nu_{00}+\nu_{01}, \nu_{11}-\nu_{10}),$$
$$\nu_{\rho,2} = \frac{1}{2}(\nu_{00}-\nu_{01}, \nu_{10}+\nu_{11}, \nu_{01}-\nu_{00}, \nu_{10}+\nu_{11}).$$

The summands lie in the corresponding subalgebras of the (G,α)-matrices

$$M_{\rho,\alpha,1} = \left\{\begin{pmatrix} a & b & a & -b \\ b & a & -b & a \\ a & b & a & -b \\ b & a & -b & a \end{pmatrix} : a,b \in \mathbb{C}\right\}, \quad M_{\rho,\alpha,2} = \left\{\begin{pmatrix} c & d & -c & d \\ d & c & d & -c \\ -c & -d & c & -d \\ -d & -c & -d & c \end{pmatrix} : c,d \in \mathbb{C}\right\},$$

for which every nonzero matrix has rank 2. By Proposition 7.1, we have that there are no (G,α)-matrices of rank 1 or 3.

9.2. The dihedral groups

The next groups with nontrivial Schur multiplier are those of order 8, of which $\mathbb{Z}_2 \times \mathbb{Z}_4$ and D_8 have Schur multiplier of order 2, and $\mathbb{Z}_2^3$ which has Schur multiplier of order 8. We consider $G = D_8 = \langle a, b : a^4 = 1, b^2 = 1, bab = a^{-1}\rangle$. In [3], it is shown that for the nontrivial cocycle α given by

$$\alpha(a^j b^k, a^\ell b^m) := i^{k\ell},$$

there are inequivalent 2-dimensional projective representations ρ_1 and ρ_2 for α given by

$$\rho_r(a^j b^k) := \begin{pmatrix} i^r & 0 \\ 0 & i^{1-r} \end{pmatrix}^j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^k.$$

We use the ordering $1, a, a^2, a^3, b, ab, a^2b, a^3b$ for G, so that

$$M_\alpha(\nu) = \begin{pmatrix} \nu_1 & \nu_a & \nu_{a^2} & \nu_{a^3} & \nu_b & \nu_{ab} & \nu_{a^2b} & \nu_{a^3b} \\ \nu_{a^3} & \nu_1 & \nu_a & \nu_{a^2} & \nu_{a^3b} & \nu_b & \nu_{ab} & \nu_{a^2b} \\ \nu_{a^2} & \nu_{a^3} & \nu_1 & \nu_a & \nu_{a^2b} & \nu_{a^3b} & \nu_b & \nu_{ab} \\ \nu_a & \nu_{a^2} & \nu_{a^3} & \nu_1 & \nu_{ab} & \nu_{a^2b} & \nu_{a^3b} & \nu_b \\ \nu_b & i\nu_{a^3b} & -\nu_{a^2b} & -i\nu_{ab} & \nu_1 & i\nu_{a^3} & -\nu_{a^2} & -i\nu_a \\ -i\nu_{ab} & \nu_b & i\nu_{a^3b} & -\nu_{a^2b} & -i\nu_a & \nu_1 & i\nu_{a^3} & -\nu_{a^2} \\ -\nu_{a^2b} & -i\nu_{ab} & \nu_b & i\nu_{a^3b} & -\nu_{a^2} & -i\nu_a & \nu_1 & i\nu_{a^3} \\ i\nu_{a^3b} & -\nu_{a^2b} & -i\nu_{ab} & \nu_b & i\nu_{a^3} & -\nu_{a^2} & -i\nu_a & \nu_1 \end{pmatrix},$$

and, e.g.,

$$\rho_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}\right).$$

Here

$$E := \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ i & 0 & 0 & 1 & -1 & 0 & 0 & -i \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ -i & 0 & 0 & 1 & -1 & 0 & 0 & i \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & i & 1 & 0 & 0 & -1 & -i & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -i & 1 & 0 & 0 & -1 & i & 0 \end{pmatrix}, \tag{9.41}$$

and $\overline{E}^* M_\alpha(\nu)\overline{E}$ is block diagonal, with 2×2 diagonal blocks $(F_\alpha\nu)_{\rho_1}^T, (F_\alpha\nu)_{\rho_1}^T, (F_\alpha\nu)_{\rho_2}^T, (F_\alpha\nu)_{\rho_2}^T$, where

