Constructing high order spherical designs as a union of two of lower order

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Abstract

We show how the variational characterisation of spherical designs can be used to take a union of spherical designs to obtain a spherical design of higher order (degree, precision, exactness) with a small number of points. The examples that we consider involve taking the orbits of two vectors under the action of a complex reflection group to obtain a weighted spherical (t,t)-design. These designs have a high degree of symmetry (compared to the number of points), and many are the first known construction of such a design, e.g., a 32 point $(9,9)$-design for $\mathbb{C}^2$, a 48 point $(4,4)$-design for $\mathbb{C}^3$, and a 400 point $(5,5)$-design for $\mathbb{C}^4$. From a real reflection group, we construct a 360 point $(9,9)$-design for $\mathbb{R}^4$ (spherical half-design of order 18), i.e., a 720 point spherical 19-design for $\mathbb{R}^4$.

Key Words: complex spherical design, harmonic Molien-Poincaré series, spherical t-designs, spherical half-designs, tight spherical designs, finite tight frames, signed frame, integration rules, cubature rules, cubature rules for the sphere,

AMS (MOS) Subject Classifications: primary 05B30, 42C15, 65D30; secondary 94A12.

1 Introduction

Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^d$ or $\mathbb{C}^d$, and σ be normalised surface area measure on $\mathbb{S}$. A **weighted spherical design** is a finite set (or sequence) of points X in $\mathbb{S}$ and **weights** $w_x \in \mathbb{R}$, $x \in X$, for which the integration (cubature) rule

$$\int_{\mathbb{S}} f \, d\sigma = \sum_{x \in X} w_x f(x), \qquad \forall f \in P, \tag{1.1}$$

holds for some finite dimensional space of functions P defined on $\mathbb{S}$ (usually a unitarily invariant polynomial space). Such configurations of points are known to exist for every choice of P (see [BT06], [SZ84]). For certain choices there is great interest in explicit constructions, especially those with a minimal number of points, e.g., the "tight spherical designs" of algebraic combinatorics [BB09]. The optimal configurations often have a high degree of symmetry, and are closely related to optimal spherical packings [MP19], [JKM19], [Via17], and points minimising a potential function on the sphere [BGM+19].

If X and Y are weighted spherical designs with weights (w_x^X) and (w_y^Y), then for any fixed $\alpha \in \mathbb{R}$ and $f \in P$, we have

$$\sum_{x \in X} (\alpha w_x^X) f(x) + \sum_{y \in Y} ((1-\alpha) w_y^Y) f(y) = \alpha \int_{\mathbb{S}} f \, d\sigma + (1-\alpha) \int_{\mathbb{S}} f \, d\sigma = \int_{\mathbb{S}} f \, d\sigma,$$

so that $X \cup Y$ is a weighted spherical design, with the "affine combination" of the weights

$$w_a^{X \cup Y, (\alpha, 1-\alpha)} := \begin{cases} \alpha w_x^X, & a = x \in X; \\ (1-\alpha) w_y^Y, & a = y \in Y. \end{cases} \tag{1.2}$$

The weights of a spherical design are usually taken to be positive, and so it would be natural to take a "convex combination" of the weights, i.e., to choose $0 < \alpha, 1-\alpha < 1$. We will call $(\beta_X, \beta_Y) = (\alpha, 1-\alpha)$ the **weighting** of the union $X \cup Y$. It is usually assumed the weights add to 1 (this follows if P contains the constants), in which case

$$\beta_X = \sum_{x \in X} w_x^{X \cup Y, (\alpha, 1-\alpha)}, \qquad \beta_Y = \sum_{y \in Y} w_y^{X \cup Y, (\alpha, 1-\alpha)}, \qquad \beta_X + \beta_Y = 1.$$

The purpose of this paper is to try and choose the weighting of a union of spherical designs to obtain one of *higher order*, i.e., for which the space P in (1.1) is enlarged. If one were to try and use (1.1) to do this, then one could increase P by just one dimension, by solving an appropriate linear equation for α.

When P is a unitarily invariant space of polynomials, (1.1) can be replaced by a single *quadratic equation* in the weights $(w_x)_{x \in X}$ with coefficients involving just (the inner products between) the points X, which comes from a variational characterisation [Wal19]. By considering this quadratic for the union of designs X and Y, and a unitarily invariant space Q, it follows that:

Lemma 1.1 *There is a quadratic equation in $\alpha = \beta_X$, which if solvable, gives a weighting for the union of spherical designs X and Y for P to be one for a larger space Q.*

This is useful only if one can choose X, Y and Q (large enough to be of interest), so that the quadratic equation has a real root, preferably with $0 < \alpha < 1$. Remarkably, we show that this approach actually works quite successfully. We will primarily consider the class of (complex) spherical (t,t)-designs. The basic properties in the milieu are:

- X and Y are chosen to have a small number of points. In practice, this means that they are an orbit of a unitary group action, with a large stabiliser.
- X and Y must have the right relationship. Clearly, we cannot take $Y = X$ and gain anything more. One could take $Y = UX$ with U unitary, but this adds additional parameters to the quadratic (making it more likely to find one which is solvable, but less tractable). Here we take X and Y to be orbits of the same group action.
- Q must be large enough to be of interest, but not so large that the quadratic has no real roots. In practice, P is polynomials up to some degree, and we take Q to be the same space for polynomials one degree larger.

Our constructions for orbits of finite complex reflection groups are summarised in §5.

2 Spherical (t,t)-designs and half-designs of order $2t$

For $t = 1, 2, \ldots$, every finite set of vectors X in $\mathbb{C}^d$ satisfies the inequality

$$\sum_{x\in X}\sum_{y\in X} |\langle x,y\rangle|^{2t} \geq c_t(\mathbb{C}^d)\Big(\sum_{x\in X}\|x\|^{2t}\Big)^2, \qquad c_t(\mathbb{C}^d) := \frac{1}{\binom{t+d-1}{t}}. \tag{2.3}$$

A set of nonzero vectors giving equality in (2.3) is called a **spherical (t,t)-design**. A spherical (t,t)-design X is a weighted spherical design for the complex sphere [Wal17], where $x \in X$ corresponds to $\hat{x} := \frac{x}{\|x\|} \in \mathbb{S}$, and the weights and polynomial space are

$$w_x = \frac{\|x\|^{2t}}{\sum_{a\in X}\|a\|^{2t}}, \qquad P = \mathrm{Hom}(t,t). \tag{2.4}$$

Here $\mathrm{Hom}(p,q)$ is the space of homogeneous polynomials in the variables $z \in \mathbb{C}^d$ and $\overline{z}$ which are of degree p in z and degree q in $\overline{z}$. The variational characterisation is

$$\sum_{x\in X}\sum_{y\in X} w_x w_y |\langle \hat{x},\hat{y}\rangle|^{2t} = c_t(\mathbb{C}^d). \tag{2.5}$$

From (2.3), it follows that a spherical (t,t)-design is a projective object, i.e., multiplying a point $x \in X$ by a unit scalar gives another such design, and so x can be identified with the complex line through x and the origin. When a spherical (t,t)-design is viewed as a collection of lines, then the term *weighted complex projective t-design* is also used [RS07]. Notable examples include *tight frames* which are the $(1,1)$-designs [Wal18] (those with the minimal number being the orthogonal bases), and *SICs* (sets of d^2 equiangular lines in $\mathbb{C}^d$) which are $(2,2)$-designs with the minimal number of vectors [ACFW18].

It is not obvious from the definition that unions of spherical (t,t)-designs are again spherical (t,t)-designs. This follows from the spherical design property (1.1).

