Technical Report
22 January 2009

On the construction of equiangular frames from graphs

Shayne Waldron

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
e–mail: waldron@math.auckland.ac.nz

ABSTRACT

We give details of the 1–1 correspondence between equiangular frames of n vectors for $\mathbb{R}^d$ and graphs with n vertices. This has been studied recently for *tight* equiangular frames because of applications to signal processing and quantum information theory. The *nontight* examples given here (which correspond to graphs with more than 2 eigenvalues) have the potential for similar applications, e.g., the frame corresponding to the 5–cycle graph is the unique Grassmannian frame of 5 vectors in $\mathbb{R}^3$. Further, the associated canonical tight frames have a *small* number of angles in many cases.

Key Words: finite frame, tight frame, Grassmannian frame, mutually unbiased basis, two angle frame, Seidel matrix, adjacency matrix, algebraic graph theory,

AMS (MOS) Subject Classifications: primary 42C15, 05C50, secondary 05C90, 52B15,

1. Introduction

Equiangular tight frames have important applications to signal processing because of their robustness to erasures (see [GKK01], [HP04], [BP05]), and to quantum information theory (see [RBSC04], [R05]). When such frames do not exist (cf [STDH07]) closely related classes of frames have been suggested as substitutes, most notably *Grassmannian frames* and *mutually unbiased bases* (see [SH03], [BK06], [KR04]). In this direction, we consider *nontight* equiangular frames of n vectors for $\mathbb{R}^d$. These are in 1–1 correspondence with graphs on n vertices. The associated dual and canonical tight frames are *not* equiangular, but are often equal–norm frames with a just a few angles.

The paper is set out as follows. Next we give the basic theory of finite frames, and define (possibly nontight) equiangular frames. We then consider the 1–1 correspondence between an equiangular frame and its so called signature matrix (which defines it up to unitary equivalence). We show if the frame is tight, then the equiangularity condition reduces to a system of quadratic equations. We solve these equations in a few cases. This leads to examples which indicate complex equiangular tight frames are likely to be more numerous than is generally believed. Understanding the subfield of the complex numbers the entries of the signature matrix can or must come from may be pivotal in understanding complex equiangular tight frames (cf [K06], [STDH07], [BPT08]).

The remainder of the paper considers the special case of real equiangular frames. Here the entries of the signature matrix are ± 1, and so it can be thought of as the Seidel (adjacency) matrix of a graph. These ideas date back to the foundations of algebraic graph theory (see [GR01], [S91]), with most attention spent on the case of *tight* equiangular frames (see [SH03], [HP04]). These correspond to graphs with 2 eigenvalues, which in turn come from a subclass of the strongly regular graphs (with certain parameters given here) with an additional point added. We conclude with a number of examples of *nontight* equiangular tight frames given by graphs with few eigenvalues, such as the strongly regular graphs themselves (which have 3 eigenvalues).

2. The basic theory of finite frames

The following definitions and observations are well known (cf [C03], [W09]). Let $\mathcal{H}$ be a real or complex Hilbert space of finite dimension d. A finite sequence of $n \geq d$ vectors $\Phi = (f_j)_{j\in J}$ in $\mathcal{H}$ is a **frame** for $\mathcal{H}$ if it spans $\mathcal{H}$, and is a **tight frame** for $\mathcal{H}$ if there is a $c > 0$ with

$$f = c \sum_{j\in J} \langle f, f_j \rangle f_j, \qquad \forall f \in \mathcal{H}. \tag{2.1}$$

The above $c > 0$ is unique, and given by

$$c = \frac{d}{\sum_{j\in J} \|f_j\|^2}. \tag{2.2}$$

A tight frame is **normalised** (the term *Parseval frame* is also used) if it has been scaled so that $c = 1$. Frames with a countable number of vectors (for a finite or infinite dimensional space) can be defined by the so called **frame bounds**, i.e., that there exists $A, B > 0$ with

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

This is easily seen to be equivalent to the definition above for J finite.

The **synthesis operator** for a finite sequence $(f_j)_{j \in J}$ in $\mathcal{H}$ is the linear map

$$V := [f_j]_{j \in J} : \ell_2(J) \to \mathcal{H} : a \mapsto \sum_{j \in J} a_j f_j,$$

and its **frame operator** is the linear map $S = VV^* : \mathcal{H} \to \mathcal{H}$ given by

$$Sf := \sum_{j \in J} \langle f, f_j \rangle f_j, \qquad \forall f \in \mathcal{H}.$$

With $I = I_{\mathcal{H}}$ the identity on $\mathcal{H}$, the tight frame condition (2.1) can be expressed as

$$S = VV^* = cI. \tag{2.3}$$

If $\Phi = (f_j)_{j \in J}$ is a frame, then S is invertible, and the **dual frame** $\tilde{\Phi} = (\tilde{f}_j)$ is defined by

$$\tilde{f}_j := S^{-1} f_j, \qquad \forall j \in J, \tag{2.4}$$

and the **canonical tight frame** $\Phi^{\text{can}} = (f_j^{\text{can}})$ by

$$f_j^{\text{can}} = S^{-\frac{1}{2}} f_j, \qquad \forall j \in J. \tag{2.5}$$

A frame and its dual satisfy the expansion

$$f = \sum_{j \in J} \langle f, f_j \rangle \tilde{f}_j = \sum_{j \in J} \langle f, \tilde{f}_j \rangle f_j, \qquad \forall f \in \mathcal{H},$$

and the canonical tight frame is a normalised tight frame, i.e.,

$$f = \sum_{j \in J} \langle f, f_j^{\text{can}} \rangle f_j^{\text{can}}, \qquad \forall f \in \mathcal{H}.$$

Frames $\Phi = (f_j)_{j \in J}$ and $\Psi = (g_j)_{j \in J}$ for $\mathcal{H}$, with the same index set J, are **unitarily equivalent** if there is a unitary transformation $\mathcal{H} \to \mathcal{H}$ with $\Psi = U\Phi := (Uf_j)_{j \in J}$, i.e.,

$$g_j = Uf_j, \qquad \forall j \in J, \tag{2.6}$$

and are **similar** if there is an invertible linear map $Q : \mathcal{H} \to \mathcal{H}$ with $\Psi = Q\Phi$. Clearly, these are equivalence relations on the set of frames for $\mathcal{H}$, indexed by a given set J. In view of definitions (2.4) and (2.5), a frame, its dual and canonical tight frame are all similar. A simple calculation shows that normalised tight frames are similar if and only if they are unitarily equivalent.

