ARTICLE IN PRESS

Linear Algebra and its Applications xxx (2010) xxx–xxx

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The symmetry group of a finite frame

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ARTICLE INFO

Article history:
Received 21 September 2009
Accepted 17 February 2010
Available online xxxx

Submitted by H. Schneider

AMS classification:
Primary 42C15, 58D19
Secondary 42C40, 52B15

Keywords:
Finite frame
Geometrically uniform frame
Gramian matrix
Harmonic frame
Maximally symmetric frames
Partition frames
Symmetry group
Tight frame

ABSTRACT

We define the *symmetry group* of a finite frame as a group of permutations on its index set. This group is closely related to the symmetry group of Vale and Waldron (2005) [12] for *tight* frames: they are isomorphic when the frame is tight and has distinct vectors. The symmetry group is the same for all similar frames, in particular for a frame, its dual and canonical tight frames. It can easily be calculated from the Gramian matrix of the canonical tight frame. Further, a frame and its complementary frame have the same symmetry group. We exploit this last property to construct and classify some classes of highly symmetric tight frames.

1. Introduction

Over the past decade there has been a rapid development of the theory and application of finite frames to areas as diverse as signal processing, quantum information theory and multivariate orthogonal polynomials, see, e.g., [10,11,14]. Key to these applications is the construction of frames with desirable properties. These often include being tight, and having a high degree of symmetry. Important examples are the *harmonic* or *geometrically uniform* frames, i.e., tight frames which are the orbit of a

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doi:10.1016/j.laa.2010.02.017

Please cite this article in press as: R. Vale, S. Waldron, The symmetry group of a finite frame, Linear Algebra Appl. (2010), doi:10.1016/j.laa.2010.02.017

single vector under an abelian group of unitary transformations (see [1,9]). A workable definition for the symmetry group is required for a full understanding of such frames.

In [12], the symmetry group of a finite *tight* frame was defined to be the group of unitary transformations which map the *set* of its vectors to itself. This led to methods for constructing and characterising tight frames with symmetries of the underlying space. For example, all harmonic frames were characterised using the theory of group representations.

Here we introduce a closely related symmetry group which maps the *sequence* of vectors of a (possibly nontight) frame to itself. In the case when a frame is tight and consists of distinct vectors these symmetry groups are isomorphic. This new symmetry group has the following key features:

- It is defined for *all* finite frames as a group of permutations on the index set.
- It is simple to calculate from the Gramian of the canonical tight frame.
- The symmetry group of a frame and all similar frames are equal. In particular, a frame, its dual frame and canonical tight frame have the same symmetry group.
- The symmetry group of various combinations of frames, such as tensor products and direct sums, are related to those of the constituent frames in a natural way.
- The symmetry group of a frame and its complementary frame are equal.

The last property leads to a simple construction of some highly symmetric tight frames, the so called *partition frames*. Further, the order of the symmetry group of a frame is $\leqslant n!$, where n is the number of vectors in the frame, and so it follows that there exist *maximally* symmetric frames of n vectors, i.e., those with the largest symmetry groups.

The paper is set out as follows. Next we give the basic theory of finite frames. We then define the symmetry group of a finite frame as a subgroup of the permutations on its index set, and prove it has the key features mentioned above. We also discuss the analogous definition if a frame is thought of as a (weighted) sequence of one-dimensional subspaces. Finally, we use it to construct the class of partition frames, and to investigate finite frames which are maximally symmetric.

2. The basics of finite frames

The following definitions and results are (mostly) well known (see, e.g., [5,16]). Let $\mathcal{H}$ be a real or complex Hilbert space of finite dimension d. A finite sequence of $n \geqslant d$ vectors $\Phi = (f_j)_{j\in J}$ in $\mathcal{H}$ is a **frame** for $\mathcal{H}$ if it spans $\mathcal{H}$, and is a **tight frame** for $\mathcal{H}$ if there is a $c > 0$ with

$$f = c\sum_{j\in J} \langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

We say a tight frame is **normalised** if it has been scaled so that $c = 1$ in (2.1), i.e.,

$$f = \sum_{j\in J} \langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H}. \tag{2.2}$$

The term *Parseval frame* is also used for a *normalised tight frame* since (2.1) is then a generalised Parseval formula. In general, frames with a countable number of vectors (for a finite or infinite dimensional space) are defined via the so called **frame bounds**, i.e., that there exist $A,\ B > 0$ with

$$A\|f\|^2 \leqslant \sum_{j\in J} |\langle f, f_j\rangle|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

This is easily seen to be equivalent to the definition above for J finite. If the vectors in the frame are distinct, then it is sometimes written as the set $\{f_j\}_{j\in J}$.

The **synthesis operator** for a finite sequence $(f_j)_{j\in J}$ in $\mathcal{H}$ is the linear map

$$V := [f_j]_{j\in J} : \ell_2(J) \to \mathcal{H} : a \mapsto \sum_{j\in J} a_j f_j,$$

and its **frame operator** is the linear map $S = VV^* : \mathcal{H} \to \mathcal{H}$ given by

$$Sf := \sum_{j\in J} \langle f, f_j\rangle f_j, \quad \forall f \in \mathcal{H}.$$

With $I = I_{\mathcal{H}}$ the identity on $\mathcal{H}$, the tight frame condition (2.1) can be expressed as

$$S = VV^* = cI. \tag{2.3}$$

If $\Phi = (f_j)_{j\in J}$ is a frame, then S is invertible, and the **dual frame** $\tilde{\Phi} = (\tilde{f}_j)_{j\in J}$ is defined by

$$\tilde{f}_j := S^{-1} f_j, \quad \forall j \in J, \tag{2.4}$$

and the **canonical tight frame** $\Phi^{\text{can}} = \left(f_j^{\text{can}}\right)_{j\in J}$ by

$$f_j^{\text{can}} = S^{-\frac{1}{2}} f_j, \quad \forall j \in J. \tag{2.5}$$

A frame and its dual satisfy the expansion

$$f = \sum_{j\in J} \langle f, f_j\rangle \tilde{f}_j = \sum_{j\in J} \langle f, \tilde{f}_j\rangle f_j, \quad \forall f \in \mathcal{H},$$

and the canonical tight frame is a normalised tight frame, i.e.,

$$f = \sum_{j\in J} \left\langle f, f_j^{\text{can}}\right\rangle f_j^{\text{can}}, \quad \forall f \in \mathcal{H}.$$

Let $\mathcal{GL}(\mathcal{H})$ be the (general linear) group of all invertible linear transformations $\mathcal{H} \to \mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ be the subgroup of unitary transformations. Following [6], we say frames $\Phi = (f_j)_{j\in J}$ and $\Psi = (g_j)_{j\in J}$ are **unitarily equivalent** if there is a $U \in \mathcal{U}(\mathcal{H})$ with $\Psi = U\Phi := (Uf_j)_{j\in J}$, i.e.,

$$g_j = Uf_j, \quad \forall j \in J, \tag{2.6}$$

and are **similar** if there is a $Q \in \mathcal{GL}(\mathcal{H})$ with $\Psi = Q\Phi$. Clearly, these are equivalence relations on the set of frames for $\mathcal{H}$, indexed by a given set J. In view of definitions (2.4) and (2.5), a frame, its dual and canonical tight frame are all similar. A simple calculation shows that normalised tight frames are similar if and only if they are unitarily equivalent.