Theorem 2.1 *If X and Y are are spherical (t,t)-designs, then so is any convex union of them, such as $X\cup Y$, and in particular*

$$\sum_{x\in X}\sum_{y\in Y}|\langle x,y\rangle|^{2t}=c_t(\mathbb{C}^d)\Bigl(\sum_{x\in X}\|x\|^{2t}\Bigr)\Bigl(\sum_{y\in Y}\|y\|^{2t}\Bigr). \tag{2.6}$$

Proof: The union $X\cup Y$, with weights given by (2.4), is given by the weighting

$$\beta_X=\frac{\sum_{a\in X}\|a\|^{2t}}{\sum_{x\in X}\|x\|^{2t}+\sum_{y\in Y}\|y\|^{2t}},\qquad \beta_Y=\frac{\sum_{b\in Y}\|b\|^{2t}}{\sum_{x\in X}\|x\|^{2t}+\sum_{y\in Y}\|y\|^{2t}}.$$

Eliminating terms for equality in (2.3) for X, Y and $X\cup Y$ gives (2.6). □

Let X and Y be finite subsets of $\mathbb{S}$ and (w_x^X) and (w_y^Y) be corresponding weights. By the variational characterisation (2.5), their union $X\cup Y$ with the weighting $(\beta_X,\beta_Y)=(\alpha,1-\alpha)$ is a spherical (t,t)-design if and only if α satisfies

$$\sum_{a\in X\cup Y}\sum_{b\in X\cup Y}w_a^{X\cup Y,(\alpha,1-\alpha)}w_b^{X\cup Y,(\alpha,1-\alpha)}|\langle a,b\rangle|^{2t}=c_t(\mathbb{C}^d),$$

which, by (1.2), expands to the following quadratic equation in α

$$\alpha^2\sum_{a\in X}\sum_{b\in X}w_a^Xw_b^X|\langle a,b\rangle|^{2t}+(1-\alpha)^2\sum_{a\in Y}\sum_{b\in Y}w_a^Yw_b^Y|\langle a,b\rangle|^{2t}$$
$$+2\alpha(1-\alpha)\sum_{a\in X}\sum_{b\in Y}w_a^Xw_b^Y|\langle a,b\rangle|^{2t}=c_t(\mathbb{C}^d). \tag{2.7}$$

This is an instance of Lemma 1.1. Here (and in general) the coefficients of the quadratic depend only on the weights and the inner products between the elements of $X\cup Y$.

We find in convenient to use the **normalised weights**

$$\hat{w}_x^X:=|X|w_x^X,$$

so that the normalised weights for X add to $|X|$, and they equal 1 when they are all the same. We now suppose the weights for X and Y are both constant (as will be the case for an orbit under a unitary action), so that the normalised weights for $X\cup Y$ have the form

$$(|X|+|Y|)w_a^{X\cup Y,(\alpha,1-\alpha)}=:\begin{cases}\hat{w}_X, & a\in X;\\ \hat{w}_Y, & a\in Y.\end{cases}$$

Since $|X|\hat{w}_X+|Y|\hat{w}_Y=|X|+|Y|$, for $\hat{w}_X,\hat{w}_Y\neq 0$, it follows from (1.2) that $X\cup Y$ with the weighting given by $z=\hat{w}_X$ is a spherical (t,t)-design if and only if it

$$z^2\sum_{a\in X}\sum_{b\in X}|\langle a,b\rangle|^{2t}+\left(\frac{|X|+|Y|-|X|z}{|Y|}\right)^2\sum_{a\in Y}\sum_{b\in Y}|\langle a,b\rangle|^{2t}$$
$$+2z\left(\frac{|X|+|Y|-|X|z}{|Y|}\right)\sum_{a\in X}\sum_{b\in Y}|\langle a,b\rangle|^{2t}=(|X|+|Y|)^2c_t(\mathbb{C}^d). \tag{2.8}$$

Once a suitable $\hat{w}_X$ has been found, the other parameters can then be calculated from

$$\hat{w}_Y=\frac{|X|+|Y|-|X|\hat{w}_X}{|Y|},\qquad \beta_X=\frac{|X|\hat{w}_X}{|X|+|Y|},\quad \beta_Y=\frac{|Y|\hat{w}_Y}{|X|+|Y|}. \tag{2.9}$$

For vectors X in $\mathbb{R}^d$, the following sharpening of (2.3) is possible (see [Wal17])

$$\sum_{x\in X}\sum_{y\in X} |\langle x,y\rangle|^{2t} \geq c_t(\mathbb{R}^d)\Big(\sum_{x\in X}\|x\|^{2t}\Big)^2, \qquad c_t(\mathbb{R}^d) := \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+2)\cdots(d+2(t-1))}. \tag{2.10}$$

The corresponding spherical designs are called **spherical half-designs** [KP11]. They integrate $P = \mathrm{Hom}(2t)$, the space of homogeneous polynomials of degree $2t$ on $\mathbb{R}^d$, and are characterised by

$$\sum_{x\in X}\sum_{y\in X} w_x w_y |\langle \hat{x},\hat{y}\rangle|^{2t} = c_t(\mathbb{R}^d). \tag{2.11}$$

Our previous discussion on spherical (t,t)-designs extends to spherical half-designs in the obvious way, i.e., replace $c_t(\mathbb{C}^d)$ by $c_t(\mathbb{R}^d)$. We will not labour the point, with a spherical (t,t)-design for $\mathbb{R}^d$ understood to be a spherical half-design of order $2t$.

3 Highly symmetric tight frames and reflection groups

Since $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$ is constant on the sphere, the space $P = \mathrm{Hom}(t,t)$ in (1.1) integrated by a spherical (t,t)-design satisfies

$$\mathrm{Hom}(p-1,q-1)|_{\mathbb{S}} \subset \mathrm{Hom}(p,q)|_{\mathbb{S}}.$$

Hence a spherical (t,t)-design is a spherical (r,r)-design for $r = 0,1,\ldots,t$. In particular, its weights add to 1 ($r=0$) and it is a tight frame ($r=1$) for $t\geq 1$. The analogous result for spherical half-designs of order $2t$ follows from the fact $\mathrm{Hom}(2(t-1))|_{\mathbb{S}} \subset \mathrm{Hom}(2t)|_{\mathbb{S}}$.

The following notion of a "highly symmetric" tight frame was given in [BW13].

> A finite frame of distinct vectors is **highly symmetric** if the action of its symmetry group is irreducible, transitive, and the stabiliser of any one vector (and hence all) is a nontrivial subgroup which fixes a space of dimension exactly one.

The upshot of this definition, is that for every unitary irreducible representation of a finite group on $\mathbb{R}^d$ or $\mathbb{C}^d$, there is a finite (possibly empty) set of highly symmetric tight frames (up to unitary equivalence) given as a group orbit, which has a nontrivial stabiliser (the number of vectors is less than the order of the group). In theory, these highly symmetric tight frames can be calculated for a given group (or representation), and this was done primarily in the case of finite complex reflection groups in [BW13].

A finite group of linear transformations on $\mathbb{R}^d$ or $\mathbb{C}^d$ is a **complex reflection group** if it is generated by complex reflections, i.e., transformations which fix a hyperplane (and have finite order). The finite irreducible complex reflection groups were classified by Shephard and Todd (see [ST54], [LT09]). There are three infinite families of imprimitive reflection groups of the type $G(m,p,n)$, $p|m$, and 31 primitive complex reflection groups $G_4,\ldots,G_{34}$ in dimensions $2,3,\ldots,8$, which are referred to as the *Shephard-Todd groups* with numbers $4,5,\ldots,34$. The complex reflection groups are a generalisation of the real reflection groups (classified by Coxeter). The Shephard-Todd classification contains the real reflection groups (numbers 23, 28, 30, 35, 36 and 37). In many presentations, the generators of the real reflection groups are given as matrices over a cyclotomic field.

The highly symmetric tight frames for the Shephard-Todd groups were calculated in [BW13]. Their strength as (t, t)-designs (the largest t can be) was calculated in [HW18b] by utilising magma software of Don Taylor to calculate the *maximal parabolic subgroups* (which stablise the vectors of a highly symmetric tight frame). Later, it was shown that in most, but not all cases, the strength of such a design was shared by all orbits (where the action is unitary), and that it could be calculated from a complex harmonic Molien-Poincaré series [RS14], [MW19]. The corresponding results for orthogonal actions on real spaces were considered earlier by [Ban79], [dlHP04]. In both the real and complex cases, we will call this the *generic* strength of an orbit.

Example 3.1 *If the unitary action of a finite group on $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$ is irreducible, i.e., every orbit of every nonzero vector spans $\mathbb{F}^d$, then every orbit of a nonzero vector is a tight frame, i.e., a $(1, 1)$-design (this is equivalent to the action being irreducible). Hence the generic strength of an orbit of an irreducible complex reflection group is at least $t = 1$.*

Our main result is the proof of concept:

The quadratic (2.8) can be solved to find a union of spherical designs with higher order.

