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The eigenstructure of the Bernstein operator

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ABSTRACT

The Bernstein operator B_n reproduces the linear polynomials, which are therefore eigenfunctions corresponding to the eigenvalue 1. We determine the rest of the eigenstructure of B_n. Its eigenvalues are

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!}\frac{1}{n^k}, \qquad k = 0, 1, \ldots, n,$$

and the corresponding monic eigenfunctions $p_k^{(n)}$ are polynomials of degree k, which have k simple zeros in $[0,1]$. By using an explicit formula, it is shown that $p_k^{(n)}$ converges as $n \to \infty$ to a polynomial related to a Jacobi polynomial. Similarly, the dual functionals to $p_k^{(n)}$ converge as $n \to \infty$ to measures that we identify. This diagonal form of the Bernstein operator and its limit, the identity (Weierstrass density theorem), is applied to a number of questions. These include the convergence of iterates of the Bernstein operator, and why Lagrange interpolation (at $n+1$ equally spaced points) fails to converge for all continuous functions whilst the Bernstein approximants do. We also give the eigenstructure of the Kantorovich operator. Previously, the only member of the Bernstein family for which the eigenfunctions were known explicitly was the Bernstein–Durrmeyer operator, which is self adjoint.

Key Words: (multivariate) Bernstein operator, diagonalisation, eigenvalues, eigenfunctions, total positivity, Stirling numbers, Jacobi polynomials, semigroup, quasi–interpolant

AMS (MOS) Subject Classifications: primary 41A10, 15A18, 38B42, secondary 33C45, 41A36

1. Introduction

It is well known that the Bernstein operator $B_n : C[0,1] \to C[0,1]$, $n = 1, 2, \ldots$, defined by

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big), \tag{1.1}$$

reproduces the linear polynomials, which are therefore eigenfunctions corresponding to the eigenvalue 1. There are a number of other forms for $B_n f$ from which the remaining eigenstructure of B_n is more apparent. The simplest of these to deal with is the expansion in terms of the monomials (see, e.g., Widder [Wi41:p.155])

$$B_n f(x) = \sum_{j=0}^{n} \binom{n}{j} x^j \Delta_{1/n}^j f(0). \tag{1.2}$$

Here Δ_h^j is the j–th order forward difference operator

$$\Delta_h^j f(x) := \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} f(x + ih),$$

which annihilates polynomials of degree $< j$. Let e_k be the monomial $x \mapsto x^k$. It follows from (1.2) that B_n maps polynomials of degree $k = 0, 1, \ldots, n$ to polynomials of degree k (is degree reducing), and so has an eigenfunction of degree k corresponding to the eigenvalue

$$\lambda_k^{(n)} = \binom{n}{k} \Delta_{1/n}^k e_k(0) = \frac{n!}{(n-k)!} \frac{1}{n^k}, \qquad k = 0, 1, \ldots, n. \tag{1.3}$$

This observation can be found in Berens and DeVore [BD80]. Let $p_k^{(n)}$ denote the corresponding monic eigenfunction of degree k, and take

$$p_0^{(n)}(x) := 1, \qquad p_1^{(n)}(x) := x - 1/2. \tag{1.4}$$

The paper is set out as follows.

In Section 2, we give an explicit description of this diagonal form of the Bernstein operator. This includes a formula for the eigenfunctions $p_k^{(n)}$ in terms of the monomial basis, and a description of the dual functionals to them, together with some symmetry properties. Previously, the only member of the Bernstein family for which the eigenfunctions were known explicitly was the Bernstein–Durrmeyer operator, which is self adjoint.

In Section 3, we use the theory of totally positive matrices to show that $p_k^{(n)}$ has k distinct real roots in the interval $[0,1]$, and to describe their location. It is observed numerically that the zeros of successive eigenfunctions of B_n interlace, but a proof of this fact using the oscillatory properties of the Bernstein kernel has yet to be given.

In Section 4, we show that $p_k^{(n)}$ converges as $n \to \infty$ to a polynomial related to a Jacobi polynomial. Limits of the dual functionals are also obtained. These results are compared with the eigenstructure of the Bernstein–Durrmeyer operator.

Sections 5, 6 and 7 contain applications of the previous sections. Simple and illuminating proofs of results for iterated Boolean sums of B_n, limits of iterates of B_n, and representations of an associated C_0–semigroup are given. The eigenstructure is used to compare the approximation properties of B_n with L_n, the operator of Lagrange interpolation at the same $n+1$ equally spaced points. This allows the possibility of defining a family Bernstein quasi–interpolants which vary from B_n to L_n. The eigenstructure of the Kantorovich operator is deduced from the eigenstructure of the Bernstein operator.

We conclude with some comments about the multivariate Bernstein operator. To simplify the presentation, a number of examples, including an alternative method for computing the dual functionals, are arranged in the appendix.

2. The diagonalisation and description of the eigenfunctions

Since B_n is degree reducing, writing the eigenfunction equation

$$B_n p_k^{(n)} = \lambda_k^{(n)} p_k^{(n)} \tag{2.1}$$

relative to a basis $\{b_0, b_1, \cdots, b_n\}$ of Π_n, with the degree of b_j equal to j, leads to an upper triangular system. We now solve this system when the b_j are the monomials e_j.

The shifted factorial function is defined by

$$(x)_j := x(x+1)\cdots(x+j-1), \quad j = 1, 2, \ldots, \qquad (x)_0 := 1,$$

and the Stirling numbers of the second kind $S(k,j)$ are defined by

$$x^k = \sum_{j=0}^{k} S(k,j)\, x(x-1)\cdots(x-j+1).$$

Note the well known identity

$$S(k,j) = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k, \quad 0 \le j \le k. \tag{2.2}$$

Theorem 2.3 (Diagonalisation). *The Bernstein operator B_n can be represented in the diagonal form*

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1], \tag{2.4}$$

with $\lambda_k^{(n)}$ and $p_k^{(n)}$ its eigenvalues and eigenfunctions, and $\mu_k^{(n)}$ the dual functionals to $p_k^{(n)}$.

The eigenvalues are given by

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!}\frac{1}{n^k} = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right), \qquad k=0,\ldots,n, \tag{2.5}$$

and they satisfy

$$1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \lambda_3^{(n)} > \cdots > \lambda_n^{(n)} > 0.$$

The eigenfunction for $\lambda_k^{(n)}$ *is a polynomial of degree* k *given by*

$$p_k^{(n)}(x) = \sum_{j=0}^{k} c(j,k,n)\,x^j = x^k - \frac{k}{2}x^{k-1} + \textit{lower order terms}, \tag{2.6}$$

where the coefficients can be computed using the recurrence formula

$$\begin{aligned} &c(k,k,n) := 1, \quad c(k-1,k,n) := -k/2, \\ &c(k-j,k,n) := \frac{1}{(n-k+1)_j - n^j}\sum_{i=0}^{j-1} n^i\, S(k-i,k-j)\, c(k-i,k,n), \quad j=2,\ldots,k. \end{aligned} \tag{2.7}$$

The dual functional $\mu_k^{(n)} \in \operatorname{span}\{f \mapsto f(j/n) : j=0,1,\ldots,n\}$, *defined on* $C[0,1]$, *satisfies*

$$\mu_k^{(n)}(p_i^{(n)}) = \delta_{ik}, \qquad \forall i,k, \tag{2.8}$$

and is given by

$$\mu_k^{(n)}(f) = \sum_{j=0}^{n} v(j,k,n) f\left(\frac{j}{n}\right), \qquad k=0,\ldots,n, \tag{2.9}$$

where the $(n+1)\times(n+1)$ *matrix of coefficients* $V := [v(j,k,n)]_{j,k=0}^n$ *is the inverse of*

$$P := [p_i^{(n)}(j/n)]_{i,j=0}^n.$$

The eigenfunctions and dual functionals have the symmetries

$$p_k^{(n)}(x) = (-1)^k p_k^{(n)}(1-x), \qquad \mu_k^{(n)}(f) = (-1)^k \mu_k^{(n)}(f\circ R), \tag{2.10}$$

where $R : x \mapsto 1-x$ *is reflection about the point* $1/2$. *The eigenfunctions of degree* ≥ 2 *can be factored as follows*

$$\begin{aligned} &p_{2j}^{(n)}(x) = x(x-1)q(x-1/2), \\ &p_{2j+1}^{(n)}(x) = x(x-1/2)(x-1)q(x-1/2), \qquad j=1,2,\ldots, \end{aligned} \tag{2.11}$$

where in each case q *is an even monic polynomial.*

Proof: We have already seen that the eigenvalues of B_n are given by (2.5), and the linear polynomials are eigenfunctions for eigenvalue $\lambda_0^{(n)} = \lambda_1^{(n)} = 1$, for which the

$p_0^{(n)}, p_1^{(n)}$ of (1.4) are clearly a basis which satisfies (2.6) and (2.7). It remains only to consider the 1–dimensional $\lambda_k^{(n)}$–eigenspace of polynomials of exact degree $k = 2, 3, \ldots, n$.