The **Gramian** of a sequence of n vectors $(f_j)_{j\in J}$ is the $n \times n$ matrix

$$\text{Gram}(\Phi) := V^*V = [\langle f_k, f_j\rangle]_{j,k\in J}.$$

Frames Φ and Ψ are unitarily equivalent if and only if their Gramians are equal, and they are similar if and only if the Gramians of their canonical tight frames are equal, i.e.,

$$\text{Gram}(\Phi^{\text{can}}) = \text{Gram}(\Psi^{\text{can}}).$$

A sequence of vectors is a normalised tight frame (for its span) if and only if its Gramian matrix P is an orthogonal projection matrix, i.e., $P^2 = P$ and $P = P^*$. Note that $P = P^*$ holds for any Gramian matrix. We say that two frames Φ and Ψ are **complementary** (or **complements** of each other) if the Gramians of the associated canonical tight frames are complementary projection matrices, i.e.,

$$\text{Gram}(\Phi^{\text{can}}) + \text{Gram}(\Psi^{\text{can}}) = I. \tag{2.7}$$

The complement of a frame is well defined up to similarity, and the complement of a tight frame in the class of normalised tight frames is well defined up to unitary equivalence.

3. The symmetry group of a sequence of vectors

Here we give two, closely related, symmetry groups of a finite frame $\Phi = (f_j)_{j\in J}$, namely

- $\text{Sym}(\Phi)$ – which gives a faithful action on the *sequence* of vectors in Φ.
- $\text{sym}(\Phi)$ (small 's') – which gives a faithful action on the *set* of vectors in Φ.

Let S_J denote the **symmetric group** on the set J, i.e., the group of all bijections $J \to J$ (called permutations) under composition. For $J = \{1, \ldots, n\}$ the notation S_n is used.

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Definition 3.1. Let $\Phi = (f_j)_{j\in J}$ be a finite frame for $\mathcal{H}$, i.e., a sequence of vectors which spans $\mathcal{H}$. Then the **symmetry group** of Φ is the group

$$\mathrm{Sym}(\Phi) := \{\sigma \in S_J : \exists L_\sigma \in \mathcal{GL}(\mathcal{H}) \text{ with } L_\sigma f_j = f_{\sigma j},\ \forall j \in J\}.$$

It is easy to check that $\mathrm{Sym}(\Phi)$ is a subgroup of S_J. Since linear maps are determined by their action on a spanning set, it follows that if $\sigma \in \mathrm{Sym}(\Phi)$, then there is unique $L_\sigma \in \mathcal{GL}(\mathcal{H})$ with

$$L_\sigma f_j = f_{\sigma j}, \quad \forall j \in J.$$

From this it follows that

$$\pi_\Phi : \mathrm{Sym}(\Phi) \to \mathcal{GL}(\mathcal{H}) : \sigma \mapsto L_\sigma \tag{3.2}$$

is a group homomorphism. We denote its image, a subgroup of $\mathcal{GL}(\mathcal{H})$, by

$$\mathrm{sym}(\Phi) := \pi_\Phi(\mathrm{Sym}(\Phi)) = \{L_\sigma : \sigma \in \mathrm{Sym}(\Phi)\} \quad \text{(note the small ‘s’)}.$$

This is the group of invertible linear transformations which map the *set* of vectors in Φ to itself. Further, π_Φ is injective if and only if the vectors in Φ are distinct, in which case

$$\mathrm{Sym}(\Phi) \approx \mathrm{sym}(\Phi) \quad \text{(group isomorphism)}.$$

If Φ is a frame of n vectors, m of which are distinct, for a space of dimension d, then

$$|\mathrm{sym}(\Phi)| \,\Big|\, |\mathrm{Sym}(\Phi)| \,\Big|\, n!, \quad |\mathrm{sym}(\Phi)| \leqslant m(m-1)\cdots(m-d+1). \tag{3.3}$$

When we refer to *the* symmetry group of Φ, without further qualification, we mean $\mathrm{Sym}(\Phi)$.

Let S_Φ denote the frame operator for Φ. If $g \in \mathcal{GL}(\mathcal{H})$, then

$$S_\Phi(gf) = (g^*)^{-1} S_{g^*\Phi}(f), \quad \forall f \in \mathcal{H}.$$

In particular, if $g \in \mathrm{sym}(\Phi)$ is unitary, then it commutes with S_Φ, i.e.,

$$S_\Phi(gf) = g S_\Phi(f), \quad \forall f \in \mathcal{H}.$$

Example 1. If Φ is a tight frame, then $\mathrm{sym}(\Phi)$ is a group of unitary transformations, and hence commutes with the frame operator S_Φ. This follows because if $L_\sigma \in \mathrm{sym}(\Phi)$ and $\Phi = (f_j)$ is tight, then $L_\sigma \Phi = (L_\sigma f_j) = (f_{\sigma j})$ is tight, and so (2.3) gives

$$cI = [L_\sigma f_j][L_\sigma f_j]^* = L_\sigma [f_j][f_j]^* L_\sigma^* = c L_\sigma L_\sigma^* \quad \Longrightarrow \quad L_\sigma L_\sigma^* = I.$$

Hence for Φ a tight frame of distinct vectors, $\mathrm{sym}(\Phi)$ is the symmetry group (of unitary transformations) defined in [12]. The essential point of difference is that $\mathrm{Sym}(\Phi)$ is larger than $\mathrm{sym}(\Phi)$ for a frame with repeated vectors, which is necessary if a frame and its complement are to have the same symmetry group (see Example 3 and Theorem 3.7).

Example 2. Many interesting examples of finite frames are given by the orbit of a single vector under the action of a finite group of (usually unitary) transformations, e.g., the n equally spaced unit vectors

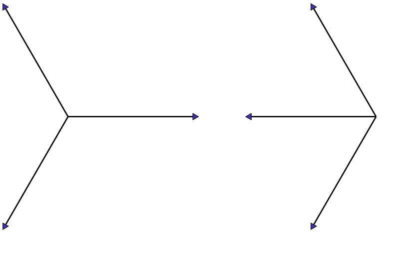

Fig. 1. The frames Φ and Φ' of Example 3, with $|\mathrm{Sym}(\Phi)| = 6$, $|\mathrm{Sym}(\Phi')| = 2$.

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in $\mathbb{R}^2$, the vertices of the platonic solids in $\mathbb{R}^3$, and the harmonic frames. Given a finite group G, a frame $\Phi = (\phi_g)_{g\in G}$ for $\mathcal{H}$, indexed by the elements of G, is a **group frame** or a G-**frame** (see [7,13]) if there exists a representation $\rho : G \to \mathcal{GL}(\mathcal{H})$ (i.e., group homomorphism) for which

$$g\phi_h := \rho(g)\phi_h = \phi_{gh}, \quad \forall g,\ h \in G.$$

If Φ is a group frame, then $\mathrm{Sym}(\Phi)$ has a subgroup isomorphic to G, and $\rho(G) \subset \mathrm{sym}(\Phi)$. Further, a given frame Φ of n distinct vectors is a group frame if and only if $\mathrm{sym}(\Phi)$ has a subgroup G of order n which acts faithfully on the sequence of vectors in Φ.