A summary of our calculations for the highly symmetric spherical (t, t)-designs for the complex reflection groups is given in Section 5. Combining these gives the following:

Theorem 3.1 *Let G be a primitive irreducible complex reflection group (these have Shephard-Todd numbers 4–34). If X and Y are different highly symmetric tight frames for G, then there is unique rational weighting for which $X \cup Y$ is a spherical (t, t)-design, where t is strictly larger than that of a generic orbit. Moreover, for every case where there are two or more highly symmetric tight frames, a pair can be chosen for which the weighting is convex, i.e., has positive entries.*

In Section 4, we give evidence to suggest that such a result also holds for *any* pair of orbits, i.e., the fact that the orbit is highly symmetric is important only in that its size is small.

We finish this section with some technical comments about our calculations.

- Our calculations were done in magma, using the software `Complements.m` of Don Taylor to calculate the maximal parabolic subgroups. Magma writes vectors as rows, and the action of a matrix group, e.g., in `Eigenspace`, is by right multiplication, and so our code must be read with this in mind.

- For an orbit of a unit vector to lie on the sphere, the group action must be unitary. The presentations of the complex reflection groups (or more generally irreducible representations) provided in magma are not all unitary. One way around this, is to consider the canonical Gramian (which can be calculated from the Gramian) of the orbit of the nonunitary representation [Wal18]. This can be done, but becomes unfeasible eventually. Another way, is to find a Hermitian matrix which gives the quadratic form under which the action is unitary (as was done in [BW13]). This works better for large examples, as the inner products in sums such as (2.8) can be created and added to the sum one by one. Thus for orbits of large size there is no need to create the Gramian.

4 The structure of the quadratic

For weighted sets X and Y of points on the sphere, let

$$b_{XY}^{(t)} := \sum_{a\in X}\sum_{b\in Y} w_a^X w_b^Y |\langle a,b\rangle|^{2t}.$$

If X and Y are spherical (t,t)-designs for $\mathbb{F}^d$, then by Theorem 2.1, we have

$$b_{XY}^{(t)} = c_t(\mathbb{F}^d),$$

so that

$$b_{XX}^{(t)} b_{YY}^{(t)} - (b_{XY}^{(t)})^2 \neq 0, \qquad b_{XX}^{(t)} + b_{YY}^{(t)} - 2b_{XY}^{(t)} \neq 0,$$

when X and Y are not both spherical (t,t)-designs.

It seems that in the many cases considered so far, when there is a root of (2.7) for a union of lower order designs, then the root is a double root, i.e., the discriminant is zero

$$b_{XX}^{(t)} b_{YY}^{(t)} - (b_{XY}^{(t)})^2 = c_t(\mathbb{C}^d)\left(b_{XX}^{(t)} + b_{YY}^{(t)} - 2b_{XY}^{(t)}\right), \tag{4.12}$$

and we have the simple formula

$$\beta_X = \frac{b_{YY}^{(t)} - b_{XY}^{(t)}}{b_{XX}^{(t)} + b_{YY}^{(t)} - 2b_{XY}^{(t)}}, \qquad \beta_Y = \frac{b_{XX}^{(t)} - b_{XY}^{(t)}}{b_{XX}^{(t)} + b_{YY}^{(t)} - 2b_{XY}^{(t)}}.$$

This seems to hold for any pair of orbits, i.e., it has nothing to do with it being a highly symmetric tight frame. Suppose that there is a unitary action of G on $\mathbb{F}^d$, and let

$$p_G^{(t)}(x,y) := \frac{1}{|G|}\sum_{g\in G}|\langle x, gy\rangle|^{2t} = \frac{1}{|G|^2}\sum_{g\in G}\sum_{h\in G}|\langle gx, hy\rangle|^{2t} = b_{Gx,Gy}^{(t)}, \tag{4.13}$$

where $Gx := (gx)_{g\in G}$. Then the condition for there to be a unique weighting for which the union of the orbits of x and y is a spherical (t,t)-design is that

$$p_G^{(t)}(\hat{x},\hat{x})p_G^{(t)}(\hat{y},\hat{y}) - \left(p_G^{(t)}(\hat{x},\hat{y})\right)^2 \neq 0, \qquad \text{(the orbits are not both } (t,t)\text{-designs)}$$

where $\hat{x} := \frac{x}{\|x\|}$, and

$$p_G^{(t)}(\hat{x},\hat{x})p_G^{(t)}(\hat{y},\hat{y}) - \left(p_G^{(t)}(\hat{x},\hat{y})\right)^2 = c_t(\mathbb{F}^d)\left(p_G^{(t)}(\hat{x},\hat{x}) + p_G^{(t)}(\hat{y},\hat{y}) - 2p_G^{(t)}(\hat{x},\hat{y})\right).$$

This condition can be written in terms of polynomials:

Theorem 4.1 *(Two orbits) Let G be a finite group with a unitary action on $\mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$. Then every generic pair of orbits has a unique weighting which is a spherical (t,t)-design if and only if the polynomial $f_G^{(t)} = f_{G,\mathbb{F}}^{(t)} : \mathbb{F}^d \times \mathbb{F}^d \to \mathbb{F}$ given by*

$$f_G^{(t)}(x,y) := p_G^{(t)}(x,x)p_G^{(t)}(y,y) - (p_G^{(t)}(x,y))^2 \tag{4.14}$$

is not identically zero, and

$$f_G^{(t)}(x,y) = c_t(\mathbb{F}^d)\left(\|y\|^{4t}p_G^{(t)}(x,x) + \|x\|^{4t}p_G^{(t)}(y,y) - 2\|x\|^{2t}\|y\|^{2t}p_G^{(t)}(x,y)\right). \tag{4.15}$$

where $p_G^{(t)}$ is given by (4.13).

Proof: Use $p_G^{(t)}(\hat{x},\hat{y}) = \frac{1}{\|x\|^{2t}}\frac{1}{\|y\|^{2t}}p_G^{(t)}(x,y)$ to rewrite the previous conditions, and then multiply by $\|x\|^{4t}\|y\|^{4t}$. □

Here the condition that the orbits $(gx)_{g\in G}$ and $(gy)_{g\in G}$ be generic is $f_G^{(t)}(x,y)\neq 0$. Clearly, $f_G^{(t)}(x,y)=0$ if the orbits are equal or if both are spherical (t,t)-designs. By way of comparison, the condition that every single orbit is a spherical (t,t)-design is that

$$p_G^{(t)}(x,x)=c_t(\mathbb{F}^d)\|x\|^{4t}.$$

We will say that "pairs of orbits give (t,t)-designs", or similar, if (4.15) holds nontrivially.

Theorem 4.1 provides a computational way to verify when a generic pair of orbits has a unique weighting giving a spherical (t,t)-design. We were able to make this computation in magma for various groups G. Our preliminary results suggest:

> *Pairs of orbits give spherical (t,t)-designs with t higher than the generic strength, for all complex reflection groups except the Coxeter group $D_4=G(2,2,4)$. This also holds for many, but not all, irreducible representations.*

The exact nature of these results is not yet clear, though it is related to the irreducible unitarily invariant subspaces $H(p,q)$ of the polynomials on $\mathbb{C}^d\cong\mathbb{R}^{2d}$ (see [Rud80]) that are integrated by the cubature rule for a generic orbit.

Since the sum in (4.13) is over all elements of the group G, and cannot be simplified, e.g., by taking a transversal giving an orbit of small size (as for highly symmetric tight frames) our calculations do not extend to all the groups considered in Section 5.

We now give some selected examples.

Example 4.1 *Let G be the dihedral group of order 6 (a reflection group) generated by*

$$a=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\quad\text{(rotation by } \tfrac{2\pi}{3}\text{)},\qquad b=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad\text{(reflection in the } x\text{-axis).}$$

This is the first (faithful) irreducible group action in more than one dimension.

