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Signed frames and Hadamard products of Gram matrices

Irine Peng, Shayne Waldron

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
e–mail: irine@math.auckland.ac.nz
e–mail: waldron@math.auckland.ac.nz (http:www.math.auckland.ac.nz/~waldron)

ABSTRACT

This paper concerns (redundant) representations in a Hilbert space H of the form

$$f = \sum_j c_j \langle f, \phi_j \rangle \phi_j, \qquad \forall f \in H.$$

These are more general than those obtained from a tight frame, and we develop a general theory based on what are called signed frames. We are particularly interested in the cases where the scaling factors c_j are unique and the geometric interpretation of negative c_j. This is related to results about the invertibility of certain Hadamard products of Gram matrices which are of independent interest, e.g., we show for almost every $v_1, \ldots, v_n \in \mathbb{C}^d$

$$\operatorname{rank}([\langle v_i, v_j \rangle^r \overline{\langle v_i, v_j \rangle}^s]) = \min\{\binom{r+d-1}{d-1}\binom{s+d-1}{d-1}, n\}, \qquad r, s \geq 0.$$

Applications include the construction of tight frames of bivariate Jacobi polynomials on a triangle which preserve symmetries, and numerical results and conjectures about the class of tight signed frames in a finite dimensional space.

Key Words: frames, wavelets, signed frames, Hadamard product, Gram matrix, generalised Hermitian forms, multivariate Jacobi polynomials, Lauricella functions

AMS (MOS) Subject Classifications: primary 05B20, 41A65, 42C15, secondary 11E39, 33C50, 33C65, 42C40

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1. Introduction

Over the last decade there has been renewed interest in frame representations because of their applications in wavelet theory (cf [D92]). Often when an orthogonal wavelet with certain desired properties doesn't exist it is possible to find a frame representation which has them. More recently the redundancy built into a frame representation has been seen to be desirable for computations (when a term in the representation is removed, not all the information associated with it is lost).

This paper concerns the question: when can a set of vectors $\{\phi_j\}$ in a Hilbert space H be scaled to obtain a tight frame $\{\alpha_j\phi_j\}$, and hence a representation of the form

$$f = \sum_j c_j \langle f, \phi_j\rangle \phi_j, \qquad \forall f \in H, \tag{1.1}$$

where $c_j = |\alpha_j|^2 > 0$? When $\dim(H) < \infty$ this is equivalent to writing the identity matrix as a linear combination of the orthogonal projections $\phi_i\phi_i^*$. Such representations are of interest because they share many features of an orthogonal expansion (which may not be available). Our motivation was the construction of tight frames of multivariate Jacobi polynomials which share the symmetries of the weight (no such orthonormal bases exist).

It turns out that representations of the form (1.1) can exist with some c_j negative, and these correspond to what we call *signed frames*. We first develop the basic theory of signed frames and give examples. Next we consider Hadamard products of Gram matrices which occur in the scaling question. Here we give a number of results of independent interest, e.g., for almost every $v_1, \ldots, v_n \in \mathbb{C}^d$

$$\operatorname{rank}([\langle v_i, v_j\rangle^r \overline{\langle v_i, v_j\rangle}^s]) = \min\{\binom{r+d-1}{d-1}\binom{s+d-1}{d-1}, n\}, \qquad r, s \geq 0.$$

We then give answers to the scaling question. For example, if H is d–dimensional, then almost every set of

$$n = \begin{cases} d(d+1)/2, & H \text{ real;} \\ d^2, & H \text{ complex} \end{cases}$$

vectors can be scaled to obtain a unique representation of the form (1.1). This includes a discussion on the particular choice of n and the geometric interpretation of negative c_j. We conclude with some applications including the construction of tight frames of bivariate Jacobi polynomials on a triangle (which preserve symmetries), and some numerical results and conjectures about the class of tight signed frames in a finite dimensional space.

2. Basic theory of signed frames

Throughout, H denotes a real or complex Hilbert space, with the linearity in the first variable of the inner product. The following motivates the definition of signed frames and provides the connection with Hadamard products of Gram matrices.

Lemma 2.1. *Let $\phi_j \in H$ and c_j be scalars. Then there exists a representation*

$$f = \sum_j c_j \langle f, \phi_j \rangle \phi_j, \qquad \forall f \in H, \tag{2.2}$$

if and only if

$$\|f\|^2 = \sum_j c_j |\langle f, \phi_j \rangle|^2, \qquad \forall f \in H. \tag{2.3}$$

If the choice of the c_j is unique for given ϕ_j, then $c_j \in \mathbb{R}, \forall j$. When H is finite–dimensional

$$\dim(H) = \sum_j c_j \|\phi_j\|^2. \tag{2.4}$$

Proof: The forward implication is immediate, and the reverse follows from the polarisation identity. If the c_j are unique, then they can be solved for by applying Gauss elimination to (a suitable subsystem of)

$$\sum_j |\langle f, \phi_j \rangle|^2 c_j = \|f\|^2, \qquad \forall f,$$

and so are real. Let (e_i) be an orthonormal basis and use Parseval's formula to obtain

$$\dim(H) = \sum_i \|e_i\|^2 = \sum_i \sum_j c_j |\langle e_i, \phi_j \rangle|^2 = \sum_j c_j \sum_i |\langle e_i, \phi_j \rangle|^2 = \sum_j c_j \|\phi_j\|^2.$$

$\square$

The condition (2.3) can be rewritten as

$$\|f\|^2 = \sum_j \sigma_j |\langle f, \psi_j \rangle|^2, \qquad \sigma_j := \operatorname{sign}(c_j), \quad \psi_j := \sqrt{|c_j|}\phi_j,$$

which motivates the following.

Definition. *A family (ψ_j) in a Hilbert space is called a* **signed frame** *with signature $\sigma = (\sigma_j)$, $\sigma_j \in \{-1, 1\}$ if there exists $A, B > 0$ with*

$$A \|f\|^2 \leq \sum_j \sigma_j |\langle f, \psi_j \rangle|^2 \leq B \|f\|^2, \qquad \forall f \in H, \tag{2.5}$$

and (ψ_j) is a Bessel set, i.e., there exists $C > 0$ with

$$\sum_j |\langle f, \psi_j \rangle|^2 \leq C \|f\|^2, \qquad \forall f \in H. \tag{2.6}$$

The **signed frame operator** $S = S^+ - S^-$ *is the self adjoint operator defined by*

$$Sf := \sum_j \sigma_j \langle f, \psi_j \rangle \psi_j, \qquad \forall f \in H, \tag{2.7}$$

where its positive and negative parts are

$$S^+ f := \sum_{\sigma_j = 1} \langle f, \psi_j \rangle \psi_j, \qquad S^- f := \sum_{\sigma_j = -1} \langle f, \psi_j \rangle \psi_j. \tag{2.8}$$

Since $\{\psi_j\}$ is a Bessel set, only countably many of the coefficients $\langle f, \psi_j \rangle$ are nonzero, and so the above sums (and those that follow) can be interpreted in the usual way.

When $A = B$ we say (ψ_j) is a **tight** signed frame, and the polarisation identity implies the representation

$$f = \frac{1}{A} \sum_j \sigma_j \langle f, \psi_j \rangle \psi_j, \qquad \forall f \in H.$$

The theory of frames (cf [HW89]) can be extended to signed frames in the obvious way.

Theorem 2.9. *The following are equivalent*
(a) (ψ_j) *is a signed frame with signature* σ *and frame bounds* A, B *and Bessel bound* C.
(b) S^+ *and* S^- *are bounded linear operators with*

$$AI \le S = S^+ - S^- \le BI, \qquad S^+ + S^- \le CI.$$

Proof: The implication (a) $\Longrightarrow$ (b) holds since

$$\langle If, f \rangle = \|f\|^2, \qquad \langle Sf, f \rangle = \sum_j \sigma_j \, |\langle f, \psi_j \rangle|^2.$$

(b) $\Longrightarrow$ (a). Consider a sequence s_n of partial sums for Sf

$$\begin{aligned}
\|s_n - s_m\|^2 &= \sup_{\|g\|=1} |\langle s_n - s_m, g \rangle|^2 = \sup_{\|g\|=1} |\langle \sum_{j=m+1}^{n} \sigma_j \langle f, \psi_j \rangle \psi_j, g \rangle|^2 \\
&= \sup_{\|g\|=1} | \sum_{j=m+1}^{n} \sigma_j \langle f, \psi_j \rangle \langle \psi_j, g \rangle|^2 \\
&\le \sup_{\|g\|=1} \Big(\sum_{j=m+1}^{n} |\langle f, \psi_j \rangle|^2 \Big) \Big(\sum_{j=m+1}^{n} |\langle \psi_j, g \rangle|^2 \Big) \qquad \text{(Cauchy–Schwartz)} \\
&\le C \sum_{j=m+1}^{n} |\langle f, \psi_j \rangle|^2 \to 0, \qquad n > m \to \infty,
\end{aligned}$$

so $Sf \in H$ is well defined, as are $S^+ f$, $S^- f$. The bounds $\|S^+\|, \|S^-\| \le \|S\| \le C$ follow from a similar calculation, and the relations $AI \le S \le BI$, $S^+ + S^- \le CI$ from the signed frame definition. □

In particular, we have the following signed frame representation.

