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Increasing the polynomial reproduction of a quasi-interpolation operator

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ABSTRACT

Quasi-interpolation is a important tool, used both in theory and in practice, for the approximation of smooth functions from univariate or multivariate spaces which contain $\Pi_m = \Pi_m(\mathbb{R}^d)$ the d–variate polynomials of degree $\le m$. In particular, the reproduction of Π_m leads to an approximation order of $m+1$. Prominent examples include Lagrange and Bernstein type approximations by polynomials, the orthogonal projection onto Π_m for some inner product, finite element methods of precision m, and multivariate spline approximations based on macroelements or the translates of a single spline.

For such a quasi-interpolation operator L which reproduces $\Pi_m(\mathbb{R}^d)$ and any $r \ge 0$, we give an explicit construction of a quasi-interpolant $R_m^{r+m}L = L+A$ which reproduces Π_{m+r}, together with an integral error formula which involves only the $(m+r+1)$–st derivative of the function approximated. The operator $R_m^{m+r}L$ is defined on functions with r additional orders of smoothness than those on which L is defined. This very general construction holds in all dimensions d. A number of representative examples are considered.

Key Words: Quasi-interpolation, Lagrange interpolation, Bernstein polynomial, finite element method, multivariate polynomial approximation, error formula, multipoint Taylor formula, divided differences, Chu–Vandermonde convolution

AMS (MOS) Subject Classifications: primary 41A80, 65D05, secondary 41A05, 41A10,

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1. Introduction

A quasi-interpolant for a space F of approximating functions is a linear map L onto F which is bounded (in some relevant norm), local, and reproduces some polynomial space, see, e.g., [S05]. When F is a univariate or multivariate space of polynomials or splines, quasi-interpolants provide useful approximations of smooth functions. These have both practical and theoretical advantages, e.g., the reproduction of the space $\Pi_m = \Pi_m(\mathbb{R}^d)$ of d–variate polynomials of degree $\leq m$ leads to an approximation order of $m+1$. Some well known examples include Lagrange and Bernstein type approximations by polynomials, the orthogonal projection onto Π_m for some inner product, finite element methods of precision m, and multivariate spline approximations based on macroelements or the translates of a single spline.

The main result of this paper is the following. For any quasi-interpolant L which reproduces $\Pi_m(\mathbb{R}^d)$ and $r \geq 0$, we explicitly construct a quasi-interpolant

$$R_m^{r+m} L = L + A$$

which reproduces Π_{m+r}, together with an integral error formula which involves only the $(m+r+1)$–st derivative of the function approximated. The quasi-interpolant $R_m^{m+r}L$ allows the order of approximation by L to be increased by r, with the trade off being that it is defined on functions with r additional orders of smoothness than those on which L is defined. The operation $L \mapsto R_m^{m+r}L$ has many nice properties, including being defined for all dimensions d, being continuous (in an appropriate sense), and satisfying

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} L = R_m^{m+r_1+r_2} L. \tag{1.1}$$

The paper is set out as follows. In the remainder of this section, we give precise definitions and establish notation. Next we give a multivariate divided difference involving two points upon which our results are based. The following section then uses this to prove the main result, and gives some representative examples. The final section establishes the remarkable formula (1.1), which requires some technical calculations.

Basic definitions and notation

The (directional) derivative of a function f in the direction $v \in \mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is denoted by

$$D_v f(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

We note that $v \mapsto D_v f(x)$ is linear. In particular, for the univariate case $d = 1$ one has

$$D_{x-y}^k f = (x-y)^k f^{(k)}, \qquad x, y \in \mathbb{R}, \tag{1.2}$$

where $f^{(k)}$ denotes the k–th derivative of a univariate function, and $D_v^k := (D_v)^k$. Let $D_j := D_{e_j}$, where e_j is the j–th standard basis vector in $\mathbb{R}^d$. Then the α–th partial derivative $D^\alpha f$ of a function f with a k–th derivative is

$$D^\alpha f := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} f, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_n = k.$$

We call $D^k f := (D^\alpha f)_{|\alpha|=k}$ the k–th derivative of f.

In this paper, a **quasi-interpolant** is defined to be a linear map L of the form

$$Lf(x) = \sum_{i=1}^{n} \lambda_i(f)\phi_i(x), \qquad \lambda_i(f) := \int q_i(x, D^{k_i} f(x))\, d\mu_i(x), \tag{1.3}$$

where q_i is some continuous function of x and $D^{k_i} f(x)$, and μ_i is some finite Borel measure on $\mathbb{R}^d$ with compact support, which reproduces $\Pi_m(\mathbb{R}^d)$, i.e., $Lf = f$, $\forall f \in \Pi_m$. In practice, the linear functionals $f \mapsto \lambda_i(f)$ can usually be taken to be a weighted integral over some simplex (which includes point evaluation) of a partial derivative $D^\alpha f$.