The **Gramian** of a sequence of n vectors $(f_j)_{j\in J}$ is the $n \times n$ matrix

$$\mathrm{Gram}(\Phi) := V^*V = [\langle f_k, f_j \rangle]_{j,k\in J}.$$

Frames Φ and Ψ are unitarily equivalent if and only if their Gramians are equal, and they are similar if and only if the Gramians of their canonical tight frames are equal, i.e.,

$$\mathrm{Gram}(\Phi^{\mathrm{can}}) = \mathrm{Gram}(\Psi^{\mathrm{can}}).$$

A sequence of vectors is a normalised tight frame (for its span) if and only if its Gramian matrix P is an orthogonal projection matrix, i.e., $P^2 = P$ and $P = P^*$. Note that $P = P^*$ holds for any Gramian matrix. We say that two frames Φ and Ψ are **complementary** (or **complements** of each other) if the Gramians of the associated canonical tight frames are complementary projection matrices, i.e.,

$$\mathrm{Gram}(\Phi^{\mathrm{can}}) + \mathrm{Gram}(\Psi^{\mathrm{can}}) = I. \tag{2.7}$$

The complement of a frame is well defined up to similarity, and the complement of a tight frame in the class of normalised tight frames is well defined up to unitary equivalence.

We say that (f_j) is an **equal–norm** frame if all its vectors have the same length, and is **equiangular** (cf [SH03]) if addition there is a $C \geq 0$ with

$$|\langle f_j, f_k \rangle| = C, \qquad \forall j \neq k.$$

The dual and canonical tight frames of a nontight equal–norm frame is in general not an equal–norm frame, but can be, see, e.g., Theorem 6.2.

3. Equiangular frames and their signature matrices

Since frames are determined up to unitary equivalence by their Gramian matrices, the Gramian of an equiangular frame with $c > 0$ has the form

$$G = \begin{pmatrix} r & cz_{12} & cz_{13} & \cdots & cz_{1n} \\ c\overline{z_{12}} & r & cz_{23} & \cdots & cz_{2n} \\ c\overline{z_{13}} & c\overline{z_{23}} & r & & \\ \vdots & \vdots & & \ddots & \\ c\overline{z_{1n}} & c\overline{z_{2n}} & & & r \end{pmatrix} = rI + c\Sigma, \qquad r > 0, \quad |z_{jk}| = 1.$$

We call any $n \times n$ Hermitian matrix Σ of the above form, i.e., with zero diagonal and off diagonal entries of modulus 1 a **signature matrix**. Let $\mathbb{F}$ stand for $\mathbb{R}$ or $\mathbb{C}$. The $n \times n$ signature matrices are in 1–1 correspondence with the equiangular frames of n vectors. For completeness, we state and proof this well known result in the frame terminology.

Theorem 3.1. *Let Σ be an $n \times n$ signature matrix (over $\mathbb{F}$), with smallest eigenvalue $-\lambda$ of multiplicity $n-d$, then*

$$A := r(I + \frac{1}{\lambda}\Sigma) = \frac{r}{\lambda}(\Sigma - (-\lambda)I), \qquad r > 0$$

is the Gramian matrix of an equiangular frame of n vectors for $\mathbb{F}^d$, and every Gramian of an equiangular frame of $n > d$ vectors for $\mathbb{F}^d$ can be constructed in this way. Further, the frame is tight if and only if Σ has (exactly) two eigenvalues.

Proof: By construction, the matrix A is positive semidefinite of rank $d > 0$, and so has a positive square root $B = A^{\frac{1}{2}}$. Since $A = B^2 = B^* B$, A is the Gramian matrix of the frame given by the columns of B (which span a d–dimensional space).

A frame of $n > d$ vectors is tight if only if its Gramian has exactly one nonzero eigenvalue, and so an equiangular frame is tight if and only if its signature matrix has exactly two eigenvalues. $\square$

If an equiangular frame is tight, then the λ above is given by

$$\lambda = \sqrt{\frac{d(n-1)}{n-d}},$$

which leads to a system of $\frac{1}{2}n(n-1)$ equations in the entries of Σ.

Corollary 3.2 ([BP05:Th. 4.2]). *Let $(z_{jk})_{1 \leq j < k \leq n}$ be scalars of modulus 1, then the signature matrix*

$$\Sigma = \begin{pmatrix} 0 & z_{12} & z_{13} & \cdots & z_{1n} \\ \overline{z_{12}} & 0 & z_{23} & \cdots & z_{2n} \\ \overline{z_{13}} & \overline{z_{23}} & 0 & & \\ \vdots & \vdots & & \ddots & \\ \overline{z_{1n}} & \overline{z_{2n}} & & & 0 \end{pmatrix}$$

gives an equiangular tight frame if and only if

$$(n-2d)\sqrt{\frac{n-1}{d(n-d)}} z_{jk} = \sum_{i=1}^{j-1} \overline{z_{ij}} z_{ik} + \sum_{i=j+1}^{k-1} z_{ji} z_{ik} + \sum_{i=k+1}^{n} z_{ji} \overline{z_{ki}}, \qquad 1 \leq j < k \leq n. \quad (3.3)$$

Proof: The signature matrix has two eigenvalues $\lambda_1 = -\lambda$ and λ_2 if and only if it satisfies the minimal polynomial

$$\Sigma^2 - (\lambda_1 + \lambda_2)\Sigma + \lambda_1 \lambda_2 I = 0. \quad (3.4)$$

In particular, by considering a diagonal entry, we must have

$$\lambda_1 \lambda_2 = -(n-1).$$

Since Σ has zero trace and λ_1 has multiplicity $n-d$, we have

$$(n-d)\lambda_1 + d\lambda_2 = 0.$$

Solving these gives

$$\lambda_1 = -\lambda = -\sqrt{\frac{d(n-1)}{n-d}}, \qquad \lambda_2 = \sqrt{\frac{(n-d)(n-1)}{d}}. \tag{3.5}$$

From the entries of the matrix equation (3.4), we therefore obtain n^2 equations in the z_{jk}, with coefficients depending only on n and d. Those from the diagonal entries hold automatically, and since the (j,k) and (k,j) entries are complex conjugates, we obtain the equivalent system

$$(\lambda_1+\lambda_2)z_{jk} = (\Sigma^2)_{jk}, \qquad 1 \le j < k \le n,$$

which can be written as (3.3). □

When $\mathbb{F}=\mathbb{R}$ the right hand side of (3.3) is the inner product between the j and k columns of Σ, and so the columns of the signature matrix of a equiangular tight frame for $\mathbb{R}^d$ are an equiangular tight frame of n vectors (which are orthogonal if $n=2d$).