By (1.2) and (2.2),

$$B_n e_k(x) = \sum_{j=0}^{k} a(j,k,n)\, x^j, \tag{2.12}$$

where

$$a(j,k,n) = \binom{n}{j} \Delta_{1/n}^{j} e_k(0) = \frac{S(k,j)n!}{n^k (n-j)!}, \qquad 0 \le j \le k \le n. \tag{2.13}$$

Note that

$$a(k,k,n) = \lambda_k^{(n)}, \qquad 0 \le k \le n. \tag{2.14}$$

Express the eigenfunctions in the form

$$p_k^{(n)}(x) = \sum_{r=0}^{k} c(r,k,n)\, x^r, \qquad c(k,k,n) := 1.$$

Then the eigenfunction equation (2.1) gives

$$\begin{aligned} \lambda_k^{(n)} \sum_{s=0}^{k} c(s,k,n) x^s &= \sum_{r=0}^{k} c(r,k,n) \sum_{s=0}^{r} a(s,r,n) x^s \\ &= \sum_{s=0}^{k} \sum_{r=s}^{k} c(r,k,n) a(s,r,n) x^s. \end{aligned}$$

Equating the coefficients of x^s above gives

$$\lambda_k^{(n)} c(s,k,n) = \sum_{r=s}^{k} c(r,k,n) a(s,r,n).$$

Into this substitute $s = k - j$ and $r = k - i$, to obtain

$$\lambda_k^{(n)} c(k-j,k,n) = \sum_{i=0}^{j} c(k-i,k,n) a(k-j,k-i,n),$$

which, for $k > 1$, can be solved for $c(k-j,k,n)$ to give

$$c(k-j,k,n) = \frac{1}{\lambda_k^{(n)} - a(k-j,k-j,n)} \sum_{i=0}^{j-1} c(k-i,k,n)\, a(k-j,k-i,n), \qquad j = 1, \ldots, k.$$

Equation (2.7) now follows from this using (2.5) and (2.13). Taking $j = 1$ in (2.7), gives

$$c(k-1,k,n) = \frac{S(k,k-1)}{-k+1} = -\frac{k}{2}, \tag{2.15}$$

which is (2.6). Using (2.9), the biorthogonality condition $\mu_k(p_i) = \delta_{ik}$ can be written as

$$\sum_{j=0}^{n} p_i^{(n)}\left(\frac{j}{n}\right) v(j,k,n) = \delta_{ik},$$

i.e., $PV = I$, and so $V = P^{-1}$.

Let R be $x \mapsto 1-x$, i.e., reflection about the point $1/2$. From (1.1), it follows that

$$B_n(f \circ R) = (B_n f) \circ R, \tag{2.16}$$

so that

$$B_n(p_k^{(n)} \circ R) = (B_n p_k^{(n)}) \circ R = \lambda_k^{(n)} (p_k^{(n)} \circ R),$$

and $p_k^{(n)} \circ R$ is a $\lambda_k^{(n)}$–eigenfunction. For $k = 0, 1$ the symmetry property of $p_k^{(n)}$ is obvious, and for $k \geq 2$ the eigenfunction $p_k^{(n)} \circ R$ must be a scalar multiple of $p_k^{(n)}$ (the eigenspace is 1–dimensional), which by equating powers of x^k must be

$$p_k^{(n)} = (-1)^k p_k^{(n)} \circ R. \tag{2.17}$$

In other words, $p_k^{(n)}$ is even (resp. odd) about the point $1/2$ when k is even (resp. odd). In particular, the zeros of $p_k^{(n)}$ are symmetric about $1/2$. Similarly, (2.16) implies that

$$\sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f \circ R) = \sum_{k=0}^{n} \lambda_k^{(n)} (p_k^{(n)} \circ R) \mu_k^{(n)}(f) = \sum_{k=0}^{n} \lambda_k^{(n)} (-1)^k p_k^{(n)} \mu_k^{(n)}(f),$$

and equating coefficients of $p_k^{(n)}$ in the above gives

$$\mu_k^{(n)}(f) = (-1)^k \mu_k^{(n)}(f \circ R). \tag{2.18}$$

Taking $j = k$ in (2.7) and using $S(m, 0) = 0$, $m \geq 1$, gives

$$c(0, k, n) = 0, \quad k \geq 2.$$

Hence, for $k \geq 2$, $x = 0$ is a zero of $p_k^{(n)}$, and by the symmetry property (2.17) so is $x = 1$. Further, when k is odd the symmetry property of the zeros implies that $x = 1/2$ must be a zero of $p_k^{(n)}$, which proves the factorisations (2.11). This completes the proof. $\square$

A list of the first few eigenfunctions and their dual functionals, and an explicit formula for $V = P^{-1}$ is given in the appendix.

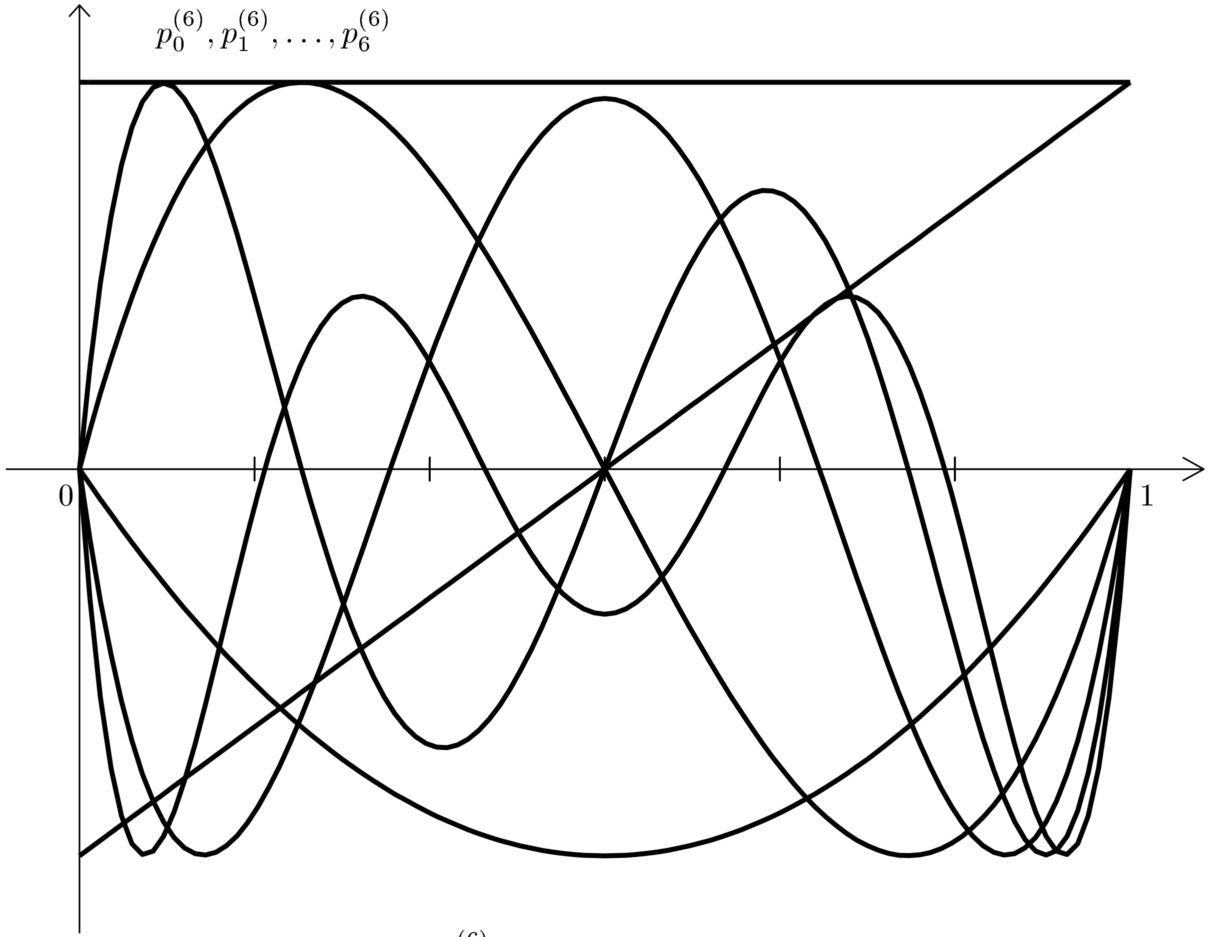

Fig. 2.1. The eigenfunctions $p_k^{(6)}$, k=0,...,6, for the Bernstein operator B_6 (scaled to have the same absolute maxima). Note the symmetries, and interlacing of the zeros.

3. Zeros of the eigenfunctions

Next we determine the distribution of roots the eigenfunctions by using the theory of oscillating kernels (total positivity). The kernel defining the Bernstein operator

$$K_n(k,x) := \binom{n}{k} x^k (1-x)^{n-k} \tag{3.1}$$

is extended totally positive $\mathrm{ETP}(x)$ in $k = 0, 1, \ldots, n$ and $0 < x < 1$ (see Karlin [K68:p.298]). The current theory of totally positive kernels (see the survey of Pinkus [P96]) cannot be applied directly, since this kernel is discrete in the first variable and continuous in the second (a case not yet considered), and it is not totally positive if the values $x = 0, 1$ are allowed. We circumvent these difficulties by considering the truncated Bernstein operator $B_n^\times$ defined by

$$B_n^\times f(x) := \sum_{k=1}^{n-1} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big), \qquad n = 2, 3, \ldots, \tag{3.2}$$

as a matrix operator.