For a given linear map L of the form (1.3), we will refer to the (largest) natural domain on which it is well defined as a space of **sufficiently smooth functions**. This common convention simplifies the presentation, and should cause no confusion. For example, if the linear functionals $f \mapsto \lambda_i(f)$ were function evaluation at the points $x_1, \ldots, x_n$, then a sufficiently smooth function would need to be defined at least at these points, and one could conveniently take the space of continuous functions.

2. A multivariate divided difference involving two points

For $x, y \in \mathbb{R}^d$ define

$$\int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,\ldots,y}_{r+1}]} f := \frac{1}{r!m!} \int_0^1 f(tx + (1-t)y)\, t^m (1-t)^r \, dt. \tag{2.1}$$

This is motivated (see [MM80] and [BHS93]) by the following instance of the Hermite–Genocchi formula for the divided difference of a univariate function f

$$[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,\ldots,y}_{r+1}]f = \int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,\ldots,y}_{r+1}]} f^{(m+r+1)}. \tag{2.2}$$

In the univariate case, (2.1) can be written

$$\int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,,\ldots,y}_{r+1}]} f = \frac{1}{r!m!} \frac{1}{(x-y)^{m+r+1}} \int_y^x (t-y)^m (x-t)^r f(t)\, dt. \tag{2.3}$$

The following can be viewed as a "lifted" version (cf [W97]) of the expansion of the divided difference in (2.2) in terms of $f(x), f'(x), \ldots, f^{(m)}(x)$ and $f(y), f'(y), \ldots, f^{(r)}(y)$.

Lemma 2.4. *If the restriction of f to the line segment between the points x and y in $\mathbb{R}^d$ is C^{m+r+1}, then*

$$\sum_{j=0}^{r} \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{D_{x-y}^{j} f(y)}{j!} - \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{m+r}{k}} \frac{D_{y-x}^{k} f(x)}{k!} = \frac{r!m!}{(m+r)!}(-1)^{m+1} \int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,\ldots,y}_{r+1}]} D_{x-y}^{m+r+1} f.$$

Proof: The divided difference of a univariate function g at the points 0 repeated $r+1$ times and 1 repeated $m+1$ times can be expressed as

$$\begin{aligned}(-1)^{m+1}[\underbrace{0,\ldots,0}_{r+1},\underbrace{1,\ldots,1}_{m+1}]g &= \sum_{j=0}^{r} \binom{m+r-j}{r-j} \frac{g^{(j)}(0)}{j!} - \sum_{k=0}^{m} \binom{m+r-k}{m-k} (-1)^k \frac{g^{(k)}(1)}{k!} \\ &= \frac{(-1)^{m+1}}{m!r!} \int_0^1 g^{(m+r+1)}(t)\, t^m (1-t)^r \, dt.\end{aligned}$$

The first expression follows from the divided difference identities, and the second is the well known Peano kernel representation in terms of a B–spline with knots $0,\ldots,0,1\ldots,1$.

Suppose without loss of generality that $x \neq y$, and let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(t) := f(tx + (1-t)y).$$

If the univariate function obtained by restricting f to the line segment from x to y is C^j, then we can differentiate g to obtain

$$g^{(j)}(t) = D_{x-y}^{j} f(tx + (1-t)y). \tag{2.5}$$

Substituting (2.5) into the formulas for the divided difference gives

$$\begin{aligned}&\sum_{j=0}^{r} \binom{m+r-j}{r-j} \frac{D_{x-y}^{j} f(y)}{j!} - \sum_{k=0}^{m} \binom{m+r-k}{m-k} (-1)^k \frac{D_{x-y}^{k} f(x)}{k!} \\ &\qquad = \frac{(-1)^{m+1}}{r!m!} \int_0^1 D_{x-y}^{(m+r+1)} f(tx+(1-t)y)\, t^m (1-t)^r \, dt.\end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\frac{r!m!}{(m+r)!}$, and using $(-1)^k D_{x-y}^k f = D_{y-x}^k f$ and (2.1) gives the desired formula. $\square$

For $r=0$ the formula of Lemma 2.4 reduces to the integral form of the error at y in Taylor interpolation of degree m to f at the point x, i.e.,

$$f(y) - \mathcal{T}_{m,x} f(y) = \mathcal{R}_{m,x} f(y),$$

where

$$\mathcal{T}_{m,x} f(y) := \sum_{k=0}^{m} \frac{D_{y-x}^{k} f(x)}{k!}, \quad \mathcal{R}_{m,x} f(y) := \int_{[\underbrace{x,\ldots,x}_{m+1},y]} D_{y-x}^{m+1} f. \tag{2.7}$$