Example 1. For $n=4$, $d=2$, (3.3) gives 6 equations. Let $z_{12}=a$, $z_{13}=b$, $z_{14}=c$. Then the $(j,k)=(1,2)$ and $(1,3)$ equations are

$$z_{13}\overline{z_{23}} + z_{14}\overline{z_{24}} = 0, \quad z_{13}z_{23} + z_{14}\overline{z_{34}} = 0 \quad \Longrightarrow \quad z_{24} = -\overline{b}cz_{23}, \quad z_{34} = -\overline{a}c\overline{z_{23}}.$$

Making the above substitutions for z_{24} and z_{34} reduces the other 4 equations to one

$$(a\overline{b}z_{23})^2 = -1 \quad \Longrightarrow \quad z_{23} = \pm i\overline{a}b.$$

Hence there is a three parameter family of unitarily inequivalent equiangular tight frames of four vectors for $\mathbb{C}^2$ given by the signature matrices

$$\Sigma = \begin{pmatrix} 0 & a & b & c \\ \overline{a} & 0 & \pm i\overline{a}b & \mp i\overline{a}c \\ \overline{b} & \mp ia\overline{b} & 0 & \pm i\overline{b}c \\ \overline{c} & \pm ia\overline{c} & \mp ib\overline{c} & 0 \end{pmatrix}, \qquad |a|=|b|=|c|=1. \tag{3.6}$$

A copy of this frame in $\mathbb{C}^2$ can be obtained by observing that (up to a scalar) the columns of the Gramian matrix $I+\frac{1}{\lambda}\Sigma = I+\frac{1}{\sqrt{3}}\Sigma$ gives such a frame. Hence, by expressing this frame in terms of the orthonormal basis obtained by applying Gram–Schmidt to its first two vectors we obtain the following unit–norm copy

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}}a \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}}b \\ \frac{\sqrt{2}}{\sqrt{3}}\zeta^{\pm 1}\overline{a}b \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}}c \\ \frac{\sqrt{2}}{\sqrt{3}}\zeta^{\mp 1}\overline{a}c \end{pmatrix} \right\}, \qquad \zeta := e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i. \tag{3.7}$$

Taking $a=b=c=-1$ in (3.7) gives the example of [STDH07].

The appearance of the third root of unity ζ here is incidental. If equiangular tight frames are thought of as sums of rank one orthogonal projections (averaging to a multiple of the identity), rather than as the vectors defining these projections, then the natural equivalence between $\Phi = (f_j)$ and $\Psi = (g_j)$ is to extend the unitary equivalence (2.6) to

$$g_j = \alpha_j U f_j, \qquad \forall j.$$

where α_j are scalars of unit modulus. This is the *type III equivalence* of [HP04] (without reordering). The frames Φ and Ψ are equivalent under this if and only if their Gramians satisfy

$$\mathrm{Gram}(\Psi) = \Lambda^* \,\mathrm{Gram}(\Phi)\Lambda, \qquad \Lambda = \mathrm{diag}(\alpha_j).$$

For equiangular tight frames, (3.4) can be used to express this condition in terms of the signature matrices

$$\Sigma_\Psi = \Lambda^* \Sigma_\Phi \Lambda.$$

Hence every equiangular tight frame is type III equivalant to one with a signature matrix of the form

$$\Lambda\Sigma\Lambda^* = \begin{pmatrix} 0 & \vec{1}^* \\ \vec{1} & \hat{\Sigma} \end{pmatrix}, \qquad \Lambda := \mathrm{diag}(1, z_{12}, z_{13}, \ldots, z_{1n}), \quad \vec{1} := (1,1,\ldots,1)^*. \tag{3.8}$$

We will call the above matrix $\hat{\Sigma}$ the **reduced signature matrix** of the frame (and its type III equivalence class). The reduced signature matrix of (3.6) is

$$\hat{\Sigma} = \begin{pmatrix} 0 & \pm i & \mp i \\ \mp i & 0 & \pm i \\ \pm i & \mp i & 0 \end{pmatrix}.$$

Thus there are just *two* equiangular tight frames of four vectors in $\mathbb{C}^2$ up to the type III equivalence. Moreover, these can be obtained from each other by taking the entrywise complex conjugate of the reduced signature matrix.

The factorisation (3.8) allows the number of variables in (3.3) to be reduced by $n-1$ to $\frac{1}{2}(n-1)(n-2)$. We express these equations in terms of the reduced signature matrix.

Proposition 3.9. *There exists an equiangular tight frame of n vectors for $\mathbb{F}^d$ with reduced signature matrix $\hat{\Sigma}$ if and only if*

$$(\lambda_1+\lambda_2)\hat{\Sigma} = \hat{\Sigma}^2 + J - (n-1)I, \qquad \lambda_1+\lambda_2 := (n-2d)\sqrt{\frac{n-1}{d(n-d)}}, \quad J := \vec{1}\vec{1}^*,$$

and $\vec{1}$ is an eigenvector of $\hat{\Sigma}$ for the eigenvalue $\lambda_1+\lambda_2$.

Proof: Substitute

$$\Sigma = \begin{pmatrix} 0 & \vec{1}^* \\ 1 & \hat{\Sigma} \end{pmatrix}$$

into (3.4), and equate the blocks. □

4. Real equiangular frames and their graphs

The **Seidel matrix** (see [GR01]) Σ of a graph G with n vertices is the $n \times n$ matrix with a -1 in the (j,k)–entry if the j and k vertices are adjacent (connected by an edge), a 1 if they are noadjacent, and 0 diagonal entries. Clearly, Seidel matrices are signature matrices over $\mathbb{R}$, and vice versa.

For $\mathbb{F} = \mathbb{R}$, there are *finitely* many possible $n \times n$ signature matrices, and hence finitely many real equiangular frames of n vectors. Each of these is in 1–1 correspondence with a graph, namely the graph whose Seidel matrix is the signature matrix of this frame.

Theorem 4.1. *Let G be a graph with n vertices, and Σ be its Seidel matrix. If $-\lambda$ is the smallest eigenvalue of Σ, and has multiplicity $n-d$, then*

$$r(I + \frac{1}{\lambda}\Sigma) = \frac{r}{\lambda}\big(\Sigma - (-\lambda)I\big)$$

is the Gramian matrix of an equiangular tight frame of n vectors for $\mathbb{R}^d$, and this frame is tight if and only if Σ has two eigenvalues. Conversely, every equiangular frame of n vectors for $\mathbb{R}^d$ can be constructed from a graph in this way.

This 1–1 correspondence between real equiangular frames and graphs has recently been studied in the case of *tight* equiangular frames (see, e.g., [SH03], [BP05], [STDH07]). We summarise the implications in the next section. After that we consider, for the first time, *nontight* equiangular frames.