Example 3 (*See Fig. 1*). Let $\Phi = (v_1, v_2, v_3)$ be the tight frame of three equally spaced unit vectors in $\mathbb{R}^2$, and $\Psi = ([1],[1],[1])$, which is the complementary tight frame for $\mathbb{R}$. Then

$$\mathrm{Sym}(\Phi) = \mathrm{Sym}(\Psi) = S_3 = S_{\{1,2,3\}}.$$

Here $\mathrm{sym}(\Phi) = \langle a, b\rangle$ is the dihedral group of order 6, which is generated by the unitary maps: $a =$ rotation through $\frac{2\pi}{3}$, and $b =$ reflection through the line spanned by v_3, while $\mathrm{sym}(\Psi)$ consists of just the identity. The tight frame $\Phi' = (v_1, v_2, -v_3)$ has

$$\mathrm{Sym}(\Phi') = \{1, (12)\}, \quad \mathrm{sym}(\Phi') = \langle b\rangle.$$

Example 4. The symmetry group of the tight frame given by n equally spaced unit vectors in $\mathbb{R}^2$ is the *dihedral group* of order $2n$, i.e.,

$$D_n := \langle a, b : a^n = 1,\ b^{-1}ab = a^{-1}\rangle,$$

where a acts as rotation through $\frac{2\pi}{n}$, and b as a reflection which maps the set of vectors onto itself. Given that all unitary transformations on $\mathbb{R}^2$ are products of rotations and reflections, it is easy to see this is the most symmetric tight frame of n distinct vectors in $\mathbb{R}^2$. By way of contrast, the frame consisting of one vector taken $n-1$ times together with another (not linearly dependent) vector has a symmetry group of order $(n-1)!$.

Example 5. The n equally spaced vectors in $\mathbb{R}^2$ are not always the most symmetric tight frame of n distinct vectors in $\mathbb{C}^2$. For n even, the (harmonic) tight frame given by the n distinct vectors

$$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\omega\\-\omega\end{pmatrix}, \begin{pmatrix}\omega^2\\\omega^2\end{pmatrix}, \begin{pmatrix}\omega^3\\-\omega^3\end{pmatrix}, \begin{pmatrix}\omega^4\\\omega^4\end{pmatrix}, \dots, \begin{pmatrix}\omega^{n-2}\\\omega^{n-2}\end{pmatrix}, \begin{pmatrix}\omega^{n-1}\\-\omega^{n-1}\end{pmatrix}\right\}, \quad \omega := e^{\frac{2\pi i}{n}}$$

has a symmetry group of order $\frac{1}{2}n^2$ (see [9] for details).

The symmetry group of a frame depends only on its similarity class:

Theorem 3.4. *If Φ and Ψ are similar finite frames, say $\Psi = Q\Phi$, then*

$$\mathrm{Sym}(\Psi) = \mathrm{Sym}(\Phi), \quad \mathrm{sym}(\Psi) = Q\,\mathrm{sym}(\Phi)Q^{-1}.$$

In particular,

$$\mathrm{Sym}(\Phi) = \mathrm{Sym}(\tilde{\Phi}) = \mathrm{Sym}(\Phi^{\mathrm{can}}).$$

Proof. Suppose that $\Phi = (f_j)_{j\in J}$ is similar to $\Psi = (g_j)_{j\in J}$, i.e., there exists an invertible Q with $g_j = Qf_j,\ \forall j \in J$. Then, for $\sigma \in \mathrm{Sym}(\Phi)$,

$$L_\sigma f_j = f_{\sigma j}, \quad \forall j \quad \Longrightarrow \quad (QL_\sigma Q^{-1})Qf_j = Qf_{\sigma j}, \quad \forall j,$$

so that $\sigma \in \mathrm{Sym}(\Psi)$, and $\mathrm{Sym}(\Phi) \subset \mathrm{Sym}(\Psi)$, $Q\,\mathrm{sym}(\Phi)Q^{-1} \subset \mathrm{sym}(\Psi)$. The reverse inclusions follow since similarity is an equivalence relation (and so Ψ is similar to Φ). $\square$

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If two frames with the same index set have the same symmetry group, then they need not be similar.

Since a finite frame Φ is determined up to similarity by $\mathrm{Gram}(\Phi^{\mathrm{can}})$, it should be possible to compute its symmetry group from this Gramian matrix. We now explain how this can be done.

Let e_j be the jth standard basis vector. There is a bijection between permutations $\sigma \in S_J$ and the $J \times J$ *permutation matrices*, given by $\sigma \mapsto P_\sigma$, where

$$P_\sigma e_j := e_{\sigma j}.$$

A symmetry $\sigma \in \mathrm{Sym}(\Phi)$ corresponds to a P_σ which satisfies the following.

Lemma 3.5. *Let $\Phi = (f_j)_{j\in J}$ be a finite frame. Then*

$$\sigma \in \mathrm{Sym}(\Phi) \iff P_\sigma^* \mathrm{Gram}(\Phi^{\mathrm{can}}) P_\sigma = \mathrm{Gram}(\Phi^{\mathrm{can}}).$$

Proof. By Theorem 3.4, $\mathrm{Sym}(\Phi) = \mathrm{Sym}(\Phi^{\mathrm{can}})$, and so we can suppose without loss of generality that Φ is a normalised tight frame. Let $V := [f_j]_{j\in J}$, and observe

$$VP_\sigma = V[e_{\sigma j}] = [Ve_{\sigma j}] = [f_{\sigma j}] = [L_\sigma f_j] = L_\sigma V. \tag{3.6}$$

($\Longrightarrow$) Suppose $\sigma \in \mathrm{Sym}(\Phi)$. By Example 1, L_σ is unitary, and so (3.6) gives

$$P_\sigma^* \mathrm{Gram}(\Phi) P_\sigma = (VP_\sigma)^* VP_\sigma = (L_\sigma V)^* L_\sigma V = V^* \left(L_\sigma^* L_\sigma\right) V = V^* V = \mathrm{Gram}(\Phi).$$

($\Longleftarrow$) Suppose $P_\sigma^* \mathrm{Gram}(\Phi) P_\sigma = \mathrm{Gram}(\Phi)$, i.e.,

$$\mathrm{Gram}((f_{\sigma j})_{j\in J}) = \mathrm{Gram}((f_j)_{j\in J}).$$

It is easy to prove (see, e.g., [12, Lem. 2.7]) that if (f_j) and (g_j) are sequences of vectors which span $\mathcal{H}$, with the same inner products, i.e., $\langle f_j, f_k\rangle = \langle g_j, g_k\rangle$, $\forall j, k$, then there is a unitary U with $Uf_j = g_j$, $\forall j$. Thus, for $g_j = f_{\sigma j}$, there is $U \in \mathcal{U}(\mathcal{H})$ with $Uf_j = f_{\sigma j}$, $\forall j$. Hence, by taking $L_\sigma = U$, we conclude $\sigma \in \mathrm{Sym}(\Phi)$. □

In words, $\sigma \in S_J$ is a symmetry of a frame $\Phi = (f_j)_{j\in J}$ if and only the corresponding permutation matrix P_σ commutes with the Gramian matrix M of the canonical tight frame, i.e., M is invariant under simultaneous permutation of its rows and columns by σ.

