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EXTREMISING THE L_p–NORM OF A MONIC POLYNOMIAL WITH ROOTS IN A GIVEN INTERVAL

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Let Θ be a multiset of n points in the interval $[a,b]$, and

$$\omega_\Theta := \prod_{\theta\in\Theta}(\cdot-\theta), \qquad \|\omega_\Theta\|_p := \|\omega_\Theta\|_{L_p[a,b]}, \quad 1\le p\le\infty.$$

In this paper we investigate the extrema of $\Theta\mapsto\|\omega_\Theta\|_p$. The main result is determining the maximum of $\|\omega_\Theta\|_p$ when some submultiset of Θ is prescribed. Applications given include L_p–error bounds for Hermite interpolation, error estimates for Gauss quadrature formulæ with multiple nodes, and certain quantitative statements about good and best approximation by polynomials of fixed degree.

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1. Introduction

Let Θ be a multiset of n points in the interval $[a,b]$, and

$$\omega_\Theta := \prod_{\theta\in\Theta}(\cdot-\theta),$$

which belongs to Π_n (the polynomials of degree $\leq n$). In this paper we investigate the size of $\|\omega_\Theta\|_p := \|\omega_\Theta\|_{L_p[a,b]}$ as a function of Θ. The constant $\|\omega_\Theta\|_p$ arises naturally in certain error bounds for *Hermite interpolation.* For example, if $H_\Theta f \in \Pi_{n-1}$ is the Hermite interpolant to f at the points Θ (counting multiplicities), then

$$\|f - H_\Theta f\|_p \leq \frac{\|\omega_\Theta\|_p}{n!}\|D^n f\|_\infty, \qquad \forall f \in W^n_\infty[a,b], \tag{1.1}$$

with equality if and only if $f \in \Pi_n$. Here $W^n_p[a,b]$ is the *Sobolev space* of functions f with $D^{n-1}f$ absolutely continuous on $[a,b]$ and $D^n f \in L_p := L_p[a,b]$.

In Section 2, we give the main result: that if some of the points in Θ are prescribed, then $\|\omega_\Theta\|_p$ is maximised by an appropriate choice of the remaining points from $\{a,b\}$. In Waldron [W97] a corollary of this result was used to obtain L_p–error bounds for Hermite interpolation in cases where some of the points in Θ are known to be from $\{a,b\}$. We apply a variation of this result to recent work of Shadrin [S95] on the derivatives of the error in Hermite interpolation.

In Section 3, we show that $\|\omega_\Theta\|_p$ is minimised when Θ is the roots of the error in the unique best L_p–approximation from Π_{n-1} to any polynomial of (exact) degree n. This classical result is closely related to Gauss quadrature formulæ with multiple nodes (via s–orthogonal polynomials), for which we give new error bounds.

Section 4 gives further applications of these results on the maximum and minimum. These include error bounds for best L_p–approximation by polynomials of fixed degree, and certain quantitative statements about good and best approximation by polynomials of fixed degree.

2. Maximising $\|\omega_\Theta\|_p$

Throughout,

$$\|\cdot\|_p := \|\cdot\|_{L_p[a,b]}, \qquad 1 \leq p \leq \infty,$$

and Θ is used for a multiset of n points from $[a,b]$.

Theorem 2.1 (Maximum of $\|\omega_\Theta\|_p$). *Let Ψ be a fixed multiset of $\leq n$ points from $[a,b]$. Then the maximum of*

$$\{\|\omega_\Theta\|_p : \Theta \supset \Psi\}$$

is attained when $\Theta \setminus \Psi$ is from $\{a,b\}$.