Graphs G_1, G_2 with Seidel matrices Σ_1, Σ_2 are **switching equivalent** if there is a diagonal matrix Λ with diagonal entries ±1 for which $\Lambda^{-1}\Sigma_1\Lambda$ is the Seidel matrix of the (unlabelled) graph G_2. The collection of graphs which are switching equivalent to a given G is called the **switching class** of G (or a **two–graph**). The equiangular frames corresponding to a switching class of graphs differ from each other (up to unitary equivalence) only by the multiplication of their vectors by ±1 (as given by Λ).

Example 2. If G is the complete graph K_n, then its Seidel matrix Σ has *two* eigenvalues: $-(n-1)$ of multiplicity 1, and 1. The corresponding tight frame of n vectors in $\mathbb{R}^{n-1}$ is given by vectors which are the vertices of a regular simplex (see Figure 1). Similarly, its complement, the empty graph, gives the equiangular tight frame for $\mathbb{R}^1$ consisting of a nonzero vector repeated n times.

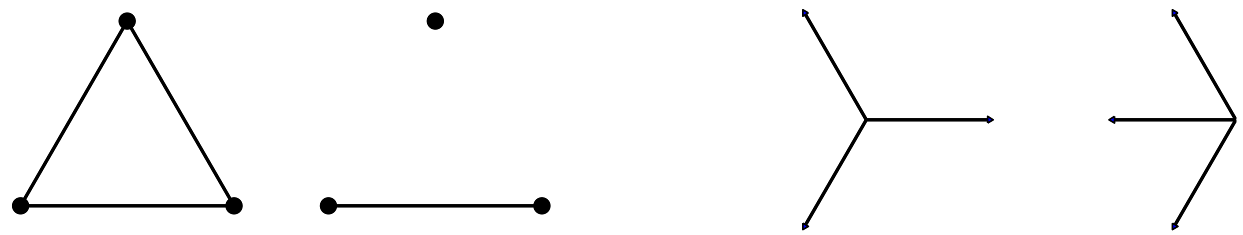

Fig. 1. The graphs in the switching class of the complete graph K_3, and the corresponding equiangular (tight) frames of three vectors in $\mathbb{R}^2$.

5. Tight equiangular frames

Let Σ be the Seidel matrix of a graph G, and $\hat{\Sigma}$ given by (3.8) be its **reduced Seidel matrix**. The condition that Σ have two eigenvalues, and hence give a real equiangular tight frame, is most easily expressed in terms of the graph with Seidel matrix $\hat{\Sigma}$ (see [HP04:Th. 3.10] for a description in terms of the switching class containing G).

A regular graph of degree k with ν vertices is said to be **strongly regular**, or a $\mathrm{srg}(\nu, k, \lambda, \mu)$, if there are integers λ and μ such that

- Every two adjacent vertices have λ common neighbours.
- Every two non-adjacent vertices have μ common neighbours.

The adjacency matrix A (which has a 1 for adjacency, and a 0 otherwise) of a strongly regular graph G which is not complete or empty is characterised by

$$AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J, \tag{5.1}$$

where $J = J_\nu$ is the $\nu \times \nu$ matrix of all 1's and $I = I_\nu$ is the identity.

Theorem 5.2. *Let Σ be the Seidel matrix of a graph G on n vertices which is not switching equivalent to the complete or empty graph, and $\hat{\Sigma}$ be given by (3.8). Then Σ has two eigenvalues (and so corresponds with an equiangular tight frame of $n > d + 1$ vectors for $\mathbb{R}^d$) if and only if $\hat{\Sigma}$ is the Seidel matrix of a strongly regular graph $\hat{G}$ of the type*

$$\mathrm{srg}(n-1, k, \lambda, \mu), \qquad \lambda = \frac{3k-n}{2}, \qquad \mu = \frac{k}{2}.$$

The n, k, d above are related as follows

$$d = \frac{1}{2}n - \frac{1}{2}\frac{n(n-2k-2)}{\sqrt{(n-2k)^2 + 8k}} > 1, \qquad k = \frac{1}{2}n - 1 + \bigl(1 - \frac{n}{2d}\bigr)\sqrt{\frac{d(n-1)}{n-d}}. \tag{5.3}$$

Proof: We adapt the relevant parts of the proof of [GR01:Th. 11.6.1]. By Proposition 3.9, Σ has two eigenvalues (i.e., gives rise to an equiangular tight frame for $\mathbb{R}^d$), if and only if $\vec{1}$ is an eigenvector of $\hat{\Sigma}$ for the eigenvalue $\lambda_1 + \lambda_2$, and

$$(\lambda_1 + \lambda_2)\hat{\Sigma} = \hat{\Sigma}^2 + J - (n-1)I, \qquad J := \vec{1}\vec{1}^*. \tag{5.4}$$

Thus $\vec{1}$ is an eigenvector of the adjacency matrix $A = \frac{1}{2}(J - I - \hat{\Sigma})$ of $\hat{G}$ for the eigenvalue

$$k = \frac{(n-1) - 1 - (\lambda_1 + \lambda_2)}{2} = \frac{1}{2}n - 1 + \bigl(1 - \frac{n}{2d}\bigr)\sqrt{\frac{d(n-1)}{n-d}}, \tag{5.5}$$

with k a positive integer (since the nonzero entries of A are 1). Hence $\hat{G}$ is a regular graph of degree k, which is not complete or empty (by our assumption). Using $\hat{\Sigma} = J - I - 2A$, $AJ = JA = kJ$ and (5.5), we can rewrite (5.4) as

$$A^2 + (\frac{n}{2} - k)A - \frac{k}{2}I = A^2 + (\frac{k}{2} - \frac{3k-n}{2})A + (\frac{k}{2} - k)I = \frac{k}{2}J,$$

which (since $AJ = JA$) is equivalent to $\hat{G}$ being a $\mathrm{srg}(n-1, k, \lambda, \mu)$, $\lambda = \frac{1}{2}(3k - n)$, $\mu = \frac{k}{2}$.

Finally, solving (5.5) for d gives

$$d = \frac{1}{2}n \pm \frac{1}{2}\frac{n(n-2k-2)}{\sqrt{(n-2k)^2 + 8k}},$$

with the choice of sign determined by the multiplicities of the eigenvalues of Σ. □

The existence and construction of equiangular tight frames in $\mathbb{R}^d$ can therefore be expressed in terms of strongly regular graphs with certain parameters.

Corollary 5.6. *There exists an equiangular tight frame of $n > d+1$ vectors for $\mathbb{R}^d$ if and only if there exists a strongly regular graph $\hat{G}$, with Seidel matrix $\hat{\Sigma}$, of the type*

$$\operatorname{srg}(n-1, k, \frac{3k-n}{2}, \frac{k}{2}), \qquad k := \frac{1}{2}n - 1 + \big(1 - \frac{n}{2d}\big)\sqrt{\frac{d(n-1)}{n-d}}.$$

Moreover, all graphs G giving an equiangular tight frame of n vectors for $\mathbb{R}^d$ have Seidel matrices of the form

$$\Lambda^{-1}\begin{pmatrix} 0 & \vec{1}^* \\ \vec{1} & \hat{\Sigma} \end{pmatrix}\Lambda, \qquad \Lambda = \operatorname{diag}(\sigma_1, \ldots, \sigma_n), \quad \sigma_j = \pm 1, \tag{5.7}$$

where $\hat{\Sigma}$ is as above. In particular, we can take G to be $\hat{G}$ together with an isolated point.