Example 6. An equal – norm finite frame $\Phi = (f_j)_{j\in J}$ is **equiangular** if there is a $C \geqslant 0$ with

$$|\langle f_j, f_k\rangle| = C, \quad \forall j \neq k.$$

These frames have applications in signal processing (see [8,2]). The equiangular frames of $n > d$ vectors for $\mathbb{R}^d$ are in 1–1 correspondence with graphs with vertices J and an edge from j to k if and only if $\langle f_j, f_k\rangle = C$ (see [15] for details). The symmetry group of such a real equiangular frame is the *automorphism group* of the corresponding graph, since σ is an automorphism of the graph if and only if P_σ commutes with its Seidel adjacency matrix (which is a linear combination of the Gramian and identity matrices).

The frames Φ and Ψ of Example 3 are complementary, and have $\mathrm{Sym}(\Phi) = \mathrm{Sym}(\Psi)$, while $\mathrm{sym}(\Phi)$ is not isomorphic to $\mathrm{sym}(\Psi)$. More generally:

Theorem 3.7. *Suppose that Φ is a finite frame, and Ψ is a complementary frame, i.e.,*

$$\mathrm{Gram}(\Phi^{\mathrm{can}}) + \mathrm{Gram}(\Psi^{\mathrm{can}}) = I,$$

then

$$\mathrm{Sym}(\Phi) = \mathrm{Sym}(\Psi).$$

It need not be the case that $\mathrm{sym}(\Phi)$ *is isomorphic to* $\mathrm{sym}(\Psi)$.

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Proof. Since the definition of frames being complementary is symmetric, it suffices to show that $\mathrm{Sym}(\Phi) \subset \mathrm{Sym}(\Psi)$. By Lemma 3.5, $\sigma \in \mathrm{Sym}(\Phi)$ if and only if $P = P_\sigma$ commutes with $\mathrm{Gram}(\Phi^{\mathrm{can}})$. Since $P^*P = I$, this implies

$$P^*\mathrm{Gram}(\Psi^{\mathrm{can}})P = P^*(I - \mathrm{Gram}(\Phi^{\mathrm{can}}))P = I - \mathrm{Gram}(\Phi^{\mathrm{can}}) = \mathrm{Gram}(\Psi^{\mathrm{can}}),$$

and so, by Lemma 3.5 again, we have $\sigma \in \mathrm{Sym}(\Psi)$. □

If a frame is constructed from its complement, or its complement is for a space of low dimension, then it is often easier to calculate its symmetry group via the complement.

Example 7. Consider the equal-norm tight frames Φ of four vectors for $\mathbb{C}^3$ with nontrivial symmetries, i.e., $|\mathrm{Sym}(\Phi)| > 1$. The complement Ψ consists of four equal-norm vectors for $\mathbb{C}$. The symmetries of Ψ (and hence Φ) are those permutations of Ψ which can be realised by multiplication by a complex number (of unit modulus), e.g., the permutation $(12)(34)$ is a symmetry of $\Psi = ([1], [-1], [1], [-1])$ corresponding to multiplication by -1. Thus, the only possibilities for these complementary frames (up to a scalar multiple) are

$$([1], [1], [1], [z]),\quad z \neq 1,\quad ([1], [1], [z], [z]),\quad z \neq \pm 1,\quad ([1], [1], [z], [w]),\quad z \neq w,\ z,\ w \neq 1,$$
$$([1], [1], [1], [1]),\quad ([1], [-1], [1], [-1]),\quad ([1], [i], [-1], [-i]),$$

and the corresponding symmetry groups are

$$S_3,\quad S_2 \times S_2,\quad S_2,\quad S_4,\quad D_4 \text{ (dihedral group of order 8)},\quad C_4.$$

The Gramian matrices for the last three (which are harmonic frames) are

$$\frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix},\quad \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix},\quad \frac{1}{4}\begin{pmatrix} 3 & -i & 1 & i \\ i & 3 & -i & 1 \\ 1 & i & 3 & -i \\ -i & 1 & i & 3 \end{pmatrix}.$$

4. Symmetries of combinations of frames

Let $\Phi = (f_j)_{j \in J}$ and $\Psi = (g_k)_{k \in K}$ be finite frames for $\mathcal{H}_1$ and $\mathcal{H}_2$. The inner product on the orthogonal direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ are given by

$$\langle (f_1, g_1), (f_2, g_2) \rangle := \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle,\quad \forall (f_1, g_1),\ (f_2, g_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2,$$
$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle := \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle,\quad \forall f_1 \otimes g_1,\ f_2 \otimes g_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

We investigate the symmetry groups of the *union*, *sums* and *tensor product* of these frames.

Example 8. The **union** $\Phi \cup \Psi$ of these frames (which is indexed by $J \cup K$)

$$\Phi \cup \Psi := \left(\begin{pmatrix} f_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_k \end{pmatrix} \right)_{j \in J, k \in K}$$

is a frame for the orthogonal direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$. If no confusion arises, then one can identify $\mathcal{H}_1$ and $\mathcal{H}_2$ as subspaces of $\mathcal{H}_1 \oplus \mathcal{H}_2$, and write f_j or $f_j + 0$ in place of $(f_j, 0)$, etc.

For $\sigma \in \mathrm{Sym}(\Phi)$ and $\tau \in \mathrm{Sym}(\Psi)$, with corresponding $L_\sigma \in \mathcal{GL}(\mathcal{H}_1)$, $L_\tau \in \mathcal{GL}(\mathcal{H}_2)$, let $L_{(\sigma,\tau)} = L_\sigma \oplus L_\tau \in \mathcal{GL}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, i.e.,

$$L_{(\sigma,\tau)} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} L_\sigma f \\ L_\tau g \end{pmatrix},\quad \forall f \in \mathcal{H}_1,\ \forall g \in \mathcal{H}_2, \tag{4.1}$$

and interpret (σ, τ) as a permutation on $J \cup K$ in the obvious way. Then

$$L_{(\sigma,\tau)} \begin{pmatrix} f_j \\ 0 \end{pmatrix} = \begin{pmatrix} L_\sigma f_j \\ 0 \end{pmatrix} = \begin{pmatrix} f_{\sigma j} \\ 0 \end{pmatrix},\quad L_{(\sigma,\tau)} \begin{pmatrix} 0 \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ L_\tau g_k \end{pmatrix} = \begin{pmatrix} 0 \\ g_{\tau k} \end{pmatrix},$$

and so the permutation $(\sigma,\tau)\in \mathrm{Sym}(\Phi\cup\Psi)$. In this way, we have

$$\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\cup\Psi),\quad \mathrm{sym}(\Phi)\oplus\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\cup\Psi).$$

These inclusions can be strict, e.g., if $\mathcal{H}_2=\mathcal{H}_1$ and $\Psi=\Phi$, then there are additional symmetries which interchange the two copies of Φ.

We say a frame is **balanced** if its vectors sum to zero.

Example 9. There are two notions of the *sum* of frames.

(i) If either Φ or Ψ is *balanced*, then the **sum** $\Phi\hat{+}\Psi$ (indexed by $J\times K$)

$$\Phi\hat{+}\Psi:=\left(\begin{pmatrix}\frac{1}{\sqrt{n_2}}f_j\\ \frac{1}{\sqrt{n_1}}g_k\end{pmatrix}\right)_{j\in J,k\in K},\quad n_1:=|J|,\ n_2:=|K|$$

is a frame for $\mathcal{H}_1\oplus\mathcal{H}_2$.