Proof. We seek the maximum of the convex function $f \mapsto \|f\|_p$ over the compact set

$$\mathcal{W} := \{\omega_\Theta : \Theta \supset \Psi\} \subset \Pi_n.$$

For $n \geq 2$ this set is not convex, e.g., $\omega_{\{a,a\}}, \omega_{\{b,b\}} \in \mathcal{W}$ while the convex combination

$$\frac{1}{2}\omega_{\{a,a\}} + \frac{1}{2}\omega_{\{b,b\}}$$

has complex roots, and so does not belong to $\mathcal{W}$. Let $\mathcal{C}$ be the convex hull of $\mathcal{W}$. Each point in $\mathcal{C} \setminus \mathcal{W}$ can be written as a (nontrivial) convex combination of two points in $\mathcal{C}$ (otherwise it would be a point in $\mathcal{W}$), and so the extreme points of $\mathcal{C}$ are contained in $\mathcal{W}$. Since $\mathcal{C} \to \mathbb{R} : f \mapsto \|f\|_p$ is a continuous convex function, it attains its maximum at an extreme point of $\mathcal{C}$ (which is a point in $\mathcal{W}$).

Suppose $\omega_\Theta \in \mathcal{W}$ is an extreme point of $\mathcal{C}$, with $\{\xi, \Psi\} \subset \Theta$, for some $\xi \in (a,b)$. Then for small ε

$$\omega_\Theta = \frac{1}{2}(\cdot - (\xi - \varepsilon))\,\omega_{\Theta\setminus\xi} + \frac{1}{2}(\cdot - (\xi + \varepsilon))\,\omega_{\Theta\setminus\xi},$$

a convex combination of points in $\mathcal{W}$, contradicting the fact ω_Θ is an extreme point of $\mathcal{C}$. Thus, the extreme points of $\mathcal{C}$ are given by ω_Θ where Θ consists of Ψ together with points from $\{a,b\}$. $\square$

We now use this result to find the maximum of $\Theta \mapsto \|\omega_\Theta\|_p$ over

$$\mathcal{A}_n(i,j) := \left\{ \begin{array}{c} \text{the set of those } \Theta \text{ containing one endpoint at least } i \text{ times} \\ \text{and the other endpoint at least } j \text{ times, where } i+j \leq n \end{array} \right\}.$$

Here are some typical examples

$$\begin{aligned} \mathcal{A}_3(0,1) &= \{\{a,\theta_1,\theta_2\}, \{b,\theta_1,\theta_2\} : \theta_1,\theta_2 \in [a,b]\}, \\ \mathcal{A}_3(1,2) &= \{\{a,b,b\}, \{a,a,b\}\}. \end{aligned}$$

Notice that $\mathcal{A}_n(i,j)$ is symmetric in i,j, that $\mathcal{A}_n(0,0)$ consists of all possible Θ, and $\mathcal{A}_n(m, n-m)$ has at most two elements.

Let B be the *beta function*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \forall x,y > 0.$$

Recall that B is symmetric, and satisfies: $0 < B(x,y) \le \min\{1, 1/\max\{x,y\}\}$, $\forall x, y > 0$. The following result was used in [W97, p.470].

Corollary 2.2. *Let* $m := \min\{i,j\}$*, and* $0^0 := 1$*. Then*

$$\max_{\Theta\in\mathcal{A}_n(i,j)} \|\omega_\Theta\|_p = (b-a)^{n+\frac{1}{p}} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m(n-m)^{n-m}/n^n, & p = \infty, \end{cases}$$

with the maximum achieved if and only if $\Theta \in \mathcal{A}_n(m, n-m)$*.*

Proof. By Theorem 2.1, the maximum occurs when all the points in Θ are from $\{a,b\}$. For $\Theta \in \mathcal{A}_n(k, n-k)$, we compute

$$\|\omega_\Theta\|_p = (b-a)^{n+\frac{1}{p}} \begin{cases} B(pk+1, p(n-k)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ k^k(n-k)^{n-k}/n^n, & p = \infty, \end{cases}$$

and then observe that the maximum of $\|\omega_\Theta\|_p$ over $m \le k \le n - \max\{i,j\}$ occurs when $k = m$. □