Theorem 3.3 (Zeros of the eigenfunctions). *The eigenfunction* $p_k^{(n)}$, $k = 0, 1, \ldots, n$ *has* k *simple real roots contained in* $[0, 1]$, *which we denote by*

$$\xi_{1,k}^{(n)} < \xi_{2,k}^{(n)} < \cdots < \xi_{k,k}^{(n)}.$$

More generally, for any nontrivial $(a_{m_1}, \ldots, a_{m_2})$, $2 \le m_1 \le m_2 \le n$,

$$m_1 \le Z\left(\sum_{k=m_1}^{m_2} a_k\, p_k^{(n)}\right) \le m_2,$$

where Z *counts the number of zeros in* $[0, 1]$. *The zeros are symmetric about* $1/2$, *i.e.,*

$$\xi_{i,k}^{(n)} + \xi_{k+1-i,k}^{(n)} = 1, \qquad \forall i. \tag{3.4}$$

For $k \ge 2$, *there are common roots of* 0 *and* 1, *i.e.,*

$$\xi_{1,k}^{(n)} = 0, \quad \xi_{k,k}^{(n)} = 1, \qquad k \ge 2, \tag{3.5}$$

and the roots inside $(0, 1)$ *satisfy the inclusions*

$$\frac{i-1}{n} < \xi_{i,k}^{(n)} < 1 - \frac{(k-i)}{n}, \quad 2 \le i \le k-1. \tag{3.6}$$

Proof: The result is clearly true for $k = 0, 1$, and (3.4), (3.5) follow from (2.11). We now consider the case $k \ge 2$. Since $p_k^{(n)}$ vanishes at 0 and 1,

$$B_n^\times p_k^{(n)} = B_n p_k^{(n)} = \lambda_k^{(n)} p_k^{(n)}, \qquad k = 2, 3, \ldots, n,$$

and so the eigenvalues of $B_n^\times$ are $\lambda_k^{(n)}$, $k = 2, \ldots, n$, with $p_k^{(n)}$ a basis for the corresponding 1–dimensional eigenspace. Consider the matrix representation of $B_n^\times$

$$A = [a_{ij}] : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} : \big(f(i/n)\big)_{i=1}^{n-1} \mapsto \big(B_n^\times f(i/n)\big)_{i=1}^{n-1} \tag{3.7}$$

which is given by

$$a_{ij} := \binom{n}{j} (i/n)^j (1 - i/n)^{n-j}.$$

A is an oscillation matrix, i.e., is totally positive and invertible, with $a_{i,i+1}, a_{i+1,i} > 0$ (see [P96:Prop.5.1]). Hence by the (Perron–Frobenius) spectral theory of such matrices (see [P96:Th.5.2]) it follows that its eigenvectors $\left(p_k^{(n)}(i/n)\right)_{i=1}^{n-1}$, $k = 2, 3, \ldots, n$ satisfy

$$r - 2 \le S^- \left(\sum_{k=r}^{s} c_k \, p_k^{(n)} \right) \le S^+ \left(\sum_{k=r}^{s} c_k \, p_k^{(n)} \right) \le s - 2, \qquad 2 \le r \le s \le n \quad (\text{some } c_k \neq 0),$$

where $S^-(f)$, $S^+(f)$ count the number of sign changes in the sequence

$$f(1/n), f(2/n), \ldots, f((n-1)/n)$$

with zero terms discarded (respectively assigned arbitrary values ± 1). In particular, the eigenvector $\left(p_k^{(n)}(i/n)\right)_{i=1}^{n-1}$ has $k-2$ strong sign changes, and so in addition to $0, 1$, the eigenfunction $p_k^{(n)}$ has $k-2$ real roots inside $(0,1)$. Clearly, $\xi_{2,k}^{(n)} > \frac{1}{n}$, $\xi_{2,k}^{(n)} > \frac{2}{n}$, …, i.e.,

$$\xi_{i,k}^{(n)} > \frac{i-1}{n}, \qquad 2 \le i \le k-1,$$

and similarly (or by symmetry) we obtain the other half of (3.6). □

Remark. When k is large with respect to n, (3.6) implies the roots of $p_k^{(n)}$ are nearly evenly spaced. For example, when $k = n$ we have $\frac{i-1}{n} < \xi_{i,n}^{(n)} < \frac{i}{n}$, for $2 \le i \le n-1$.

Interlacing of zeros

Numerical evidence suggests that the zeros of the eigenfunctions interlace, that is,

$$\xi_{j,k+1}^{(n)} < \xi_{j,k}^{(n)} < \xi_{j+1,k+1}^{(n)}, \quad 1 < j < k. \tag{3.8}$$

The classical theorems for interlacing of eigenvectors and eigenfunctions of totally positive matrices and kernels can not be applied here, where the kernel (3.1) is discrete in the first variable and continuous in the second (a case not yet considered), and is not totally positive if the values $x = 0, 1$ are allowed. By Ando [A87:Th.6.3], it follows that the nodes of consecutive eigenvectors $\left(p_k^{(n)}(i/n)\right)_{i=1}^{n-1}$ are interlacing, in the sense that zeros of the piecewise interpolants indicated in Fig. 3.1 are. Unfortunately, this is not enough to conclude that the roots of consecutive $p_k^{(n)}$ are interlacing. This is the subject of further investigation.

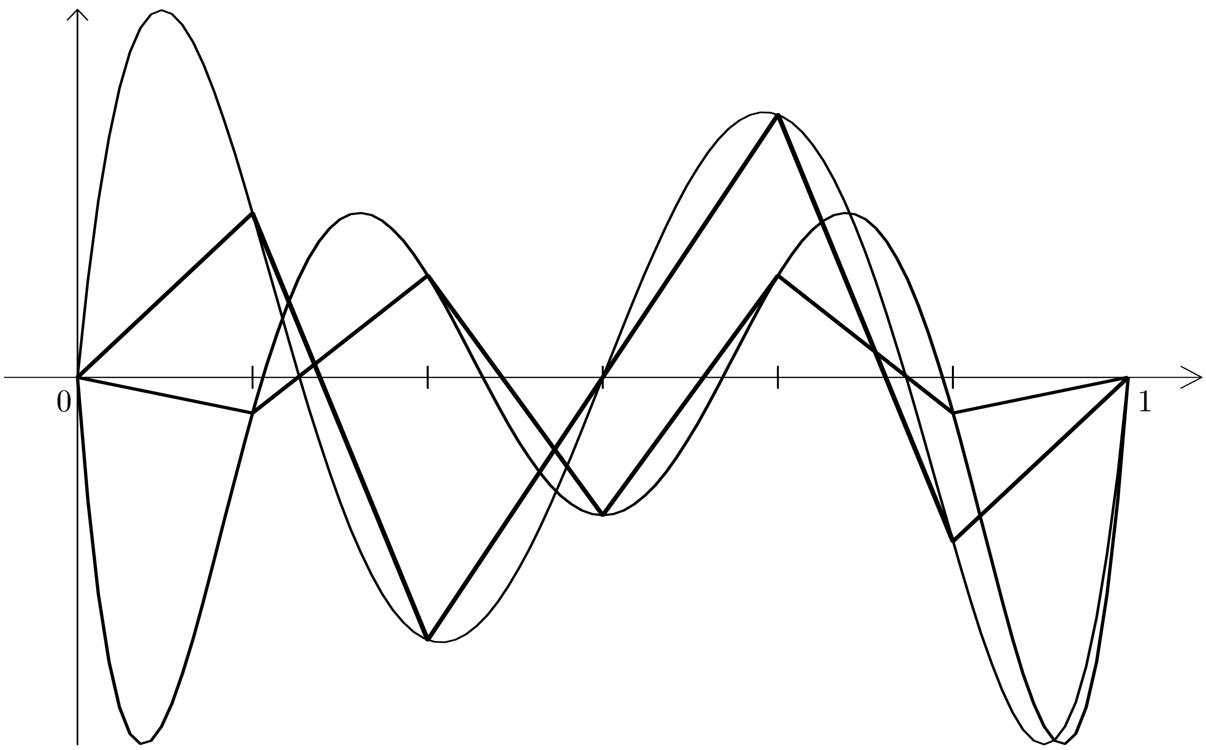

Fig. 3.1. The interlacing of the zeros of the piecewise linear interpolants to $p_5^{(6)}, p_6^{(6)}$.

4. Asymptotics of the eigenfunctions and their dual functionals

The limiting eigenfunctions

We now show that the sequence of eigenfunctions $p_k^{(n)}$ converges as $n \to \infty$.

Theorem 4.1 (Limits of the eigenfunctions). *For $0 \leq j \leq k$,*

$$\lim_{n\to\infty} c(j,k,n) = c^*(j,k), \tag{4.2}$$

where

$$c^*(0,1) := -\frac{1}{2}, \qquad c^*(j,k) := \prod_{i=1}^{k-j} \frac{(k+1-i)(k-i)}{i(i-2k+1)} \quad j \neq 0, k \neq 1. \tag{4.3}$$

In other words, $p_k^{(n)}$ converges uniformly on $[0,1]$ to $p_k^* \in \Pi_k$ as $n \to \infty$, where

$$p_k^*(x) := \sum_{j=0}^{k} c^*(j,k)\, x^j = x^k - \frac{k}{2}x^{k-1} + \frac{k(k-1)(k-2)}{4(2k-3)}x^{k-2} - \cdots. \tag{4.4}$$

Proof: Since $p_0^{(n)}(x) = 1 = p_0^*(x)$, $p_1^{(n)}(x) = x - 1/2 = p_1^*(x)$, it is sufficient to prove the result for $k \geq 2$. To show this we prove by strong induction on j and $k \geq 2$ that

$\lim_{n\to\infty} c(k-j,k,n)$ exists and is given by (4.3). Since $c(k,k,n)=1$, this result holds for $j=0$ and all values of k (where as usual the empty product in (4.3) is interpreted as 1). Suppose it is true for $\lim_{n\to\infty} c(k-i,k,n)$, $i=0,\ldots,j-1$, where $0<j\le k$. Since

$$(n-k+1)_j - n^j = \frac{1}{2} j(j-2k+1)\, n^{j-1} + \text{lower order powers of } n, \qquad j>0,$$

taking the limit as $n\to\infty$ of both sides of

$$c(k-j,k,n) = \sum_{i=0}^{j-1} \frac{n^i\, S(k-i,k-j)\, c(k-i,k,n)}{(n-k+1)_j - n^j},$$

and using the induction hypothesis gives

$$\begin{aligned}
\lim_{n\to\infty} c(k-j,k,n) &= \frac{2S(k-j+1,k-j)}{j(j-2k+1)} c^*(k-j+1,k)\\
&= \frac{(k+1-j)(k-j)}{j(j-2k+1)} \prod_{i=1}^{j-1} \frac{(k+1-i)(k-i)}{i(i-2k+1)}\\
&= \prod_{i=1}^{j} \frac{(k+1-i)(k-i)}{i(i-2k+1)},
\end{aligned}$$

which completes the induction. □

We now show the p_k^* are closely related to the Jacobi polynomials $P_k^{(\alpha,\beta)}$. These are by definition the orthogonal polynomials with respect to the weight $(1-t)^\alpha(1+t)^\beta$ on the interval $t\in[-1,1]$, see, e.g., [E53:vol.2,p.168–173].