If G acts on $\mathbb{R}^2$, then every orbit is a $(2,2)$-design, so that

$$f_{G,\mathbb{R}}^{(1)}=f_{G,\mathbb{R}}^{(2)}=0,$$

and pairs of orbits give (t,t)-designs for $\mathbb{R}^2$ for $t=3,4,5$. Here

$$f_{G,\mathbb{R}}^{(3)}(x,y)=10\prod_{U\in\mathcal{U}}(\langle x,Uy\rangle)^2,$$

where $\mathcal{U}$ is the set of unitary matrices

$$\mathcal{U}:=\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}\frac{\sqrt{3}}{2}&\frac{1}{2}\\ \frac{1}{2}&-\frac{\sqrt{3}}{2}\end{pmatrix},\begin{pmatrix}\frac{\sqrt{3}}{2}&\frac{1}{2}\\ -\frac{1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix},\begin{pmatrix}\frac{\sqrt{3}}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix},\begin{pmatrix}\frac{\sqrt{3}}{2}&-\frac{1}{2}\\ -\frac{1}{2}&-\frac{\sqrt{3}}{2}\end{pmatrix}\right\},$$

and

$$f_{G,\mathbb{R}}^{(4)}(x,y)=\tfrac{7}{4}\|x\|^4\|y\|^4 f_{G,\mathbb{R}}^{(3)}(x,y),\qquad f_{G,\mathbb{R}}^{(5)}(x,y)=(\tfrac{7}{4}\|x\|^4\|y\|^4)^2 f_{G,\mathbb{R}}^{(3)}(x,y).$$

It is not obvious from the definition (4.14) that these polynomials should be squares (or have common factors), or how the matrices in $\mathcal{U}$ relate the elements of G. If G acts on $\mathbb{C}^2$, then every orbit is a $(1,1)$-design, and pairs of orbits give $(2,2)$-designs for $\mathbb{C}^2$, where

$$f^{(2)}_{G,\mathbb{C}}(x,y) = \tfrac{1}{8}\big(\|x\|^2\alpha(y)+\|y\|^2\beta(x)\big)^2\big(\|x\|^2\alpha(y)+\|y\|^2\beta(\overline{x})\big)^2,$$

with

$$\alpha(y) := y_1\overline{y_2} - \overline{y_1}y_2, \qquad \beta(x) := x_1\overline{x_2} - \overline{x_1}x_2.$$

The lines in a spherical (t,t)-design for $\mathbb{C}^d$ which is an orbit depend only the the matrices in the action group of the representation up to unit scalar multiples. Hence for the purpose of calculation, it suffices to take a representative set of such matrices. A convenient way to do this, is to take the associated group obtained by normalising the matrices to have determinant 1 (and taking all d such choices). This subgroup of $\mathcal{SU}(\mathbb{C}^d)$ (as an abstract group) was called a *canonical abstract error group* in [CW17].

The finite subgroups of $\mathcal{SU}(\mathbb{C}^2)$ are given by the ADE classification: the binary tetrahedral, octahedral and icosahedral groups, together with the binary dihedral groups $\mathcal{D}_{2m}$ of order $4m$, which are generated by the matrices

$$a = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega := e^{\frac{2\pi i}{2m}}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Except for $\mathcal{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, these are all irreducible (see Theorem 5.14 of [LT09] for details). A summary of our calculations for these groups is given in Table 1.

Table 1: The unions of pairs of orbits for the irreducible subgroups of $\mathcal{SU}(\mathbb{C}^2)$ (these correspond to all irreducible representations). Here t_{generic} is the strength of a generic orbit, and t_{pairs} is the range of t for which pairs of orbits give spherical (t,t)-designs.

Subgroup of $\mathcal{SU}(\mathbb{C}^2)$	order	#lines	t_{generic}	t_{pairs}	comments
Binary tetrahedral group $\mathcal{T}$	24	12	1-2	3	ST 4-7 (type $\mathcal{T}$)
Binary octahedral group $\mathcal{O}$	48	24	1-3	4-5	ST 8-15 (type $\mathcal{O}$)
Binary icosahedral group $\mathcal{I}$	120	60	1-5	6-9	ST 16-22 (type $\mathcal{I}$)
Binary dihedral group $\mathcal{D}_4$	8	4	1	{}	associated real group
Binary dihedral group $\mathcal{D}_6$	12	6	1	2	associated real group
Binary dihedral group $\mathcal{D}_{2m}$	$4m \geq 16$	$2m$	1	2-3	associated real group

The binary dihedral groups come from real representations, and the corresponding pairs of real orbits (see Table 2) give real spherical (t,t)-designs. Let $D_{2m} = G(m,m,2)$ be the dihedral group of order $2m$ generated by

$$a = \begin{pmatrix} \cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{pmatrix}, \quad \text{(rotation by } \tfrac{2\pi}{m}\text{)}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(reflection in the } x\text{-axis)},$$

and $R_m = \langle a \rangle$ be the rotation subgroup. Since b is a reflection, it does not have determinant 1. Multiplying it by the scalar i gives a matrix in $\mathcal{SU}(\mathbb{C}^2)$. The subgroup $\langle a, ib \rangle$ of $\mathcal{SU}(\mathbb{C}^2)$ is conjugate to $\mathcal{D}_{2m}$ for m odd, and is conjugate to $\mathcal{D}_m$ for m even.

Table 2: The unions of pairs of orbits for the irreducible subgroups of $\mathcal{O}(\mathbb{R}^2)$ (these correspond to all irreducible represention).

Subgroup of $\mathcal{O}(\mathbb{R}^2)$	order	#lines	t_{generic}	t_{pairs}	comments
Dihedral group D_{2m}	$2m$ (m odd)	$2m$	$1,..,(m-1)$	$m,..,(2m-1)$	$m \geq 3$
Dihedral group D_{2m}	$2m$ (m even)	m	$1,..,(\frac{m}{2}-1)$	$\frac{m}{2},..,(m-1)$	$m \geq 4$
Rotation group R_m	m (m odd)	m	$m-1$	$\{\}$	$m \geq 3$
Rotation group R_m	m (m even)	$\frac{m}{2}$	$m-1$	$\{\}$	$m \geq 4$

We now list some additional calculations (Table 3). These include the Heisenberg group $\mathcal{H}_d$ in d dimensions, which is generated by a cyclic shift S and the modulation Ω, where $Se_j = e_{j+1}$, $\Omega e_j = \omega^j e_j$, $j \in \mathbb{Z}_d$.

Table 3: The unions of pairs of orbits for selected groups. Here $G = G(m,p,n)$, $p \mid m$ ($|G| = m^n n!/p$), is the infinite family in Shephard-Todd classification, and the groups B_d, D_d, H_3, F_4 in brackets are from the Coxeter classification of real reflection groups.

Group	d	order	#lines	t_{generic}	t_{pairs}	comments
G_{23} (ST 23, H_3)	3	120	60	1-2	3-4	real group
G_{24} (ST 24)	3	336	168	1-2	3	complex group
G_{25} (ST 25)	3	648	216	1-2	3	complex group
G_{26} (ST 26)	3	1296	216	1-2	3	complex group
G_{28} (ST 28, F_4)	4	1152	576	1-2	3	real group
$G(2,2,4)$ (D_4)	4	192	96	1	$\{\}$	Example 4.2
$G(2,2,d)$ (D_d)	d	$2^{d-1}d!$		1	2	$d \neq 4$, $3 \leq d \leq 7$
$G(2,1,4)$ (B_4)	2	348	192	1	2-3	
$G(2,1,d)$ (B_d)	d	$2^d d!$	$2^{d-1}d!$	1	2	$d \neq 2$, $2 \leq d \leq 6$
$\mathcal{H}_2 = D_8$	2	8	4	1	2-3	$G(2,1,2) \cong G(4,4,2)$
$\mathcal{H}_d$	d	d^3	d^2	1	$\{\}$	$d \geq 3$, Example 4.2

Example 4.2 *The real reflection group $G(2,2,4)$ was the only complex reflection group we considered for which pairs of orbits do not give spherical (t,t)-designs. Even in this case, some pairs of highly symmetric tight frames still give higher order (t,t)-designs (see Table 9). Also, the Heisenberg groups $\mathcal{H}_d$, $d \geq 3$ (which are not complex reflection groups) do not have the property that pairs of orbits give spherical (t,t)-designs.*

5 Summary of calculations

In the following tables we summarise our calculations to find a weighting (β_X, β_Y) so that a union of highly symmetric tight frames X and Y for a complex reflection group G is a spherical (t,t)-design with t larger than the generic strength of an orbit.