This improves upon the weaker result of Agarwal [A91, p.253], that

$$\max_{\Theta\in\mathcal{A}_n(i,j)} \|\omega_\Theta\|_p \le (b-a)^{n+\frac{1}{p}} (2B_{1/2}(pm+1, p(n-m)+1))^{\frac{1}{p}}, \qquad 1 \le p < \infty. \tag{2.3}$$

where $B_{1/2}$ is the *incomplete beta function*

$$B_{1/2}(x,y) := \int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt, \qquad \forall x, y > 0.$$

Observe that $B_{1/2}$ is not symmetric, and satisfies $B(x,y) \le 2B_{1/2}(x,y)$, $\forall\, 1 \le x \le y$, with strict inequality unless $x = y$. Thus, Corollary 2.2 gives better bounds than (2.3) whenever $m \ne n-m$, and the same bounds otherwise.

L_p–error bounds for Hermite interpolation

Let $1 \le p, q \le \infty$, and $H_\Theta f \in \Pi_{n-1}$ be the Hermite interpolant to f at Θ (counting multiplicities). Recently, the author has shown (see [W97, Th. 4.1]) that:

$$\|f - H_\Theta f\|_p \le \mathrm{const}_{n,p,q,\Theta} (b-a)^{n+\frac{1}{p}-\frac{1}{q}} \|D^n f\|_q, \qquad \forall f \in W_q^n, \tag{2.4}$$

where

$$\mathrm{const}_{n,p,q,\Theta} := \frac{n^{\frac{1}{q}}}{n!} \Big\| x \mapsto \frac{\omega_\Theta(x)}{(\mathrm{diam}\{x,\Theta\})^{1/q}} \Big\|_p (b-a)^{-(n+\frac{1}{p}-\frac{1}{q})}.$$

Here **diam** denotes the diameter of a (multi)set of points. Corollary 2.2 can be used to estimate the constants $\mathrm{const}_{n,p,q,\Theta}$.

Corollary 2.5 (Hermite error bounds). *Let* $\Theta \in \mathcal{A}_n(i,j)$*, with* $m := \min\{i,j\} > 0$*. Then*

$$\mathrm{const}_{n,p,q,\Theta} \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m (n-m)^{n-m}/n^n, & p = \infty. \end{cases}$$

Proof. Since $m > 0$, $\mathrm{diam}\{x, \Theta\} = b - a$, and we obtain

$$\mathrm{const}_{n,p,q,\Theta} = \frac{n^{\frac{1}{q}}}{n!} \|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}.$$

Now apply Corollary 2.2. □

This improves upon the bounds in [A91], which involve $B_{1/2}$. In the case $m = 0$, the above argument can be modified, by observing that

$$\left\| x \mapsto \frac{\omega_\Theta(x)}{(\mathrm{diam}\{x,\Theta\})^{1/q}} \right\|_p \le \left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})} \right)^{1-\frac{1}{nq}}, \tag{2.6}$$

See [W97] for a full discussion, including the cases of equality in (2.4) and mention of some related inequalities of Brink [B72].

Application to the solution of ordinary differential equations

The Hermite interpolation error bounds of Corollary 2.5 can be applied to the analysis of the boundary value problem: $D^n f = g$, with *Hermite multipoint conditions* given by $H_\Theta f = 0$ (see, e.g., Agarwal and Wong [AW93]).

L_∞–bounds for the error in Hermite interpolation

Shadrin [S95] proved the following sharp bound for the j–th derivative of the error in Hermite interpolation (where $0 \le j < n$)

$$\|D^j(f - H_\Theta f)\|_\infty \le \frac{1}{n!} \|D^j \omega_\Theta\|_\infty \|D^n f\|_\infty, \qquad \forall f \in W_\infty^n.$$

For $j = 0$ the result is well known and easily proved, but for $j > 0$ it is highly nontrivial. In light of this, one might seek to maximise $\|D^j \omega_\Theta\|_\infty$ to obtain an universal bound. A simple modification of the proof of Theorem 2.1 gives the following extension.