Theorem 4.5 (Identification of the p_k^*). *It is immediate that*

$$p_0^*(x) = 1, \qquad p_1^*(x) = x - 1/2. \tag{4.6}$$

Moreover,

$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!} x(x-1) P_{k-2}^{(1,1)}(2x-1), \qquad k\ge 2. \tag{4.7}$$

Proof: Suppose that $k\ge 2$. Then

$$c^*(j,k) = \prod_{i=1}^{k-j} \frac{(k+1-i)(k-i)}{i(i-2k+1)} = \frac{(-k)_{k-j}(1-k)_{k-j}}{(k-j)!(2-2k)_{k-j}},$$

and so, since $c^*(0,k)=0$,

$$\begin{aligned}
p_k^*(x) &= \sum_{j=0}^{k} c^*(j,k) x^j = \sum_{j=1}^{k} \frac{(-k)_{k-j}(1-k)_{k-j}}{(k-j)!(2-2k)_{k-j}} x^j\\
&= x \sum_{j=0}^{k-1} \frac{(-k)_{k-1-j}(1-k)_{k-1-j}}{(k-1-j)!(2-2k)_{k-1-j}} x^j.
\end{aligned}$$

Next use

$$(a)_{n-j} = \frac{(a)_n(-1)^j}{(1-a-n)_j}$$

with $n = k-1$ to obtain

$$\begin{aligned} p_k^*(x) &= x\frac{(-k)_{k-1}(1-k)_{k-1}}{(k-1)!(2-2k)_{k-1}} \sum_{j=0}^{k-1} \frac{(1-k)_j(k)_j}{(2)_j j!} x^j \\ &= x(-1)^{k-1}\frac{k!(k-1)!}{(2k-2)!}\, {}_2F_1(1-k,k;2;x). \end{aligned}$$

Apply Euler's transformation [E53:vol.1,p.64(23)], to get

$$\begin{aligned} p_k^*(x) &= x(x-1)(-1)^k\frac{k!(k-1)!}{(2k-2)!}\, {}_2F_1(2-k,k+1;2;x) \\ &= x(x-1)\frac{k!(k-2)!}{(2k-2)!} P_{k-2}^{(1,1)}(2x-1). \end{aligned} \tag{4.8}$$

This proves the result. □

It is interesting to compare this result with the spectral properties of the Bernstein–Durrmeyer operator, which is a self adjoint operator on $L_2[0,1]$, defined by

$$M_n f(x) := \sum_{k=0}^n p_{k,n}(x)\,(n+1)\int_0^1 f(t)p_{k,n}(t)\,dt, \qquad 0 \le x \le 1,$$

where

$$p_{k,n}(x) := \binom{n}{k} x^k(1-x)^{n-k}, \qquad 0 \le k \le n.$$

Derriennic [D81] showed that the eigenvalues of M_n are

$$\lambda_{k,n} = \frac{n!}{(n-k)!}\frac{(n+1)!}{(n+k+1)!}, \qquad k = 0,1,\ldots,n,$$

and the corresponding eigenfunctions are the Legendre polynomials

$$P_k(2x-1) := P_k^{(0,0)}(2x-1), \qquad k=0,1,\ldots,n.$$

Notice that these are independent of n. Similarly, in [BX91] it was shown for Bernstein–Durrmeyer operators with Jacobi weights $w^{(\alpha,\beta)}(x) := x^\alpha(1-x)^\beta$, $\alpha,\beta > -1$, the eigenfunctions are the Jacobi polynomials $P_k^{(\alpha,\beta)}(2x-1)$.

The limiting dual functionals

By (1.3) and Theorem 4.5, it follows that

$$\lambda_k^{(n)} \to 1, \quad p_k^{(n)} \to p_k^*, \qquad n \to \infty,$$

whilst the Weierstrass density theorem implies for all $f \in C[0,1]$ that

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f) \to f, \qquad n \to \infty.$$

We now use these facts to investigate the limiting behaviour of $\mu_k^{(n)}(f)$ as $n \to \infty$. Let L denote the operator of linear interpolation at 0 and 1, i.e.,

$$Lf(x) := (1-x)f(0) + xf(1). \tag{4.9}$$

Lemma 4.10 (Orthogonal expansion). *Each $f \in C[0,1]$ satisfying*

$$\int_0^1 (f(x) - Lf(x))^2 \frac{dx}{x(1-x)} < \infty, \tag{4.11}$$

or equivalently

$$\int_0^1 \frac{(f(x)-f(0))^2}{x} dx < \infty, \quad \int_0^1 \frac{(f(x)-f(1))^2}{1-x} dx < \infty, \tag{4.12}$$

can be uniquely represented by a series of the form

$$f = \sum_{k=0}^{\infty} p_k^* \mu_k^*(f), \tag{4.13}$$

where the convergence of $\sum_{k=2}^{\infty} p_k^* \mu_k^*(f)$ above is in the $L_2(dx/x(1-x))$–norm, and the linear functionals μ_k^* are defined by

$$\mu_0^*(f) := (f(0)+f(1))/2, \qquad \mu_1^*(f) := f(1) - f(0), \tag{4.14}$$

$$\mu_k^*(f) := \frac{1}{2}\binom{2k}{k}\left\{(-1)^k f(0) + f(1) - k\int_0^1 f(x) P_{k-2}^{(1,1)}(2x-1)dx\right\}, \qquad k \geq 2. \tag{4.15}$$

If f is differentiable on $[0,1]$, then

$$\mu_k^*(f) = \frac{1}{2}\binom{2k}{k}\int_0^1 f'(x) P_{k-1}(2x-1)dx, \qquad k \geq 2, \tag{4.16}$$

where $\{P_k(x)\}_{k\geq 0}$ are the Legendre polynomials.

Proof: Define inner products by

$$\langle g,h\rangle := \int_0^1 g(x)h(x)\,dx, \qquad \langle\!\langle g,h\rangle\!\rangle := \int_0^1 g(x)h(x)\,x(1-x)dx.$$

Suppose that f satisfies the hypotheses of the lemma, and let

$$g(x) := \frac{f(x) - Lf(x)}{x(1-x)}.$$

Then $g \in L_2\big(x(1-x)dx, [0,1]\big)$, and so has a unique representation

$$g = \sum_{j=0}^{\infty} \langle\!\langle g, g_j \rangle\!\rangle g_j,$$

where

$$g_j(x) := \sqrt{\frac{(j+2)(2j+3)}{j+1}} P_j^{(1,1)}(2x-1), \qquad j = 0, 1, 2, \ldots$$

are orthonormal Jacobi polynomials with respect to the weight function $x(1-x)$ on $[0,1]$. This can be rewritten as

$$\begin{aligned}
\frac{f(x) - Lf(x)}{x(1-x)} &= \sum_{j=0}^{\infty} \langle\!\langle g, g_j \rangle\!\rangle g_j(x) = \sum_{j=0}^{\infty} \langle f - Lf, g_j \rangle g_j(x) \\
&= \sum_{j=0}^{\infty} \frac{(j+2)(2j+3)}{j+1} \langle f - Lf, P_j^{(1,1)}(2\cdot -1) \rangle P_j^{(1,1)}(2x-1) \\
&= \frac{1}{x(x-1)} \sum_{j=0}^{\infty} \frac{j+2}{2} \binom{2j+4}{j+2} \langle f - Lf, P_j^{(1,1)}(2\cdot -1) \rangle p_{j+2}^*(x),
\end{aligned}$$

which gives

$$f(x) = Lf(x) - \sum_{k=2}^{\infty} \frac{k}{2} \binom{2k}{k} \langle f - Lf, P_{k-2}^{(1,1)}(2\cdot -1) \rangle p_k^*(x). \tag{4.17}$$

Since

$$Lf(x) = \frac{f(0) + f(1)}{2} + \big(f(1) - f(0)\big)(x - 1/2) = \mu_0^*(f)\, p_0^*(x) + \mu_1^*(f)\, p_1^*(x),$$

and

$$\begin{aligned}
\langle Lf, P_{k-2}^{(1,1)}(2\cdot -1) \rangle &= f(0) \int_0^1 (1-x) P_{(k-2)}^{(1,1)}(2x-1)dx + f(1) \int_0^1 x P_{k-2}^{(1,1)}(2x-1)dx \\
&= f(0) \frac{(-1)^k}{k} + \frac{f(1)}{k},
\end{aligned}$$

we obtain (4.13) from (4.17), with the convergence as asserted. Equation (4.16) follows from (4.15) using

$$P_{k-2}^{(1,1)}(2x-1) = \frac{2}{k} P_{k-1}'(2x-1), \qquad k \geq 2,$$

and integrating by parts. $\square$

Clearly, condition (4.11) is satisfied when f is differentiable at 0 and 1, and so we have

$$f = \sum_{k=0}^{s} p_k^* \mu_k^*(f), \qquad \forall f \in \Pi_s, \tag{4.18}$$

and

$$\mu_k^*(\Pi_{k-1}) = 0, \qquad k = 1, 2, \ldots. \tag{4.19}$$

These facts are now used to investigate the limiting behaviour of the dual functionals.

Theorem 4.20 (Limits of the dual functionals). *For every* $f \in \Pi$,

$$\lim_{n\to\infty} \mu_k^{(n)}(f) = \mu_k^*(f). \tag{4.21}$$

Proof: We prove (4.21) holds for $f \in \Pi_{k+r}$, $r = 0, 1, 2, \ldots$, by strong induction on r (with the result holding for all k). Recall that

$$\lambda_k^{(n)} \to 1, \qquad n \to \infty. \tag{4.22}$$

First suppose $f \in \Pi_k$ ($r = 0$). Because B_n is degree reducing, we have

$$B_n f = \sum_{j=0}^{k} \lambda_j^{(n)} p_j^{(n)} \mu_j^{(n)}(f) \to f = \sum_{j=0}^{k} p_j^* \mu_j^*(f), \qquad n \to \infty. \tag{4.23}$$

Since the convergence in (4.23) takes place in the finite–dimensional space Π_k, we may equate coefficients of x^k to obtain

$$\lambda_k^{(n)} \mu_k^{(n)}(f) \to \mu_k^*(f), \qquad n \to \infty, \tag{4.24}$$

which by (4.22) gives (4.21).