Here ST is the Shephard-Todd number of G acting on $\mathbb{R}^d, \mathbb{C}^d$, t is the strength of the union $X \cup Y$ above that of a generic orbit, and n is the number of lines in the union. We note that there is at least one highly symmetric tight frame for each group. Such frames are identified by their number of lines, with 4_2 in Table 4 indicating that either of the two highly symmetric tight frames of 4 points/lines can be taken.

5.1 Primitive complex reflection groups

Table 4: The unions of pairs of the highly symmetric tight frames for $\mathbb{C}^2$ given by the complex reflection groups with Shephard-Todd numbers 4-22 which are (t,t)-designs. The groups $ST = 4, 5, 8, 12, 16, 20, 22$ have only one highly symmetric tight frame.

ST	d	order	t	n	$\|X\|,\ \|Y\|$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
6	2	48	3	6	4, 6	$0,\ 1$	$0,\ 1$
7	2	144	3	8	4, 4	$\frac{1}{2},\ \frac{1}{2}$	$1,\ 1$
			3	6	4_2, 6	$0,\ 1$	$0,\ 1$
9	2	192	4-5	18	6, 12	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
10	2	288	4-5	14	6, 8	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
11	2	576	4-5	14	6, 8	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
			4-5	18	6, 12	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
			4-5	20	8, 12	$-\frac{3}{5},\ \frac{8}{5}$	$-\frac{3}{2},\ \frac{8}{3}$
13	2	96	4-5	18	6, 12	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
14	2	144	4-5	20	8, 12	$-\frac{3}{5},\ \frac{8}{5}$	$-\frac{3}{2},\ \frac{8}{3}$
15	2	288	4-5	14	6, 8	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
			4-5	18	6, 12	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
			4-5	20	8, 12	$-\frac{3}{5},\ \frac{8}{5}$	$-\frac{3}{2},\ \frac{8}{3}$
17	2	1200	6-9	42	12, 30	$\frac{5}{21},\ \frac{16}{21}$	$\frac{5}{6},\ \frac{16}{15}$
18	2	1800	6-9	32	12, 20	$\frac{5}{14},\ \frac{9}{14}$	$\frac{20}{21},\ \frac{36}{35}$
19	2	3600	6-9	32	12, 20	$\frac{5}{14},\ \frac{9}{14}$	$\frac{20}{21},\ \frac{36}{35}$
			6-9	42	12, 30	$\frac{5}{21},\ \frac{16}{21}$	$\frac{5}{6},\ \frac{16}{15}$
			6-9	50	20, 30	$-\frac{9}{7},\ \frac{16}{7}$	$-\frac{45}{14},\ \frac{80}{21}$
21	2	720	6-9	50	20, 30	$-\frac{9}{7},\ \frac{16}{7}$	$-\frac{45}{14},\ \frac{80}{21}$

Table 5: The unions of pairs of the highly symmetric tight frames for $\mathbb{C}^d$ given by the complex reflection groups with Shephard-Todd numbers in the range 23-37 which are (t,t)-designs. The groups $ST = 23, 28, 30, 35, 36, 37$ are real reflection groups.

ST	d	order	t	n	$\lvert X\rvert,\ \lvert Y\rvert$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
24	3	336	3	49	21, 28	$\frac{26}{35},\ \frac{9}{35}$	$\frac{26}{15},\ \frac{9}{20}$
25	3	648	3	21	9, 12	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
26	3	1296	3	21	9, 12	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
			3	45	9, 36	$-\frac{1}{5},\ \frac{6}{5}$	$-1,\ \frac{3}{2}$
			3-4	48	12, 36	$\frac{1}{5},\ \frac{4}{5}$	$\frac{4}{5},\ \frac{16}{15}$
27	3	2160	4	81	36, 45	$\frac{5}{9},\ \frac{4}{9}$	$\frac{5}{4},\ \frac{4}{5}$
			4	96	$36,\ 60_2$	$\frac{5}{8},\ \frac{3}{8}$	$\frac{5}{3},\ \frac{3}{5}$
			4	105	$45,\ 60_2$	$4,\ -3$	$\frac{28}{3},\ -\frac{21}{4}$
29	4	7680	3	60	20, 40	$\frac{1}{3},\ \frac{2}{3}$	$1,\ 1$
			3	100	$20,\ 80_2$	$-\frac{1}{3},\ \frac{4}{3}$	$-\frac{5}{3},\ \frac{5}{3}$
			3	120	$40,\ 80_2$	$\frac{1}{3},\ \frac{2}{3}$	$1, 1$
			3	180	20, 160	$-\frac{7}{11},\ \frac{18}{11}$	$-\frac{63}{11},\ \frac{81}{44}$
			3	200	40, 160	$\frac{7}{16},\ \frac{9}{16}$	$\frac{35}{16},\ \frac{45}{64}$
			3	240	$80_2,\ 160$	$\frac{14}{5},\ -\frac{9}{5}$	$\frac{42}{5},\ -\frac{27}{10}$
31	4	46080	4-5	540	60, 480	$\frac{5}{21},\ \frac{16}{21}$	$\frac{15}{7},\ \frac{6}{7}$
			4-5	1020	60, 960	$\frac{1}{28},\ \frac{27}{28}$	$\frac{17}{28},\ \frac{459}{448}$
			4-5	1440	480, 960	$-\frac{16}{119},\ \frac{135}{119}$	$-\frac{48}{119},\ \frac{405}{238}$
32	4	155520	4-5	400	40, 360	$\frac{1}{7},\ \frac{6}{7}$	$\frac{10}{7},\ \frac{20}{21}$
33	5	51840	3	85	40, 45	$\frac{3}{7},\ \frac{4}{7}$	$\frac{51}{56},\ \frac{68}{63}$
			3	256	40, 216	$\frac{3}{28},\ \frac{25}{28}$	$\frac{24}{35},\ \frac{200}{189}$
			3	261	45, 216	$-\frac{4}{21},\ \frac{25}{21}$	$-\frac{116}{105},\ \frac{725}{504}$
			3	580	40, 540	$-\frac{15}{49},\ \frac{64}{49}$	$-\frac{435}{98},\ \frac{1856}{1323}$
			3	585	45, 540	$\frac{5}{21},\ \frac{16}{21}$	$\frac{65}{21},\ \frac{52}{63}$
			3	756	216, 540	$\frac{125}{189},\ \frac{64}{189}$	$\frac{125}{54},\ \frac{64}{135}$
34	6	39191040	4	672	$672,\ \lvert Y\rvert$	$1,\ 0$	$1,\ 0$
			4	9072	$9072,\ \lvert Y\rvert$	$1,\ 0$	$1,\ 0$
			4-5	3528	126, 3402	$\frac{1}{9},\ \frac{8}{9}$	$\frac{28}{9},\ \frac{224}{243}$
			4	5166	126, 5040	$-\frac{1}{4},\ \frac{5}{4}$	$-\frac{41}{4},\ \frac{41}{32}$
			4	8442	3402, 5040	$\frac{8}{13},\ \frac{5}{13}$	$\frac{536}{351},\ \frac{67}{104}$
			4-5	9744_2	$672,\ 9072_2$	$-\frac{1}{5},\ \frac{6}{5}$	$-\frac{29}{10},\ \frac{58}{45}$
			4	18144	9072, 9072	$\beta,\ 1-\beta$	$2\beta, 2(1-\beta)$
			≥ 4	27342	126, 27216	$-\frac{13}{612},\ \frac{625}{612}$	$-\frac{2821}{612},\ \frac{135625}{132192}$
			≥ 4	30618	3402, 27216	$\frac{104}{729},\ \frac{625}{729}$	$\frac{104}{81},\ \frac{625}{648}$
			≥ 4	32256	5040, 27216	$-\frac{13}{112},\ \frac{125}{112}$	$-\frac{26}{35},\ \frac{250}{189}$
			≥ 4	45486	126, 45360	$-\frac{3}{37},\ \frac{40}{37}$	$-\frac{1083}{37},\ \frac{361}{333}$

Table 6: The unions of pairs of the highly symmetric tight frames for $\mathbb{R}^d$ given by the real reflection groups G with Shephard-Todd numbers $23, 28, 30, 35, 36, 37$ which are spherical (t,t)-designs. The Coxeter classification names are included under G.