Theorem 2.1*. *Let Ψ be a fixed multiset of $\le n$ points from $[a,b]$ and $0 \le j < n$. Then the maximum of*

$$\{\|D^j\omega_\Theta\|_p : \Theta \supset \Psi\}$$

is attained when $\Theta \setminus \Psi$ is from $\{a,b\}$.

Combining these two observations we have

Corollary 2.7 (Sharp Hermite error bounds). *For $0 \le j < n$, the error in Hermite interpolation satifies*

$$\|D^j(f - H_\Theta f)\|_\infty \le \frac{(b-a)^{n-j}}{(n-j)!}\|D^n f\|_\infty, \qquad \forall f \in W_\infty^n.$$

This inequality is best possible in the sense that equality is attained if and only if H_Θ is Taylor interpolation at one of the endpoints and f is a polynomial of degree n.

Proof. By Theorem 2.1*, the maximum of $\|D^j\omega_\Theta\|_\infty$ occurs for some Θ consisting of endpoints. A simple calculation shows it is attained when one endpoint is repeated n times, giving

$$\max_\Theta \|D^j\omega_\Theta\|_\infty = \|D^j((\cdot - a)^n)\|_\infty = \frac{n!}{(n-j)!}(b-a)^{n-j}.$$

□

3. Minimising $\|\omega_\Theta\|_p$

We now show that $\Theta \mapsto \|\omega_\Theta\|_p$ has a unique minimum, using the classical result that:

Theorem 3.1. *If $P \subset C[a,b]$ is an n–dimensional Haar space, then g^*, the unique best L_p–approximation to $f \in C[a,b]$ from P, interpolates f at n distinct points in (a,b).*

For $1 \le p < \infty$, the characterisation theorem for best L_p–approximation gives that g^* is uniquely determined by

$$\int_a^b |f-g^*|^{p-1}\mathrm{sign}(f-g^*)\, g = 0, \qquad \forall g \in P, \tag{3.2}$$

where **sign** denotes the signum function. Taking $P = \Pi_{n-1}$, and $f = (\cdot)^n$, we obtain:

Proposition 3.3 (Minimum of $\|\omega_\Theta\|_p$). *There is a unique choice of Θ which minimises $\|\omega_\Theta\|_p$, namely the n distinct points in (a,b) that are the roots of $M_{n,p} \in \Pi_n$, which is defined to be the error in the unique best L_p–approximation to $(\cdot)^n$ from Π_{n-1}. We have*

$$\frac{1}{4^n}(b-a)^{n+\frac{1}{p}} \leq \min_\Theta \|\omega_\Theta\|_p = \|M_{n,p}\|_p \leq \frac{2}{4^n}(b-a)^{n+\frac{1}{p}},$$

with equality only when $p = 1, \infty$, respectively. In addition

$$\min_\Theta \|\omega_\Theta\|_2 = \|M_{n,2}\|_2 = \frac{(n!)^2}{(2n)!\sqrt{2n+1}}(b-a)^{n+\frac{1}{2}}.$$

Proof. Taking $P = \Pi_{n-1}$, and $f = (\cdot)^n$ in Theorem 3.1 gives that $M_{n,p}$ (the error in best approximation) is of the form $M_{n,p} = \omega_\Theta$ for a certain Θ consisting of distinct points in (a,b). Thus, this choice of Θ uniquely minimises $\|\omega_\Theta\|_p$ (even when Θ is not restricted to lie within $[a,b]$).

From Hölder's inequality, it follows that

$$p \mapsto C_p := \|M_{n,p}\|_p (b-a)^{-(n+\frac{1}{p})} = \min_\Theta \|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}$$

is strictly increasing.