Now suppose $f \in \Pi_{k+r}$. Since B_n is degree reducing, we have

$$B_n f = \sum_{j=0}^{k+r} \lambda_j^{(n)} p_j^{(n)} \mu_j^{(n)}(f) \to f = \sum_{j=0}^{k+r} p_j^* \mu_j^*(f), \qquad n \to \infty.$$

Since the convergence above is in Π_{k+r}, equating coefficients of x^k gives

$$\lambda_k^{(n)} \mu_k^{(n)}(f) + \sum_{j=1}^{r} \lambda_{k+j}^{(n)} c(k, k+j, n) \mu_{k+j}^{(n)}(f) \to \mu_k^*(f) + \sum_{j=1}^{r} c^*(k, k+j) \mu_{k+j}^*(f), \qquad n \to \infty. \tag{4.25}$$

By the inductive hypothesis together with (4.2) and (4.22), we have

$$\sum_{j=1}^{r} \lambda_{k+j}^{(n)} c(k, k+j, n) \mu_{k+j}^{(n)}(f) \to \sum_{j=1}^{r} c^*(k, k+j) \mu_{k+j}^*(f),$$

and so (4.25) gives (4.24) as before. This completes the induction. □

The first few μ_k^* are listed in the appendix.

Remark. We conjecture that (4.21) holds for all $f \in C[0,1]$, which is equivalent to the sequence $(\|\mu_k^{(n)}\|)_{n=0}^\infty$ being bounded.

5. Application to iterates of the Bernstein operator

There have been a number of papers dealing with iterates of the Bernstein operator: Kelisky and Rivlin [KR67], Karlin and Ziegler [KZ70], Micchelli [M73], da Silva [Si85], Gonska and Zhou [GZ94], Sevy [Se95] and Wenz [W97]. We now investigate those results relevant to this work in terms of the diagonal form of B_n. By Theorem 2.3,

$$B_n^j f = \sum_{k=0}^n (\lambda_k^{(n)})^j \, p_k^{(n)} \, \mu_k^{(n)}(f), \qquad \forall f \in C[0,1], \quad j = 0, 1, 2, \ldots. \tag{5.1}$$

Theorem 1 of [KR67] is that

$$\lim_{j \to \infty} B_n^j f = Lf, \qquad \forall f \in C[0,1],$$

where Lf is defined by (4.9). This follows immediately from (5.1) since $\lambda_k^{(n)} < 1$, $k \geq 2$. More generally we have:

Corollary 5.2 (Limits for n fixed). *Suppose (g_j) is a sequence of polynomials, with*

$$\lim_{j \to \infty} g_j(\lambda_k^{(n)}) = G(k,n), \qquad k = 0, 1, \ldots, n,$$

then

$$\lim_{j \to \infty} g_j(B_n) \, f = \sum_{k=0}^n G(k,n) p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1], \tag{5.3}$$

with the convergence above in the uniform norm.

Proof: The appropriate linear combination of (5.1) gives

$$g_j(B_n) f = \sum_{k=0}^n g_j(\lambda_k^{(n)}) \, p_k^{(n)} \, \mu_k^{(n)}(f), \qquad \forall f \in C[0,1]. \tag{5.4}$$

Taking the limit $j \to \infty$ then gives (5.3). □

Let L_n denote the operator of Lagrange interpolation at the equally spaced points $\{0, 1/n, 2/n, \ldots, 1\}$,

$$L_n f(x) := \sum_{k=0}^{n} \Big\{ \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{x - \frac{j}{n}}{\frac{k}{n} - \frac{j}{n}} \Big\} f\Big(\frac{k}{n}\Big).$$

The biorthogonality condition (2.8) implies that $f \mapsto \sum_{k=0}^{n} p_k^{(n)} \mu_k^{(n)}(f)$ reproduces Π_n, and so

$$L_n f = \sum_{k=0}^{n} p_k^{(n)} \mu_k^{(n)}(L_n f) = \sum_{k=0}^{n} p_k^{(n)} \mu_k^{(n)}(f). \tag{5.5}$$

In Sevy [Se95] the particular sequence of polynomials

$$g_j(x) := 1 - (1-x)^j, \qquad j = 1, 2, \ldots$$

was considered. For these

$$\lim_{j\to\infty} g_j(\lambda_k^{(n)}) = 1, \tag{5.6}$$

(since $0 \le 1 - \lambda_k^{(n)} < 1$) so that Corollary 5.2, together with (5.5), gives

$$\lim_{j\to\infty} \big(1 - (1 - B_n)^j\big) f = \sum_{k=0}^{n} p_k^{(n)} \mu_k^{(n)}(f) = L_n f, \qquad \forall f \in C[0,1], \tag{5.7}$$

which is [Se95:Th.1] (also see Wenz [W97:Th.3] for a generalisation to the Bernstein–Schoenberg operator). The above operator has been studied by many, sometimes viewed as an iterated Boolean sum of B_n

$$\oplus^j B_n = g_j(B_n) = 1 - (1 - B_n)^j$$

(see Gonska and Zhou [GZ94] and the references therein). Since

$$1 - \lambda_k^{(n)} = O\big(\frac{1}{n}\big), \qquad n \to \infty,$$

using (5.4) and (5.5) one obtains that $\forall f \in C[0,1]$

$$\|L_n f - \oplus^j B_n f\|_\infty = \|\sum_{k=0}^{n} (1 - \lambda_k^{(n)})^j p_k^{(n)} \mu_k^{(n)}(f)\|_\infty = \|L_n f\|_\infty O\big(\frac{1}{n^j}\big), \qquad n \to \infty. \tag{5.8}$$

From (5.8), and its pointwise analog, it is then possible to obtain approximation order results for $\oplus^j B_n$ (large n) by appropriately modifying those for L_n (cf. [GZ94:Th.1]).

Lemma 5.9 (Limits of powers of the eigenvalues). *Suppose that j_n is a sequence of positive integers with*

$$\lim_{n\to\infty} \frac{j_n}{n} = t, \tag{5.10}$$

then

$$\lim_{n\to\infty} (\lambda_k^{(n)})^{j_n} = e^{-\frac{1}{2}k(k-1)t}, \qquad \forall k, \quad 0 \le t < \infty, \tag{5.11}$$

and

$$\lim_{n\to\infty} (\lambda_k^{(n)})^{j_n} = 0, \qquad \forall k \ge 2, \quad t = \infty. \tag{5.12}$$

Proof: Let

$$y = (\lambda_k^{(n)})^{j_n - nt} = \big((1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})\big)^{j_n-nt}.$$

Then

$$\begin{aligned}\log y &= (j_n - nt)\big(\log(1-\frac{1}{n}) + \log(1-\frac{2}{n}) + \cdots \log(1-\frac{k-1}{n})\big)\\ &= \big(\frac{j_n}{n} - t\big)\big(-\frac{k(k-1)}{2} + O(\frac{1}{n})\big)\\ &\to 0, \qquad (n\to\infty).\end{aligned}$$

Therefore

$$\lim_{n\to\infty} (\lambda_k^{(n)})^{j_n-nt} = \lim_{n\to\infty} y = 1. \tag{5.13}$$

But

$$\lim_{n\to\infty} (\lambda_k^{(n)})^{nt} = \lim_{n\to\infty} (1-\frac{1}{n})^{nt}(1-\frac{2}{n})^{nt}\cdots(1-\frac{k-1}{n})^{nt} = e^{-\frac{1}{2}k(k-1)t}. \tag{5.14}$$

Combining (5.13) and (5.14) gives (5.11). Let $k \to \infty$ to obtain (5.12). □

Theorem 2 of [KR67] shows that if (5.10) holds, then $B_n^{j_n}(e_s)$ converges to a polynomial of degree s which is given explicity. We offer the following extension of this result.

Corollary 5.15 (Limits for $0 \le t \le \infty$). *Suppose that*

$$\lim_{n\to\infty} \frac{j_n}{n} = t,$$

then for $0 \le t < \infty$

$$\begin{aligned}\lim_{n\to\infty} B_n^{j_n} f &= \sum_{k=0}^{s} e^{-\frac{1}{2}k(k-1)t} p_k^* \mu_k^*(f)\\ &= \sum_{k=0}^{\infty} e^{-\frac{1}{2}k(k-1)t} p_k^* \mu_k^*(f), \qquad \forall f \in \Pi_s,\end{aligned} \tag{5.16}$$

and for $t = \infty$

$$\lim_{n\to\infty} B_n^{j_n} f = Lf = \sum_{k=0}^{1} p_k^* \mu_k^*(f) \qquad \forall f \in \Pi, \tag{5.17}$$

with the convergence in (5.16) *and* (5.17) *being in the uniform norm.*

Proof: Suppose that $f \in \Pi_s$. Since B_n is degree reducing (5.1) gives

$$B_n^{j_n} f = \sum_{k=0}^{s} (\lambda_k^{(n)})^{j_n} p_k^{(n)} \mu_k^{(n)}(f), \qquad n \geq s.$$

Take the limit as $n \to \infty$ in the above, and use Lemma 5.9 , Theorem 4.1 and Theorem 4.20 to obtain (5.17) and the first equality in (5.16). The second equality in (5.16) follows from (4.19).
□

Since the operators $B_n^{j_n} : C[0,1] \to C[0,1]$ have norm 1 and the polynomials are dense in $C[0,1]$, the limits in (5.16) and (5.17) exist for all $f \in C[0,1]$. It has been shown that there exists a semigroup $\{\mathcal{B}_t : t \geq 0\}$ of class (C_0) on $C[0,1]$ such that

$$\lim_{n\to\infty} B_n^{j_n} f = \mathcal{B}_t f, \qquad \text{whenever} \quad \lim_{n\to\infty} \frac{j_n}{n} = t, \tag{5.18}$$

for all $t \geq 0$ and $f \in C[0,1]$, and $\{\mathcal{B}_t : t \geq 0\}$ is a positive contraction semigroup (see Karlin and Ziegler, [KZ70] and Micchelli [M73:Th.3.1]). Corollary 5.15 gives an explicit representation of this semigroup on the polynomials. This representation can be extended trivially to $C[0,1]$ in the case (5.17) and in the case (5.16) as follows.