We note that the switching class of a graph G giving an equiangular tight frame may contain graphs consisting of an isolated point together various nonisomorphic strongly regular graphs. (this is sometimes used as a method to construct strongly regular graphs). Thus it seems that the "reduced signature matrix" of an equiangular frame is not unique.

The graph G consisting of a strongly regular graph $\hat{G}$ together with an isolated point is switching equivalent to $\hat{G}$ together with a point which is adjacent to all points of $\hat{G}$. This is illustrated in Figure 2, which shows the switching class of the graph G obtained from $\hat{G}$ the 5–cycle, which is the unique $\operatorname{srg}(5,2,0,1)$. The corresponding tight frame of $n = 6$ vectors for $\mathbb{R}^3$ consists of vectors which are on the six diagonals of the icosahedron.

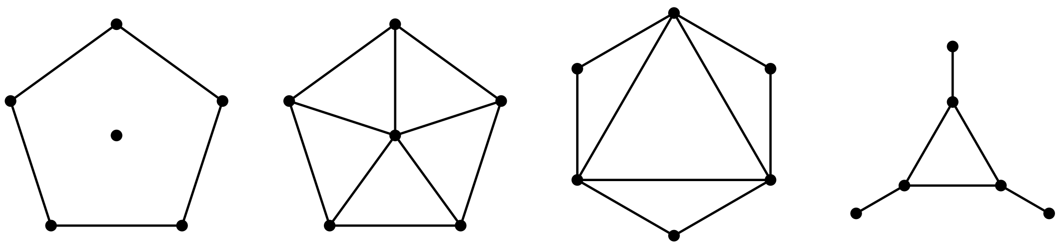

Fig. 2. The switching class of the graph G consisting of a 5–cycle and an isolated point.

Example 3. For $n = 2d$, i.e., $k = \frac{1}{2}(n-2)$, the parameters in Corollary 5.6 are those of a **conference graph** on $n-1$ vertices, and the matrices Σ of (5.7) are the associated (symmetric) **conference matrices** of size n, i.e., (3.4) becomes

$$\Sigma^2 = (n-1)I.$$

The **Paley graphs** on $n - 1 = q = p^m$ vertices, q a prime power with $q \equiv 1 \pmod 4$, are an important family of conference graphs. A necessary condition for the existence of such a conference graph is that d be odd, and $2d-1$ be a sum of squares (cf [STDH07:Th. 17]).

Example 4. For $n \neq 2d$ one obtains necessary conditions on n, d from those for the existence of a strongly regular graph $\hat{G}$ with the given parameters. Let $A = \frac{1}{2}(J - I - \hat{\Sigma})$ be the adjacency matrix of $\hat{G}$, which has eigenvalues k, $\frac{-1-\lambda_1}{2}$, $\frac{-1-\lambda_2}{2}$, where λ_1, λ_2 are given by (3.5). The eigenvalues of A must be integers, and hence λ_1, λ_2 are odd integers (cf [STDH07]). The other conditions which follow from the theory of strongly regular graphs do not appear in the frame literature, and so we give a formal statement.

Corollary 5.8. *Suppose $n \neq 2d$. Then the following are necessary conditions for an equiangular tight frame of $n > d + 1$ vectors for $\mathbb{R}^d$ to exist*

(a) $\sqrt{\frac{d(n-1)}{n-d}}$, $\sqrt{\frac{(n-d)(n-1)}{d}}$ *are odd integers.*

(b) $\frac{1}{4}\frac{n^2(n-1)}{d(n-d)}$ *is a perfect square.*

(c) $n - 1$ *is odd, but not a prime.*

Proof: We have already discussed (a). By [GR01:Lem. 10.3.3], the eigenvalues of A satisfy

$$\Bigl(\frac{-1-\lambda_2}{2} - \frac{-1-\lambda_1}{2}\Bigr)^2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2 = \frac{1}{4}\frac{n^2(n-1)}{d(n-d)} \quad \text{is a perfect square,}$$

which gives (b). Since λ_1 and λ_2 are odd integers, so is their product $-(n-1)$. Further, by [GR01:Lem. 10.3.4], the only strongly regular graphs $\hat{G}$ with $n-1$ prime are conference graphs, i.e., those with

$$2k + (n-2)\Bigl(\frac{3k-n}{2} - \frac{k}{2}\Bigr) = 0 \quad \Longleftrightarrow \quad n = 2d,$$

and we obtain (c). Clearly, (b),(c) also follow directly from (a) and the fact $-\lambda_1, \lambda_2 > 1$. $\square$

The n, d must also satisfy the **Krein bounds** for a strongly regular graph (see [GR01:Th. 10.7.1]), i.e.,

$$\theta\tau^2 - 2\theta^2\tau - \theta^2 - k\theta + k\tau^2 + 2k\tau \geq 0,$$

where θ, τ are $\frac{-1-\lambda_1}{2}$, $\frac{-1-\lambda_2}{2}$ (in either order), and we recall λ_1, λ_2 given by (3.5) satisfy

$$\lambda_1\lambda_2 = -(n-1), \qquad k = \frac{1}{2}n - 1 - \frac{1}{2}(\lambda_1 + \lambda_2).$$

It is not clear whether the following necessary condition (cf [CRT08])

$$\frac{2d + 1 + \sqrt{8d+1}}{2} \leq n \leq \frac{1}{2}d(d+1) \tag{5.9}$$

implies the Krein bounds. It is conjectured in [STDH07] that (a) together with (5.9) implies the existence of a corresponding equiangular tight frame, and there is no evidence to refute this. However, the case $n = 76$, $d = 19$, which satisfies both bounds, is equivalent to the existence of a srg$(75, 32, 10, 16)$ which is a long standing open question in algebraic graph theory, and so this conjecture is unlikely to be decided without considerable insight.

Since the tables of equiangular tight frames given in [STDH07] and [CRT08] contain omissions and additions, in Table 1 we give all equiangular tight frames known to exist for $d \leq 50$. This was determined from the literature on strongly regular graphs as summarised in Brouwer [B07], and the associated internet page

`http://www.win.tue.nl/~aeb/graphs/srg/`

In particular, we note that whether there exist conference graphs srg$(65, 32, 15, 16)$ and srg$(85, 42, 20, 21)$ are still open questions. In Table 1, the # column gives the number of strongly regular graphs $\hat{G}$ known to exist, following the notation of [B07]: ! (exactly one exists), + (at least one exists), ? (existence unknown), m! (m exist).