(ii) If $K=J$, and the frames are *disjoint*, i.e., satisfy

$$\sum_{j\in J}\langle f,f_j\rangle g_j=0,\quad\forall f\in\mathcal{H}_1\quad\Longleftrightarrow\quad\sum_{j\in J}\langle g,g_j\rangle f_j=0,\quad\forall g\in\mathcal{H}_2,$$

then the **direct sum** $\Phi\oplus\Psi$ (indexed by J)

$$\Phi\oplus\Psi:=\left(\begin{pmatrix}f_j\\ g_j\end{pmatrix}\right)_{j\in J}$$

is a frame for $\mathcal{H}_1\oplus\mathcal{H}_2$.

For the first sum, we may apply the transformation $L_{(\sigma,\tau)}$ of Example 8, to get

$$L_{(\sigma,\tau)}\begin{pmatrix}\frac{1}{\sqrt{n_2}}f_j\\ \frac{1}{\sqrt{n_1}}g_k\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{n_2}}f_{\sigma j}\\ \frac{1}{\sqrt{n_1}}g_{\tau k}\end{pmatrix}\in\Phi\hat{+}\Psi,$$

and hence conclude

$$\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\hat{+}\Psi),\quad \mathrm{sym}(\Phi)\oplus\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\hat{+}\Psi),$$

with strict inclusion possible as before. For the second, it is easy to verify

$$\mathrm{Sym}(\Phi)\cap\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\oplus\Psi).$$

Example 10. The **tensor product** $\Phi\otimes\Psi$ of these frames (which is indexed by $J\times K$)

$$\Phi\otimes\Psi:=(f_j\otimes g_k)_{j\in J,k\in K}$$

is a frame for the tensor product Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2$. Similarly to Example 8, we can define $L_{(\sigma,\tau)}\in\mathcal{GL}(\mathcal{H}_1\otimes\mathcal{H}_2)$ by $L_{(\sigma,\tau)}=L_\sigma\otimes L_\tau$. Then

$$L_{(\sigma,\tau)}(f_j\otimes g_k)=(L_\sigma f_j)\otimes(L_\tau f_k)=f_{\sigma j}\otimes g_{\tau k}$$

and so we obtain

$$\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\otimes\Psi),\quad \mathrm{sym}(\Phi)\otimes\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\otimes\Psi).$$

These inclusions can be strict, e.g., take the tensor product of orthonormal bases for spaces of dimensions d_1 and d_2 to obtain an orthonormal basis for a space of dimension d_1d_2.

We summarise the results of these examples.

Proposition 4.2. *The symmetry groups of a finite frame satisfy*

$$\begin{aligned}&\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\cup\Psi),\quad \mathrm{sym}(\Phi)\oplus\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\cup\Psi),\\ &\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\hat{+}\Psi),\quad \mathrm{sym}(\Phi)\oplus\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\hat{+}\Psi),\\ &\mathrm{Sym}(\Phi)\times\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\otimes\Psi),\quad \mathrm{sym}(\Phi)\otimes\mathrm{sym}(\Psi)\subset\mathrm{sym}(\Phi\otimes\Psi),\\ &\mathrm{Sym}(\Phi)\cap\mathrm{Sym}(\Psi)\subset\mathrm{Sym}(\Phi\oplus\Psi).\end{aligned}$$

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In some situations these inclusions become equalities, e.g., we have:

Proposition 4.3. *Suppose that Φ_j is an equal-norm tight frame of n_j vectors for a space of dimension d_j, $j = 1, 2$. If $d_1/n_1 \neq d_2/n_2$, then*

$$\mathrm{Sym}(\Phi_1) \times \mathrm{Sym}(\Phi_2) \cong \mathrm{Sym}(\Phi_1 \cup \Phi_2).$$

Proof. For $\sigma \in \mathrm{Sym}(\Phi_1)$ and $\tau \in \mathrm{Sym}(\Phi_2)$, with corresponding $L_\sigma \in \mathrm{sym}(\Phi_1)$ and $L_\tau \in \mathrm{sym}(\Phi_2)$, let $L_{(\sigma,\tau)} \in \mathrm{sym}(\Phi_1 \cup \Phi_2)$ be defined by (4.1). Let J_1 and J_2 be the index sets of Φ_1 and Φ_2, and denote by $\theta(\sigma, \tau)$ the permutation on $J_1 \cup J_2$ given by

$$(\theta(\sigma,\tau))j := \begin{cases} \sigma j, & j \in J_1; \\ \tau j, & j \in J_2. \end{cases}$$

Then $\theta(\sigma, \tau) \in \mathrm{Sym}(\Phi_1 \cup \Phi_2)$ (as detailed in Example 8). It is easy to check that

$$\theta : \mathrm{Sym}(\Phi_1) \times \mathrm{Sym}(\Phi_2) \to \mathrm{Sym}(\Phi_1 \cup \Phi_2)$$

is an injective group homomorphism, and so it suffices to show that θ is onto.

In view of Theorem 3.4, we may suppose without loss of generality that Φ_1 and Φ_2 are normalised tight frames, in which case $\Phi_1 \cup \Phi_2$ is also. It follows that all 'sym' groups consist of unitary transformations (Example 1). Taking the trace of (2.3), then gives that the norm of each vector in Φ_j is $\sqrt{d_j/n_j}$. Since $d_1/n_1 \neq d_2/n_2$, any unitary transformation which permutes the vectors in $\Phi_1 \cup \Phi_2$ must permute the vectors in Φ_1 and Φ_2 individually, and so is of the form $L_{(\sigma,\tau)}$ for some σ and τ, as required. □

We now generalise Example 7. Recall that a group G is a *semidirect* product of a normal subgroup N and a subgroup H, written $G = N \rtimes H$, if $G = NH$ and $N \cap H = \{1\}$.

Theorem 4.4. *Let Φ be a tight frame of n vectors for $\mathbb{C}^{n-1}$ (or $\mathbb{C}$). Then*

$$\mathrm{Sym}(\Phi) \approx S_m \times [(S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r})^\ell \rtimes C_\ell], \tag{4.5}$$

where $m \geqslant 0$, $\ell \geqslant 1$, $a_1, \ldots, a_r \geqslant 1$ are integers satisfying

$$\ell(a_1 + \cdots + a_r) + m = n,$$

with $m = 0$ when Φ is an equal-norm frame. Moreover, for any integers satisfying the above conditions there is a tight frame Φ for $\mathbb{C}^{n-1}$ with symmetry group given by (4.5).

Proof. By Theorem 3.7, we may suppose Φ is a tight frame for $\mathbb{C}^1$. Let m be the number of zero vectors in Φ, so that it may be written $\Phi = (\phi_1, \ldots, \phi_{n-m}, 0, \ldots, 0)$ a union of the tight frames $(0, \ldots, 0)$ for $\mathbb{C}^0$ and $(\phi_1, \ldots, \phi_{n-m})$ for $\mathbb{C}^1$. By a slight variation of the argument for Proposition 4.3 (zero vectors must map to zero vectors under a symmetry), we have

$$\mathrm{Sym}(\Phi) \approx \mathrm{Sym}(0, \ldots, 0) \times \mathrm{Sym}(\phi_1, \ldots, \phi_{n-m}) = S_m \times \mathrm{Sym}(\phi_1, \ldots, \phi_{n-m}).$$

Thus it suffices to consider only the case when $m = 0$ and Φ consists of nonzero vectors.