Corollary 5.19 (Representation of $\{\mathcal{B}_t\}$). *The semigroup* $\{\mathcal{B}_t : t \geq 0\}$ *defined by* (5.18) *has the representation*

$$\mathcal{B}_t f(x) = Lf(x) + x(1-x) \int_0^1 G_t(x,y)(f - Lf)(y)\, dy, \qquad \forall f \in C[0,1], \tag{5.20}$$

where the kernel G_t *is given by*

$$G_t(x,y) := \sum_{k=2}^{\infty} \frac{k(2k-1)}{k-1} e^{-\frac{1}{2}k(k-1)t} P_{k-2}^{(1,1)}(2x-1) P_{k-2}^{(1,1)}(2y-1). \tag{5.21}$$

Proof: Suppose first that $f \in \Pi$. From (4.14), $\mu_0^*(f - Lf) = \mu_1^*(f - Lf) = 0$, while by (4.19), $\mu_k^*(Lf) = 0$, $k \geq 2$. Therefore (5.16) gives

$$\begin{aligned}
\mathcal{B}_t f &= \mathcal{B}_t(Lf) + \mathcal{B}_t(f - Lf) \\
&= \sum_{k=0}^{\infty} e^{-\frac{1}{2}k(k-1)t} p_k^* \mu_k^*(Lf) + \sum_{k=0}^{\infty} e^{-\frac{1}{2}k(k-1)t} p_k^* \mu_k^*(f - Lf) \\
&= \sum_{k=0}^{1} p_k^* \mu_k^*(Lf) + \sum_{k=2}^{\infty} e^{-\frac{1}{2}k(k-1)t} p_k^* \mu_k^*(f - Lf).
\end{aligned}$$

By (4.14) and (4.15), this becomes

$$\mathcal{B}_t f = Lf + \sum_{k=2}^{\infty} \frac{-k}{2}\binom{2k}{k} e^{-\frac{1}{2}k(k-1)t} p_k^*(x) \int_0^1 P_{k-2}^{(1,1)}(2y-1)(f-Lf)(y)dy.$$

Using the dominated convergence theorem and Theorem 4.5 , this gives

$$\begin{aligned}\mathcal{B}_t f(x) &= Lf(x)\\ &+ x(1-x)\int_0^1 \sum_{k=2}^{\infty} \frac{k(2k-1)}{k-1} e^{-\frac{1}{2}k(k-1)t} P_{k-2}^{(1,1)}(2x-1) P_{k-2}^{(1,1)}(2y-1)(f-Lf)(y)dy.\end{aligned}$$

This proves the result for $f \in \Pi$. The extension of (5.20) to all $f \in C[0,1]$ now follows from the density of Π in $C[0,1]$. $\square$

This representation of the semigroup $\{\mathcal{B}_t\}$ was given in [KZ70:(4.7)] and da Silva [Si85] (where total positivity properties of the kernel G_t are investigated).

The infinitesimal generator of the semigroup $\{\mathcal{B}_t\}$ is

$$Af := \lim_{t\to 0^+} \frac{\mathcal{B}_t f - f}{t},$$

whenever this limit exists. On the polynomials (4.13) and (5.16) give

$$Af = \lim_{t\to 0^+} \sum_{k=2}^{\infty} \frac{e^{-\frac{1}{2}k(k-1)t} - 1}{t} p_k^* \mu_k^*(f) = \sum_{k=2}^{\infty} -\frac{1}{2}k(k-1) p_k^* \mu_k^*(f), \qquad \forall f \in \Pi. \tag{5.22}$$

In [KZ70] it is shown that

$$Af(x) = \frac{1}{2}x(1-x)D^2 f(x), \qquad \forall f \in C^2[0,1]. \tag{5.23}$$

Since

$$x(x-1)D^2 p_k^*(x) = k(k-1)p_k^*(x), \qquad \forall k \geq 2,$$

it is clear (5.22) and (5.23) are consistent. It is possible to extend (5.22) to $C[0,1]$ in the following way.

Corollary 5.24 (Infinitesimal generator of $\{\mathcal{B}_t\}$). *The infinitesimal generator A of the semigroup $\{\mathcal{B}_t : t \geq 0\}$ defined by (5.18) has the representation*

$$Af(x) = \frac{1}{2}x(x-1)\int_0^1 G(x,y)(f-Lf)(y)\,dy, \qquad \forall f \in C[0,1], \tag{5.25}$$

where the kernel G is given by

$$G(x,y) := \sum_{k=2}^{\infty} k^2(2k-1) P_{k-2}^{(1,1)}(2x-1) P_{k-2}^{(1,1)}(2y-1). \tag{5.26}$$

It can also be represented by (5.22) *and* (5.23).

Proof: Suppose first that $f \in \Pi$. It follows from (5.22) that

$$Af = A(Lf) + A(f - Lf) = -\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)p_k^*\mu_k^*(f - Lf),$$

and so Theorem 4.5 and (4.15) give

$$\begin{aligned} &Af(x) \\ &= -\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)p_k^*(x)\frac{-k}{2}\binom{2k}{k}\int_0^1 P_{k-2}^{(1,1)}(2y-1)(f-Lf)(y)\,dy \\ &= \frac{1}{2}x(x-1)\int_0^1 \sum_{k=2}^{\infty} k^2(2k-1)P_{k-2}^{(1,1)}(2x-1)P_{k-2}^{(1,1)}(2y-1)(f-Lf)(y)\,dy. \end{aligned}$$

This gives (5.25) for $f \in \Pi$. The extension of (5.25) to all $f \in C[0,1]$ now follows from the density of Π in $C[0,1]$. □

Iterates of all orders

As in [KR67:§4], it is possible to use (5.1) to define the iterates of B_n of all orders $-\infty < t < \infty$, in a manner consistent with the case when t is a nonnegative integer, namely

$$B_n^t f := \sum_{k=0}^{n} (\lambda_k^{(n)})^t p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1].$$

By (5.5), the iterate B_n^0 eqauls L_n (Lagrange interpolation at equally spaced points, which is the identity on Π_n). The inverse B_n^{-1} restricted to Π_n has been studied by Sablonnière [S92] (see next section).

6. Application to Bernstein quasi–interpolants

By (5.5), the operator of Lagrange interpolation at equally spaced points can be written as

$$L_n f = \sum_{k=0}^{n} p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1],$$

while the Bernstein operator is

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1].$$

In this way the Bernstein operator can be thought of as being obtained from the Lagrange interpolant by 'damping out' the $p_k^{(n)}$ coefficient (frequency) by the amount $0 < \lambda_k^{(n)} < 1$, $k \geq 2$. The failure of Lagrange interpolation at n equally spaced points to converge for all continuous functions (whilst the Bernstein approximants do) might then be explained by its failure to sufficiently damp out the highly oscillatory polynomials $p_k^{(n)}$ (cf (3.6)). It is then natural to consider operators of the form

$$A_{n,\alpha} f = \sum_{k=0}^{n} \alpha_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1], \tag{6.1}$$

for other amounts of damping $\alpha_k^{(n)} \in \mathbb{R}$. Remember that $A_{n,\alpha} f$ depends only on the values $f(0), f(1/n), \ldots, f(1)$. These operators are automatically degree reducing, and reproduce the linear polynomials if and only if $\alpha_0^{(n)} = \alpha_1^{(n)} = 1$. Indeed (6.1) is the diagonalised form of these operators. The linear combinations of iterates of the Bernstein operator considered in Section 6 (including iterated Boolean sums) are operators of this type. Presumably, by choosing the quantities $\alpha_k^{(n)}$ appropriately it should be possible to construct approximation processes inheriting some of the desirable properties of L_n (such as interpolation) and of B_n (like convergence as $n \to \infty$ for all $f \in C[0,1]$). We now suggest a few possibilities.

1. Continuous families. The eigenvalues of L_n could be continuously transformed via some parameter $0 \leq t \leq 1$ into those for B_n, giving a family of operators $A_{n,t}$, which depends continuously on t, with endpoints $A_{n,0} = L_n$ and $A_{n,1} = B_n$. Depending on the properties of the operator desired (or the smoothness of f) an appropriate value of t could then be chosen. A couple of such schemes for changing the eigenvalues (and corresponding operator) are

$$A_{n,t} f := \sum_{k=0}^{n} ((1-t) + t\lambda_k^{(n)}) p_k^{(n)} \mu_k^{(n)}(f) = (1-t) L_n f + t B_n f, \qquad \forall f \in C[0,1],$$

and

$$A_{n,t} f := \sum_{k=0}^{n} (\lambda_k^{(n)})^t p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1].$$

2. Polynomial reproduction. The first $j+1$ $(0 \leq j \leq n)$ eigenvalues could be set to 1. This then gives an operator which reproduces Π_j, e.g.,

$$A_{n,j} f := \sum_{k=0}^{j} p_k^{(n)} \mu_k^{(n)}(f) + \sum_{k=j+1}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f), \qquad \forall f \in C[0,1].$$

For this choice $A_{n,0} = A_{n,1} = B_n$ and $A_{n,n} = L_n$. This property, together with reproduction of Π_j is shared with the left Bernstein quasi–interpolant of order j of Sablonnière [S92],

$$B_n^{(j)} := A_n^{(j)} \circ B_n,$$

where $A_n^{(j)}$ is a truncated version of B_n^{-1} thought of as a differential operator on Π_n. It is not clear to the authors at this point how these two similar operators are related.

Operators which reproduce Π_j can also be obtained by taking affine combinations of Bernstein operators B_n of various degrees n, see, e.g., [B53], [DT87] and [Z95].