ST	G	d	order	t	n	$\|X\|,\ \|Y\|$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
23	H_3	3	120	3-4	16	6, 10	$\frac{5}{14},\ \frac{9}{14}$	$\frac{20}{21},\ \frac{36}{35}$
				3-4	21	6, 15	$\frac{5}{21},\ \frac{16}{21}$	$\frac{5}{6},\ \frac{16}{15}$
				3-4	25	10, 15	$-\frac{9}{7},\ \frac{16}{7}$	$-\frac{45}{14},\ \frac{80}{21}$
28	F_4	4	1152	3	24	12, 12	$\frac{1}{2},\ \frac{1}{2}$	$1,\ 1$
				3	60_2	12, 48	$-\frac{1}{8},\ \frac{9}{8}$	$-\frac{5}{8},\ \frac{45}{32}$
				3	60_2	12, 48	$\frac{1}{10},\ \frac{9}{10}$	$\frac{1}{2},\ \frac{9}{8}$
				3	96	48, 48	$\frac{1}{2},\ \frac{1}{2}$	$1,\ 1$
30	H_4	4	14400	6-9	360	60, 300	$\frac{5}{21},\ \frac{16}{21}$	$\frac{10}{7},\ \frac{32}{35}$
				6-9	420	60, 360	$\frac{3}{28},\ \frac{25}{28}$	$\frac{3}{4},\ \frac{25}{24}$
				6-9	660	60, 600	$\frac{65}{308},\ \frac{234}{308}$	$\frac{65}{28},\ \frac{243}{280}$
				6-9	660	300, 360	$-\frac{48}{77},\ \frac{125}{77}$	$-\frac{48}{35},\ \frac{125}{42}$
				6-9	900	300, 600	$-\frac{208}{35},\ \frac{243}{35}$	$-\frac{624}{35},\ \frac{729}{70}$
				6-9	960	360, 600	$\frac{1625}{896},\ -\frac{729}{896}$	$\frac{1625}{336},\ -\frac{729}{560}$
35	E_6	6	51840	3	63	27, 36	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
				3	243	27, 216	$\frac{2}{27},\ \frac{25}{27}$	$\frac{2}{3},\ \frac{25}{24}$
				3	252	36, 216	$-\frac{3}{22},\ \frac{25}{22}$	$-\frac{21}{22},\ \frac{175}{132}$
				3	387	27, 360	$\frac{2}{11},\ \frac{9}{11}$	$\frac{86}{33},\ \frac{387}{440}$
				3	396	36, 360	$-\frac{1}{2},\ \frac{3}{2}$	$-\frac{11}{2},\ \frac{33}{20}$
				3	576	216, 360	$\frac{25}{16},\ -\frac{9}{16}$	$\frac{25}{6},\ -\frac{9}{10}$
36	E_7	7	2903040	3	91	28, 63	$\frac{3}{11},\ \frac{8}{11}$	$\frac{39}{44},\ \frac{104}{99}$
				3	316	28, 288	$\frac{6}{55},\ \frac{49}{55}$	$\frac{474}{385},\ \frac{3871}{3960}$
				3	351	63, 288	$-\frac{16}{33},\ \frac{49}{33}$	$-\frac{208}{77},\ \frac{637}{352}$
				3	406	28, 378	$-\frac{9}{55},\ \frac{64}{55}$	$-\frac{261}{110},\ \frac{1856}{1485}$
				3	441	63, 378	$\frac{3}{11},\ \frac{8}{11}$	$\frac{21}{11},\ \frac{28}{33}$
				3	666	288, 378	$\frac{147}{275},\ \frac{128}{275}$	$\frac{5439}{4400},\ \frac{4736}{5775}$
				3	1036	28, 1008	$\frac{7}{55},\ \frac{48}{55}$	$\frac{259}{55},\ \frac{148}{165}$
				3	1071	63, 1008	$-\frac{7}{11},\ \frac{18}{11}$	$-\frac{119}{11},\ \frac{153}{88}$
				3	1296	288, 1008	$\frac{343}{55},\ -\frac{288}{55}$	$\frac{3087}{110},\ -\frac{2592}{385}$
				3	1386	378, 1008	$\frac{28}{55},\ \frac{27}{55}$	$\frac{28}{15},\ \frac{27}{40}$
				3	2044	28, 2016	$\frac{2}{77},\ \frac{75}{77}$	$\frac{146}{77},\ \frac{1825}{1848}$
				3	2079	63, 2016	$-\frac{16}{209},\ \frac{225}{209}$	$-\frac{48}{19},\ \frac{675}{608}$
				3	2304	288, 2016	$-\frac{49}{176},\ \frac{225}{176}$	$-\frac{49}{22},\ \frac{225}{154}$
				3	2394	378, 2016	$\frac{128}{803},\ \frac{675}{803}$	$\frac{2432}{2409},\ \frac{12825}{12848}$
				3	3024	1008, 2016	$-\frac{32}{143},\ \frac{175}{143}$	$-\frac{96}{143},\ \frac{525}{286}$
				3	5068	28, 5040	$\frac{17}{209},\ \frac{192}{209}$	$\frac{3077}{209},\ \frac{2896}{3135}$
				3	5103	63, 5040	$-\frac{17}{55},\ \frac{72}{55}$	$-\frac{1377}{55},\ \frac{729}{550}$

				3	5328	288, 5040	$-\frac{883}{319}, \frac{1152}{319}$	$-\frac{30821}{638}, \frac{42624}{11165}$
				3	5418	378, 5040	$\frac{17}{44}, \frac{27}{44}$	$\frac{731}{132}, \frac{1161}{1760}$
				3	6048	1008, 5040	$-\frac{17}{11}, \frac{28}{11}$	$-\frac{102}{11}, \frac{168}{55}$
				3	7056	2016, 5040	$\frac{425}{297}, -\frac{128}{297}$	$\frac{2975}{594}, -\frac{896}{1485}$
37	E_8	8	696729600	4-5	1200	120, 1080	$\frac{1}{7}, \frac{6}{7}$	$\frac{10}{7}, \frac{20}{21}$
				4-5	3480	120, 3360	$-\frac{1}{8}, \frac{9}{8}$	$-\frac{29}{8}, \frac{261}{224}$
				4-5	4440	1080, 3360	$\frac{2}{5}, \frac{3}{5}$	$\frac{74}{45}, \frac{111}{140}$
				4-5	8760	120, 8640	$\frac{3}{35}, \frac{32}{35}$	$\frac{219}{35}, \frac{292}{315}$
				4-5	9720	1080, 8640	$-\frac{9}{7}, \frac{16}{7}$	$-\frac{81}{7}, \frac{18}{7}$
				4-5	12000	3360, 8640	$\frac{27}{59}, \frac{32}{59}$	$\frac{675}{413}, \frac{400}{531}$
				4-5	30360	120, 30240	$\frac{1}{55}, \frac{54}{55}$	$\frac{23}{5}, \frac{69}{70}$
				4-5	31320	1080, 30240	$-\frac{1}{8}, \frac{9}{8}$	$\frac{29}{8}, \frac{261}{224}$
				4-5	33600	3360, 30240	$\frac{1}{7}, \frac{6}{7}$	$\frac{10}{7}, \frac{20}{21}$
				4-5	34680	120, 34560	$\frac{33}{376}, \frac{343}{376}$	$\frac{9537}{376}, \frac{99127}{108288}$
				4-5	35640	1080, 34560	$-\frac{198}{145}, \frac{343}{145}$	$-\frac{6534}{145}, \frac{11319}{4640}$
				4-5	37920	3360, 34560	$\frac{297}{640}, \frac{343}{640}$	$\frac{23463}{4480}, \frac{27097}{46080}$
				4-5	38880	8640, 30240	$-\frac{16}{65}, \frac{81}{65}$	$-\frac{72}{65}, \frac{729}{455}$
				4-5	43200	8640, 34560	$\frac{352}{9}, -\frac{343}{9}$	$\frac{1760}{9}, -\frac{1715}{36}$
				4-5?	64800	30240, 34560	$\frac{1782}{1439}, -\frac{343}{1439}$	$\frac{26730}{10073}, -\frac{5145}{11512}$
				4-5?	121080	120, 120960	$\frac{3}{53}, \frac{50}{53}$	$\frac{3027}{53}, \frac{25225}{26712}$
				4-5?	122040	1080, 120960	$-\frac{9}{16}, \frac{25}{16}$	$-\frac{1017}{16}, \frac{2825}{1792}$
				4-5?	124320	3360, 120960	$\frac{27}{77}, \frac{50}{77}$	$\frac{999}{77}, \frac{925}{1386}$

5.2 Imprimitive complex reflection groups

Table 7: Selected examples for the Coxeter groups $A_d = G(1,1,d+1) \cong S_{d+1}$, $d \geq 2$. These n vector spherical $(2,2)$-designs give rise to spherical 5-designs with $2n$ vectors.