Theorem 2.10 (Signed frame representation).
(a) *S is invertible with*
$$(1/B)\, I \le S^{-1} \le (1/A)\, I.$$
(b) *Let $\tilde{\psi}_j := S^{-1}\psi_j$, then $(\tilde{\psi}_j)$ is a signed frame with signature σ and frame bounds $1/A, 1/B$ and Bessel bound C/A^2, which we call the dual signed frame.*
(c) *Each $f \in H$ can be represented*
$$f = \sum_j \sigma_j \, \langle f, \tilde{\psi}_j \rangle \psi_j = \sum_j \sigma_j \, \langle f, \psi_j \rangle \tilde{\psi}_j.$$

Proof: Since $AI \le S \le BI$, $\|I - (1/B)S\| \le (B-A)/B < 1$, so S is invertible, and it is positive since
$$\langle S^{-1}f, f\rangle = \langle S^{-1}f, S(S^{-1}f)\rangle \ge A\|S^{-1}f\|^2 \ge 0, \quad \forall f.$$
Multiplying $AI \le S \le BI$ by S^{-1} (which commutes with I and S) gives (a). Since S^{-1} is self adjoint,
$$\tilde{S}f := \sum_j \sigma_j \langle f, \tilde{\psi}_j\rangle \tilde{\psi}_j = S^{-1}\Big(\sum_j \sigma_j \langle S^{-1}f, \psi_j\rangle \psi_j\Big) = S^{-1}S(S^{-1}f) = S^{-1}f,$$
$$(\tilde{S}^+ + \tilde{S}^-)f := \sum_j \langle f, \tilde{\psi}_j\rangle \tilde{\psi}_j = S^{-1}\Big(\sum_j \langle S^{-1}f, \psi_j\rangle \psi_j\Big) = S^{-1}(S^+ + S^-)S^{-1}f.$$
Hence
$$(1/B)\, I \le \tilde{S} \le (1/A)\, I, \qquad \tilde{S}^+ + \tilde{S}^- \le (C/A^2)\, I,$$
and we obtain (b) from Theorem 2.9. Part (c) follows by expanding
$$f = S(S^{-1}f) = S^{-1}(Sf).$$
□

Corollary 2.11 (Equivalence). *Let $c_j \in \mathbb{R}$ and $\phi_j \in H$. The following are equivalent*
a) *There exists a representation*
$$f = \sum_j c_j \, \langle f, \phi_j \rangle \, \phi_j, \qquad \forall f \in H.$$
b) *$(\sqrt{|c_j|}\phi_j)$ is a tight signed frame with signature $\sigma = \operatorname{sign}(c)$ and frame bound $A = 1$.*

Proof: The forward implication follows since
$$\|f\|^2 = \langle \sum_j c_j \langle f, \phi_j\rangle \phi_j, f\rangle = \sum_j c_j |\langle f, \phi_j\rangle|^2 = \sum_j \sigma_j |\langle f, \sqrt{|c_j|}\phi_j\rangle|^2.$$
Conversely, taking $\psi_j := \sqrt{|c_j|}\phi_j$ in Theorem 2.10 gives
$$f = \sum_j \sigma_j \langle f, \sqrt{|c_j|}\phi_j\rangle \sqrt{|c_j|}\phi_j = \sum_j c_j \langle f, \phi_j\rangle \phi_j.$$
□

Example 1 (Frames). A signed frame with zero negative part, i.e., $\sigma_j = 1$, $\forall j$, is frame in the usual sense (and conversely). Here $B = C$ and the Bessel property (2.6) is a consequence of (2.5). Also the positive part of a signed frame $\{\phi_j\}_{\sigma_j=1}$ is a frame.

Example 2 (Nonharmonic Fourier signed frames). A system of complex exponentials $e_{\lambda_j} : t \mapsto e^{i\lambda_j t}$, $\lambda_j \in \mathbb{C}$ is a signed frame with signature (σ_j) for $L_2[-\pi,\pi]$ if

$$A\int_{-\pi}^{\pi} |f|^2 \le \sum_j \sigma_j \left|\int_{-\pi}^{\pi} f e_{\lambda_j}\right|^2 \le B\int_{-\pi}^{\pi} |f|^2, \quad \sum_j \left|\int_{-\pi}^{\pi} f e_{\lambda_j}\right|^2 \le C\int_{-\pi}^{\pi} |f|^2, \quad \forall f.$$

By the Paley–Weiner theorem, this is equivalent to

$$A\int_{-\infty}^{\infty} |g|^2 \le \sum_j \sigma_j \,|g(\lambda_j)|^2 \le B\int_{-\infty}^{\infty} |g|^2, \quad \sum_j |g(\lambda_j)|^2 \le C\int_{-\infty}^{\infty} |g|^2,$$

for every function g from the Paley–Weiner space (cf [Y80]).

Example 3. Take any three unit vectors in $\mathbb{R}^2$ none of which are multiples of each other. These can be scaled in a unique way (up to ± 1) to a tight signed frame, with the c_j for a vector given by

$$c_j = \frac{\cos(\beta-\alpha)}{\sin\alpha\,\sin\beta},$$

where $-\pi/2 \le \alpha < \beta \le \pi/2$ are the (acute) angles from the subspace spanned by this vector to those spanned by the other two. This is negative if $\alpha < 0$, $\beta > 0$, $\beta - \alpha < \pi/2$, i.e., the subspace generated by the vector lies in the region between the acute angle made by the other two.

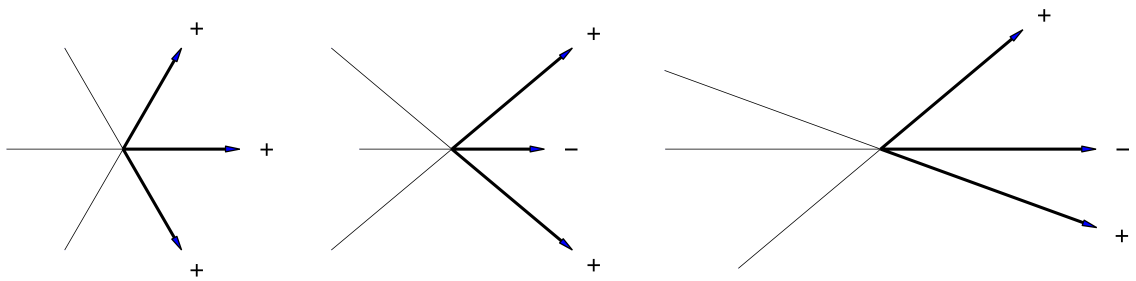

Fig. 1. Tight signed frames of three vectors in $\mathbb{R}^2$ with the signature indicated.

Example 4. Almost all choices of four unit vectors in $\mathbb{C}^2$ can be scaled uniquely to a tight signed frame. The possible signatures are $++++$ (a frame), $+++-$ and $++--$.

Examples 3 and 4 are special cases of the scaling results in Section 4.

Example 5 (Associated tight signed frame). Given a signed frame (ψ_j) with signature σ, let $v_j := S^{-1/2}\psi_j$. Then (v_j) is a tight signed frame with signature σ and frame bound 1, since

$$\sum_j \sigma_j \langle f, v_j\rangle v_j = S^{1/2}\sum_j \sigma_j \langle S^{-1/2} f, \psi_j\rangle \tilde{\psi}_j = S^{1/2}S^{-1/2} f = f, \qquad \forall f \in H.$$

We call (v_j) the **associated tight signed frame**.

Example 6 (Possible signatures). Since the positive part of a signed frame is a frame, the signature σ of a signed frame in $H = \mathbb{R}^d, \mathbb{C}^d$ must have at least d positive entries, say $\sigma_1 = \cdots = \sigma_d = 1$. A tight signed frame can have any signature σ which satisfies this restriction. For example, let $(\psi_j)_{j=1}^d$ be any orthonormal basis, then

$$\sum_{j=1}^{d} \sigma_j |\langle f, \psi_j \rangle|^2 = \|f\|^2, \qquad \Big| \sum_{j=d+1}^{n} \sigma_j |\langle f, \psi_j \rangle|^2 \Big| \le \Big(\sum_{j=d+1}^{n} \|\psi_j\|^2 \Big) \|f\|^2,$$

and so any choice of the remaining ψ_j with $\sum_{j=d+1}^{n} \|\psi_j\|^2 < 1$ will give a signed frame with signature σ. Now take the associated tight signed frame (which has the same signature).