Since Φ is a tight frame, $\mathrm{sym}(\Phi)$ consists of unitary maps on $\mathbb{C}^1$ (cf. Example 1), i.e., is a subgroup of the unit modulus complex numbers under multiplication. Thus $\mathrm{sym}(\Phi)$ is cyclic of order ℓ, say generated by z. We now show how to construct a $\tau \in \mathrm{Sym}(\Phi)$ of order ℓ with $\pi_\Phi(\tau) = z$, where π_Φ is the group homomorphism given by (3.2). Let $\langle z \rangle$ act on the set of vectors in Φ. The vectors in Φ consist of the orbits under this action (which have size dividing ℓ) repeated some number of times. Choose some fixed way of doing this. For a given appearance of an orbit of size t in Φ as

$$\phi_{j_1}, \quad \phi_{j_2} = z\phi_{j_1}, \quad \phi_{j_3} = z^2\phi_{j_1}, \ldots, \phi_{j_t} = z^{t-1}\phi_{j_1},$$

define τ on the set of indices for the vectors above to be the cycle $(j_1 j_2 \cdots j_t)$ (so τ is a product of such cycles). For example, if $\Phi = (1, \omega, \omega^2, 2, 2\omega, 2\omega^2, 2, 2\omega, 2\omega^2)$, with $z = \omega = e^{\frac{2\pi i}{3}}$, then one could take $\tau = (123)(486)(759)$, amongst other choices.

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Let $\sigma \in \mathrm{Sym}(\Phi)$. Then $\pi_\Phi(\sigma) = z^b$, for some b, and $\pi_\Phi(\sigma\tau^{-b}) = z^b z^{-b} = 1$ implies $\sigma\tau^{-b} \in \ker(\pi_\Phi)$, so that $\mathrm{Sym}(\Phi) = \ker(\pi_\Phi)\langle\tau\rangle$. Further, if $\tau^b \in \ker(\pi_\Phi)$, then $z^b = 1$, so that ℓ divides b, which gives $\tau^b = 1$. Hence $\ker(\pi_\Phi) \cap \langle\tau\rangle = \{1\}$, and we have

$$\mathrm{Sym}(\Phi) \approx \ker(\pi_\Phi) \rtimes \langle\tau\rangle \approx \ker(\pi_\Phi) \rtimes C_\ell.$$

It remains only to show that $\ker(\pi_\Phi) = (S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r})^\ell$.

For a given vector ϕ in Φ, the number of times ϕ and $z\phi$ appear in Φ are the same. Further, the vectors $\phi, z\phi, \ldots, z^{\ell-1}\phi$ are all distinct, since otherwise

$$z^a\phi = \phi, \quad 0 < a < \ell \quad \Longrightarrow \quad z^a = 1 \quad (\text{since } \phi \neq 0).$$

Thus if ϕ appears exactly a_j times in Φ, then so do $z\phi, z^2\phi, \ldots, z^{\ell-1}\phi$. A symmetry $\sigma \in \mathrm{Sym}(\Phi)$ is in $\ker(\pi_\Phi)$ if and only if it permutes all the vectors of Φ which are equal, and so $\ker(\pi_\Phi)$ is isomorphic to $S_{a_1}^\ell \times \cdots \times S_{a_r}^\ell = (S_{a_1} \times \cdots \times S_{a_r})^\ell$. □

The semidirect product in (4.5), which does not define a unique group until the multiplication is specified, can (with a little work) be written as a *wreath product*. In this way (4.5) can be expressed as

$$\mathrm{Sym}(\Phi) \approx S_m \times ((S_{a_1} \times S_{a_2} \times \cdots \times S_{a_r}) \wr C_\ell),$$

where $\wr$ denotes the wreath product of permutation groups.

Example 11. For the equal-norm tight frames of four vectors in $\mathbb{C}^3$ given in Example 7, $m = 0$, and the groups in the [ ] in (4.5) are (respectively)

$$(S_3 \times S_1)^1 \rtimes C_1 \approx S_3, \quad (S_2 \times S_2)^1 \rtimes C_1 \approx S_2 \times S_2, \quad (S_2 \times S_1 \times S_1)^1 \rtimes C_1 \approx S_2,$$
$$(S_4)^1 \rtimes C_1 \approx S_4, \quad (S_2)^2 \rtimes C_2 \approx S_2 \wr C_2 \approx D_4, \quad (S_1)^4 \rtimes C_4 \approx C_4.$$

5. Symmetries of frames thought of as sums of projections

In applications, the interest is in the decomposition (2.1), i.e., writing the identity as

$$I = \sum_{j\in J} c_j P_j, \quad c_j := c\|f_j\|^2, \quad P_j f := \frac{\langle f, f_j\rangle}{\langle f_j, f_j\rangle} f_j,$$

and the particular (unit modulus) scalar multiple of f_j used to define the orthogonal projection P_j is unimportant. Indeed, there may not be a natural choice (cf. [11]). In this case, the *unitary equivalence* of (2.6) is replaced by *type III equivalence*, i.e.,

$$g_j = \alpha_j U f_j, \quad \forall j \in J,$$

where α_j are scalars of unit modulus (see [8]).

Correspondingly, one would define a permutation σ to be a (type III) symmetry if there exists $L_\sigma \in \mathcal{GL}(\mathcal{H})$ and unit modular scalars α_j (possibly not unique), with

$$L_\sigma f_j = \alpha_j f_{\sigma j}, \quad \forall j \in J.$$

This type III symmetry group is larger than our symmetry group, *but*, for $\mathcal{H}$ a complex space, it cannot be computed in the same way, since for a possible symmetry σ and $L_\sigma : f_j \mapsto \alpha_j f_{\sigma j}$ there are infinitely many possible choices for α_j. The analogue of (3.5) is

$$(P_\sigma \Lambda)^* \mathrm{Gram}(\Phi^{\mathrm{can}}) P_\sigma \Lambda = \mathrm{Gram}(\Phi^{\mathrm{can}}), \quad \Lambda := \mathrm{diag}(\alpha_j),$$

and so to find type III symmetries one must solve a system of quadratic equations in the complex variables α_j, $\overline{\alpha_j}$, $j \in J$. Such systems of equations define Heisenberg frames, and are extremely difficult to solve in general (see [4]). However, in some cases of interest the scalars α_j come from a *finite* set, e.g., $\{-1, 1\}$ for frames in $\mathbb{R}^d$ or the cube roots of unity (see [3]), and so this type III symmetry group can be calculated.