ST	d	order	t	n	$\lvert X\rvert,\ \lvert Y\rvert$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
(1,1,4)	3	24	2	7	3, 4	$\frac{2}{5}, \frac{3}{5}$	$\frac{14}{15}, \frac{21}{20}$
(1,1,5)	4	120	2	15	5, 10	$\frac{2}{5}, \frac{3}{5}$	$\frac{6}{5}, \frac{9}{10}$
(1,1,6)	5	720	2	16	6, 10	$\frac{5}{14}, \frac{9}{14}$	$\frac{20}{21}, \frac{36}{35}$
			2	21	6, 15	$\frac{5}{21}, \frac{16}{21}$	$\frac{5}{6}, \frac{16}{15}$
(1,1,7)	6	5040	2	28	7, 21	$\frac{3}{28}, \frac{25}{28}$	$\frac{3}{7}, \frac{25}{21}$
			2	42	7, 35	$\frac{2}{7}, \frac{5}{7}$	$\frac{12}{7}, \frac{6}{7}$
(1,1,8)	7	40320	2	28	28, $\lvert Y\rvert$	1, 0	1, 0
			2	43	8, 35	$\frac{7}{27}, \frac{20}{27}$	$\frac{301}{216}, \frac{172}{189}$
			2	64	8, 56	$\frac{7}{32}, \frac{250}{32}$	$\frac{7}{4}, \frac{25}{28}$
(1,1,9)	8	362880	2	45	9, 36	$-\frac{4}{45}, \frac{49}{45}$	$-\frac{4}{9}, \frac{49}{36}$
			2	93	9, 84	$\frac{4}{25}, \frac{21}{25}$	$\frac{124}{75}, \frac{93}{100}$
(1,1,10)	9	3628800	2	55	10, 45	$-\frac{9}{55}, \frac{64}{55}$	$-\frac{9}{10}, \frac{64}{45}$
			2	130	10, 120	$\frac{6}{55}, \frac{49}{55}$	$\frac{78}{55}, \frac{637}{660}$
(1,1,11)	10	39916800	2	66	11, 55	$-\frac{5}{22}, \frac{27}{22}$	$-\frac{15}{11}, \frac{81}{55}$
			2	176	11, 165	$\frac{5}{77}, \frac{72}{77}$	$\frac{80}{77}, \frac{384}{385}$

Table 8: Selected examples for the Coxeter groups $B_d = G(2,1,d)$, $d \geq 2$, for orbits of the vectors $x = e_1 + e_2 + \cdots + e_k$, $1 \leq k \leq d$.

ST	d	order	t	n	$\lvert X\rvert,\ \lvert Y\rvert$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
(2,1,2)	2	8	2-3	4	2, 2	$\frac{1}{2},\ \frac{1}{2}$	$1,\ 1$
(2,1,3)	3	48	2	7	3, 4	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
			2	9	3, 6	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
(2,1,4)	4	384	2	12	4, 8	$\frac{1}{3},\ \frac{2}{3}$	$1,\ 1$
			2	12	12, $\lvert Y\rvert$	$1,\ 0$	$1,\ 0$
			2	20	4, 16	$\frac{1}{4},\ \frac{3}{4}$	$\frac{5}{4},\ \frac{15}{16}$
(2,1,5)	5	3840	2	21	5, 16	$\frac{2}{7},\ \frac{5}{7}$	$\frac{6}{5},\ \frac{15}{16}$
			2-3	45	5, 40	$\frac{1}{7},\ \frac{6}{7}$	$\frac{9}{7},\ \frac{27}{28}$
(2,1,6)	6	46080	2	36	6, 30	$-\frac{1}{4},\ \frac{5}{4}$	$-\frac{3}{2},\ \frac{3}{2}$
			2	38	6, 32	$\frac{1}{4},\ \frac{3}{4}$	$\frac{19}{12},\ \frac{57}{64}$
(2,1,7)	7	645120	2	49	7, 42	$-\frac{1}{3},\ \frac{4}{3}$	$-\frac{7}{3},\ \frac{14}{9}$
			2	71	7, 64	$\frac{2}{9},\ \frac{7}{9}$	$\frac{142}{63},\ \frac{497}{576}$
(2,1,8)	8	10321920	2	64	8, 56	$-\frac{2}{5},\ \frac{7}{5}$	$-\frac{16}{5},\ \frac{8}{5}$
			2	136	8, 128	$\frac{1}{5},\ \frac{4}{5}$	$\frac{17}{5},\ \frac{17}{20}$
			2-3	184	56, 128	$\frac{7}{15},\ \frac{8}{15}$	$\frac{23}{15},\ \frac{23}{30}$
			2-3	568	8, 560	$\frac{1}{15},\ \frac{14}{15}$	$\frac{71}{15},\ \frac{71}{75}$
(2,1,9)	9	185794560	2	81	9, 72	$-\frac{5}{11},\ \frac{16}{11}$	$-\frac{45}{11},\ \frac{18}{11}$
			2	265	9, 256	$\frac{2}{11},\ \frac{9}{11}$	$\frac{530}{99},\ \frac{2385}{2816}$
(2,1,10)	10	3715891200	2	100	10, 90	$-\frac{1}{2},\ \frac{3}{2}$	$-5,\ \frac{5}{3}$
			2	490	10, 480	$-\frac{1}{8},\ \frac{9}{8}$	$-\frac{49}{48},\ \frac{147}{128}$
			2	522	10, 512	$\frac{1}{6},\ \frac{5}{6}$	$\frac{87}{10},\ \frac{435}{512}$
			2-3	7770	90, 7680	$\frac{3}{10},\ \frac{7}{10}$	$\frac{259}{10},\ \frac{1813}{2560}$

Table 9: Selected examples for the Coxeter groups $D_d = \mathrm{G}(2,2,d)$, $d \geq 3$, for orbits of the vectors $x = e_1 + e_2 + \cdots + e_k$, $1 \leq k \leq d$. Note that $G(2,2,2)$ is not irreducible, and $G(2,2,3) \cong G(1,1,4)$.