3. The main result

We now give the main result. The truncated power function $(\cdot)_+^k$ is defined by

$$(x)_+^k := \begin{cases} x^k, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Theorem 3.1. *Fix $r \geq 0$. Let L be a quasi-interpolant which reproduces $\Pi_m(\mathbb{R}^d)$. Then for all sufficiently smooth functions f, we have*

$$f(x) - Lf(x) - Af(x) = E(f,x), \tag{3.2}$$

where Af is the function in $\operatorname{span}\{\Pi_r \operatorname{ran}(L)\}$ *given by*

$$Af(x) := \sum_{j=1}^{r} \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} \big(L(D_{x-\cdot}^{j} f)\big)(x), \tag{3.3}$$

and the error $E(f,x)$ can be expressed as the following integral of $D^{m+r+1}f$

$$\begin{aligned} E(f,x) &:= \frac{r!m!}{(m+r)!}(-1)^m L\Big(\int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{\cdot,\ldots,\cdot}_{r+1}]} D_{x-\cdot}^{m+r+1} f\Big)(x) \\ &= \frac{1}{(m+r)!}(-1)^m \int_0^1 L\big(D_{x-\cdot}^{m+r+1} f(tx+(1-t)\cdot)\big)(x)\, t^m (1-t)^r \, dt. \end{aligned} \tag{3.4}$$

In the univariate case $E(f,x)$ has the Peano kernel representation

$$E(f,x) = \int_a^b f^{(m+r+1)}(t) K(t)\, dt, \qquad K(t) := \frac{(x-t)^r}{(m+r)!}\big((1-L)(\cdot - t)_+^m\big)(x). \tag{3.5}$$

Proof: By Lemma 2.4, for x fixed, and f sufficiently smooth, we may write

$$f + Bf - Pf = Rf, \tag{3.6}$$

where

$$Bf := \sum_{j=1}^{r} \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{D_{x-\cdot}^{j} f}{j!}, \qquad Pf := \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{m+r}{k}} \frac{D_{\cdot-x}^{k} f(x)}{k!},$$

and

$$Rf := \frac{r!m!}{(m+r)!}(-1)^{m+1} \int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{\cdot,\ldots,\cdot}_{r+1}]} D_{x-\cdot}^{m+r+1} f.$$

The key features of the decomposition (3.6) are:

(i) P maps into Π_m.

(ii) $(Pf)(x) = f(x)$.

(iii) Rf is an integral of the $(m+r+1)$–st derivative of f.

Apply $-L$ to (3.6), and use (i), to get

$$-Lf - LBf + Pf = -LRf,$$

then evaluate at x, using (ii), to obtain

$$-Lf(x) - LBf(x) + f(x) = -LRf(x).$$

This gives (3.2), where $Af(x) := LBf(x)$ and $E(f,x) := -LRf(x)$. The second formula for $E(f,x)$ given in (3.4) follows from the interchange of the integral given by (2.1) and the linear functional $\lambda : f \mapsto (Lf)(x)$ which is justified as per [W99].

Using the multinomial and binomial expansions, the function $D^j_{x-\cdot}f$ can be expanded in terms of the partial derivatives $D^\alpha f$, $|\alpha| = j$, $\alpha \in \mathbb{Z}^d_+$ as follows

$$D^j_{x-t}f(t) = \sum_{|\alpha|=j} \binom{j}{\alpha} D^\alpha f(t)(x-t)^\alpha = \sum_{|\alpha|=j} \binom{j}{\alpha} D^\alpha f(t) \sum_{\beta\le\alpha} \binom{\alpha}{\beta} x^\beta (-t)^{\alpha-\beta},$$

where the above uses standard multiindex notation. Therefore

$$\big(L(D^j_{x-\cdot}f)\big)(x) = \sum_{|\alpha|=j} \binom{j}{\alpha} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} x^\beta \big(L((-\cdot)^{\alpha-\beta} D^\alpha f)\big)(x), \tag{3.7}$$

and so $Af \in \operatorname{span}\{\Pi_r \operatorname{ran}(L)\}$.

Finally, for the univariate case ($d = 1$) we compute the Peano kernel representation of $E(f,x)$ from (3.4). From (2.2), (2.3) and

$$\int_y^x g(t)\,dt = \int_a^b \big((x-t)^0_+ - (y-t)^0_+\big) g(t)\,dt, \qquad a \le x, y \le b \tag{3.8}$$

we obtain

$$\int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{y,\ldots,y}_{r+1}]} D^{m+r+1}_{x-y} f = \frac{(-1)^m}{r!m!} \int_a^b \big((x-t)^0_+ - (y-t)^0_+\big)(y-t)^m (x-t)^r f^{(m+r+1)}(t)\,dt.$$

