SICs and the elements of order three in the Clifford group

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Abstract

For over a decade, there has been intensive work on the numerical and analytic construction of SICs (d^2 equiangular lines in $\mathbb{C}^d$) as an orbit of the Heisenberg group. The Clifford group, which consists of the unitary matrices which normalise the Heisenberg group, plays a key role in these constructions. All of the known fiducial (generating) vectors for such SICs are eigenvectors of symplectic operations in the Clifford group with canonical order 3. Here we describe the Clifford group and the subgroup of symplectic operations in terms of a natural set of generators. From this, we classify all its elements of canonical order three. In particular, we show (contrary to prior claims) that there are symplectic operations of canonical order 3 for $d \equiv 6 \bmod 9$. It is as yet unknown whether these give rise to SICs.

Key Words: finite tight frames, SIC (symmetric informationally complete positive operator valued measure), ghost SIC, Heisenberg group, Clifford group, symplectic operation, complex equiangular lines, quadratic Gauss sum,

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1 Introduction

A set of d^2 unit vectors in $\mathbb{C}^d$ (or the lines they determine) is said to be **equiangular** if

$$|\langle v_j, v_k\rangle|^2 = \frac{1}{d+1}, \qquad j \neq k.$$

In the quantum information theory community, the corresponding rank one orthogonal projections $P_j = v_j v_j^*$ are said to be a **symmetric informationally complete positive operator valued measure** (**SIC** or **SIC-POVM** for short). The existence of a SIC for every dimension d is known as *Zauner's conjecture* or the *SIC problem* [FHS17].

Since Zauner's thesis in 1999 (see [Zau10]), there has been constant progress on the SIC problem. With one exception (the Hoggar lines in $\mathbb{C}^8$), all the known SICs appear as the orbit of a single (fiducial) vector/projection under the action of the Heisenberg group (this is sometimes referred to as a strong form of Zauner's conjecture). The search for such a fiducial vector dramatically reduces the number of unknown variables from the order of d^3 to d, and has led to high accuracy numerical SICs [RBKSC04], [SG10], [Sco17], which in turn has led to analytic SICs [ACFW18] i.e., a proof Zauner's conjecture for various dimensions d.

A key feature of these SIC fiducials is that they are mapped to each other by elements of the Clifford group, i.e., the normaliser of the Heisenberg group in the unitary matrices (see especially Appleby [App05]), and there is a fiducial (on a given Clifford group orbit) which is an eigenvector of a "symplectic" operation in the Clifford group with canonical order 3. This further reduces the number of unknowns in a SIC fiducial to the order of $\frac{d}{3}$. Here, we consider the structure of the Clifford group, and in particular, its elements of order 3. The main points are:

- We determine the diagonal elements of the Clifford group, and thereby show that it is generated by the Fourier matrix F and a diagonal matrix R (together with generators of the Heisenberg group).
- The generator R above can be replaced by the Zauner matrix Z (of order 3).
- We show that the subgroup of symplectic operations is generated by F and R.
- We give an alternative to the Appleby indexing of the elements of the Clifford group, which is 1–1, in all cases.
- We use our indexing to determine the Clifford operations of order 3, and in particular, those with Clifford trace -1 (canonical order 3). This includes a family for $d \equiv 6 \bmod 9$, which was previously overlooked.
- We show that for d even there are nontrivial symplectic operations which are not displacement free, i.e., belong to the Heisenberg group.
- We show explicitly how to write the permutation matrices as words in F and R.

The techniques involved include the calculation of certain quadratic Gauss sums, and the analysis of certain binary quadratic forms over $\mathbb{Z}_d$. Along the way we prove Conjecture 4 of [Fla06].