Table 1. The equiangular tight frames of n vectors for $\mathbb{R}^d$ $(d \leq 50)$, for $n \neq 2d$ and $n = 2d$. Here # gives the number of associated strongly regular graphs $\hat{G}$.

d	n	#	λ_1	λ_2	$\hat{G}$
6	16	!	-3	5	srg$(15, 6, 1, 3)$
10	16	!	-5	3	srg$(15, 8, 4, 4)$
7	28	!	-3	9	srg$(27, 10, 1, 5)$
21	28	!	-9	3	srg$(27, 16, 10, 8)$
15	36	+	-5	7	srg$(35, 16, 6, 8)$
21	36	+	-7	5	srg$(35, 18, 9, 9)$
19	76	?	-5	15	srg$(75, 32, 10, 16)$
20	96	?	-5	19	srg$(95, 40, 12, 20)$
21	126	+	-5	25	srg$(125, 52, 15, 26)$
22	176	+	-5	35	srg$(175, 72, 20, 36)$
23	276	!	-5	55	srg$(275, 112, 30, 56)$
28	64	+	-7	9	srg$(63, 30, 13, 15)$
36	64	+	-9	7	srg$(63, 32, 16, 16)$
35	120	+	-7	17	srg$(119, 54, 21, 27)$
37	148	?	-7	21	srg$(147, 66, 25, 33)$
41	246	?	-7	35	srg$(245, 108, 39, 54)$
42	288	?	-7	41	srg$(287, 126, 45, 63)$
43	344	+	-7	49	srg$(343, 150, 53, 75)$
45	100	+	-9	11	srg$(99, 48, 22, 24)$
45	540	?	-7	77	srg$(539, 234, 81, 117)$
46	736	?	-7	105	srg$(735, 318, 109, 159)$

d	n	#	$\hat{G}$
3	6	!	Paley(5)
5	10	!	Paley(9)
7	14	!	Paley(13)
9	18	!	Paley(17)
13	26	15!	Paley(25)
15	30	41!	Paley(29)
19	38	+	Paley(37)
21	42	+	Paley(41)
23	46	+	Conference
25	50	+	Paley(49)
27	54	+	Paley(53)
31	62	+	Paley(61)
33	66	?	Conference
37	74	+	Paley(73)
41	82	+	Paley(81)
43	86	?	Conference
45	90	+	Paley(89)
49	98	+	Paley(97)

6. Nontight equiangular frames

For a frame (f_j) of n equal–norm vectors, we define its **angles** to be the set

$$\{|\langle f_j, f_k \rangle| : j \neq k\}.$$

In this terminolgy, an equiangular frame is characterised by having *one angle*, and a set of (more than one) mutually unbiased bases has *two angles*. The **minimal angle** between (the lines defined by) a set $\Phi = (f_j)$ of nonzero vectors is

$$\theta(\Phi) := \min_{j\neq k} \cos^{-1}(|\langle f_j, f_k\rangle|) \in [0, \frac{1}{2}\pi]. \tag{6.1}$$

For a nontight equal–norm frame the dual and canonical tight frames do not have equal norms (in general). Here we will show that certain classes of equiangular frames do have this property, and in addition they have a small number of angles.

By way of motivation, consider the equiangular frame Φ for $\mathbb{R}^3$ given by the 5–cycle, which has Seidel matrix

$$\Sigma = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix},$$

with eigenvalues $-\sqrt{5}, -\sqrt{5}, 0, \sqrt{5}, \sqrt{5}$. This has Gramian matrix $I + \frac{1}{\sqrt{5}}\Sigma$, and is *not* tight. It is the first such example of a *Grassmannian frame*, i.e., a frame of unit vectors maximising (6.1) the minimal angle (see [BK06]). The Gramian matrix of the dual frame is obtained by taking the pseudoinverse

$$(I + \frac{1}{\sqrt{5}}\Sigma)^\dagger = \begin{pmatrix} \frac{2}{5} & a & b & b & a \\ a & \frac{2}{5} & a & b & b \\ b & a & \frac{2}{5} & a & b \\ b & b & a & \frac{2}{5} & a \\ a & b & b & a & \frac{2}{5} \end{pmatrix}, \qquad a := \frac{3-\sqrt{5}}{20}, \quad b := \frac{3+\sqrt{5}}{20},$$

and the Gramian of the canonical tight frame is

$$(I + \frac{1}{\sqrt{5}}\Sigma)^\dagger (I + \frac{1}{\sqrt{5}}\Sigma) = \begin{pmatrix} \frac{3}{5} & -a & b & b & -a \\ -a & \frac{3}{5} & -a & b & b \\ b & -a & \frac{3}{5} & -a & b \\ b & b & -a & \frac{3}{5} & -a \\ -a & b & b & -a & \frac{3}{5} \end{pmatrix}, \qquad a := \frac{\sqrt{5}-1}{10}, \quad b := \frac{\sqrt{5}+1}{10}.$$

Thus the dual and canonical tight frames have two angles.

The minimal angles for the sets of vectors Φ, $\tilde{\Phi}$ and Φ^{can} are

$$\cos^{-1}\frac{1}{\sqrt{5}} \approx 63.4349^\circ, \qquad \cos^{-1}\frac{3+\sqrt{5}}{8} \approx 49.1176^\circ, \qquad \cos^{-1}\frac{1+\sqrt{5}}{6} \approx 57.3610^\circ.$$

It is easy to verify that Φ consists of vectors that lie in five of the six diagonals of the iscosahedron, and that the tight frame Φ^{can} is the harmonic frame given by the lifted fifth roots of unity (see [W09]).

The above properties follow from the fact that the 5–cycle is a strongly regular graph – the unique srg$(5, 2, 0, 1)$.

Theorem 6.2. *Let G be a strongly regular graph* $\mathrm{srg}(\nu, k, a, c)$*, and Φ the real equiangular frame of ν vectors for $\mathbb{R}^d$ that it determines. Then Φ is tight if and only if*

$$a - c - 2k + \nu = \pm\sqrt{(a-c)^2 + 4(k-c)}. \tag{6.3}$$

Otherwise, either

$$\sqrt{(a-c)^2 + 4(k-c)} > c - a + 2k - \nu, \tag{6.4}$$

and the dual and the canonical tight frames are equal–norm frames with two angles, where

$$d = \frac{1}{2}\Big(\nu + 1 + \frac{2k + (\nu-1)(a-c)}{\sqrt{(a-c)^2 + 4(k-c)}}\Big), \tag{6.5}$$

or $d = \nu - 1$.