Since $(y-t)^0_+(y-t)^m = (y-t)^m_+$, substituting the above into (3.4) gives

$$\begin{aligned}
E(f,x) &= \frac{1}{(m+r)!} L\Big(\int_a^b \big((x-t)^0_+(\cdot-t)^m - (\cdot-t)^m_+\big)(x-t)^r f^{(m+r+1)}(t)\,dt\Big)(x) \\
&= \frac{1}{(m+r)!} \int_a^b L\big((x-t)^0_+(\cdot-t)^m - (\cdot-t)^m_+\big)(x)(x-t)^r f^{(m+r+1)}(t)\,dt,
\end{aligned}$$

where the interchange of the integral and the linear functional $\lambda : f \mapsto (Lf)(x)$ is justified as per [W99]. Hence the Peano kernel is given by

$$K(t) := \frac{(x-t)^r}{(m+r)!} L\big((x-t)_+^0 (\cdot - t)^m - (\cdot - t)_+^m\big)(x).$$

This can be written in the form (3.5), by using the fact L reproduces $(\cdot - t)^m$, to calculate

$$\begin{aligned} L\big((x-t)_+^0(\cdot - t)^m - (\cdot - t)_+^m\big)(x) &= (x-t)_+^0 (x-t)^m - L\big((\cdot - t)_+^m\big)(x) \\ &= (x-t)_+^m - L\big((\cdot - t)_+^m\big)(x) = (1-L)\big((\cdot - t)_+^m\big)(x). \end{aligned}$$

□

Since the formula (3.4) for $f - (Lf + Af)$ involves only $D^{m+r+1}f$, it follows that

$$R_m^{m+r} L := L + A$$

is a quasi-interpolant which reproduces Π_{m+r}. With $e_\alpha : x \mapsto x^\alpha$, (3.7) can be expanded

$$\begin{aligned} R_m^{m+r} Lf &= \sum_{j=0}^r \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} \sum_{|\alpha|=j} \binom{j}{\alpha} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} e_\beta (-1)^{\alpha-\beta} L(e_{\alpha-\beta} D^\alpha f) \\ &= \sum_{|\alpha| \le r} \frac{\binom{r}{|\alpha|}}{\binom{m+r}{|\alpha|}} \sum_{\beta \le \alpha} \frac{(-1)^{\alpha-\beta}}{\beta!(\alpha-\beta)!} e_\beta L(e_{\alpha-\beta} D^\alpha f). \end{aligned} \tag{3.9}$$

Thus $R_m^{m+r}L$ is defined on functions with r additional orders of smoothness than required for L, and the operation $L \mapsto R_m^{m+r}L$ is continuous (for an appropriate norm).

As in [H03], the formula (3.2) can be interpreted as an asymptotic expansion of the error in approximation by L, i.e.,

$$f(x) - Lf(x) = Af(x) + E(f,x).$$

Example 1. Han [H03] considers linear operators on $C[a,b]$ of the form

$$Lf(x) := \sum_{i=0}^n f(x_i)\phi_i(x), \qquad a = x_0 < x_1 < \cdots < x_n = b, \quad \phi_i \in C[a,b], \tag{3.10}$$

which reproduce Π_m. For this choice, (3.3) becomes

$$Af(x) = \sum_{j=1}^r \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} \sum_{i=0}^n D_{x-x_i}^j f(x_i)\phi_i(x) = \sum_{i=0}^n \phi_i(x) \sum_{j=1}^r \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} (x-x_i)^j f^{(j)}(x_i).$$

Han denotes the operator $L + A$ by H_{nr}. The error formula for H_{nr} given by (3.4) is

$$E(f,x) = \frac{r!m!}{(m+r)!}(-1)^m \sum_{i=0}^n \phi_i(x) \int_{[\underbrace{x,\ldots,x}_{m+1},\underbrace{x_i,\ldots,x_i}_{r+1}]} D_{x-x_i}^{m+r+1} f.$$

Using (1.2), (2.2) and (2.3), this can be written

$$E(f,x)=\frac{1}{(m+r)!}\sum_{i=0}^{n}\phi_i(x)\int_{x_i}^{x}(x_i-t)^m(x-t)^r f^{(m+r+1)}(t)\,dt,$$

which is the Theorem 1 of [H03].

Example 2. Han's results were extended from L of the form (3.10) to an arbitrary bounded linear map on $C[a,b]$ by [B05]. The expansion for A is that of (3.3), i.e.,

$$Af(x)=\sum_{j=1}^{r}\frac{\binom{r}{j}}{\binom{m+r}{j}}\big(L\big(\frac{(x-\cdot)^j}{j!}f^{(j)}\big)\big)(x). \tag{3.11}$$

By Theorem 3.1, this further extends to any linear map defined a space of sufficiently smooth functions, e.g., on $C^k[a,b]$.