3. Hadamard products of Gram matrices

It follows from (2.3) of Lemma 2.1 that a necessary condition for a scaling of $\{\phi_j\}$ to a tight signed frame to exist is that there are c_j satisfying

$$\sum_j |\langle \phi_i, \phi_j \rangle|^2 c_j = \|\phi_i\|^2, \qquad \forall i. \tag{3.1}$$

Thus we are interested in the matrix

$$A := [|\langle \phi_i, \phi_j \rangle|^2] = B \circ \overline{B}, \qquad B := [\langle \phi_i, \phi_j \rangle]. \tag{3.2}$$

Here $\circ$ denotes the Hadamard (pointwise) product

$$(S \circ T)_{ij} := s_{ij} t_{ij}.$$

The positive semidefinite matrix $B := [\langle \phi_i, \phi_j \rangle]$ is commonly known as the *Gram matrix*.

We will use the Schur product theorem (cf [HJ91]).

Theorem (Schur product). *If A and B are positive semidefinite, then so is $A \circ B$. If, in addition, B is positive definite and A has no diagonal entry equal to zero, then $A \circ B$ is positive definite. In particular, if both A and B are positive definite, then so is $A \circ B$.*

We now provide general results about the rank of Hadamard products of the Gram matrix and its conjugate, of which we will use the particular case (3.2).

Suppose H is d–dimensional, and let

$$\begin{aligned} S_r &:= S_r(H) := \text{the symmetric } r\text{–linear mappings on } H, \\ \Pi_r^0 &:= \Pi_r^0(H) := \text{the homogeneous polynomials of degree } r \text{ on } H, \\ \mathcal{H}_r &:= \mathcal{H}_r(H) := \text{the restrictions of } p \in \Pi_r^0 \text{ to the sphere.} \end{aligned}$$

These spaces are isomorphic via the association of the symmetric r–linear map L with the homogeneous polynomial $p : x \mapsto L(x, \ldots, x)$ and the restriction of p to the sphere $\{x \in H : \|x\| = 1\}$.

Lemma 3.3. *Let $u_1,\ldots,u_n$ be unit vectors in a real or complex Hilbert space H of dimension d, where*

$$n := \binom{d+r-1}{r} = \dim(S_r) = \dim(\Pi_r^0) = \dim(\mathcal{H}_r), \qquad r \geq 0.$$

Then the following are equivalent

(a) *The points $\{u_i\}$ are in general position on the sphere, by which we mean that no nonzero $p \in \mathcal{H}_r$ vanishes at all of them.*
(b) *There is a unique $p \in \mathcal{H}_r$ which interpolates arbitrary data at the points $\{u_i\}$.*
(c) *The $n \times n$ positive semidefinite matrix*

$$A := [\langle u_i, u_j\rangle^r]$$

is invertible.
(d) *The polynomials $\{\langle \cdot, u_i\rangle^r\}$ are a basis for Π_r^0 and $\mathcal{H}_r$.*
(e) *The functionals $\{f \mapsto f(u_i)\}$ are a basis for the dual spaces of Π_r^0 and $\mathcal{H}_r$.*
(f) *The symmetric r–linear mappings on H have a basis given by*

$$(x_1, x_2, \ldots, x_r) \mapsto \langle x_1, u_i\rangle\langle x_2, u_i\rangle \cdots \langle x_r, u_i\rangle, \qquad i = 1,\ldots,n.$$

(g) *The functionals $L \mapsto L(u_i,\ldots,u_i)$ are a basis for the dual space of S_r.*

Proof: The positive semidefiniteness of $A = B \circ \cdots \circ B$ follows from the Schur product theorem. The equivalence of (a),(b),...,(e) is the standard conditions for unique linear interpolation from $V = \mathrm{span}\{\langle \cdot, u_i\rangle\}$ to the linear functionals $f \mapsto f(u_i)$. The implications (d) $\iff$ (f), (e) $\iff$ (g) follow from the isomorphism between S_r and Π_r^0. $\square$

Remark 1. Lemma 3.3 also holds with the inner product replaced by the dot product $x \cdot y := \sum_i x_i y_i$ on $\mathbb{C}^n$, in which case (c) becomes A is an invertible symmetric matrix.

In the following we use Lebesgue measure on $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ and $\mathbb{C}^d \times \cdots \times \mathbb{C}^d$, and remind the reader that the zero set of a nonzero polynomial has measure zero.

Theorem 3.4. *For almost every $v_1,\ldots,v_n \in \mathbb{R}^d$ or $\mathbb{C}^d$*

$$\mathrm{rank}([\langle v_i, v_j\rangle^r]) = \min\{n, \binom{d+r-1}{r}\}, \qquad r \geq 0. \tag{3.5}$$

Proof: This matrix is the r times Hadamard product of the Gram matrix

$$A := [\langle v_i, v_j\rangle^r] = \underbrace{B \circ B \circ \cdots \circ B}_{r \text{ times}}, \qquad B = V^*V, \qquad V := [v_1,\ldots,v_n].$$

Since $B = V^*V$ is positive semidefinite, it follows from the Schur product theorem that A is also. Almost every choice of $\{v_i\}_{i=1}^n$ is in general position, and so we may assume without loss of generality that they are chosen so.

First suppose $n \le d$. Then the $\{v_i\}$ are linearly independent, so B is positive definite, and by the Schur product theorem A is positive definite, giving $\operatorname{rank}(A) = n$, as asserted.

Hence it suffices to suppose $n > d$. Clearly, $\operatorname{rank}(A) \le n$. Since B, V have the same kernel and $\operatorname{rank}(V) = d$, the positive semidefinite matrix B has rank d, and so can be written

$$B = \sum_{i=1}^{d} u_i u_i^*,$$

where $\{u_1, \ldots, u_d\}$ is an orthogonal basis for the range of B. Now

$$A = B \circ \cdots \circ B = \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_r=1}^{d} (u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_r})(u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_r})^*,$$

a sum of at most $\binom{d+r-1}{r}$ rank one matrices ($\circ$ is commutative), giving

$$\operatorname{rank}(A) \le \binom{d+r-1}{r}.$$

Thus, by considering principal submatrices, it suffices to show $\operatorname{rank}(A) = n$, where

$$n = \binom{d+r-1}{r}.$$

Since $\det(A)$ is a polynomial in $v_1, \ldots, v_n$ it will be nonzero for almost every choice of $\{v_i\}$ (giving the result) provided it is nonzero for some choice. Using equivalence with (c) in Lemma 3.3, it is easy to see such choices exist. For example, use (d) and the well known fact that the polynomials Π_r^0 have a basis of ridge functions $\{\langle \cdot, u_i \rangle^r\}$. $\square$

Example 1. In three dimensions ($d = 3$), let $r = 2$. Then the matrix $[\langle v_i, v_j \rangle^2]$ is invertible for almost every choice of $\{v_1, \ldots, v_6\}$. If we take v_1, v_2, v_3 to be an orthonormal basis and

$$v_4 := v_1 + v_2, \quad v_5 := v_2 + v_3, \quad v_6 := v_4 + v_5 = v_1 + 2v_2 + v_3,$$

then these $\{v_i\}$ are not in general position (since $v_6 = v_4 + v_5$), and satisfy

$$|\det([\langle v_i, v_j \rangle^2])| = 8.$$

Thus, the configurations of points $\{v_i\}$ which give (3.5) are not simply those which are in general position. In Example 2 we give an example where this is the case.

We now give the counterparts to Lemma 3.3 and Theorem 3.4 for complex matrices

$$A = [\langle v_i, v_j \rangle^r \overline{\langle v_i, v_j \rangle}^s], \qquad r, s \ge 0.$$

This requires a generalisation of Hermitian forms and the associated polynomial algebra. We can not find a reference to this in the literature, and so provide the basic results.

Suppose H is a complex Hilbert space. Then a map $L : H^r \times H^s \to \mathbb{C}$ is called a **Hermitian** (r,s)–**form** on H if it is symmetric r–linear in the first r variables and symmetric s–conjugate–linear in the last s variables. Let

$$S_{r,s} := S_{r,s}(H) := \text{the real vector space of all Hermitian } (r,s)\text{–forms}.$$

The map which associates $L \in S_{r,s}$ with $x \mapsto L(x,\ldots,x;x,\ldots,x)$ is an isomorphism onto

$$\Pi^0_{r,s} := \Pi^0_{r,s}(H) := \Pi^0_r \otimes \overline{\Pi^0_s}$$

($\overline{f}(z) := \overline{f(z)}$), and the restriction of $\Pi^0_{r,s}$ to the sphere is an isomorphism onto

$$\mathcal{H}_{r,s} := \mathcal{H}_{r,s}(H) := \mathcal{H}_r \otimes \overline{\mathcal{H}_s}.$$

Lemma 3.6. *Let $u_1,\ldots,u_n$ be unit vectors Hilbert space H of dimension d, where*

$$n := \binom{r+d-1}{d-1}\binom{s+d-1}{d-1} = \dim(S_{r,s}) = \dim(\Pi^0_{r,s}) = \dim(\mathcal{H}_{r,s}), \qquad r,s \geq 0.$$

Then the following are equivalent

(a) *No nonzero $p \in \mathcal{H}_{r,s}$ vanishes at all the points $\{u_i\}$.*

(b) *There is a unique $p \in \mathcal{H}_{r,s}$ which interpolates arbitrary data at the points $\{u_i\}$.*

(c) *The $n \times n$ positive semidefinite matrix*

$$A := [\langle u_i, u_j\rangle^r \langle u_j, u_i\rangle^s]$$

is invertible.