2 The Heisenberg group and Weyl–Heisenberg SICs

Throughout, let ω and μ be the primitive d–th and $2d$–th roots of unity

$$\omega := e^{\frac{2\pi i}{d}}, \qquad \mu := e^{\frac{2\pi i}{2d}},$$

and take the indices for elements of $\mathbb{C}^d$ and $\mathbb{C}^{d\times d}$ from $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$. Let $S \in \mathbb{C}^{d\times d}$ be the **cyclic shift matrix**, and $\Omega \in \mathbb{C}^{d\times d}$ be the **modulation matrix** given by

$$(S)_{jk} := \delta_{j,k+1}, \qquad (\Omega)_{jk} := \omega^j \delta_{j,k}. \tag{2.1}$$

These have order d, and satisfy the commutativity relation

$$\Omega^k S^j = \omega^{jk} S^j \Omega^k. \tag{2.2}$$

Thus the group generated by the unitary matrices S and Ω is

$$H := \langle S, \Omega \rangle = \{\omega^r S^j \Omega^k : r, j, k \in \mathbb{Z}_d\}. \tag{2.3}$$

This is called the **Heisenberg group**[1] (for $\mathbb{Z}_d$), as is the group

$$\hat{H} := \{ch : c \in \mathbb{T}, h \in H\} \subset \mathcal{U}(\mathbb{C}^d), \qquad \mathbb{T} := \{c \in \mathbb{C} : |c| = 1\}. \tag{2.4}$$

The map $(j, k) \mapsto S^j \Omega^k$ is a faithful irreducible projective representation of $\mathbb{Z}_d \times \mathbb{Z}_d$. In particular, the unitary action of H on $\mathbb{C}^d$ is irreducible, and so $(hv)_{h\in H}$ and $(S^j \Omega^k v)_{j,k\in\mathbb{Z}_d}$ are tight frames for $\mathbb{C}^d$ for any $v \neq 0$ (see [VW05]), i.e.,

$$f = \frac{1}{d} \sum_{(j,k)\in\mathbb{Z}_d^2} \langle f, S^j \Omega^k v \rangle S^j \Omega^k v, \qquad \forall f \in \mathbb{C}^d.$$

Every SIC is a tight frame. A SIC (equiangular tight frame of d^2 vectors for $\mathbb{C}^d$) is said to be a **Weyl-Heisenberg SIC** for $\mathbb{C}^d$ if (up to projective unitary equivalence) it has the form

$$\Phi_v := (S^j \Omega^k v)_{j,k\in\mathbb{Z}_d},$$

where the unit vector $v \in \mathbb{C}^d$ (or the projection $\Pi = vv^*$) is called a **fiducial**.

A Weyl-Heisenberg SIC for $\mathbb{C}^d$ is generated from a single fiducial vector v by applying S (translation) and Ω (frequency shift). Thus, it is a discrete analogue of a Gabor system (Weyl–Heisenberg frame) with good time–frequency localisation. In this analogy the fiducial vector v corresponds to the *mother wavelet*. From now on, we consider only Weyl-Heisenberg SICs (which we refer to as SICs).

3 The Clifford group

Let $[U] := \{cU : c \in \mathbb{T}\} = \{e^{it}U : t \in \mathbb{R}\}$, so that $[I]$ is the unitary scalar matrices. The normaliser of the Heisenberg group $\hat{H}$ in the group of unitary matrices is called

[1] It is also known as the **generalised Pauli** or **Weyl–Heisenberg group**.

the **Clifford group**, and it is denoted by $\mathrm{C}(d)$. The **projective Clifford group** is $\mathrm{PC}(d) := \mathrm{C}(d)/[I]$ (its elements are called **Clifford operations**).

There is a natural action of $\mathrm{C}(d)$ on the SIC fiducial vectors v, and of $\mathrm{PC}(d)$ on SIC fiducial projectors $\Pi = vv^*$ given by

$$a \cdot v := av, \qquad [a] \cdot \Pi := (av)(av)^* = a\Pi a^{-1}.$$

This maps SICs to SICs, since if $[a] \in \mathrm{PC}(d)$ and v is a SIC fiducial vector, then

$$|\langle S^{j_1}\Omega^{k_1}av, S^{j_2}\Omega^{k_2}av\rangle|^2 = |\langle a^{-1}S^{j_1-j_2}\Omega^{k_1-k_2}av, v\rangle|^2 = \frac{1}{d+1}, \qquad (j_1,k_1) \neq (j_2,k_2),$$

because $a^{-1}S^j\Omega^k a$, $(j,k) \neq (0,0)$, is a nonscalar element of $\hat{H}$.