The Peano kernel representation (3.5) extends the corresponding results of [H03:Th. 2] and [B05]. It is interesting to observe that the Peano kernel of the error (3.4) given by the general theory (cf [W99]) has the form

$$K(t)=\big((1-L-A)\frac{(\cdot-t)_+^{m+r}}{(m+r)!}\big)(x).$$

The simplified form of $K(t)$ given by (3.5) is convenient for determining the sign of the Peano kernel from the error in approximation by $L+A$ from that for L (cf [H03:Th. 3]).

Example 3. For $r=0$, and d arbitrary, (3.4) gives the error formula of [W98:Th. 3.15], i.e.,

$$f(x)-Lf(x)=-(L\mathcal{R}_{m,x}f)(x),$$

where $\mathcal{R}_{m,x}f$ is given by (2.7). This work also explores (for $r=0$) other error formulas that can be obtained by taking other maps P in the proof of Theorem 3.1 which satisfy (i),(ii),(iii), and formulas for the derivatives of the error which can be applied to $E(f,x)$.

Example 4. Let L be Lagrange interpolation at the points $0,1\in\mathbb{R}$, i.e.,

$$Lf(x)=f(0)(1-x)+f(1)x.$$

Since $R_1^{1+r}L\subset\Pi_{1+r}$, it follows from (3.4) that $R_1^{1+r}L$ is a linear projector onto Π_{1+r}. The first couple of quasi-interpolants with raised polynomial reproduction are given by

$$R_1^2Lf(x)=Lf(x)+\frac{1}{2}\{f'(0)x(1-x)+f'(1)(x-1)x\},$$
$$R_1^3Lf(x)=Lf(x)+\frac{2}{3}x(x-1)\{f'(1)-f'(0)\}+\frac{1}{6}\{f''(0)x^2(1-x)+f''(1)(x-1)^2x\}.$$

The interpolation conditions for these are

$$\begin{aligned}
R_1^1L:&\quad f(0),\ f(1)\\
R_1^2L:&\quad f(0),\ f(1),\ f'(1)-f'(0)\\
R_1^3L:&\quad f(0),\ f(1),\ 4f'(1)-4f'(0)-f''(1),\ f''(1)-f''(0)
\end{aligned}$$

We observe there is no interpolation condition for $R_1^3 L$ which involves only $f'(0)$ and $f'(1)$.

Example 5. As Example 4 indicates, the operator $R_m^{m+r} L$ may not preserve interpolation conditions of L. However, if L interpolates at a point $\theta \in \mathbb{R}^d$, then so does $R_m^{m+r} L$, since

$$(R_m^{m+r} L f)(\theta) = \sum_{j=0}^{r} \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} \big(L(D_{\theta-\cdot}^j f)\big)(\theta) = \sum_{j=0}^{r} \frac{\binom{r}{j}}{\binom{m+r}{j}} \frac{1}{j!} D_{\theta-\theta}^j f(\theta) = f(\theta).$$

Thus if L is Lagrange interpolation at set of points $\Theta \subset \mathbb{R}^d$, then $R_m^{m+r} L$ interpolates at Θ and matches various derivatives of up to order r at the points Θ, and so can be thought of as a (multivariate) Hermite interpolation operator.

Example 6. Let L be bilinear interpolation at the vertices of the square, i.e.,

$$Lf(x,y) := f(0,0)(1-x)(1-y) + f(1,0)x(1-y) + f(0,1)(1-x)y + f(1,1)xy.$$

The quasi-interpolant $R_1^2 L$ is given by

$$\begin{aligned} R_1^2 L f(x,y) = Lf(x,y) &+ \big(xD_1 f(0,0) + yD_2 f(0,0)\big)(1-x)(1-y) \\ &+ \big((x-1)D_1 f(1,0) + yD_2 f(1,0)\big)x(1-y) \\ &+ \big(xD_1 f(0,1) + (y-1)D_2 f(0,1)\big)(1-x)y \\ &+ \big((x-1)D_1 f(1,1) + (y-1)D_2 f(1,1)\big)xy. \end{aligned}$$

This is in fact a projector onto $\Pi_2 \oplus \mathrm{span}\{x^2 y, y^2 x\}$, with interpolation conditions $f(0,0)$, $f(1,0)$, $f(0,1)$, $f(1,1)$, together with

$$D_1 f(1,0) - D_1 f(0,0),\ D_1 f(1,1) - D_1 f(0,1),\ D_2 f(0,1) - D_2 f(0,0),\ D_2 f(1,1) - D_2 f(1,0).$$

Further examples of multivariate Bernstein and Lagrange operators (including sharp error estimates) are explored in [GNS06].