(d) *The polynomials $\{\langle\cdot, u_i\rangle^r \langle u_i, \cdot\rangle^s\}$ are a basis for $\Pi^0_{r,s}$ and $\mathcal{H}_{r,s}$.*

(e) *The functionals $\{f \mapsto f(u_i)\}$ are a basis for the dual spaces of $\Pi^0_{r,s}$ and $\mathcal{H}_{r,s}$.*

(f) *The Hermitian (r,s)–forms on H have a basis given by*

$$(x_1,\ldots,x_r,y_1,\ldots,y_s) \mapsto \langle x_1,u_i\rangle \cdots \langle x_r,u_i\rangle\langle u_i,y_1\rangle \cdots \langle u_i,y_s\rangle, \qquad i=1,\ldots,n.$$

(g) *The functionals $L \mapsto L(u_i,\ldots,u_i)$ are a basis for the dual space of $S_{r,s}$.*

Proof: The proof is similar to that of Lemma 3.6. □

In particular, a Hermitian $(1,1)$–form is a Hermitian form.

Theorem 3.7. *For almost every $v_1,\ldots,v_n \in \mathbb{C}^d$*

$$\operatorname{rank}([\langle v_i,v_j\rangle^r \overline{\langle v_i,v_j\rangle}^s]) = \min\{n, \binom{d+r-1}{r}\binom{d+s-1}{s}\}, \qquad r,s \geq 0. \tag{3.8}$$

Proof: The proof is similar to that of Theorem 3.4, with

$$A := [\langle v_i,v_j\rangle^r \overline{\langle v_i,v_j\rangle}^s] = \underbrace{B \circ B \circ \cdots \circ B}_{r \text{ times}} \circ \underbrace{\overline{B} \circ \overline{B} \circ \cdots \circ \overline{B}}_{s \text{ times}}.$$

This leads to

$$A = \sum_{i_1=1}^d \cdots \sum_{i_r=1}^d \sum_{j_1=1}^d \cdots \sum_{j_s=1}^d (u_{i_1} \circ \cdots \circ u_{i_r} \circ \overline{u_{j_1}} \circ \cdots \circ \overline{u_{j_s}})(u_{i_1} \circ \cdots \circ u_{i_r} \circ \overline{u_{j_1}} \circ \cdots \circ \overline{u_{j_s}})^*$$

a sum of at most $\binom{d+r-1}{r}\binom{d+s-1}{s}$ rank one matrices. □

We now give an explicit formula for the determinant of $[\langle v_i, v_j\rangle^r]$ in two dimensions.

Lemma 3.9. *Let $v_1, \ldots, v_n$ be vectors in $\mathbb{C}^2$, where $n = r+1$. Then*

$$\det([\langle v_i, v_j\rangle^r]) = C(r) \prod_{1\le i<j\le r+1} |\det([v_i, v_j])|^2,$$

$$\det([(v_i \cdot v_j)^r]) = C(r) \prod_{1\le i<j\le r+1} \det([v_i, v_j])^2,$$

where

$$C(r) := \prod_{k=0}^{r} \binom{r}{k}.$$

Proof: Let $A := [\langle v_i, v_j\rangle^r]$ and $v_i = (v_{i1}, v_{i2})^T$. Then a binomial expansion gives

$$\begin{aligned} a_{ij} = (\overline{v_{i1}}v_{j1} + \overline{v_{i2}}v_{j2})^r &= \sum_{k=0}^{r} \binom{r}{k} (\overline{v_{i1}}v_{j1})^k (\overline{v_{i2}}v_{j2})^{r-k} \\ &= \sum_{k=0}^{r} (\overline{v_{i1}})^k (\overline{v_{i2}})^{r-k} \binom{r}{k} (v_{j1})^k (v_{j2})^{r-k}, \end{aligned}$$

i.e., $A = B^* D B$, where

$$b_{ij} := (v_{j1})^{i-1}(v_{j2})^{r-i+1}, \qquad D = \operatorname{diag}\{\binom{r}{i-1}, i = 1, \ldots, n\}.$$

Similarly, with $M := [(v_i \cdot v_j)^r]$, we have $M = B^T D B$. Taking determinants gives

$$\det(A) = C(r)\,|\det(B)|^2, \qquad \det(M) = C(r)\,\det(B)^2,$$

and so it remains only to compute the determinant of B.

By unitary invariance we may assume that $v_{j2} \ne 0$, $\forall j$. Divide row j of B by $(v_{j2})^r$ to obtain a Vandermonde matrix in the variable v_{j1}/v_{j2}, giving

$$\det(B) = \prod_{1\le j\le n} (v_{j2})^{n-1} \det\left(\left(\frac{v_{j1}}{v_{j2}}\right)^{i-1} \right) = \prod_{1\le i<j\le n} (v_{i1}v_{j2} - v_{i2}v_{j1}).$$

□

Example 2. When $n = r + 1$, $d = 2$ the conditions of Lemma 3.3 are equivalent to $u_1, \ldots, u_n$ being in general position, since by Lemma 3.9

$$\begin{aligned}[\langle v_i, v_j\rangle^r] \text{ is invertible} &\iff \det([\langle v_i, v_j\rangle^r]) \neq 0 \\ &\iff \det([v_i, v_j]) \neq 0, \quad 1 \leq i < j \leq n \\ &\iff v_1, \ldots, v_n \text{ are in general position.}\end{aligned}$$

Example 1 shows that this is not the case for $d \geq 3$.

4. Scaling to obtain a tight signed frame

In this section, we investigate when a set of unit vectors $\{u_j\}$ in H can be scaled

$$\psi_j := \alpha_j u_j,$$

to obtain a tight signed frame $\{\psi_j\}$, and hence a representation of the form

$$f = \sum_j \sigma_j \langle f, \psi_j\rangle \psi_j = \sum_j c_j \langle f, u_j\rangle u_j, \qquad \forall f \in H. \tag{4.1}$$

where $c_j := \sigma_j |\alpha_j|^2$. Clearly, multiplying the α_j by scalars of unit modulus gives a signed frame with the same signature and bounds. Thus we say there is a **unique scaling** if there is a unique signature σ and $|\alpha_j|$ giving a tight signed frame, i.e., there is a unique choice of the c_j. If a more than one scaling exists, then there are infinitely many since the set of such $c = (c_j)$ is affine. Here we consider a finite set $\{u_1, \ldots, u_n\}$ where H has dimension d.

A necessary and sufficient condition for such a scaling to exist is that

$$\sum_j \langle e_{i_1}, u_j\rangle \langle u_j, e_{i_2}\rangle c_j = \langle e_{i_1}, e_{i_2}\rangle, \qquad \forall i = (i_1, i_2) \in I,$$

where $(e_i)_{i=1}^d$ is an orthonormal basis of H, and I is the index set

$$\begin{aligned} I &:= \{(i_1, i_2) : 1 \leq i_1 \leq i_2 \leq d\} \quad (H \text{ real}), \\ I &:= \{(i_1, i_2) : 1 \leq i_1, i_2 \leq d\} \quad (H \text{ complex}). \end{aligned}$$

This can be written in matrix form

$$Mc = b, \qquad m_{ij} := \langle e_{i_1}, u_j\rangle \langle u_j, e_{i_2}\rangle, \qquad b_i := \langle e_{i_1}, e_{i_2}\rangle, \tag{4.2}$$

where M is an $I \times n$ matrix. The normal equation for this system

$$M^* M c = M^* b \tag{4.3}$$

is the necessary condition (3.1) in matrix form, i.e.,

$$Ac = [1], \qquad A := [|\langle u_i, u_j\rangle|^2]. \tag{4.4}$$

This follows from the calculations

$$\begin{aligned}(M^*M)_{st} = \sum_{i\in I} \overline{m}_{is} m_{it} &= \sum_{i\in I} \overline{\langle e_{i_1}, u_s\rangle\langle u_s, e_{i_2}\rangle}\langle e_{i_1}, u_t\rangle\langle u_t, e_{i_2}\rangle \\ &= \Bigl(\sum_{i_1}\langle u_s, e_{i_1}\rangle\langle e_{i_1}, u_t\rangle\Bigr)\Bigl(\sum_{i_2}\overline{\langle u_s, e_{i_2}\rangle}\,\overline{\langle e_{i_2}, u_t\rangle}\Bigr) = |\langle u_s, u_t\rangle|^2\end{aligned}$$

$$(M^*b)_j = \sum_{i\in I} \overline{m}_{ij} b_i = \sum_{i\in I}\langle u_j, e_{i_1}\rangle\langle e_{i_2}, u_j\rangle\langle, e_{i_1}, e_{i_2}\rangle = \sum_{i_1}\langle u_j, e_{i_1}\rangle\langle e_{i_1}, u_j\rangle = \|u_j\|^2 = 1,$$

which show $M^*M = A$, $M^*b = [1]$.