Proof: Let G be a $\mathrm{srg}(\nu, k, a, c)$, with adjacency and Seidel matrices A and Σ. Then the eigenvalues of A are

$$\theta = \frac{a - c + \sqrt{\Delta}}{2}, \quad \tau = \frac{a - c - \sqrt{\Delta}}{2}, \qquad \Delta := (a-c)^2 + 4(k-c),$$

with multiplicities

$$m_\theta = \frac{1}{2}\Big(\nu - 1 - \frac{2k + (\nu-1)(a-c)}{\sqrt{\Delta}}\Big), \qquad m_\tau = \frac{1}{2}\Big(\nu - 1 + \frac{2k + (\nu-1)(a-c)}{\sqrt{\Delta}}\Big), \tag{6.6}$$

and k with eigenvector $\vec{1}$ (see [GR01]). Thus $\Sigma = J - I - 2A$, $J := \vec{1}\vec{1}^*$, has eigenvalues

$$-1 - 2\theta, \quad -1 - 2\tau, \quad \nu - 1 - 2k$$

with the corresponding eigenspaces. These are distinct, unless $-1-2\theta$ or $-1-2\tau$ equals $\nu - 1 - 2k$, which is equivalent to (6.3), and by Theorem 4.1 we have that Φ is tight.

Otherwise, the minimal eigenvalue of Σ is $-\lambda = -1 - 2\theta$ (with multiplicity m_θ) when $-1 - 2\theta < \nu - 1 - k$, i.e., (6.4) holds. In this case, the spectral decomposition of the symmetric matrix Σ is

$$\Sigma = -\lambda P_\theta + (-1 - 2\tau)P_\tau + (\nu - 1 - 2k)\frac{J}{\nu},$$

where P_θ and P_τ are the orthogonal projections onto the θ and τ eigenspaces of A. Hence the Gramian of the associated equiangular frame for $\mathbb{R}^d$, $d = n - m_\theta$, has the form

$$I + \frac{1}{\lambda}\Sigma = \alpha P_\tau + \beta\frac{J}{n}, \qquad \alpha = \frac{2\theta - 2\tau}{1 + 2\theta}, \quad \beta = \frac{2\theta - \nu - 2k}{1 + 2\theta}.$$

In particular, we observe P_τ has a constant diagonal, and off diagonal entries taking two possible values (all entries of J are 1). The dual and canonical tight frames have Gramians

$$(I + \frac{1}{\lambda}\Sigma)^\dagger = \frac{1}{\alpha}P_\tau + \frac{1}{\beta}\frac{J}{n}, \qquad (I + \frac{1}{\lambda}\Sigma)^\dagger(I + \frac{1}{\lambda}\Sigma) = P_\tau + \frac{J}{n},$$

and so are equal–norm frames with two angles (indeed the off diagonal entries of their Gramians take just two values). The only remaining case is when $\nu - 1 - k < -1 - 2\theta$, and the mimimal eigenvalue $\nu - 1 - k$ has multiplicity one, and so $d = \nu - 1$. $\square$

The above argument can be extended to any connected regular graph with three eigenvalues – here the eigenvalue corresponding to $\vec{1}$ is simple.

The list of all the equiangular frames of $n \leq 50$ vectors that can be constructed from strongly regular graphs by Theorem 6.2 are given in Table 2.

Table 2. The equiangular frames of $n \leq 50$ vectors for $\mathbb{R}^d$ constructed from strongly regular graphs G. Here type refers to the three cases in Theorem 6.2.

n	d	type	G
5	3	two	srg$(5, 2, 0, 1)$
9	5	two	srg$(9, 4, 1, 2)$
10	5	tight	srg$(10, 3, 0, 1)$
10	5	tight	srg$(10, 6, 3, 4)$
13	7	two	srg$(13, 6, 2, 3)$
15	6	two	srg$(15, 6, 1, 3)$
15	10	two	srg$(15, 8, 4, 4)$
16	6	tight	srg$(16, 5, 0, 2)$
16	10	tight	srg$(16, 10, 6, 6)$
16	10	tight	srg$(16, 6, 2, 2)$
16	6	tight	srg$(16, 9, 4, 6)$
17	9	two	srg$(17, 8, 3, 4)$
21	7	two	srg$(21, 10, 3, 6)$
21	15	two	srg$(21, 10, 5, 4)$
25	17	two	srg$(25, 8, 3, 2)$
25	24	-	srg$(25, 16, 9, 12)$
25	13	two	srg$(25, 12, 5, 6)$
26	13	tight	srg$(26, 10, 3, 4)$
26	13	tight	srg$(26, 15, 8, 9)$
27	7	two	srg$(27, 10, 1, 5)$
27	21	two	srg$(27, 16, 10, 8)$
28	21	tight	srg$(28, 12, 6, 4)$
28	7	tight	srg(28,15,6,10)
29	15	two	srg(29,14,6,7)
35	15	two	srg(35,16,6,8)
35	21	two	srg(35,18,9,9)

n	d	type	G
36	26	two	srg(36,10,4,2)
36	35	-	srg(36,25,16,20)
36	15	tight	srg(36,14,4,6)
36	21	tight	srg(36,21,12,12)
36	28	two	srg(36,14,7,4)
36	35	-	srg(36,21,10,15)
36	21	tight	srg(36,15,6,6)
36	15	tight	srg(36,20,10,12)
37	19	two	srg(37,18,8,9)
40	16	two	srg(40,12,2,4)
40	39	-	srg(40,27,18,18)
41	21	two	srg(41,20,9,10)
45	25	two	srg(45,12,3,3)
45	44	-	srg(45,32,22,24)
45	36	two	srg(45,16,8,4)
45	44	-	srg(45,28,15,21)
45	23	two	srg(45,22,10,11)
49	37	two	srg(49,12,5,2)
49	48	-	srg(49,36,25,30)
49	31	two	srg(49,18,7,6)
49	48	-	srg(49,30,17,20)
49	25	two	srg(49,24,11,12)
50	22	two	srg(50,7,0,1)
50	49	-	srg(50,42,35,36)
50	25	two	srg(50,21,8,9)
50	25	tight	srg(50,28,15,16)

Example 5. For $n \leq 50$ there are 29 equal–norm tight frames with two angles that can be constructed by Theorem 6.2. There also many equiangular tight frames that can be constructed in this way, e.g., for $n = 10$ the unique graphs srg$(10, 3, 0, 1)$ (the *Petersen graph*) and srg$(10, 6, 3, 4)$ (the *Triangular graph* T_5) give a set of 10 equiangular vectors in $\mathbb{R}^5$. Since there a unique such frame up to switching equivalence of the graphs, it follows these two graphs are switching equivalent to that obtained by taking the Paley graph Paley(9) and adding an isolated vertex (see Figure 3).