4. Successive increases of the polynomial reproduction

If the polynomial reproduction of a quasi-interpolant L is raised in successive stages, then it is natural to ask whether the resulting quasi-interpolant is the same as that obtained by doing it all together, i.e., whether or not

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} L = R_m^{m+r_1+r_2} L. \tag{4.1}$$

Let Q_j be defined by $Q_j f(x) := \big(L(D_{x-\cdot}^j f)\big)(x)$, then

$$Q_j Q_k f \neq Q_{j+k} f,$$

in general, e.g., for $Lf(x) := f(0)$, take $f(x) = x$, for which $Q_1 Q_1 f = f \neq 0 = Q_2 f$.

It is therefore somewhat remarkable, and consequently nontrivial to prove, that (4.1) holds. To show this we need the following multivariate forms of the Chu–Vandermonde convolution.

Lemma 4.2. *For multi-indices ξ, η, α with $\xi - \eta \geq 0$,*

$$\sum_{\beta \leq \alpha} \binom{\eta}{\beta} \binom{\xi - \eta}{\alpha - \beta} = \binom{\xi}{\alpha}, \tag{4.3}$$

and for ξ a multi-index, $j, \ell \geq 0$ integers,

$$\sum_{|\alpha|=j} \sum_{|\epsilon|=\ell} \binom{\xi}{\alpha} \binom{\alpha}{\epsilon} = \binom{|\xi|}{j} \binom{j}{\ell}. \tag{4.4}$$

Proof: By the Chu–Vandermonde convolution

$$\begin{aligned}
\sum_{\beta \leq \alpha} \binom{\eta}{\beta} \binom{\xi - \eta}{\alpha - \beta} &= \sum_{\beta_1 \leq \alpha_1} \cdots \sum_{\beta_d \leq \alpha_d} \binom{\eta_1}{\beta_1} \binom{\xi_1 - \eta_1}{\alpha_1 - \beta_1} \cdots \binom{\eta_d}{\beta_d} \binom{\xi_d - \eta_d}{\alpha_d - \beta_d} \\
&= \sum_{\beta_1 \leq \alpha_1} \binom{\eta_1}{\beta_1} \binom{\xi_1 - \eta_1}{\alpha_1 - \beta_1} \cdots \sum_{\beta_d \leq \alpha_d} \binom{\eta_d}{\beta_d} \binom{\xi_d - \eta_d}{\alpha_d - \beta_d} \\
&= \binom{\xi_1}{\alpha_1} \cdots \binom{\xi_d}{\alpha_d} = \binom{\xi}{\alpha},
\end{aligned}$$

which is (4.3). It follows by induction on d, and the Chu–Vandermonde convolution, that

$$\begin{aligned}
\sum_{|\alpha|=j} \binom{\xi}{\alpha} &= \sum_{\alpha_d=0}^{j} \binom{\xi_d}{\alpha_d} \sum_{\alpha_1+\cdots+\alpha_{d-1}=j-\alpha_d} \binom{\xi_1, \ldots, \xi_{d-1}}{\alpha_1, \ldots, \alpha_{d-1}} \\
&= \sum_{\alpha_d=0}^{j} \binom{\xi_d}{\alpha_d} \binom{\xi_1 + \cdots + \xi_{d-1}}{j - \alpha_d} = \binom{\xi_1 + \cdots + \xi_d}{j} = \binom{|\xi|}{j}.
\end{aligned}$$

Applying this twice gives

$$\sum_{|\alpha|=j} \sum_{|\epsilon|=\ell} \binom{\xi}{\alpha} \binom{\alpha}{\epsilon} = \sum_{|\alpha|=j} \binom{\xi}{\alpha} \sum_{|\epsilon|=\ell} \binom{\alpha}{\epsilon} = \sum_{|\alpha|=j} \binom{\xi}{\alpha} \binom{j}{\ell} = \binom{|\xi|}{j} \binom{|\alpha|}{j},$$

and we obtain (4.4). $\square$

Lemma 4.5. *Let $m, r_1, r_2 \geq 0$ and $0 \leq i \leq r_1 + r_2$, then*

$$\sum_{\substack{k,j,\ell \geq 0 \\ j+k-\ell=i}} \frac{\binom{r_2}{k}}{\binom{m+r_1+r_2}{k}} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} (-1)^\ell \binom{i}{j} \binom{j}{\ell} = \frac{\binom{r_1+r_2}{i}}{\binom{m+r_1+r_2}{i}}.$$