Morever, the c satisfying (4.4) give a representation which is closest to (4.1) in the following sense

Theorem 4.5 (Best approximation property). *The c which minimise the Frobenius (matrix) norm*

$$\|I - \sum_{j=1}^n c_j P_j\|_F, \qquad P_j f := \langle f, u_j\rangle u_j, \quad If := f \tag{4.6}$$

are given by $Ac = [1]$.

Proof: Recall the Frobenius inner product is given by

$$\langle A, B\rangle_F := \mathrm{trace}(AB^*) = \textstyle\sum_{i,j} a_{ij}\overline{b}_{ij},$$

and so

$$\langle P_j, P_i\rangle_F = \mathrm{trace}(u_j u_j^* u_i u_i^*) = |\langle u_i, u_j\rangle|^2, \qquad \langle I, P_i\rangle_F = \mathrm{trace}(u_i u_i^*) = \|u_i\|^2 = 1.$$

The minimum (least squares solution) of (4.6) occurs when (the error) $I - \sum_j c_j P_j$ is orthogonal to all the P_i, i.e., $\forall i$

$$I - \textstyle\sum_j c_j P_j \perp P_i \quad\Longleftrightarrow\quad \sum_j c_j \langle P_j, P_i\rangle_F = \langle I, P_i\rangle_F \quad\Longleftrightarrow\quad \sum_j c_j |\langle u_i, u_j\rangle|^2 = 1.$$

□

When $|I|=n$ the matrix M is square and the necessary condition $Ac=[1]$ becomes necessary and sufficient for a unique scaling to exist.

By a **Hermitian form** on H we mean a symmetric bilinear map when H is a real space and a $(1,1)$–Hermitian form when H is complex, i.e., one satisfying the conditions of an Hermitian form. This is a real vector space of dimension

$$n=|I|=\begin{cases}\frac{1}{2}d(d+1), & H \text{ real};\\ d^2, & H \text{ complex}.\end{cases} \tag{4.7}$$

Theorem 4.8 (Equivalence). *Let $u_1,\ldots,u_n$ be unit vectors in a Hilbert space H of dimension d, where*

$$n=\begin{cases}\frac{1}{2}d(d+1), & H \text{ real};\\ d^2, & H \text{ complex}.\end{cases}$$

Then the following are equivalent

(a) *The $n\times n$ positive semidefinite matrix*

$$A:=[|\langle u_i,u_j\rangle|^2]$$

is invertible.

(b) *The vectors $u_1,\ldots,u_n$ have a unique scaling which gives a tight signed frame, with the c of* (4.1) *given by*

$$c=A^{-1}[1],\qquad A:=[|\langle u_i,u_j\rangle|^2].$$

(c) *The Hermitian forms on H have a basis given by*

$$(f,g)\mapsto\langle f,u_i\rangle\langle u_i,g\rangle,\qquad i=1,\ldots,n.$$

(d) *The functionals $L\mapsto L(u_i,u_i)$ are a basis for the dual space of the Hermitian forms.*

(e) *The self adjoint operators on H have a basis given by the rank* 1 *orthogonal projections*

$$P_i: f\mapsto\langle f,u_i\rangle u_i,\qquad i=1,\ldots,n.$$

Proof: The equivalence of (a),(c),(d) is a special case of Lemmas 3.3 and 3.6. Since $|I|=n$, there is a unique scaling (given by $Mc=b$) iff the $I\times n$ matrix M is invertible iff $A=M^*M$ is invertible. Since c is then given by (4.4) this gives (a)$\Longleftrightarrow$(b). The self adjoint (Hermitian) operator corresponding to $(f,g)\mapsto\langle f,u_i\rangle\langle u_i,g\rangle$ is P_i, which gives (c)$\Longleftrightarrow$(e). $\square$

Corollary 4.9 (Scaling to a tight frame). *Let H be a Hilbert space of dimension d, and*

$$n=\begin{cases}\frac{1}{2}d(d+1), & H \text{ real};\\ d^2, & H \text{ complex}.\end{cases}$$

Then almost every choice of unit vectors $\{u_1,\ldots,u_n\}$ in H has a unique scaling that gives a tight signed frame, with the constants c_j in (4.1) *given by*

$$c=A^{-1}[1],\qquad A:=[|\langle u_i,u_j\rangle|^2]. \tag{4.10}$$

The signature and the scaling factors of the tight signed frame so obtained satisfy

$$\sigma=\operatorname{sign}(c),\qquad |\alpha_j|^2=|c_j|,\quad\forall j,\qquad \textstyle\sum_j c_j=d. \tag{4.11}$$

Proof: Since $\det(A)$ is a nonzero polynomial in $u_1,\ldots,u_n$, A is invertible for almost every choice of $\{u_i\}$. The equations (4.11) follow from $c_j=\sigma_j|\alpha_j|^2$ and (2.4). $\square$

For $d = 1$ the result is trivial. The examples of three vectors in $\mathbb{R}^2$ (being in general position implies there is unique scaling) and four vectors in $\mathbb{C}^2$ have already been discussed.

Example 1. If $n = d(d+1)/2$ unit vectors are in general position on the sphere in $\mathbb{R}^d$, i.e., no homogeneous quadratic vanishes at all of them, then there is a unique scaling of them giving a tight signed frame.

Example 2. With the exception of three vectors in $\mathbb{R}^2$, it is possible to construct a set of n vectors in general position for which more than one scaling to a tight signed frame exists. For example, take two different orthonormal bases (possible for $d \geq 2$, H complex and $d \geq 3$, H real) whose union is in general position and enlarge this to a set of n vectors in general position. Then this can be scaled to a tight frame (in two different ways) by taking the weights corresponding to one of the orthonormal bases to be 1, and all the others to be zero.

Example 3. It is also possible to construct a set of n vectors for which no scaling to a tight frame exists. This can be done by taking the vectors from a basis which is not orthogonal. Examples where the vectors are in general position also exist, e.g., in $\mathbb{C}^2$ take

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -i \\ e^{5i} \end{pmatrix}, \quad \begin{pmatrix} e^{-5i} \\ \sqrt{3}+2 \end{pmatrix}, \quad \begin{pmatrix} e^{\frac{\pi}{3}i} \\ e^{5i} \end{pmatrix}, \tag{4.12}$$

and in $\mathbb{R}^3$ take

$$\begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ \sqrt{5} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ \sqrt{10} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ \sqrt{17} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 5 \\ \sqrt{26} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 6 \\ \sqrt{37} \end{pmatrix}. \tag{4.13}$$

The considerations which led to these choices are discussed in the appendix.

Example 4. When H is real a specific choice of $\{u_i\}$ for which A is invertible is

$$(e_k + e_l)/\sqrt{2}, \qquad 1 \leq k \leq l \leq d,$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis. When H is complex add to this

$$(e_k + i\, e_l)/\sqrt{2}, \qquad 1 \leq k < l \leq d,$$

to get such a choice.

5. Tight frames of Jacobi polynomials on a triangle

Here we construct tight frames of bivariate orthogonal Jacobi polynomials which share the symmetries of the weight. Though primarily interested in the bivariate case, we give the definitions for $\mathbb{R}^s$ (which are no more complicated).

Let V be a set of $s+1$ affinely independent points in $\mathbb{R}^s$, i.e., the vertices of an s–simplex which we denote by T. Let $\xi = (\xi_v)_{v\in V}$ be the corresponding **barycentric coordinates**, i.e., the unique linear polynomials that satisfy

$$\sum_{v\in V} \xi_v(a) = 1, \qquad \sum_{v\in V} \xi_v(a)v = a, \quad \forall a \in \mathbb{R}^s.$$

For the (standard) triangle with vertices 0, $e_1 = (1,0)$, $e_2 = (0,1)$, these are

$$\xi_0(x,y) = 1 - x - y, \qquad \xi_{e_1}(x,y) = x, \qquad \xi_{e_2}(x,y) = y.$$

We will use standard multi–index notation for indices, so, for example,

$$\xi^\mu := \prod_{v\in V} \xi_v^{\mu_v}, \quad \mu \in \mathbb{R}^V, \qquad \beta! := \prod_{v\in V} \beta_v!, \quad \beta \in \mathbb{Z}_+^V.$$

For functions defined on T, we define an inner product by

$$\langle f, g\rangle_\mu := \int_T f g\, \xi^\mu, \qquad \mu > -1.$$

The condition $\mu_v > -1$ ensures the nonnegative weight ξ^μ is integrable over T.