Since $H \subset \mathrm{C}(d)$, the action of $\mathrm{C}(d)$ on $\mathbb{C}^d$ is irreducible, and its centre is $[I]$. Since

$$S^* = S^T = S^{-1}, \qquad \Omega^* = \Omega^{-1}, \quad \Omega^T = \Omega, \tag{3.5}$$

the Heisenberg group and the Clifford group are closed under taking the transpose and Hermitian transpose, and hence also entrywise conjugation $\overline{A} = (A^*)^T$. Therefore entrywise conjugation maps a given Heisenberg SIC fiducial to another. The group generated by entrywise conjugation and $\mathrm{C}(d)$ is the **extended Clifford group** $\mathrm{EC}(d)$, and the **extended projective Clifford group** is $\mathrm{PEC}(d) := \mathrm{EC}(d)/[I]$. These map SICs to SICs. The counting of (Weyl-Heisenberg) SICs is usually done up to projective unitary equivalence and the extended Clifford orbit it lies on (see [Wal18]). In addition to (entrywise) complex conjugation, certain Galois automorphisms of the SIC field (the field generated by μ and entries of a fiducial projector Π) have been shown to map SICs to SICs. Counting SICs up to the orbit under the extension of the Clifford group by these (pointwise) automorphisms gives a so called **multiplet** (union of extended Clifford orbits) [ACFW18].

Elements of the Clifford group include the **Fourier matrix** F, the diagonal matrix R, and the permutation matrices P_σ, $\sigma \in \mathbb{Z}_d^*$ (the units modulo d), which are given by

$$(F)_{jk} := \frac{1}{\sqrt{d}}\omega^{jk}, \tag{3.6}$$

$$(R)_{jk} := \mu^{j(j+d)}\delta_{jk}, \tag{3.7}$$

$$(P_\sigma)_{jk} := \delta_{j,\sigma k}, \quad \sigma \in \mathbb{Z}_d^*. \tag{3.8}$$

We observe that R is well defined, i.e., the value of $j(j+d)$ depends only on the integer j mod d. The entry $\mu^{j(j+d)}$ has many alternative descriptions, e.g.,

$$\mu^{j(j+d)} = \mu^{j^2}(-1)^j = \mu^{j^2}(-1)^{j^2} = (-\mu)^{j^2} = \mu^{(d+1)j^2}.$$

Indeed, elementary computations (see [Wal18]) give:

Lemma 3.1 *The unitary matrices F, R, P_a belong to the Clifford group. Indeed*

$$F(S^j\Omega^k)F^{-1} = \omega^{-jk}S^{-k}\Omega^j, \tag{3.9}$$

$$R(S^j\Omega^k)R^{-1} = \mu^{j(j+d)}S^j\Omega^{j+k}, \tag{3.10}$$

$$P_\sigma(S^j\Omega^k)P_\sigma^{-1} = S^{\sigma j}\Omega^{\sigma^{-1}k}, \tag{3.11}$$

where σ^{-1} is the multiplicative inverse of $\sigma \in \mathbb{Z}_d^*$.

The appearance of R can be explained (it was first observed by [BW07] for d odd). It appears in a direct search for diagonal matrices in the normaliser of $\hat{H}$.