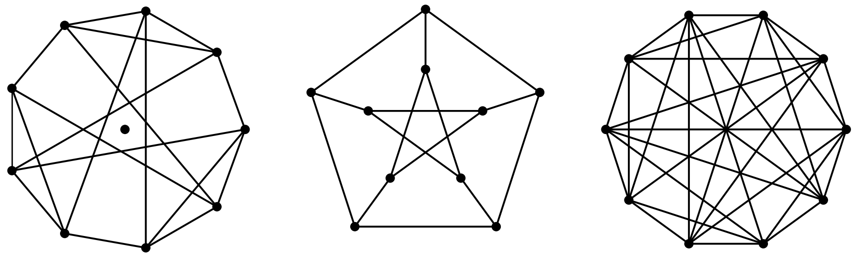

Fig. 3. Three switching equivalent graphs that give 10 lines in $\mathbb{R}^5$: The Paley graph on 9 vertices and a point, the Petersen graph, and the triangular graph T_5.

Example 6. For some parameters there exist many strongly regular graphs. For example, there are 3854 strongly regular graphs on 35 points which give an equal–norm tight frame of 35 vectors for $\mathbb{R}^{15}$ (or $\mathbb{R}^{21}$) with *two angles* and there are 32548 strongly regular graphs which give *equiangular* tight frames of 36 vectors for $\mathbb{R}^{15}$ (or $\mathbb{R}^{21}$).

The 6–cycle is a regular, but *not* strongly regular graph. Its Seidel matrix has three eigenvalues, and gives an equiangular frame of six vectors for $\mathbb{R}^4$. The dual and canonical tight frames have equal norms (and more than two angles). This is a consequence of the 6–cycle being a *circulant graph*.

Let C be a subset of $\mathbb{Z}_n$ which is closed under taking additive inverses, i.e, $-c \in C$, $\forall c \in C$. Then the **circulant graph** G with **connection set** C is the graph with vertices $\mathbb{Z}_n$ and an edge from j to k if $j - k \in C$. The choice $C = \{-1, 1\}$ gives the n–cycle.

Theorem 6.7. *Let G be a circulant graph, and Φ be the real equiangular frame that it determines. Then the dual frame $\tilde{\Phi}$ and canonical tight frame Φ^{can} are equal–norm frames.*

Proof: Since G is a circulant graph, the Gramian of the equiangular frame it determines is a circulant matrix, and hence is diagonalised by the Fourier matrix F, i.e.,

$$F^{-1}(I + \frac{1}{\lambda}\Sigma)F = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \qquad F := \frac{1}{\sqrt{n}}[\omega^{jk}]_{j,k\in\mathbb{Z}_n}, \ \omega := e^{\frac{2\pi i}{n}}.$$

Since F is unitary, we can write this spectral decomposition as

$$I + \frac{1}{\lambda}\Sigma = \sum_j \lambda_j P_j, \qquad P_j := f_j f_j^*, \quad f_j := \frac{1}{\sqrt{n}}(\omega^{jk})_{k\in\mathbb{Z}_n}.$$

The rank one projection matrices P_j have constant diagonal entries (equal to $\frac{1}{n}$), and so (as in the proof of Theorem 6.2) the dual and canoncial tight frames have equal–norms. $\square$

Example 7. A simple calculation shows that the Seidel matrix of the n–cycle has a minimal eigenvalue $-1 - 4\cos\frac{\pi}{n}$ of multiplicity 2 with corresponding eigenvectors f_{-1}, f_1. The corresponding equiangular frame is of n vectors for $\mathbb{R}^{n-2}$, and its complement is the tight frame of n equally spaced vectors for $\mathbb{R}^2$ (up to similarity).

Acknowledgement

The author would like to thank a number of colleagues for helpful email discussions: Gordon Royle and Andries Brouwer on the strongly regular graphs, and Vern Paulsen on the equivalence of tight frames.

References

[BK06] J. J. Benedetto and J. D. Kolesar, Geometric Properties of Grassmannian Frames for $\mathbb{R}^2$ and $\mathbb{R}^3$, *EURASIP J. Appl. Signal Process.* **2006** (2006), 1–17.

[BP05] B. G. Bodmann and V. I. Paulsen, Frames, graphs and erasures, *Linear Algebra Appl.* **404** (2005), 118–146.

[BPT08] B. G. Bodmann, V. I. Paulsen, and M. Tomforde, Equiangular tight frames from complex Seidel matrices containing cube roots of unity, Preprint (University of Houston), 2008.

[B07] A. E. Brouwer, Strongly regular graphs, *in* "Handbook of Combinatorial designs" (Colbourn, C. J., Dinitz, J. H. Eds.), pp. 852–868, Chapman and Hall, Boca Raton, 2007.

[CRT08] P.G. Casazza, D. Redmond, and J.C. Tremain, Real equiangular frames, *Information Sciences and Systems (42nd Annual Conference on)* (2008), 715–720.

[C03] O. Christensen, "An introduction to frames and Riesz bases", Birkhäuser, Boston, 2003.

[GR01] C. Godsil and G. Royle, "Algebraic graph theory", Interscience, Netherlands, 1960. Springer-Verlag (New York); 2001;

[GKK01] V. K. Goyal, J. Kovačević, and J. A. Kelner, Quantized Frame Expansions with Erasures, *Appl. Comput. Harmonic Anal.* **10** (2001), 203–233.

[HP04] R. B. Holmes and V. I. Paulsen, Optimal frames for erasures, *Linear Algebra Appl.* **377** (2004), 31–51.

[K06] D. Kalra, Complex equiangular cyclic frames and erasures, *Linear Algebra Appl.* **419** (2006), 373–399.

[KR04] A. Klappenecker and M. Rötteler, Constructions of mutually unbiased bases, *Lecture Notes in Comput. Sci* **2948** (2004), 137–144.

[R05] J. M. Renes, Equiangular spherical codes in quantum cryptography, *Quantum Inf. Comput.* **5** (2005), 81–92.

[RBSC04] J. M. Renes, R. Blume–Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, *J. Math. Phys.* **45** (2004), 2171–2180.

[S91] J. J. Seidel, "Geometry and combinatorics: Selected works of J. J. Seidel", Academic Press, Boston, 1991.

[STDH07] M. A. Sustik, J. A. Tropp, I. S. Dhillon, and R. W.Heath, On the existence of equiangular tight frames, *Linear Algebra Appl.* **426** (2007), 619–635.

[SH03] T. Strohmer and R. W. Heath, Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmonic Anal.* **14** (2003), 257–275.

[W09] S. Waldron, An Introduction to Finite Tight frames, Birkhäuser (Boston), 2009.