For $p = 1$, $M_{n,p}$ is, up to an affine change of variables equal to U_n, the *Chebyshev polynomial of the second kind*, and we calculate

$$C_1 = \|2^{-n}U_n\|_1 \, 2^{-(n+\frac{1}{1})} = \frac{1}{4^n}.$$

Similarly $M_{n,2}$, $M_{n,\infty}$ are P_n, T_n. i.e., the *Legendre*, *Chebyshev polynomials*, respectively, and

$$\begin{aligned} C_2 &= \left\| \frac{2^n (n!)^2}{(2n)!} P_n \right\|_2 2^{-(n+\frac{1}{2})} = \frac{(n!)^2}{(2n)!\sqrt{2n+1}}, \\ C_\infty &= \|2^{-(n-1)}T_n\|_\infty \, 2^{-(n+\frac{1}{\infty})} = \frac{2}{4^n}. \end{aligned}$$

The facts about U_n, P_n, T_n that we have used above can be found in any standard book on orthogonal polynomials (see, e.g., Szegö [Sz59, p.41]). □

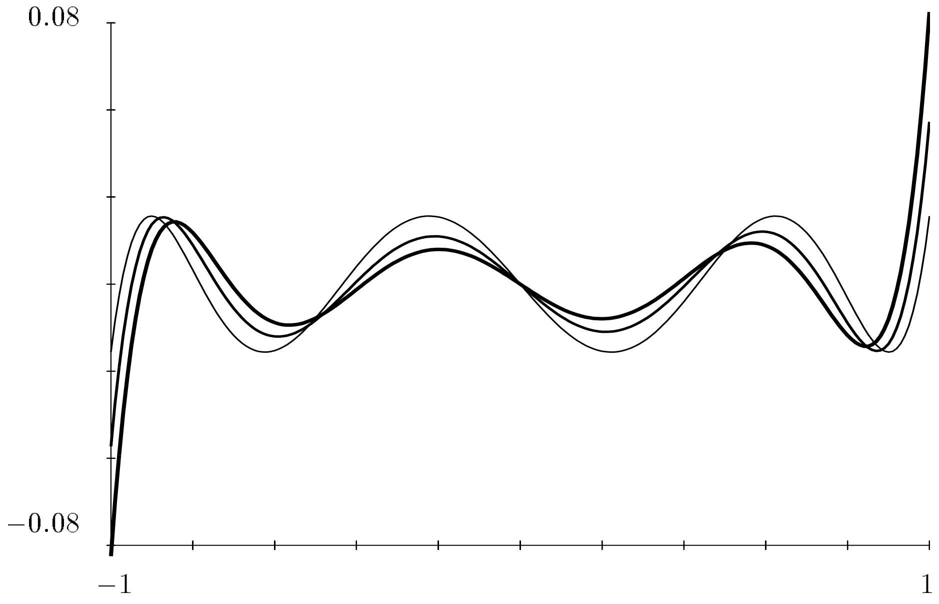

Fig 3.1 Graphs of the polynomials
$M_{7,1}$ (thick), $M_{7,2}$ (medium), and $M_{7,\infty}$ (thin)

As mentioned in the proof, when $p = 1, 2, \infty$, the $M_{n,p}$ are well known orthogonal polynomials. For other values of p, no recurrence relations are known for $M_{n,p}$. By (3.2), for $1 \leq p < \infty$, $M_{n,p}$ is the unique $m \in \Pi_n$ with leading term $(\cdot)^n$ and

$$\int_a^b |m|^{p-1}\operatorname{sign}(m)\, g = 0, \qquad \forall g \in \Pi_{n-1}. \tag{3.4}$$

It is possible to view (3.4) as a nonlinear system of equations in the roots of $M_{n,p}$ (with a unique solution), and solve it numerically. For two different iterative schemes, together with sample results, see Burgoyne [Bu67], and Vincenti [V86].