Let S_V be the **symmetry group** of the simplex T with vertices V, i.e., the group of affine maps of T onto T. This is (isomorphic to) the symmetric group on V since an affine map $\mathbb{R}^s \to \mathbb{R}^s$ is uniquely determined by its action on $s+1$ affinely independent points (such as V). Let $S \in S_V$ act on functions f defined on T via $S \cdot f := f \circ S^{-1}$. Then S permutes the barycentric coordinates ξ_v, and so if all the μ_v are equal, the inner product has the symmetries

$$\langle S\cdot f, S\cdot g\rangle_\mu = \langle f, g\rangle_\mu, \qquad S \in S_V.$$

We say that $f \in \Pi_k(\mathbb{R}^s)$ is a **Jacobi polynomial** (of degree k) for the simplex T with weight ξ^μ (cf [DX01]) if it satisfies the orthogonality condition

$$\langle f, p\rangle_\mu = \int_T f p\, \xi^\mu = 0, \qquad \forall p \in \Pi_{k-1}(\mathbb{R}^s).$$

Such a polynomial of exact degree k is uniquely determined by its **leading term** $f_\uparrow$, i.e., the homogeneous polynomial of degree k for which $\deg(f - f_\uparrow) < k$, via

$$f = f_\uparrow - P_{\Pi_{k-1}}(f_\uparrow), \qquad P_{\Pi_{k-1}} := \text{orthogonal projection onto } \Pi_{k-1}(\mathbb{R}^s).$$

Thus the space $\mathcal{P}_k^\mu$ of Jacobi polynomials of (exact) degree k has

$$\dim(\mathcal{P}_k^\mu) = \dim(\Pi_k^0(\mathbb{R}^s)) = \binom{k+s-1}{s-1}.$$

There exist explicit formulae for an orthogonal basis of this space (see [P57] and [KMT91]), and also biorthogonal systems (see [AK26] and [FL74]). But these do not share the symmetries of the weight, i.e., they are not invariant under the action of S_V when $\mu_v = \mu_0$, $\forall v \in V$. We now use the scaling results to construct a tight frame of Jacobi polynomials with these symmetries for the triangle.

Let $p^\mu_{\xi^\beta}$, $|\beta| = k$ denote the Jacobi polynomial of degree k with leading term

$$(\xi^\beta)_\uparrow = \prod_{v\in V} (\xi_{v\uparrow})^{\beta_v} \in \Pi^0_k.$$

Then $\{p^\mu_{\xi^\beta} : |\beta| = k, \beta \in \mathbb{Z}^V_+\} \subset \mathcal{P}^\mu_k$ is an S_V–invariant family when $\mu_v = \mu_0$, $\forall v$. In the bivariate case ($s = 2$), this consists of $(k+1)(k+2)/2$ Jacobi polynomials of degree k, and so, by Theorem 4.8, they have a unique scaling that gives a tight signed frame provided the matrix

$$A := [|\langle p^\mu_{\xi^\alpha}, p^\mu_{\xi^\beta}\rangle_\mu|^2]_{|\alpha|,|\beta|=k} \tag{5.1}$$

is invertible. We first give examples where A was inverted and the scaling factors computed exactly, then give the general result suggested by these calculations. Normalise the $p^\mu_{\xi^\beta}$ so that the tight signed representation is

$$f = \sum_{\substack{|\beta|=k\\ \beta\in\mathbb{Z}^V_+}} c^\mu_\beta \frac{\langle f, p^\mu_{\xi^\beta}\rangle_\mu}{\langle p_{\xi^\beta}, p_{\xi^\beta}\rangle_\mu} p^\mu_{\xi^\beta}, \qquad \forall f \in \mathcal{P}^\mu_k, \tag{5.2}$$

where, by (2.4),

$$\sum_{|\beta|=k} c^\mu_\beta = k+1. \tag{5.3}$$

Example 1 (Quadratics). For quadratic Jacobi polynomials the β have two forms: $(1,1,0)$ and $(2,0,0)$ (three of each). The c_β for selected $\mu = (\mu_0, \mu_0, \mu_0)$ are (respectively)

$$c^{(0,0,0)}_\beta = \frac{7}{10}, \frac{3}{10}, \qquad c^{(1,1,1)}_\beta = \frac{9}{14}, \frac{5}{14}, \qquad c^{(2,2,2)}_\beta = \frac{34}{55}, \frac{21}{55}, \qquad c^{(3,3,3)}_\beta = \frac{55}{91}, \frac{36}{91}.$$

The Jacobi polynomials are

$$p^\mu_{\xi_v\xi_w} = \xi_v\xi_w - \frac{\mu_0+1}{3\mu_0+5}(\xi_v+\xi_w) + \frac{(\mu_0+1)^2}{(3\mu_0+4)(3\mu_0+5)}, \qquad v \neq w,$$
$$p^{(0,0,0)}_{\xi_v^2} = \xi_v^2 - \frac{2(\mu_0+2)}{3\mu_0+5}\xi_v + \frac{(\mu_0+1)(\mu_0+2)}{(3\mu_0+4)(3\mu_0+5)}.$$

Example 2 (Cubics). For cubics the β have three forms $(1,1,1)$, $(2,1,0)$ and $(3,0,0)$ (1,6,3 of each). The c_β for selected $\mu = (\mu_0, \mu_0, \mu_0)$ are (respectively)

$$c^{(0,0,0)}_\beta = \frac{24}{35}, \frac{52}{105}, \frac{4}{35}, \qquad c^{(1,1,1)}_\beta = \frac{3}{5}, \frac{29}{60}, \frac{1}{6},$$

$$c_\beta^{(2,2,2)} = \frac{80}{143}, \frac{68}{143}, \frac{28}{143}, \qquad c_\beta^{(3,3,3)} = \frac{15}{28}, \frac{79}{168}, \frac{3}{14}.$$

The Jacobi polynomials with constant weight (referred to as **Legendre** polynomials) are

$$\begin{aligned}
p^{(0,0,0)}_{\xi_u\xi_v\xi_w} &= \xi_u\xi_v\xi_w - \frac{1}{7}(\xi_u\xi_v + \xi_u\xi_w + \xi_v\xi_w) + \frac{2}{105}, \\
p^{(0,0,0)}_{\xi_v^2\xi_w} &= \xi_v^2\xi_w - \frac{4}{7}\xi_v\xi_w - \frac{1}{7}\xi_v^2 + \frac{2}{21}\xi_v + \frac{1}{21}\xi_w - \frac{1}{105}, \\
p^{(0,0,0)}_{\xi_v^3} &= \xi_v^3 - \frac{9}{7}\xi_v^2 + \frac{3}{7}\xi_v - \frac{1}{35}.
\end{aligned}$$

We now give an explicit formula for a general c_β^μ. Define a multivariate hypergeometric function with arguments c a scalar, and β, γ, x vectors from $\mathbb{R}^V$ by

$$F(c,\beta;\gamma;x) := \sum_{\alpha\in\mathbb{Z}_+^V} (c)_{|\alpha|} \frac{(\beta)_\alpha}{(\gamma)_\alpha} \frac{x^\alpha}{\alpha!}, \qquad c \in \mathbb{R}, \quad \beta,\gamma,x \in \mathbb{R}^V,$$

where $(\beta)_\alpha$ is the *multivariate shifted factorial*

$$(\beta)_\alpha := \prod_{v\in V} (\beta_v)_{\alpha_v}, \qquad (\beta_v)_{\alpha_v} := \beta_v(\beta_v+1)\cdots(\beta_v+\alpha_v-1).$$

This is the *Lauricella function F_A*. Note that $F(c,-\beta;\gamma;\xi)$ is a polynomial of degree $|\beta|$ in ξ, i.e.,

$$F(c,-\beta;\gamma;\xi) = \sum_{\alpha\le\beta} (c)_{|\alpha|} \frac{(-1)^{|\alpha|}}{(\gamma)_\alpha} \frac{\beta!}{(\beta-\alpha)!} \frac{\xi^\alpha}{\alpha!}.$$

In [AK26] it was shown how in two variables this relates to the Jacobi polynomials with a restricted class of weights (no weight on the third barycentric coordinate), and the general result can be found in [FL74], namely

$$p^\mu_{\xi^\beta} := \frac{(-1)^{|\beta|}(\mu+1)_\beta}{(|\beta|+|\mu|+s)_{|\beta|}} q^\mu_\beta, \qquad q^\mu_\beta := F(|\beta|+|\mu|+s, -\beta; \mu+1; \xi),$$

where $\mu+1 := (\mu_v+1)_{v\in V}$, $|\mu| := \sum_v \mu_v$.