Example 12. The simplest example of a Heisenberg frame (see [11,4]) is the equiangular tight frame $\Phi := (v, Sv, \Omega v, S\Omega v)$ of four vectors for $\mathbb{C}^2$, where

$$v := \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{\frac{\pi}{4}i}\sqrt{3-\sqrt{3}} \end{pmatrix}, \quad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Omega := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Since $S\Omega = -\Omega S$, multiplication by S and Ω induces type III symmetries, namely $(12)(34)$ and $(13)(24)$. Further, the unitary matrix

$$B := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

induces a type III symmetry (132), and so the type III symmetry group of Φ contains

$$A_4 = \langle (12)(34), (13)(24), (132) \rangle, \quad |A_4| = 12.$$

6. Partition frames

The simplest way to obtain a highly symmetric frame is to repeat the vectors in a basis a number of times. By Theorem 3.7, the complement of such a frame is also highly symmetric, and it turns out consists of distinct vectors. We call these *partition frames*.

Let $(e_j)_{j=1}^k$ be an orthonormal basis for $\mathbb{R}^k$.

Definition. Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$ be a partition of n, i.e.,

$$n = \alpha_1 + \cdots + \alpha_k, \quad 1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k.$$

Then the normalised tight frame of n vectors which is complementary to the frame

$$\Bigg(\underbrace{\frac{e_1}{\sqrt{\alpha_1}}, \ldots, \frac{e_1}{\sqrt{\alpha_1}}}_{\alpha_1 \text{ times}}, \ldots, \underbrace{\frac{e_k}{\sqrt{\alpha_k}}, \ldots, \frac{e_k}{\sqrt{\alpha_k}}}_{\alpha_k \text{ times}} \Bigg) \tag{6.1}$$

for $\mathbb{R}^k$ is called the α-**partition frame**, and this is **proper** if $\alpha_1 \geqslant 2$.

By (2.7), the Gramian of the α-partition frame is the block diagonal $n \times n$ matrix

$$P = \begin{bmatrix} B_1 & & & & \\ & \ddots & & & \\ & & B_j & & \\ & & & \ddots & \\ & & & & B_k \end{bmatrix}, \quad B_j := \begin{bmatrix} \frac{\alpha_j - 1}{\alpha_j} & \frac{-1}{\alpha_j} & \frac{-1}{\alpha_j} & \cdots & \frac{-1}{\alpha_j} \\ \frac{-1}{\alpha_j} & \frac{\alpha_j - 1}{\alpha_j} & \frac{-1}{\alpha_j} & \cdots & \frac{-1}{\alpha_j} \\ \frac{-1}{\alpha_j} & \frac{-1}{\alpha_j} & \frac{\alpha_j - 1}{\alpha_j} & & \frac{-1}{\alpha_j} \\ \vdots & \vdots & & \ddots & \frac{-1}{\alpha_j} \\ \frac{-1}{\alpha_j} & \frac{-1}{\alpha_j} & \frac{-1}{\alpha_j} & \frac{-1}{\alpha_j} & \frac{\alpha_j - 1}{\alpha_j} \end{bmatrix}, \tag{6.2}$$

where the above B_j is a $\alpha_j \times \alpha_j$ orthogonal projection matrix of rank $\alpha_j - 1$. Since each normalised tight frame is isomorphic to the projection of an orthonormal basis, namely the columns of its Gramian matrix, it follows that the vectors in a partition frame are distinct, except those corresponding to $\alpha_j = 1$, which are all zero (hence the term *proper*).

We can easily determine the basic properties of a partition frame from its Gramian.

Proposition 6.3. *Let Φ be a proper α-partition frame, $\alpha = (\alpha_1, \ldots, \alpha_k)$. Then*

(a) *Φ is a normalised tight frame of $n := \alpha_1 + \cdots + \alpha_k$ distinct vectors for $\mathbb{R}^d$, $d := n - k$.*
(b) *Φ is a harmonic frame (and a G-frame) if and only if $\alpha_1 = \cdots = \alpha_k$.*

(c) *The symmetry group of* Φ *has order*

$$|\mathrm{Sym}(\Phi)| = \prod_{m\in\{\alpha_j\}} m_\#!(m!)^{m_\#} > n, \quad m_\# := |\{j : \alpha_j = m\}|. \tag{6.4}$$

Proof. The α-partition frame is (Hilbert space) isomorphic to the normalised tight frame given by the columns of its Gramian matrix P, which is given by (6.2).

(a) We observed that this implies a *proper* α-partition frame has distinct vectors. Moreover, the frame (6.1) of $n = \alpha_1 + \cdots + \alpha_k$ vectors spans a real k-dimensional space, and so by (2.7) its complement is a normalised tight frame for a space of dimension $d = n - k$.

(b) A tight G-frame (see Example 2) has equal length vectors, and a tight frame has equal length vectors if and only if its complement does. Thus Φ can be a G-frame only if $\alpha_1 = \cdots = \alpha_k$. A frame is G-frame if and only if its complement is (see [13]), and so it suffices to show that (6.1) is a G–frame for some abelian group G. Let S be the shift operator on $\mathbb{R}^k$, i.e.,

$$Se_j := \begin{cases} e_{j+1}, & 1 \leqslant j < k; \\ e_1, & j = k. \end{cases}$$

It is easy to check that (6.1) reordered as

$$\left(\frac{e_1}{\sqrt{\alpha}}, \frac{e_2}{\sqrt{\alpha}}, \ldots, \frac{e_k}{\sqrt{\alpha_1}}, \ldots, \frac{e_1}{\sqrt{\alpha}}, \frac{e_2}{\sqrt{\alpha}}, \ldots, \frac{e_k}{\sqrt{\alpha}}\right), \quad \alpha := \alpha_1 = \cdots = \alpha_k$$

is a $\mathbb{Z}_n$-frame, via the representation $\rho : \mathbb{Z}_n \to \mathcal{GL}(\mathbb{R}^k) : j \mapsto S^j$.

(c) By Theorem 3.7, $\mathrm{Sym}(\Phi) = \mathrm{Sym}(\Psi)$, where Ψ is the complementary normalised tight frame Ψ given by (6.1). Since Ψ is tight, each symmetry $\sigma \in \mathrm{Sym}(\Psi)$ corresponds a unitary map L_σ which permutes the vectors in Ψ. Since unitary maps preserve vector length, the only possible symmetries are those that map the vectors of length $m \in \{\alpha_j\}$ to themselves. Since the subspaces $\mathcal{H}_m := \mathrm{span}\{e_j : \alpha_j = m\} \subset \mathbb{R}^k$ are orthogonal to each other, the symmetry group $\mathrm{Sym}(\Phi)$ is the product of the symmetry groups for the equal norm tight frames of the $m \cdot m_\#$ vectors contained in $\mathcal{H}_m$. There are $m_\#!$ unitary maps which map these vectors to themselves. For each of these maps, the image of the m copies of each of the $m_\#$ vectors $\frac{e_j}{\sqrt{m}} \in \mathcal{H}_m$ (which are equal) can be reordered in $m!$ ways, giving

$$|\mathrm{Sym}(\Phi)| = \prod_{m\in\{\alpha_j\}} m_\#!(m!)^{m_\#} \geqslant \prod_{m\in\{\alpha_j\}} 2^{m_\#} = 2^n > n. \qquad \square$$

Example 13. The $(d+1)$-partition frame is the vertices of the regular simplex in $\mathbb{R}^d$, which has $\mathrm{Sym}(\Phi) = (d+1)!$. It follows from the block diagonal form (6.2) of its Gramian matrix that a proper α-partition frame can be decomposed as the union over $1 \leqslant j \leqslant k$ of the normalised tight frames given by the α_j vertices of the simplex $\mathbb{R}^{\alpha_j - 1}$.