ST	d	order	t	n	$\|X\|,\ \|Y\|$	$\beta_X,\ \beta_Y$	$\hat{w}_X,\ \hat{w}_Y$
(2,2,3)	3	24	2	7	3, 4	$\frac{2}{5},\ \frac{3}{5}$	$\frac{14}{15},\ \frac{21}{20}$
			2	9	3, 6	$\frac{1}{5},\ \frac{4}{5}$	$\frac{3}{5},\ \frac{6}{5}$
			2	10	4, 6	$-\frac{3}{5},\ \frac{8}{5}$	$-\frac{3}{2},\ \frac{8}{3}$
(2,2,4)	4	192	{}	8	4, 4		
			2	12	12, $\|Y\|$	1, 0	1, 0
			2	20	4, 16	$\frac{1}{4},\ \frac{3}{4}$	$\frac{5}{4},\ \frac{15}{16}$
			{}	20	4, 16		
(2,2,5)	5	1920	2	21	5, 16	$\frac{2}{7},\ \frac{5}{7}$	$\frac{6}{5},\ \frac{15}{16}$
(2,2,6)	6	23040	2	22	6, 16	$\frac{1}{4},\ \frac{3}{4}$	$\frac{11}{12},\ \frac{33}{32}$
(2,2,7)	7	322560	2	49	7, 42	$-\frac{1}{3},\ \frac{4}{3}$	$-\frac{7}{3},\ \frac{14}{9}$
			2	71	7, 64	$\frac{2}{9},\ \frac{7}{9}$	$\frac{142}{63},\ \frac{497}{576}$
			2	140	140, $\|Y\|$	1, 0	1, 0
(2,2,8)	8	5160960	2	64	8, 56	$-\frac{2}{5},\ \frac{7}{5}$	$-\frac{16}{5},\ \frac{8}{5}$
			2	72	8, 64	$\frac{1}{5},\ \frac{4}{5}$	$\frac{9}{5},\ \frac{9}{10}$
			2-3	120	56, 64	$\frac{7}{15},\ \frac{8}{15}$	1, 1
			2-3	568	8, 560	$\frac{1}{15},\ \frac{14}{15}$	$\frac{71}{15},\ \frac{71}{75}$
(2,2,9)	9	92897280	2	81	9, 72	$-\frac{5}{11},\ \frac{16}{11}$	$-\frac{45}{11},\ \frac{18}{11}$
			2	265	9, 256	$\frac{2}{11},\ \frac{9}{11}$	$\frac{530}{99},\ \frac{2385}{2816}$
(2,1,10)	10	1857945600	2	100	10, 90	$-\frac{1}{2},\ \frac{3}{2}$	$-5,\ \frac{5}{3}$
			2	266	10, 256	$\frac{1}{6},\ \frac{5}{6}$	$\frac{133}{30},\ \frac{665}{768}$
			2	1680	1680, $\|Y\|$	1, 0	1, 0
			2-3	7770	90, 7680	$\frac{3}{10},\ \frac{7}{10}$	$\frac{259}{10},\ \frac{1813}{2560}$

5.3 Observations and examples

We first observe that it is possible to have a weighting (β_X, β_Y) with a *negative value* that gives a union $X \cup Y$ which is a (t,t)-design of higher order. We will refer to a (t,t)-design with some negative weights as a **signed** (t,t)-design. Signed $(1,1)$-designs were first studied in [PW02], where they were called *signed tight frames* and defined as systems with

$$f = \sum_j c_j \langle f, \phi_j \rangle \phi_j, \qquad \forall f \in \mathbb{F}^d,$$

where $c_j \in \mathbb{R}$ and $\phi_j \in \mathbb{S}$. The equivalence of these two notions is easily proved.

In some cases, pairs of orbits can give (t,t)-designs with strength $t_{\text{pairs}} > t_{\text{generic}} + 1$. We illustrate the mechanism for this with an example. Consider the Shephard-Todd groups numbered 9 to 15. For these, an orbit gives a cubature rule for $P = \text{Hom}(t,t)$, $t = 1, 2, 3, 5$, as does any union of orbits. Thus a union of orbits is a $(4,4)$-design if and only if it is a $(5,5)$-design. There are also examples, such as the Shephard-Todd group 26, where some, but not all, pairs of orbits have strength greater than $t_{\text{generic}} + 1$.

In [HW18b] a numerical study was done to find "putatively optimal" (t,t)-designs. We now consider our constructions in relation to the table in [HW18b] (and [Wal18]).

Example 5.1 *For* $\mathbb{C}^2$ *the putatively optimal* (t,t)*-designs for* $t = 3, 4, 5$ *come as highly symmetric tight frames (one orbit). For* $t = 8, 9$ *the putatively optimal number of vectors was estimated to be* 37 *and* 44*. Since we have constructed a* $(9,9)$*-design of* 32 *vectors for* $\mathbb{C}^2$ *as a union of orbits of size* 12 *and* 20*, these numbers can be improved.*

Example 5.2 *For* $\mathbb{C}^3$ *the putatively optimal* $(3,3)$*-design had* 22 *vectors, and we give one with* 21 *vectors. The putatively optimal* $(4,4)$*-design had* 47 *vectors, and we give one with* 48 *vectors.*

Example 5.3 *For* $\mathbb{C}^4$ *the putatively optimal* $(4,4)$*-design had more than* 85 *vectors, and there was no estimate for* $(5,5)$*-designs. Here we give a* $(5,5)$*-design with* 400 *vectors.*

Example 5.4 *For* $\mathbb{C}^5$ *the putatively optimal* $(3,3)$*-design had more than* 100 *vectors. Here we give a* $(3,3)$*-design of* 85 *vectors for* $\mathbb{C}^5$*. This design was found by [BGM+19] by optimizing a potential, and then presented explicitly (in terms of root vectors of* G_{33}*).*

Example 5.5 *For* $\mathbb{C}^6$ *the highly symmetric tight frame of* 672 *vectors for the group* G_{34} *was identified as a* $(4,4)$*-design (higher strength than a generic orbit), and a pair of orbits gives a* $(5,5)$*-design of* 3528 *vectors.*

We now consider examples for real reflection groups. We note that if X is a spherical (t,t)-design of n vectors for $\mathbb{R}^d$, then $X \cup -X$ (with the same weight on x and $-x$) is a spherical $(2t+1)$-design of $2n$ vectors for $\mathbb{R}^d$ (see [HW18a]).

Example 5.6 *For the Shephard-Todd group* G_{23} *a union of orbits of size* 6 *and* 10 *gives a spherical* $(4,4)$*-design for* $\mathbb{R}^3$ *(with normalised weights* $\frac{20}{21}$*,* $\frac{36}{35}$*). For the Shephard-Todd group* $G(1,1,6)$ *acting on five dimensional space a union of orbits of size* 6 *and* 10 *gives a spherical* $(2,2)$*-design for* $\mathbb{R}^5$ *(with normalised weights* $\frac{20}{21}$*,* $\frac{36}{35}$*). These putatively optimal spherical half-designs were given in in [HW18a].*

Example 5.7 *(Tables 6 and 8) A union of pairs of highly symmetric tight frames for real reflection groups gives* $(3,3)$*-designs of* 4 *vectors for* $\mathbb{R}^2$*,* 16 *vectors for* $\mathbb{R}^3$*,* 24 *vectors for* $\mathbb{R}^4$*,* 45 *vectors for* $\mathbb{R}^5$*,* 63 *vectors for* $\mathbb{R}^6$*,* 91 *vectors for* $\mathbb{R}^7$*, and* 184 *vectors for* $\mathbb{R}^8$*.*

Example 5.8 *For the Shephard-Todd group* G_{30} *a generic orbit is a* $(5,5)$*-design for* $\mathbb{R}^4$*. A union of highly symmetric tight frames with* 60 *and* 300 *vectors gives a* $(9,9)$*-design for* $\mathbb{R}^4$ *(with normalised weights* $\frac{10}{7}$*,* $\frac{32}{35}$*). By taking these vectors and their negatives one obtains a* 720 *vector spherical* 19*-design for* $\mathbb{R}^4$*. It has been shown [BB09] that there is a single orbit of* $G_{30} = W(H_4)$ *which gives a spherical* 19*-design for* $\mathbb{R}^4$*. The vectors* x *giving such orbits are the roots of the harmonic polynomial of degree* 12 *which is invariant under the action of* G_{30}*, and the orbit size is nominally* 14400 *vectors.*

There is ongoing work of [BGM$^+$19] on minimising a p-frame energy on a sphere. They present various tables of putatively optimal spherical (t,t)-designs that they have collected from the literature and calculated (see Example 5.4). Many of these a clearly examples of our general construction (by a comparision of number of vectors and weights). These include a 24-point $(3,3)$-design for $\mathbb{R}^4$, a 22-point $(2,2)$-design for $\mathbb{R}^6$, a 63-point $(3,3)$-design for for $\mathbb{R}^6$, a 91-point $(3,3)$-design for for $\mathbb{R}^7$, and a 21-point $(3,3)$-design for for $\mathbb{C}^3$.

5.4 Conclusion

We have demonstrated that it is possible to take a union of two orbits to obtain a spherical (t,t)-design of higher strength than that of a generic orbit, i.e., t_{generic}, and some of these designs have a minimal number of vectors. Given that $t_{\text{generic}} \leq t_{\max}(d)$ (for some function $t_{\max}$) for every group acting on $\mathbb{R}^d$, $d \geq 3$, it is not possible to find arbitrary strong (t,t)-designs as a single orbit (by selecting a sufficiently large group), and so this technique might be useful for finding designs with strength $t > t_{\max}(d)$. We note that $t_{\max}$ has not yet been determined.

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