Proposition 3.1 *The group of diagonal unitary matrices in* $\mathrm{C}(d)$ *is generated by the unitary scalar matrices,* Ω*, and the matrix* R*.*

Proof: Suppose that $\Lambda = \mathrm{diag}(\lambda_j)$ normalises $\hat{H}$, and $\lambda_d := \lambda_0$. Then

$$\Lambda S \Lambda^{-1} = \begin{pmatrix} 0 & 0 & \cdots & \frac{\lambda_d}{\lambda_{d-1}} \\ \frac{\lambda_1}{\lambda_0} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_2}{\lambda_1} & \cdots & 0 \\ \vdots & & & \vdots \end{pmatrix} = cS\Omega^k, \qquad \text{i.e.,} \quad \frac{\lambda_{j+1}}{\lambda_j} = c\,\omega^{jk},\ \forall j,$$

where $c \in \mathbb{C}$, $k \in \mathbb{Z}$. Solving this recurrence gives

$$\lambda_j = \lambda_0 c^j \omega^{\frac{1}{2}j(j-1)k} = \lambda_0 c^j \mu^{j(j-1)k} = \lambda_0 \mu^{j(j+d)k} (c\mu^{-k(d+1)})^j.$$

Since $\lambda_d = \lambda_0$, this gives

$$(c\mu^{-k(d+1)})^d = (c\mu^{-k(d+1)})^0 = 1 \quad \Longrightarrow \quad c\mu^{-k(d+1)} = \omega^m,$$

and so $\Lambda = \lambda_0 R^k \Omega^m$. □

We will see (Theorem 4.1) that R along with F and H generate the Clifford group. To this end, we now consider the structure of $\mathrm{C}(d)$, by using a variation of the arguments of [App05]. Let

$$U_{(j,k)} := S^j \Omega^k, \qquad (j,k) \in \mathbb{Z}_d^2. \tag{3.12}$$

If $a \in \mathrm{C}(d)$, then

$$a U_\lambda a^{-1} = z_a(\lambda) U_{\psi_a(\lambda)}, \qquad \forall \lambda \in \mathbb{Z}_d^2, \tag{3.13}$$

which defines functions $\psi_a : \mathbb{Z}_d^2 \to \mathbb{Z}_d^2$ and $z_a : \mathbb{Z}_d^2 \to \mathbb{T}$, since no U_λ is a scalar multiple of another. For example, (3.9) and (3.10) give

$$\psi_F \begin{pmatrix} j \\ k \end{pmatrix} = \begin{pmatrix} -k \\ j \end{pmatrix}, \quad z_F \begin{pmatrix} j \\ k \end{pmatrix} = \omega^{-jk}, \qquad \psi_R \begin{pmatrix} j \\ k \end{pmatrix} = \begin{pmatrix} j \\ j+k \end{pmatrix}, \quad z_R \begin{pmatrix} j \\ k \end{pmatrix} = \mu^{j(j+d)}.$$

We now show the elements of the Clifford group factored by $\hat{H}$ can be indexed by the elements of $SL_2(\mathbb{Z}_d)$ For a 2×2 matrix A, we define a symmetric matrix σ_A by

$$\sigma_A := \begin{pmatrix} \alpha\gamma & \beta\gamma \\ \beta\gamma & \beta\delta \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{3.14}$$

The map $A \mapsto \sigma_A$ is not 1–1, e.g., $\sigma_A = 0$ for all diagonal matrices.

Lemma 3.2 *Let* ψ_a *and* z_a *be given by* (3.13)*. Then the map*

$$\psi : \mathrm{C}(d) \to SL_2(\mathbb{Z}_d) : a \mapsto \psi_a \tag{3.15}$$

is a group homomorphism with kernel $\hat{H}$*, and* z_a *satisfies*

$$z_a(p+q) = \omega^{p^T \sigma_A q} z_a(p) z_a(q), \qquad p, q \in \mathbb{Z}_d^2 \tag{3.16}$$

where $A = \psi_a$ *and* σ_A *is given by (3.14).*

Proof: By (2.2), we have $U_pU_q = \omega^{p_2q_1}U_{p+q}$. and so

$$\omega^{p_2q_1}(aU_{p+q}a^{-1}) = aU_pU_qa^{-1} = (aU_pa^{-1})(aU_qa^{-1}),$$

which gives

$$\begin{aligned}\omega^{p_2q_1}z_a(p+q)U_{\psi_a(p+q)} &= z_a(p)U_{\psi_a(p)}z_a(q)U_{\psi_a(q)}\\ &= z_a(p)z_a(q)\omega^{\psi_a(p)_2\psi_a(q)_1}U_{\psi_a(p)+\psi_a(q)}.\end{aligned}$$