In [WX01] a technical proof, which uses the orthogonal basis of Proriol [P57] and the Hahn polynomials, is given for the following bivariate result. Let Γ be the *multivariate gamma function.*

Theorem 5.4 (Tight frame of Jacobi polynomials on a triangle). *On the triangle there is a unique scaling of $\{p^\mu_{\xi^\beta} : |\beta| = k\}$ that gives a tight signed frame for $\mathcal{P}^\mu_k$, with the scalars of* (5.2) *given by*

$$c^\mu_\beta = C^\mu_k \frac{(\mu+1)_\beta}{\beta!} \langle q^\mu_\beta, q^\mu_\beta\rangle_\mu > 0, \qquad |\beta| = k, \quad \beta \in \mathbb{Z}_+^V,$$

where

$$C_k^\mu := \frac{(|\mu|+s+1)_{2k}}{(k+|\mu|+s)_k^2}\frac{\Gamma(|\mu|+s+1)}{\Gamma(\mu+1)}, \qquad s=2,$$

and so this is a frame. The representation (5.2) *can be written in the compact form*

$$f = C_k^\mu \sum_{|\beta|=k} \frac{(\mu+1)_\beta}{\beta!}\langle f, q_\beta^\mu\rangle_\mu q_\beta^\mu, \qquad \forall f \in \mathcal{P}_k^\mu. \tag{5.5}$$

This was first observed, by chance, for the Legendre polynomials, i.e., when $\mu=(0)$, and (5.5) simplifies to

$$f = (2k+2)\frac{((k+1)!)^2}{(2k+1)!}\sum_{|\beta|=k}\langle f, q_\beta^0\rangle_\mu q_\beta^0, \qquad \forall f \in \mathcal{P}_k^0.$$

It was then extended whilst proving this case. In [WX01] it is also shown this result holds in all dimensions, where now (5.3) becomes

$$\sum_{|\beta|=k} c_\beta^\mu = \dim(\mathcal{P}_k^\mu) = \binom{k+s-1}{s-1}.$$

In contrast to the bivariate result, our abstract scaling results do not suggest that this should be the case, and the result was proved without determining whether or not the matrix A of (5.1) is invertible.

Since $\{p_{\xi^\beta}\}_{|\beta|=k}$ spans $\mathcal{P}_k^\mu$ any scalar multiples of these functions forms a frame. The determination of the dual frame (which shares any symmetries) is still an open question in all but the above (most interesting) case.

6. Numerical results and conjectures

Consider the c of minimal norm giving the best approximation of Theorem 4.5, i.e., the least squares solution of the necessary condition (4.4) given by taking the (Moore–Penrose) pseudoinverse

$$c = c(u_1,\ldots,u_n) := A^+[1], \qquad A := [|\langle u_i, u_j\rangle|^2]. \tag{6.1}$$

This is a continuous function of $u_1,\ldots,u_n$ except at those points where the number of singular values of A changes (a set of measure zero). By Corollary 4.9, for n greater than or equal to the value (4.7), it has constant trace, i.e., $\sum_j c_j = d$ for almost all choices of $u_1,\ldots,u_n$. When a scaling to a tight signed frame exists this value of c gives the scaling factors with minimal $\sum_j c_j^2$. In particular, the scaling gives a frame if this is possible.

Thus, one could imagine finding a set of vectors $u_1,\ldots,u_n$ for which $c(u_1,\ldots,u_n)$ takes some specified (and allowable) value c^* by taking an initial guess, computing c, then comparing it with the value obtained for some appropriately sized (random) perturbation of $u_1, \ldots, u_n$, and keeping whichever set of vectors gives a value closest to c^*. Using `MATLAB` we implemented this naive scheme. A number of interesting, now mostly proved, conjectures arose from the computations we undertook.

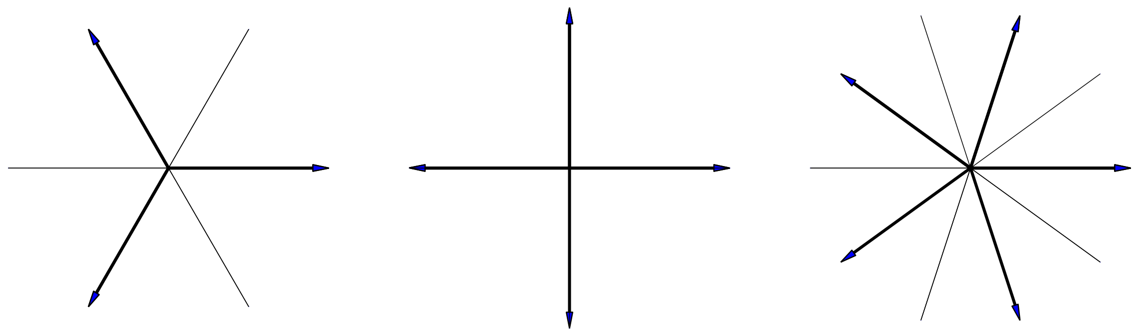

Fig. 1. Tight frames of vectors in $\mathbb{R}^2$ which are equally spaced on the circle.

In $H = \mathbb{R}^2, \mathbb{C}^2$ the standard examples of a tight frame of $n \geq 2$ vectors are

$$u_j := \begin{pmatrix} \cos \frac{2\pi j}{n} \\ \sin \frac{2\pi j}{n} \end{pmatrix}, \qquad u_j := \frac{1}{\sqrt{2}} \begin{pmatrix} w^j \\ \overline{w}^j \end{pmatrix}, \quad w := e^{\frac{2\pi i}{n}}, \qquad j = 1, \ldots, n.$$

For each of these the frame representation is of the form

$$x = \frac{2}{n} \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \qquad \forall x \in H.$$

Moreover, the vectors $\{u_j\}$ in $\mathbb{R}^2$ are equally spaced on the circle. Thus, it is natural to ask whether there exist frames with all the c_i equal in higher dimensions (other than the orthonormal bases), and whether they can be interpreted as points which are equally spaced on the sphere. The answers to these questions are yes and probably not.

Theorem 6.2 (Isometric tight frames). *For each $n \geq d$, there exist unit vectors $u_1, \ldots, u_n$ in general position in $H = \mathbb{R}^d, \mathbb{C}^d$ for which*

$$x = \frac{d}{n} \sum_{i=1}^{n} \langle x, u_i \rangle u_i, \qquad \forall x \in H,$$

i.e., there exists a tight frame consisting of n vectors of equal length.

This was supported by all our calculations. For example, in $\mathbb{R}^3$ we obtained the following vectors $U = [u_1, \ldots, u_n]$, $n = 4, 5, 6$ which give a tight frame with equal c_j (to 4 sf).

$$U = \begin{pmatrix} -0.5742 & -0.4972 & -0.5799 & -0.6569 \\ -0.7015 & 0.3905 & 0.7496 & -0.3424 \\ -0.4221 & -0.7748 & 0.3191 & 0.6718 \end{pmatrix}, \qquad c_i = 0.7500$$

$$U = \begin{pmatrix} 0.4771 & 0.4732 & 0.7153 & 0.6624 & 0.4955 \\ 0.6468 & -0.8745 & 0.5849 & -0.2450 & -0.2994 \\ 0.5950 & 0.1061 & -0.3825 & -0.7079 & 0.8153 \end{pmatrix}, \qquad c_i = 0.6000$$

$$U = \begin{pmatrix} 0.5767 & 0.8003 & 0.6376 & -0.0293 & -0.5824 & 0.5294 \\ -0.5587 & 0.4885 & 0.1064 & 0.7417 & -0.5180 & -0.7871 \\ 0.5961 & -0.3475 & 0.7630 & 0.6701 & 0.6265 & -0.3165 \end{pmatrix}, \qquad c_i = 0.5000$$

Here is an example of 8 vectors in $\mathbb{R}^5$ ($c_i = 0.6250$)

$$U = \begin{pmatrix} 0.6257 & -0.3562 & -0.2393 & 0.4430 & 0.4650 & 0.0081 & 0.5352 & 0.5522 \\ 0.4655 & 0.1264 & -0.9514 & -0.4910 & -0.2354 & 0.2406 & 0.0612 & -0.3293 \\ -0.0407 & 0.7345 & 0.0480 & 0.4079 & -0.5840 & 0.5357 & 0.3845 & 0.3407 \\ 0.5238 & 0.0028 & 0.1466 & 0.5980 & -0.4689 & -0.5520 & 0.0466 & -0.6521 \\ -0.3403 & 0.5636 & -0.1175 & -0.1966 & 0.4092 & -0.5918 & 0.7482 & -0.2130 \end{pmatrix}$$

In the complex case our naive algorithm converges only when the perturbation of $u_1, \ldots, u_n$ is taken to be real (here accuracy of 4 sf is typically obtained within 1000 iterations). As yet, we have been unable to explain why this is so in term of the underlying geometry of $c(u_1, \ldots, u_n)$. It is also observed that the c of iterates tend to approach c^* (equal entries) from below in the cases where the trace of c is need not be d. Here are examples of isometric tight frames of 4 vectors in $\mathbb{C}^2$ and $\mathbb{C}^3$ obtained from our calculations

$$U = \begin{pmatrix} 0.5587 + 0.0842i & -0.0848 + 0.0482i & -0.8080 + 0.1602i & -0.8242 - 0.5600i \\ 0.8225 - 0.0657i & -0.9889 + 0.1119i & 0.5467 - 0.1505i & 0.0832 + 0.0157i \end{pmatrix}$$

$$U = \begin{pmatrix} -0.2995 + 0.2150i & 0.5050 - 0.1116i & 0.9105 + 0.0821i & -0.0492 + 0.3040i \\ 0.6115 + 0.3047i & -0.6113 + 0.3512i & 0.3641 + 0.0847i & -0.3063 + 0.3687i \\ 0.6101 - 0.1581i & 0.4626 - 0.1466i & 0.1271 - 0.0910i & -0.5943 - 0.5677i \end{pmatrix}$$

A proof of Theorem 6.2, which is based on a simple inductive construction, is given in Reams and Waldron [RW01]. After proving this, it was pointed out to us by Matthew Fickus that this result was proved by Goyal, Vetterli and Thao in [GVT98] using discrete overcomplete Fourier series. Further discussion is given Fickus's thesis [F01], which uses the term 'normalized tight frame' for what we call an 'isometric tight frame'.