Gauss quadrature formulæ with multiple nodes

The polynomials $M_{n,p}$ have the following interesting connection with quadrature (see Turan [T50], also Ghizzetti and Ossicini [GO70, p.74]).

If $p = 2s+2$, $s = 0, 1, 2, \ldots$, then (3.2) reduces to

$$\int_a^b m^{2s+1} g = 0, \qquad \forall g \in \Pi_{n-1}.$$

The corresponding m $(= M_{n,2s+2})$ is called *s-orthogonal* (with weight dx).

There is a quadrature formula of the form

$$Q(f) := \sum_{i=0}^{2s} \sum_{v \in \Theta} w(i,v) D^i f(v), \tag{3.5}$$

for the integral $I(f) := \int_a^b f$, of precision $(2s+2)n-1$, if and only if Θ is the zeros of $M_{n,2s+2}$. In keeping with the special case $s = 0$, such a Q is referred to as a *Gauss formulæ with multiple nodes*, or simply as a *s–Gauss* formula, and $M_{n,2s+2}$ is called a *Legendre s-polynomial.*

The s–Gauss formulæ are *interpolatory*, i.e. $Q(f) = I(H_{\Theta^*} f)$, where Θ^* is any set of $\leq n(2s+2)$ points, which contains each zero of $M_{n,2s+2}$ with multiplicity at least $2s+1$. This allows us to estimate the error for these formulæ.

Corollary 3.6 (Error bound for s–Gauss formulæ). *Let Θ be the zeros of $M_{n,2s+2}$. Then*

$$|I(f) - Q(f)| \leq \frac{1}{(n(2s+2))!} \Big(\|\omega_\Theta\|_{2s+2} \Big)^{2s+2} \|D^{n(2s+2)} f\|_\infty, \quad \forall f \in W_\infty^{n(2s+2)},$$

with equality for all $f \in \Pi_{n(2s+2)}$. In addition

$$\Big(\|\omega_\Theta\|_{2s+2} \Big)^{2s+2} < \left(\frac{2}{4^n} \right)^{2s+2} (b-a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$.

Proof. With Θ^* as above, by (1.1)

$$\begin{aligned} |I(f) - Q(f)| &= |I(f - H_{\Theta^*} f)| \leq \|f - H_{\Theta^*} f\|_1 \\ &\leq \frac{1}{(n(2s+2))!} \|\omega_{\Theta^*}\|_1 \|D^{n(2s+2)} f\|_\infty. \end{aligned} \tag{3.7}$$

Let Θ^* consist of the points Θ, each with multiplicity $2s+2$. For this choice,

$$\|\omega_{\Theta^*}\|_1 = \Big(\|\omega_\Theta\|_{2s+2} \Big)^{2s+2}.$$

Further, if $f \in \Pi_{n(2s+2)}$, then $f - H_{\Theta^*} f$ is a scalar multiple of ω_Θ^{2s+2}, which is nonnegative, and so equality holds in (3.7). Finally, by Proposition 3.3,

$$\Big(\|\omega_\Theta\|_{2s+2} \Big)^{2s+2} < \left(\frac{2}{4^n} (b-a)^{n + \frac{1}{2s+2}} \right)^{2s+2} = \left(\frac{2}{4^n} \right)^{2s+2} (b-a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$. $\square$

Only when $s = 0$ is this result known (see, e.g., Davis and Rabinowitz [DR75, p.98]). In this case $\|\omega_\Theta\|_2$ is the L_2–norm of a Legendre polynomial, and can be computed exactly. For a full account of s–Gauss formulæ, including other error estimates, see the survey article of Gautschi [G81].

By using (2.4) and (2.6), it is possible to run through the above argument, to get error bounds for s–Gauss formulæ in terms of $\|D^m f\|_q$, where $n(2s+1) \le m \le n(2s+2)$.