Proof: We calculate

$$
\begin{aligned}
S(m,r_1,r_2,i) &:= \sum_{\substack{k,j,\ell\geq 0\\ j+k-\ell=i}} \frac{\binom{r_2}{k}}{\binom{m+r_1+r_2}{k}} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} (-1)^\ell \binom{i}{j}\binom{j}{\ell} \\
&= \sum_{j\geq 0} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} \binom{i}{j} \sum_{\ell\geq 0} \frac{(-j)_\ell}{\ell!} \frac{\binom{r_2}{\ell+i-j}}{\binom{m+r_1+r_2}{\ell+i-j}} \\
&= \sum_{j\geq 0} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} \binom{i}{j} \frac{\binom{r_2}{i-j}}{\binom{m+r_1+r_2}{i-j}} \sum_{\ell\geq 0} \frac{(-j)_\ell}{\ell!} \frac{(i-j-r_2)_\ell}{(i-j-m-r_1-r_2)_\ell} \\
&= \sum_{j\geq 0} \frac{(-r_1)_j}{(-m-r)_j} \binom{i}{j} \frac{\binom{r_2}{i-j}}{\binom{m+r_1+r_2}{i-j}} \frac{(-m-r_1)_j}{(i-j-m-r_1-r_2)_j} \\
&= \sum_{j\geq 0} \binom{i}{j} \frac{\binom{r_2}{i-j}}{\binom{m+r_1+r_2}{i-j}} \frac{(-r_1)_j}{(i-j-m-r_1-r_2)_j} \\
&= \sum_{j\geq 0} \frac{(-1)^j(-i)_j}{j!} \frac{\binom{r_2}{i}(m+r_1+r_2-i+1)_j}{\binom{m+r_1+r_2}{i}(r_2-i+1)_j} \frac{(-r_1)_j}{(i-j-m-r_1-r_2)_j} \\
&= \frac{\binom{r_2}{i}}{\binom{m+r_1+r_2}{i}} \sum_j \frac{(-i)_j(-r_1)_j}{j!(r_2-i+1)_j} \\
&= \frac{\binom{r_2}{i}}{\binom{m+r_1+r_2}{i}} \frac{(r_1+r_2-i+1)_i}{(r_2-i+1)_i} \\
&= \frac{\binom{r_1+r_2}{i}}{\binom{m+r_1+r_2}{i}},
\end{aligned}
$$

which uses the following identities

$$
(-1)^\ell \binom{j}{\ell} = \frac{(-j)_\ell}{\ell!}, \qquad \binom{a}{\ell+b} = \binom{a}{b}(-1)^\ell \frac{(b-a)_\ell}{(b+1)_\ell},
$$

$$
\sum_{\ell\geq 0} \frac{(-j)_\ell}{\ell!}\frac{(a)_\ell}{(b)_\ell} = \frac{(b-a)_j}{(b)_j} \qquad \text{(Chu–Vandermonde).}
$$

Here $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. □

Theorem 4.6. *The polynomial reproduction raising operator satifies*

$$
R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} L = R_m^{m+r_1+r_2} L, \qquad m, r_1, r_2 \geq 0.
$$

Proof: From (3.9), we obtain

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} Lf = \sum_{|\gamma|\le r_2} \frac{\binom{r_2}{|\gamma|}}{\binom{m+r_1+r_2}{|\gamma|}} \sum_{\delta\le\gamma} \frac{(-1)^{\gamma-\delta}}{\delta!(\delta-\gamma)!} e_\delta (R_m^{m+r_1} L)(e_{\gamma-\delta} D^\gamma f)$$

$$= \sum_{|\gamma|\le r_2} \frac{\binom{r_2}{|\gamma|}}{\binom{m+r_1+r_2}{|\gamma|}} \sum_{\delta\le\gamma} \frac{(-1)^{\gamma-\delta}}{\delta!(\delta-\gamma)!} e_\delta \sum_{|\alpha|\le r_1} \frac{\binom{r_1}{|\alpha|}}{\binom{m+r_1}{|\alpha|}} \sum_{\beta\le\alpha} \frac{(-1)^{\alpha-\beta}}{\beta!(\alpha-\beta)!} e_\beta$$
$$\times L(e_{\alpha-\beta} D^\alpha (e_{\gamma-\delta} D^\gamma f)).$$

Using the multivariate Leibniz formula

$$D^\alpha(e_{\gamma-\delta}D^\gamma f) = \sum_{\epsilon\le\alpha} \binom{\alpha}{\epsilon} (D^{\alpha-\epsilon} D^\gamma f)(D^\epsilon e_{\gamma-\delta}) = \sum_{\epsilon\le\alpha} \binom{\alpha}{\epsilon}\binom{\gamma-\delta}{\epsilon} \epsilon! e_{\gamma-\delta-\epsilon} D^{\alpha-\epsilon+\gamma} f,$$