Now we consider the question of whether or not a (tight) frame with all c_j equal can be interpreted as a set of points which are equally spaced on the sphere. For three vectors in $\mathbb{R}^2$ this is the case. Here the intersection of the three subspaces spanned by the $\{u_i\}$ with the circle gives six equally spaced points. However, for four or more points there exist frames where this is not the case. For example, all frames of four vectors with equal c_j can be obtained by taking the union of two orthonormal bases. This gives equally spaced points only when the axes corresponding to the bases can be mapped onto each other by rotation through $\pi/4$.

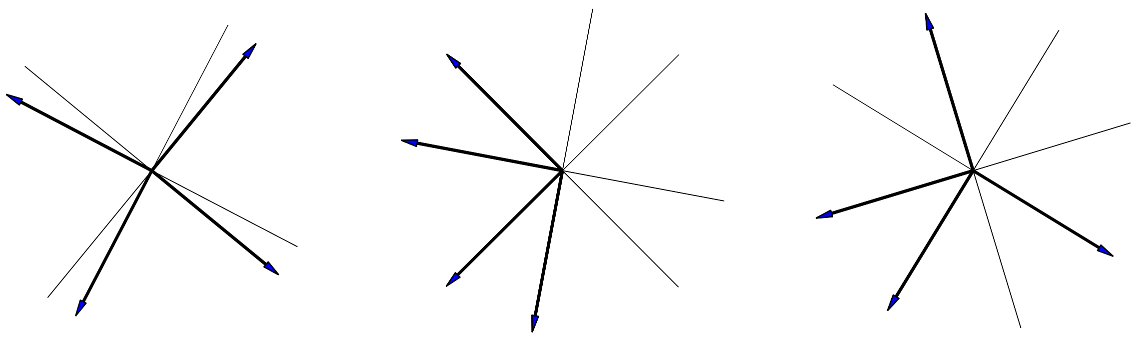

Fig. 2. Isometric tight frames of four vectors in $\mathbb{R}^2$.

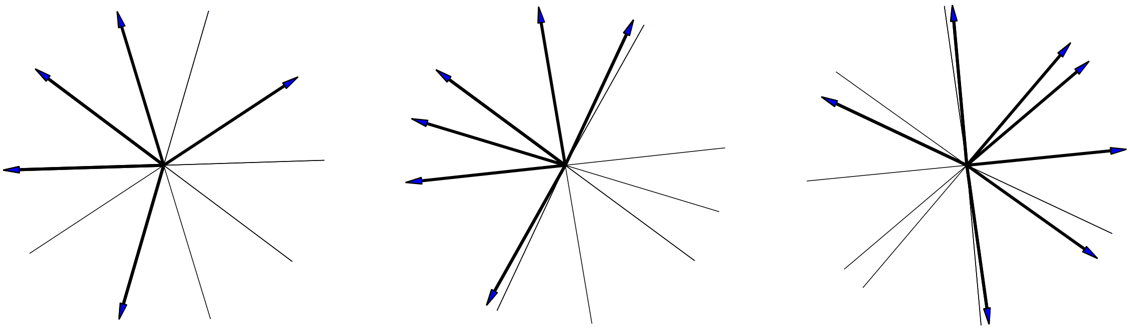

Fig. 3. Isometric tight frames of five, six and seven vectors in $\mathbb{R}^2$.

An even more extreme example is given by $H = \mathbb{C}$. Here $u_1, \ldots, u_n$ can be any complex numbers of unit modulus. These can be placed anywhere on the circle, even taken to be all the same. Thus it appears that, except for a few special cases, a randomly generated isometric tight frame can not be interpreted as points which are equally spaced. Many nice examples, such as the roots of unity (in $\mathbb{R}^2$), the vertices of the five Platonic solids in $\mathbb{R}^3$ and normalised Eutactic stars (see Coxeter [C63]) and the closely related spherical 2–designs (see [DGS77]) do exist, but the authors can think of no systematic way of finding them.

7. Appendix

Here we provide details on the constructions of Example 3 of Section 4, i.e., we find vectors $u_1, \ldots, u_n$ which are in general position for which (4.2) has no solution.

To find four vectors in general position in $\mathbb{C}^2$ for which no scaling to a tight frame exists it suffices to consider ones of the form

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \cos t_2 \\ \sin t_2 \, e^{i\theta_2} \end{pmatrix}, \quad u_3 = \begin{pmatrix} \cos t_3 \\ \sin t_3 \, e^{i\theta_3} \end{pmatrix}, \quad u_4 = \begin{pmatrix} \cos t_4 \\ \sin t_4 \, e^{i\theta_4} \end{pmatrix},$$

where $t_j, \theta_j \in \mathbb{R}$. For these the determinant of the matrix M in (4.2) is scalar multiple of

$$\sin(2t_2)\sin(2t_3)\sin(2t_4)\{\tan t_2 \, \sin(\theta_3 - \theta_4) + \tan t_3 \, \sin(\theta_4 - \theta_2) + \tan t_4 \, \sin(\theta_2 - \theta_3)\},$$

provided $\cos t_j \neq 0$. It is easy to choose t_j, θ_j so that the second factor above is zero, and so there is not a unique scaling to a tight signed frame. Moreover, a choice can be made so that there is no solution to $Mc = b$ (hence no scaling), and the $\{u_j\}$ are in general position. One such choice is $t_2 = \pi/4$, $t_3 = \tan^{-1}(\sqrt{3}+2)$, $t_4 = 5\pi/4$, $\theta_3 = 5$, $\theta_2 = \pi/2+5$, $\theta_4 = 5 - \pi/3$, which gives (4.12) up to a scalar factor.

The second example (4.13) is a special case of the following.

Proposition 7.1. *There exist $n := d(d+1)/2$ vectors in general position in $\mathbb{R}^d$, $d \geq 3$ for which no scaling to a tight signed frame exists.*

Proof: Let $V = [u_1, \ldots, u_n] \in \mathbb{R}^{d \times n}$ and $\{e_i\}$ be the standard basis vectors in $\mathbb{R}^d$. With $I := \{(i_1, i_2) : 1 \leq i_1 \leq i_2 \leq d\}$, the condition (4.2) becomes

$$Mc = b, \qquad m_{ij} := \langle e_{i_1}, u_j \rangle \langle u_j, e_{i_2} \rangle = v_{i_1 j} v_{i_2 j}, \qquad b_i := \langle e_{i_1}, e_{i_2} \rangle.$$

The system $Mc = b$ (which gives the scalings to a tight signed frame) has no solution, i.e., $b \notin \operatorname{ran}(M) = \ker(M^*)^\perp$, if we can find a vector $a \in \mathbb{R}^I$ with $M^* a = 0$ and $\langle a, b \rangle \neq 0$. Let

$$a_i := \begin{cases} 1, & i \in \{(1,1),(2,2)\}; \\ -1, & i = (3,3); \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle a, b \rangle = 1 \neq 0$, and the condition $M^* a = 0$ is

$$v_{1j}^2 + v_{2j}^2 = v_{3j}^2, \qquad j = 1, \ldots, n. \tag{7.2}$$

Thus it suffices to find a $V \in \mathbb{R}^{d \times n}$ whose first three rows satisfy (7.2), and whose columns are in general position. Let

$$v_{ij} := \begin{cases} j^{i-1}, & i \neq 3; \\ \sqrt{j^2+1}, & i = 3. \end{cases}$$

Then this satisfies (7.2), and it columns are in general position since any $d \times d$ submatrix is a Vandermonde matrix (for distinct integer points) with the third row modified in such a way that it can not be written as a linear combination of the others. $\square$

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The authors would like to thank Brad Baxter for emphasizing the connection between the invertibility of Hadamard products of Gram matrices and multivariate polynomial interpolation. The referee provided helpful comments including the nice proof for $\det(B)$ in the proof of Lemma 3.9.

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