$$(F_\alpha\nu)_{\rho_1} = \begin{pmatrix} \nu_1 - i\nu_a - \nu_{a^2} + i\nu_{a^3} & \nu_b + \nu_{ab} + \nu_{a^2b} + \nu_{a^3b} \\ \nu_b - i\nu_{ab} - \nu_{a^2b} + i\nu_{a^3b} & \nu_1 + \nu_a + \nu_{a^2} + \nu_{a^3} \end{pmatrix},$$

$$(F_\alpha\nu)_{\rho_2} = \begin{pmatrix} \nu_1 - \nu_a + \nu_{a^2} - \nu_{a^3} & \nu_b + i\nu_{ab} - \nu_{a^2b} - i\nu_{a^3b} \\ \nu_b - \nu_{ab} + \nu_{a^2b} - \nu_{a^3b} & \nu_1 + i\nu_a - \nu_{a^2} - i\nu_{a^3} \end{pmatrix}.$$

Thus $\det(M_\alpha(\nu))$ factors as $\det((F_\alpha\nu)_{\rho_1})^2 \det((F_\alpha\nu)_{\rho_2})^2$. It also happens that $E^* M_\alpha(\nu) E$ is block diagonal, i.e., the $U_{\rho,\alpha,j}$ are invariant subspaces of the (G,α)-matrices. This is because conjugation permutes the subspaces $U_{\rho,\alpha,j}$, i.e., from (9.41) it is apparent that

$$V_{\rho_1,\alpha,1} = U_{\rho_2,\alpha,2}, \quad V_{\rho_1,\alpha,2} = U_{\rho_1,\alpha,2}, \quad V_{\rho_2,\alpha,1} = U_{\rho_2,\alpha,1}, \quad V_{\rho_2,\alpha,2} = U_{\rho_1,\alpha,1}. \tag{9.42}$$

The diagonal blocks of $E^* M_\alpha(\nu) E$ are

$$\begin{pmatrix} \nu_1 + i\nu_a - \nu_{a^2} - i\nu_{a^3} & \nu_b + i\nu_{ab} - \nu_{a^2b} - i\nu_{a^3b} \\ \nu_b - \nu_{ab} + \nu_{a^2b} - \nu_{a^3b} & \nu_1 - \nu_a + \nu_{a^2} - \nu_{a^3} \end{pmatrix}, \begin{pmatrix} \nu_1 - i\nu_a - \nu_{a^2} + i\nu_{a^3} & \nu_b - i\nu_{ab} - \nu_{a^2b} + i\nu_{a^3b} \\ \nu_b + \nu_{ab} + \nu_{a^2b} + \nu_{a^3b} & \nu_1 + \nu_a + \nu_{a^2} + \nu_{a^3} \end{pmatrix},$$

$$\begin{pmatrix} \nu_1 - \nu_a + \nu_{a^2} - \nu_{a^3} & \nu_b - \nu_{ab} + \nu_{a^2b} - \nu_{a^3b} \\ \nu_b + i\nu_{ab} - \nu_{a^2b} - i\nu_{a^3b} & \nu_1 + i\nu_a - \nu_{a^2} - i\nu_{a^3} \end{pmatrix}, \begin{pmatrix} \nu_1 + \nu_a + \nu_{a^2} + \nu_{a^3} & \nu_b + \nu_{ab} + \nu_{a^2b} + \nu_{a^3b} \\ \nu_b - i\nu_{ab} - \nu_{a^2b} + i\nu_{a^3b} & \nu_1 - i\nu_a - \nu_{a^2} + i\nu_{a^3} \end{pmatrix}.$$

We observe that these are all different.

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10. Other (G,α)-matrices

The definition (3.10) for $M_\alpha(\nu)$ and that of the Fourier transform F_α are motivated by our analysis of projective group frames. Since these notions are so new, we now provide the tools to compare the variants. As the theory evolves, perhaps it will become apparent if there are ones which are best.

In (1.3), we define the Gramian of $(gv)_{g\in G}$ so that it factors V^*V, where $V=[gv]_{g\in G}$. In [3] the transpose (or equivalently the complex conjugate) of this is considered (for unitary representations), i.e., the matrix with (g,h)-entry

$$\langle \phi_g,\phi_h\rangle=\overline{\langle \phi_h,\phi_g\rangle}=\frac{\langle v,\rho(g^{-1}h)v\rangle}{\overline{\alpha}(g,g^{-1}h)}=\alpha(g,g^{-1}h)\langle v,\rho(g^{-1}h)v\rangle.$$