and the linearity of L, this gives

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} Lf = \sum_{\substack{|\alpha|\le r_1 \\ |\gamma|\le r_2}} \frac{\binom{r_1}{|\alpha|}}{\binom{m+r_1}{|\alpha|}} \frac{\binom{r_2}{|\gamma|}}{\binom{m+r_1+r_2}{|\gamma|}} \sum_{\substack{\beta\le\alpha \\ \delta\le\gamma}} \frac{1}{\beta!(\alpha-\beta)!\delta!(\gamma-\delta)!}$$
$$\sum_{\epsilon\le\alpha} \binom{\alpha}{\epsilon}\binom{\gamma-\delta}{\epsilon} \epsilon!(-1)^\epsilon (\beta+\delta)!(\alpha-\epsilon+\gamma-\delta-\beta)!$$
$$\times \frac{(-1)^{\gamma-\delta+\alpha-\beta-\epsilon}}{(\beta+\delta)!(\alpha-\epsilon+\gamma-\delta-\beta)!} e_{\beta+\delta} L(e_{\alpha-\beta+\gamma-\delta-\epsilon} D^{\alpha-\epsilon+\gamma} f).$$

Thus

$$R_{m+r_1}^{m+r_1+r_2} R_m^{m+r_1} Lf = \sum_{|\xi|\le r_1+r_2} \sum_{\eta\le\xi} c(\eta,\xi) \frac{1}{\eta!(\xi-\eta)!} e_\eta (1-)^{\xi-\eta} L(e_{\xi-\eta} D^\xi f),$$

where

$$C(\eta,\xi) := \sum_{\substack{\alpha,\gamma,\epsilon \\ \alpha+\gamma-\epsilon=\xi}} \frac{\binom{r_1}{|\alpha|}}{\binom{m+r_1}{|\alpha|}} \frac{\binom{r_2}{|\gamma|}}{\binom{m+r_1+r_2}{|\gamma|}} \binom{\alpha}{\epsilon} (-1)^\epsilon \epsilon! \sum_{\substack{\beta\le\alpha \\ \delta\le\gamma \\ \beta+\delta=\eta}} \frac{(\beta+\delta)!(\alpha-\epsilon+\gamma-\delta-\beta)!}{\beta!(\alpha-\beta)!\delta!(\gamma-\delta)!} \binom{\gamma-\delta}{\epsilon}.$$

In view of (3.9), it therefore suffices to show that

$$C(\eta,\xi) = \frac{\binom{r_1+r_2}{|\xi|}}{\binom{m+r_1+r_2}{|\xi|}}.$$

The terms in $C(\eta,\xi)$ involving δ and β can be summed by (4.3)

$$\sum_{\substack{\beta\le\alpha \\ \delta\le\gamma \\ \beta+\delta=\eta}} \frac{(\beta+\delta)!(\alpha-\epsilon+\gamma-\delta-\beta)!}{\beta!(\alpha-\beta)!\delta!(\gamma-\delta)!} \binom{\gamma-\delta}{\epsilon} = \frac{1}{\epsilon!} \sum_{\substack{\beta\le\alpha \\ \delta\le\gamma-\epsilon \\ \beta+\delta=\eta}} \frac{(\beta+\delta)!}{\beta!\delta!} \frac{(\alpha-\epsilon+\gamma-\delta-\beta)!}{(\alpha-\beta)!(\gamma-\delta-\epsilon)!}$$
$$= \frac{1}{\epsilon!} \sum_{\beta\le\alpha} \binom{\eta}{\beta}\binom{\xi-\eta}{\alpha-\beta} = \frac{1}{\epsilon!}\binom{\xi}{\alpha}.$$

Thus, by (4.4) and Lemma 4.5,

$$\begin{aligned}
C(\eta,\xi) &= \sum_{\substack{\alpha,\gamma,\epsilon\\ \alpha+\gamma-\epsilon=\xi}} \frac{\binom{r_1}{|\alpha|}}{\binom{m+r_1}{|\alpha|}} \frac{\binom{r_2}{|\gamma|}}{\binom{m+r_1+r_2}{|\gamma|}} \binom{\alpha}{\epsilon}(-1)^{\epsilon}\binom{\xi}{\alpha} \\
&= \sum_{\substack{j,k,\ell\geq 0\\ j+k-\ell=|\xi|}} \frac{\binom{r_2}{k}}{\binom{m+r_1+r_2}{k}} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} (-1)^{\ell} \sum_{|\alpha|=j}\binom{\xi}{\alpha} \sum_{|\epsilon|=\ell}\binom{\alpha}{\epsilon} \\
&= \sum_{\substack{j,k,\ell\geq 0\\ j+k-\ell=|\xi|}} \frac{\binom{r_2}{k}}{\binom{m+r_1+r_2}{k}} \frac{\binom{r_1}{j}}{\binom{m+r_1}{j}} (-1)^{\ell} \binom{|\xi|}{j}\binom{j}{\ell} \\
&= \frac{\binom{r_1+r_2}{|\xi|}}{\binom{m+r_1+r_2}{|\xi|}},
\end{aligned}$$

which completes the proof. □

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