4. Further applications

Good approximation by polynomials

Combining Corollary 2.2 and Proposition 3.3, we obtain:

$$\frac{1}{2}\frac{4^n}{(np+1)^{1/p}} \le \frac{\max_\Theta \|\omega_\Theta\|_p}{\min_\Theta \|\omega_\Theta\|_p} \le \frac{4^n}{(np+1)^{1/p}},$$

where $(np+1)^{1/p} := 1$, when $p = \infty$. Thus, a good choice of Θ can greatly improve the size of the constant $\|\omega_\Theta\|_p$ occurring in (1.1) over that for a poor choice.

For example, with Θ_{Eq} consisting of points with equal spacing $h := (b-a)/(n-1)$, and Θ_{Ch} the Chebyshev points, Isaacson and Keller [IK66, p.267] provide the estimate:

$$\frac{\|\omega_{\Theta_{\mathrm{Eq}}}\|_\infty}{\|\omega_{\Theta_{\mathrm{Ch}}}\|_\infty} > \frac{\sqrt{2}}{n-1}\left(\frac{4}{e}\right)^{n-1},$$

for large n, in support of doing Lagrange interpolation at the Chebyshev points.

Best approximation by polynomials

By Theorem 3.1, the unique best L_p–approximation to $f \in C[a,b]$ from Π_{n-1} is obtained by Lagrange interpolation at n points in (a,b). Thus, in view of (1.1), we expect some relation between $\min_\Theta \|\omega_\Theta\|_p$, and the error

$$E_{n,p}(f) := \inf_{g\in\Pi_{n-1}} \|f-g\|_p$$

in best L_p–approximation. The main result in this direction is the following.

Theorem 4.1 (Phillips [P70]). *If $f \in C^n[a,b]$, then $\exists\, \xi \in [a,b]$ such that*

$$E_{n,p}(f) = \frac{\|M_{n,p}\|_p}{n!} |D^n f(\xi)| \le \frac{\|M_{n,p}\|_p}{n!} \|D^n f\|_\infty,$$

with equality if and only if $f \in \Pi_n$.

Along the same lines, Fink [F77] defines $B(n,p,q)$ as the smallest constant such that

$$E_{n,p}(f) \le B(n,p,q)(b-a)^{n+\frac{1}{p}-\frac{1}{q}} \|D^n f\|_q, \qquad \forall f \in W_q^n,$$

and gives some equivalent definitions.

Since best approximations are given by Lagrange interpolation, we might hope to estimate $B(n,p,q)$ by interpolating f at some Θ, as does Phillips in Theorem 4.1, where he shows:

$$B(n,p,\infty) = \frac{\|M_{n,p}\|_p}{n!} (b-a)^{-(n+\frac{1}{p})}. \tag{4.2}$$

Pursuing this idea, we are able to estimate $B(n,p,q)$ to within a factor of $8n$.

Corollary 4.3 (Estimate for Fink's constant).

$$\frac{1}{n!}\frac{1}{4^n} \le B(n,p,q) \le \frac{n^{\frac{1}{q}}}{n!} \left(\frac{2}{4^n}\right)^{1-\frac{1}{nq}} \le 8n \frac{1}{n!}\frac{1}{4^n}.$$

Proof. Let $b-a=1$. First the lower bound. Since $M_{n,p}$ is the error in approximating $f = (\cdot)^n$, which has $D^n f = n!$, we must have

$$B(n,p,q) \ge \frac{\|M_{n,p}\|_p}{n!} \ge \frac{1}{n!}\frac{1}{4^n}.$$

By (2.4) and (2.6):

$$B(n,p,q) \le \frac{n^{\frac{1}{q}}}{n!} \|\omega_\Theta^{1-\frac{1}{nq}}\|_p = \frac{n^{\frac{1}{q}}}{n!} \left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})}\right)^{1-\frac{1}{nq}}.$$

Choosing $\omega_\Theta = M_{n,p(1-1/nq)}$, then applying Proposition 3.3 to (3.7), we obtain the upper bound. □

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