For α unitary or not, we say that $A\in\mathbb{C}^{G\times G}$ a $[G,\alpha]$**-matrix** if it has this form, i.e.,

$$a_{g,h}=M(\nu)_{g,h}:=\alpha(g,g^{-1}h)\nu(g^{-1}h),\qquad \nu\in\mathbb{C}^G. \tag{10.43}$$

In [3], the formula (10.43) is written as

$$M(\nu)_{g,h}=\alpha(g,g^{-1}h)\nu(g^{-1}h)=\frac{\alpha(g,g^{-1})\alpha(1,1)}{\alpha(g^{-1},h)}\nu(g^{-1}h). \tag{10.44}$$

We observe that $1/\alpha$ is cocycle, and that a $[G,\alpha]$-matrix is a $(G,1/\alpha)$-matrix, i.e.,

$$M(\nu)=M_{1/\alpha}(\nu). \tag{10.45}$$

Moreover, for α unitary, the complex conjugate of a (G,α)-matrix is a $[G,\alpha]$-matrix, i.e.,

$$\overline{M_\alpha(\nu)}=M(\overline{\nu}). \tag{10.46}$$

Proposition 10.1. *The (G,α)-matrices and $[G,\alpha]$-matrices are the transposes of each other, i.e.,*

$$M_\alpha(\nu)^T=M(\mu),\qquad \mu(g):=\frac{\nu(g^{-1})}{\alpha(1,1)\alpha(g,g^{-1})},$$
$$M(\nu)^T=M_\alpha(\mu),\qquad \mu(g):=\alpha(1,1)\alpha(g,g^{-1})\nu(g^{-1}).$$

Proof. Since $\alpha(g,g^{-1}h)\alpha(h,h^{-1}g)=\alpha(g,1)\alpha(g^{-1}h,h^{-1}g)$, we calculate

$$(M_\alpha(\nu)^T)_{g,h}=M_\alpha(\nu)_{h,g}=\frac{\nu(h^{-1}g)}{\alpha(h,h^{-1}g)}=\alpha(g,g^{-1}h)\frac{\nu((g^{-1}h)^{-1})}{\alpha(1,1)\alpha(g^{-1}h,(g^{-1}h)^{-1})}.$$

The other follows similarly, or by a change of variables. □

Example 10.1. Taking the transpose of the diagonalisation of Theorem 7.2 gives

$$E^*M_\alpha(\nu)^TE=\overline{E}^TM_\alpha(\nu)^T(\overline{E}^*)^T=\operatorname{diag}\big((F_\alpha\nu)_\rho:\rho\in R,1\le k\le d_\rho\big),$$

so that the $U_{\rho,\alpha,j}$ are invariant subspaces of $M_\alpha(\nu)^T$, and hence of the $[G,\alpha]$-matrices.

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Group matrices can also be defined to have (g,h)-entries of the form $\nu(gh^{-1})$ [15]. To transform these to matrices of the above type, we consider the unitary involution J given by $Je_h := e_{h^{-1}}$, i.e., $(J)_{g,h} = \delta_{g,h^{-1}}$. Then

$$(JAJ)_{g,h} = a_{g^{-1},h^{-1}},$$

and so for $A = M_\alpha(\nu)$ and $A = M(\nu)$, we have

$$(JM_\alpha(\nu)J)_{g,h} = \frac{\nu(gh^{-1})}{\alpha(g^{-1},gh^{-1})}, \qquad (JM(\nu)J)_{g,h} = \alpha(g^{-1},gh^{-1})\nu(gh^{-1}),$$

which would provide the natural definitions for (G,α)-matrices of this type.

Example 10.2. For ordinary representations, Theorem 7.1 gives

$$(JE)^*(JM_\alpha(\nu)J)(JE) = \operatorname{diag}\big((F_\alpha\nu)_\rho : \rho \in R, 1 \le k \le d_\rho\big),$$

so the "group matrices" $JM_\alpha(\nu)J$ are block diagonalised by JE, which has (ρ,k)-blocks

$$\sqrt{\tfrac{d_\rho}{|G|}}[J\rho_{k1}, J\rho_{k2}, \ldots, J\rho_{kd_\xi}] = \sqrt{\tfrac{d_\rho}{|G|}}[\overline{\rho_{1k}}, \overline{\rho_{2k}}, \ldots, \overline{\rho_{d_\xi k}}],$$

since

$$(J\rho_{jk})_g = \rho_{jk}(g^{-1}) = (\rho(g^{-1}))_{jk} = (\rho(g)^*)_{jk} = \overline{\rho_{kj}(g)}.$$

This is the Theorem 61 of [15].

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