Example 14. The proper partition frames in $\mathbb{R}^2$ and $\mathbb{R}^3$ are as follows.

Partition	n	*Description of* α*–partition frame* Φ	$\lvert\mathrm{Sym}(\Phi)\rvert$
(3)	3	Three equally spaced vectors in $\mathbb{R}^2$	6
(2, 2)	4	Four equally spaced vectors in $\mathbb{R}^2$	8
(4)	4	Vertices of the tetrahedron in $\mathbb{R}^3$	24
(2, 3)	5	Vertices of the trigonal bipyramid in $\mathbb{R}^3$	12
(2, 2, 2)	6	Vertices of the octahedron in $\mathbb{R}^3$	48

In four dimensions the possible choices for α are (5), (2, 4), (3, 3) and (2, 2, 3) (see Fig. 2).

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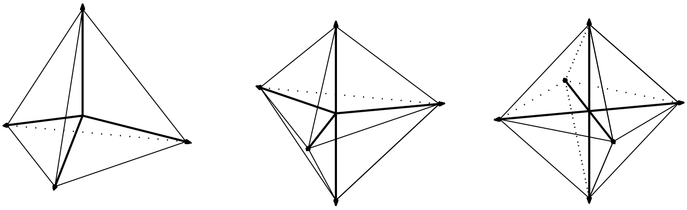

Fig. 2. The proper α-partition frames in $\mathbb{R}^3$ for $\alpha = (4), (2, 3)$ and $(2, 2, 2)$, respectively, i.e., the vertices of the tetrahedron, trigonal bipyramid and octagon.

Example 15. One can modify the construction of partition frames, e.g., the sequence of repeated vectors in (6.1) could be replaced by

$$\underbrace{\frac{e_j}{\sqrt{\alpha_j}}, \frac{\omega e_j}{\sqrt{\alpha_j}}, \ldots, \frac{\omega^{j-1} e_j}{\sqrt{\alpha_j}}}_{\alpha_j \text{ times}}, \quad \omega := e^{\frac{2\pi i}{j}},$$

for some or all choices of j, which gives a tight frame with $|\mathrm{Sym}(\Phi)| > n$ ($\alpha_1 \geqslant 2$).

7. Maximally symmetric frames

Since the order of the symmetry group of a frame of n vectors (a subgroup of S_n) divides $n!$, there are maximally symmetric frames in any class of such frames.

Definition 7.1. Let $\mathcal{C}$ be a class of frames of n vectors, e.g., the tight frames or equal norm tight frames in $\mathbb{R}^d$ and $\mathbb{C}^d$. We say that $\Phi \in \mathcal{C}$ is **maximally symmetric** if

$$|\mathrm{Sym}(\Phi)| = \max_{\Psi \in \mathcal{C}} |\mathrm{Sym}(\Psi)|.$$

This definition should be treated with a little caution for frames with repeated vectors. For example, the frame of n vectors for $\mathbb{R}^2$ consisting of e_1 repeated $n-1$ times and e_2 has symmetry group of order $(n-1)!$, whilst that of the n equally spaced unit vectors has order $2n$.

Example 16. The only cases when a frame Φ of n vectors for $\mathbb{F}^d$ can have maximal symmetry by virtue of $|\mathrm{Sym}(\Phi)| = n!$ are for $n = d$, i.e., when it is a *basis* (all bases are similar), and for $n = d + 1$ when it is (similar to) the vertices of the *regular d-simplex*.

Example 17. In Examples 4 and 5, we observed that the n equally spaced unit vectors in $\mathbb{R}^2$ give the most symmetric tight frame of n *distinct* vectors in $\mathbb{R}^2$, but that this need not be the most symmetric such frame in $\mathbb{C}^2$.

Example 18 (*See Fig. 3*). We determine the maximally symmetric tight frames Φ of five distinct vectors in $\mathbb{R}^3$. By Theorem 3.7, these can be described via the complementary normalised tight frames Ψ of five vectors in $\mathbb{R}^2$. By (3.3), the order of $\mathrm{Sym}(\Phi)$ divides $5!$. The complement of a unit vector is a zero vector, and so the most symmetric Φ with a zero vector is given by the $(1, 4)$-partition frame

$$\Psi = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right\}, \quad |\mathrm{Sym}(\Phi)| = 4! = 24,$$

which is the vertices of the *tetrahedron* and a zero vector. If $\mathrm{Sym}(\Phi)$ does not have an element of order 5, then the next most symmetric is the $(2, 3)$-partition frame

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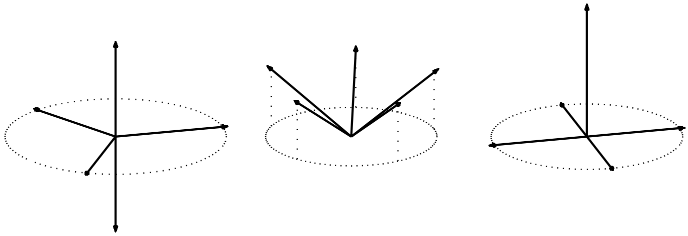

Fig. 3. The most symmetric tight frames of five distinct nonzero vectors in $\mathbb{R}^3$. The trigonal bipyramid (12 symmetries), five equally spaced vectors lifted (10 symmetries), and four equally spaced vectors and one orthogonal (8 symmetries).

$$\Psi = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}, \quad |\mathrm{Sym}(\Phi)| = 2!3! = 12,$$

which is the vertices of the *trigonal bipyramid*, i.e.,

$$\Phi = \left\{ \sqrt{\frac{2}{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \sqrt{\frac{2}{3}} \begin{pmatrix} \cos \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} \\ 0 \end{pmatrix}, \sqrt{\frac{2}{3}} \begin{pmatrix} \cos \frac{4\pi}{3} \\ \sin \frac{4\pi}{3} \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\},$$

followed by

$$\Psi = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad |\mathrm{Sym}(\Phi)| = 8,$$

which is *four equally spaced vectors and one orthogonal*, i.e.,

$$\Phi = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

If $\mathrm{Sym}(\Phi)$ has an element of order 5, then Ψ must be five equally spaced vectors

$$\Psi = \left\{ \sqrt{\frac{2}{5}} \begin{pmatrix} \cos \frac{2\pi j}{5} \\ \sin \frac{2\pi j}{5} \end{pmatrix} : j = 1, \dots, 5 \right\} \quad |\mathrm{Sym}(\Phi)| = |D_5| = 10,$$

and Φ is the *lifted five equally spaced vectors*, i.e.,

$$\Phi = \left\{ \sqrt{\frac{2}{5}} \begin{pmatrix} \cos \frac{2\pi j}{5} \\ \sin \frac{2\pi j}{5} \\ \frac{1}{\sqrt{2}} \end{pmatrix} : j = 1, \dots, 5 \right\}.$$

Motivated by examples such as these, we end with the following conjecture.

Conjecture. *A maximally symmetric tight frame is a union of group frames.*

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Please cite this article in press as: R. Vale, S. Waldron, The symmetry group of a finite frame, Linear Algebra Appl. (2010), doi:10.1016/j.laa.2010.02.017