3. Adaptive methods. Since $p_k^{(n)}$ is an eigenfunction of (6.1) with eigenvalue $\alpha_k^{(n)}$ the limiting properties of $p_k^{(n)}$ and $\mu_k^{(n)}$ imply that for $A_{n,\alpha}f$ converge as $n \to \infty$ for all continuous f (p_k^* in particular) it is necessary that $\alpha_k^{(n)} \to 1$. If the rate of convergence of $\alpha_k^{(n)} \to 1$ is too fast (as in the case of Lagrange interpolation L_n, when $\alpha_k^{(n)} = 1$) then $A_{n,\alpha}f$ fails to converge for some f. Hence it seems the approximation properties of $A_{n,\alpha}$ (large n) are controlled by the rates at which $\alpha_k^{(n)} \to 1$, as $n \to \infty$.

A more detailed analysis of these questions seems worthy of further study.

7. The spectrum of the Kantorovich operator

Recall the Kantorovich operator $K_n : L_1[0,1] \to C[0,1]$, $n = 1, 2, \ldots$, which is defined by

$$K_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt.$$

It satisfies

$$K_n(Df) = D(B_{n+1}f), \qquad \forall f \in C^1[0,1],$$

and so in particular

$$K_n(Dp_{k+1}^{(n+1)}) = D(B_{n+1}p_{k+1}^{(n+1)}) = \lambda_{k+1}^{(n+1)}(Dp_{k+1}^{(n+1)}), \qquad k = 0, 1, \ldots, n,$$

i.e., $\lambda_{k+1}^{(n+1)}$ is an eigenvalue of K_n with corresponding eigenfunction $Dp_{k+1}^{(n+1)}$.

Corollary 7.1 (Eigenstructure of K_n). *The eigenvalues of the Kantorovich operator K_n are*

$$\nu_k^{(n)} := \lambda_{k+1}^{(n+1)} = \frac{n!}{(n-k)!} \frac{1}{(n+1)^k}, \qquad k = 0, 1, \ldots, n,$$

and the corresponding eigenfunctions are polynomials of exact degree k given by

$$q_k^{(n)} := Dp_{k+1}^{(n+1)} \qquad \textit{(these have leading coefficient } k+1\textit{)}.$$

The eigenvalues of K_n are distinct

$$\nu_0^{(n)} = 1 > \nu_1^{(n)} = \frac{n}{(n+1)} > \nu_2^{(n)} = \frac{n(n-1)}{(n+1)^2} > \cdots > \nu_n^{(n)} = \frac{n!}{(n+1)^n} > 0.$$

Many of the previous results for the Bernstein operator can now be adapted to the Kantorovich operator. For example, its diagonal form is

$$K_n f = \sum_{k=0}^{n} \nu_k^{(n)} \, q_k^{(n)} \mu_{k+1}^{(n+1)}(D^{-1}f), \qquad \forall f \in L_1[0,1],$$

where

$$D^{-1}f(x) := \int_0^x f,$$

and the analogue of (5.7) is

$$\lim_{j\to\infty} \left(1-(1-K_n)^j\right) f = D\left(L_{n+1}(D^{-1}f)\right), \qquad \forall f \in L_1[0,1]. \tag{7.2}$$

The limit in (7.2) is the area matching map which interpolates from Π_n to the data

$$\int_0^{1/(n+1)} f, \quad \int_{1/(n+1)}^{2/(n+1)} f, \quad \int_{2/(n+1)}^{3/(n+1)} f, \quad \ldots, \quad \int_{(n-1)/(n+1)}^{n/(n+1)} f, \quad \int_{n/(n+1)}^{1} f.$$

8. Multivariate Bernstein operators

The eigenstructure of the tensor product Bernstein operators (see [L53:p.51]) can be deduced from that of the univariate operator in the usual way. For simplicity, we illustrate this for the bivariate tensor product Bernstein operator $B_n \otimes B_m : C([0,1]^2) \to C([0,1]^2)$, which is defined by

$$(B_n \otimes B_m)f(x) := \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} x^i (1-x)^{n-i} y^j (1-y)^{m-j}.$$

Corollary 8.1 (Eigenstructure of $B_n \otimes B_m$). *The eigenvalues of $B_n \otimes B_m$ are*

$$\lambda_{i,j}^{(n,m)} := \lambda_i^{(n)} \lambda_j^{(m)} = \frac{n!}{(n-i)!}\frac{m!}{(m-j)!}\frac{1}{n^i m^j}, \qquad i = 0,1,\ldots,n, \quad j = 0,1,\ldots,m,$$

and the corresponding eigenfunctions $p_{i,j}^{(n,m)}$ are given by

$$p_{i,j}^{(n,m)}(x,y) := p_i^{(n)}(x) p_j^{(m)}(y).$$

It follows from the analogue of (1.2), that the multivariate Bernstein operator B_n on a simplex $S \subset \mathbb{R}^d$ (see [L53:p.51]) has the same spectra as the univariate operator, and the $\lambda_k^{(n)}$–eigenspaces has dimension $\binom{k+d-1}{d-1}$. Thus it is diagonalisable. More detailed results, including computational formulæ and symmetry properties of the eigenspaces can be found in [CW99].

9. Appendix

The eigenfunctions

The first 4 eigenfunctions are independent of n

$$p_0^{(n)}(x) = 1, \qquad p_1^{(n)}(x) = x-1/2, \qquad p_2^{(n)}(x) = x(x-1), \qquad p_3^{(n)}(x) = x(x-1)(x-1/2),$$

and the others depend on n. Here are the next 3 eigenfunctions in the factored form (2.11)

$$\begin{aligned}
p_4^{(n)}(x) &= x(x-1)\left((x-1/2)^2 + \frac{2-n}{4(5n-6)}\right), \\
p_5^{(n)}(x) &= x(x-1)(x-1/2)\left((x-1/2)^2 + \frac{8-3n}{4(7n-12)}\right), \\
p_6^{(n)}(x) &= x(x-1)\Biggl((x-1/2)^4 + \frac{10-3n}{2(9n-20)}(x-1/2)^2 \\
&\qquad\qquad + \frac{(n-2)(n-4)(6n^2-23n+40)}{16(9n-20)(14n^3-71n^2+154n-120)}\Biggr).
\end{aligned} \tag{9.1}$$

Using (2.7) the first few coefficients in (2.6) are

$$\begin{aligned}
c(k,k,n) &= 1, \qquad 0 \le k \le n, \\
c(k-1,k,n) &= -\frac{k}{2}, \qquad 1 \le k \le n, \\
c(k-2,k,n) &= \frac{1}{24}\frac{k(k-1)(k-2)(6n+5-3k)}{-k^2+3k+2nk-2-3n}, \qquad 2 \le k \le n, \\
c(k-3,k,n) &= -\frac{1}{48}\frac{k(k-1)(k-2)(k-3)(2n+2-k)}{-k^2+3k+2nk-2-3n}, \qquad 3 \le k \le n.
\end{aligned} \tag{9.2}$$

The dual functionals

The matrix P of Theorem 2.3 can be inverted as follows.

Lemma 9.3 (Finding V). *The j–th row of the matrix $V = P^{-1}$ of Theorem 2.3, whose k–th column gives the coefficients of the dual functional $\mu_k^{(n)}$, i.e.,*

$$\mu_k^{(n)}(f) = \sum_{j=0}^{n} v(j,k,n) f\left(\frac{j}{n}\right), \qquad k = 0,\ldots,n,$$

can be calculated using the recurrence

$$\begin{aligned}
v(j,n,n) &= (-1)^{n-j}\frac{n^n}{j!(n-j)!}, \\
v(j,n-k,n) &= \frac{(-1)^{n-j-k}n^{n-k}}{(n-j-k)!j!} \\
&\qquad - \sum_{s=0}^{k-1}\frac{k!}{s!}n^{s-k}\, v(j,n-s,n)\, c(n-k,n-s,n), \quad k = 1,\ldots,n,
\end{aligned} \tag{9.4}$$

where

$$\frac{1}{(n-k-j)!} := 0, \qquad j > n-k.$$

Proof: Let $\ell_j^{(n)}$ be the (Lagrange) polynomial of degree n satisfying $\ell_j^{(n)}(i/n) = \delta_{ij}$, i.e.,

$$\ell_j^{(n)}(x) := \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{(x-i/n)}{(j/n-i/n)}.$$

Apply B_n in the form (1.1) and in the diagonal form (2.4) to $\ell_j^{(n)}$, and equate the results to obtain

$$\binom{n}{j} x^j(1-x)^{n-j} = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)}(x) v(j,k,n), \qquad j=0,\ldots,n. \tag{9.5}$$

Equating the coefficients of x^{n-k}, $k=0,\ldots,n$ in (9.5) gives

$$\begin{aligned}&(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{n-j-k}\\ &= \sum_{s=0}^{k-1} \lambda_{n-s}^{(n)}\, v(j,n-s,n)\, c(n-k,n-s,n) + \lambda_{n-k}^{(n)}\, v(j,n-k,n).\end{aligned}$$

For $k=0$ this gives the first equation in (9.4), and for $k=1,\ldots,n$ this can be solved for $v(j,n-k,n)$ as follows

$$\begin{aligned}&v(j,n-k,n)\\ &= \frac{1}{\lambda_{n-k}^{(n)}}\left\{(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{n-j-k} - \sum_{s=0}^{k-1}\lambda_{n-s}^{(n)}\, v(j,n-s,n)\, c(n-k,n-s,n)\right\}\\ &= \frac{(-1)^{n-j-k}n^{n-k}}{j!(n-k-j)!} - \sum_{s=0}^{k-1}\frac{k!}{s!}n^{s-k}\, v(j,n-s,n)\, c(n-k,n-s,n).\end{aligned}$$

giving (9.4). The columns of V correspond to the dual functionals μ_k, and the rows of P to the eigenfunctions p_k. $\square$

The first few matrices V are

$$\begin{pmatrix}1/2 & -1\\ 1/2 & 1\end{pmatrix}, \begin{pmatrix}1/2 & -1 & 2\\ 0 & 0 & -4\\ 1/2 & 1 & 2\end{pmatrix}, \begin{pmatrix}1/2 & -1 & 9/4 & -9/2\\ 0 & 0 & -9/4 & 27/2\\ 0 & 0 & -9/4 & -27/2\\ 1/2 & 1 & 9/4 & 9/2\end{pmatrix}$$

$$\begin{pmatrix}1/2 & -1 & 50/21 & -16/3 & 32/3\\ 0 & 0 & -32/21 & 32/3 & -128/3\\ 0 & 0 & -12/7 & 0 & 64\\ 0 & 0 & -32/21 & -32/3 & -128/3\\ 1/2 & 1 & 50/21 & 16/3 & 32/3\end{pmatrix}$$

$$\begin{pmatrix} 1/2 & -1 & 375/152 & -1625/276 & 625/48 & -625/24 \\ 0 & 0 & -175/152 & 2375/276 & -625/16 & 3125/24 \\ 0 & 0 & -25/19 & 250/69 & 625/24 & -3125/12 \\ 0 & 0 & -25/19 & -250/69 & 625/24 & 3125/12 \\ 0 & 0 & -175/152 & -2375/276 & -625/16 & -3125/24 \\ 1/2 & 1 & 375/152 & 1625/276 & 625/48 & 625/24 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & -1 & 2681/1060 & -63/10 & 252/17 & -162/5 & 324/5 \\ 0 & 0 & -981/1060 & 36/5 & -594/17 & 648/5 & -1944/5 \\ 0 & 0 & -225/212 & 9/2 & 108/17 & -162 & 972 \\ 0 & 0 & -115/106 & 0 & 468/17 & 0 & -1296 \\ 0 & 0 & -225/212 & -9/2 & 108/17 & 162 & 972 \\ 0 & 0 & -981/1060 & -36/5 & -594/17 & -648/5 & -1944/5 \\ 1/2 & 1 & 2681/1060 & 63/10 & 252/17 & 162/5 & 324/5 \end{pmatrix}$$

For example,

$$\mu_3^{(4)}(f) = -16/3f(0) + 32/3f(1/4) + 0f(1/2) - 32/3f(3/4) + 16/3f(1).$$

The dual functionals can also be computed using the symmetry properties of $\mu_k^{(n)}$ and $p_k^{(n)}$. For example, the dual functional $\mu_{n-1}^{(n)}$, $n \geq 1$ must come from the 2–dimensional subspace (of span$\{f \mapsto f(j/n) : j = 0, 1, \ldots, n\}$) consisting of those functionals which annihilate Π_{n-2}. A basis for this space is

$$\{[0, 1/n, \ldots, (n-2)/n, (n-1)/n], [1/n, 2/n, \ldots, (n-1)/n, 1]\}.$$

Here $[x_0, x_1, \ldots, x_k]$ denotes the divided difference at the points $x_0, x_1, \ldots, x_k$, which if equally spaced equals

$$[x, x+h, x+2h, \ldots, x+kh]f = \frac{1}{k!h^k}\Delta_h^k f(x) = \frac{1}{k!h^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+ih). \quad (9.6)$$

By the symmetry conditions (2.10) and (9.6), $\mu_{n-1}^{(n)}$ must be a scalar multiple of

$$[0, 1/n, \ldots, (n-2)/n, (n-1)/n] + [1/n, 2/n, \ldots, (n-1)/n, 1], \quad (9.7)$$

which (by the symmetry conditions) annihilates $p_n^{(n)}$. For (9.7) to satisfy the condition $\mu_{n-1}^{(n)}(p_{n-1}^{(n)}) = 1$, its scalar multiplier must be $1/2$. Continuing to argue along these lines leads to the following.

Theorem 9.8. *The dual functionals in (2.4) can be expressed as follows*

$$
\begin{aligned}
\mu_0^{(n)}(f) &= \frac{1}{2}\left(f(0)+f(1)\right), \qquad n \ge 0,\\
\mu_1^{(n)}(f) &= f(1)-f(0), \qquad n \ge 1,\\
\mu_{n-3}^{(n)}(f) &= \frac{1}{8}\frac{n(n+1)}{n^2-6}[0,\frac{1}{n},\ldots,\frac{n-3}{n}]f\\
&\quad+\frac{1}{8}\frac{(3n+8)(n-3)}{n^2-6}[\frac{1}{n},\frac{2}{n},\ldots,\frac{n-2}{n}]f\\
&\quad+\frac{1}{8}\frac{(3n+8)(n-3)}{n^2-6}[\frac{2}{n},\frac{3}{n},\ldots,\frac{n-1}{n}]f,\\
&\quad+\frac{1}{8}\frac{n(n+1)}{n^2-6}[\frac{3}{n},\frac{4}{n},\ldots,1]f, \qquad n \ge 3,\\
\mu_{n-2}^{(n)}(f) &= \frac{1}{12}\frac{(n+1)(3n-2)}{n^2-2}[0,\frac{1}{n},\ldots,\frac{n-2}{n}]f\\
&\quad+\frac{1}{6}\frac{(3n+5)(n-2)}{n^2-2}[\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n}]f\\
&\quad+\frac{1}{12}\frac{(n+1)(3n-2)}{n^2-2}[\frac{2}{n},\frac{3}{n},\ldots,1]f, \qquad n \ge 2,\\
\mu_{n-1}^{(n)}(f) &= \frac{1}{2}\left([0,\frac{1}{n},\ldots,\frac{n-1}{n}]f+[\frac{1}{n},\frac{2}{n},\ldots,1]f\right), \qquad n \ge 1,\\
\mu_n^{(n)}(f) &= [0,\frac{1}{n},\frac{2}{n},\ldots,1]f, \qquad n \ge 0.
\end{aligned}
$$

Formulæ for $\mu_{n-4}^{(n)}$, $\mu_{n-5}^{(n)},\ldots$ can, in principle, also be obtained by this method, but are more complicated. There appears to be no simple closed form.

The limiting dual functionals

Using (4.14) and (4.15) to compute the first few μ_k^* gives

$$
\begin{aligned}
\mu_0^*(f) &= \frac{1}{2}f(0)+\frac{1}{2}f(1),\\
\mu_1^*(f) &= -f(0)+f(1),\\
\mu_2^*(f) &= 3f(0)+3f(1)-6\int_0^1 f(x)\,dx,\\
\mu_3^*(f) &= -10f(0)+10f(1)-120\int_0^1 f(x)(x-1/2)\,dx,\\
\mu_4^*(f) &= 35f(0)+35f(1)-2100\int_0^1 f(x)(x^2-x+1/5)\,dx,\\
\mu_5^*(f) &= -126f(0)+126f(1)-35280\int_0^1 f(x)(x^3-3/2x^2+9/14x-1/14)\,dx,\\
\mu_6^*(f) &= 462f(0)+462f(1)-582120\int_0^1 f(x)(x^4-2x^3+4/3x^2-1/3x+1/42)\,dx.
\end{aligned}
$$

References

[A87] T. Ando, Totally positive matrices, *Linear Algebra Appl.* **90** (1987), 165–219.

[BD80] H. Berens and R. DeVore, A characterisation of Bernstein polynomials, *in* "Approximation Theory III" (E. W. Cheney Ed.), pp. 213–219, Academic Press, New York, 1980.

[BX91] H. Berens and Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights, *in* "Approximation theory and functional analysis" (Chui, C. K. Ed.), pp. 25–46, Academic Press, Boston, 1991.

[B53] P. L. Butzer, Linear combinations of Bernstein polynomials, *Canad. J. Math.* **5** (1953), 559–567.

[CW99] S. Cooper and S. Waldron, The diagonalisation of the multivariate Bernstein operator, manuscript, 1999.

[D81] M. M. Derriennic, Sur l'approximation de fonctions intégrables sur $[0,1]$ par des polynômes de Bernstein modifies, *J. Approx. Theory* **31** (1981), 325–343.

[DT87] Z. Ditzian and V. Totik, "Moduli of smoothness", Springer–Verlag, New York, 1987.

[E53] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Higher Transcendental Functions", McGraw-Hill, New York, 1953.

[GZ94] H. H. Gonska and Xin Long Zhou, Approximation theorems for the iterated Boolean sums of Bernstein operators, *J. Comput. Appl. Math.* **53** (1994), 21–31.

[K68] S. Karlin, "Total Positivity", Stanford Univ. Press, Stanford, 1968.

[KZ70] S. Karlin and Z. Ziegler, Iteration of positive approximation operators, *J. Approx. Theory* **3** (1970), 310–339.

[KR67] R. P. Kelisky and T. J. Rivlin, Iterates of Bernstein polynomials, *Pacific J. Math.* **21(3)** (1967), 511–520.

[L53] G. G. Lorentz, "Bernstein Polynomials", Toronto Press, Toronto, 1953.

[M73] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory* **8** (1973), 1–18.

[P96] A. Pinkus, Spectral properties of totally positive kernels and matrices, *in* "Total positivity and its applications" (Gasca, M., Micchelli, C. A. Eds.), pp. 477–511, Kluwer Acad. Publ., Dordrecht, 1996.

[S92] P. Sablonnière, A family of Bernstein quasi-interpolants on $[0,1]$, *Approx. Theory Appl.* **8(3)** (1992), 62–76.

[Se95] J. C. Sevy, Lagrange and least-squares polynomials as limits of linear combinations of iterates of Bernstein and Durrmeyer polynomials, *J. Approx. Theory* **80(2)** (1995), 267–271.

[Si85] M. R. da Silva, Nonnegative order iterates of Bernstein polynomials and their limiting semigroup, *Portugal. Math.* **42(3)** (1985), 225–248.

[W97] H.-J. Wenz, On the limits of (linear combinations of) iterates of linear operators, *J. Approx. Theory* **89** (1997), 219–237.

[Wi41] D. V. Widder, "The Laplace transform", Princeton University Press, Princeton, 1941.

[Z95] D. X. Zhou, On smoothness characterized by Bernstein type operators, *J. Approx. Theory* **81(3)** (1995), 303–315.