and hence

$$\psi_a(p+q) = \psi_a(p) + \psi_a(q), \tag{3.17}$$

$$\omega^{p_2q_1}z_a(p+q) = z_a(p)z_a(q)\omega^{\psi_a(p)_2\psi_a(q)_1}. \tag{3.18}$$

For $p = p_1e_1 + p_2e_2 \in \mathbb{Z}_d^2$, from (3.17) we obtain

$$\psi_a(p) = p_1\psi_a(e_1) + p_2\psi_a(e_2) = [\psi_a(e_1), \psi_a(e_2)]p,$$

i.e., ψ_a can be represented by the 2×2 matrix $[\psi_a(e_1), \psi_a(e_2)]$.

Let $[p', q'] = [\psi_a(p), \psi_a(q)] = \psi[p, q]$, so that $\det([p', q']) = \det(\psi_a)\det([p, q])$. Since the quotient $z_a(p)z_a(q)/z_a(p+q)$ is symmetric in p and q, (3.18) gives

$$\begin{aligned}\omega^{p_2q_1-p_2'q_1'} = \omega^{q_2p_1-q_2'p_1'} \quad &\Longrightarrow \quad p_1'q_2' - q_1'p_2' = p_1q_2 - q_1p_2\\ &\Longrightarrow \quad \det([p', q']) = \det([p, q]),\\ &\Longrightarrow \quad \det(\psi_a) = 1,\end{aligned} \tag{3.19}$$

i.e., $\psi_a \in SL_2(\mathbb{Z}_d)$. Using this, (3.18) can be written as (3.16).

Since $(ab)U_\lambda(ab)^{-1} = a(bU_\lambda b^{-1})a^{-1}$, we have

$$z_{ab}(\lambda)U_{\psi_{ab}(\lambda)} = a(z_b(\lambda)U_{\psi_b(\lambda)})a^{-1} = z_b(\lambda)z_a(\psi_b(\lambda))U_{\psi_a(\psi_b(\lambda))}, \tag{3.20}$$

so that $\psi_{ab}(\lambda) = \psi_a(\psi_b(\lambda))$, i.e., $a \mapsto \psi_a$ is a homomorphism.

We now determine the kernel of ψ. By (2.2), $\hat{H} \subset \ker\psi$. Suppose $\psi_a = I$, so that $aSa^{-1} = z_a(1,0)S$ and $a\Omega a^{-1} = z_a(0,1)\Omega$. Since $S^d = \Omega^d = I$, this implies that $z_a(1,0)$ and $z_a(0,1)$ are d–th roots of unity, say

$$aSa^{-1} = \omega^\alpha S, \qquad a\Omega a^{-1} = \omega^\beta\Omega. \tag{3.21}$$

If $a \in \hat{H}$, then (3.21) implies that a is a scalar multiple of $S^{-\beta}\Omega^\alpha$. Hence, we consider the unitary matrix $b = (S^{-\beta}\Omega^\alpha)^{-1}a$. By (3.21) and repeated application of (2.2), we have that

$$\begin{aligned}b(S^j\Omega^k)b^{-1} &= \Omega^{-\alpha}S^\beta(aSa^{-1})^j(a\Omega a^{-1})^kS^{-\beta}\Omega^\alpha\\ &= \Omega^{-\alpha}S^\beta(\omega^\alpha S)^j(\omega^\beta\Omega)^kS^{-\beta}\Omega^\alpha = S^j\Omega^k.\end{aligned}$$

Since b commutes with the basis $(S^j\Omega^k)_{j,k\in\mathbb{Z}_d}$ for $\mathbb{C}^{d\times d}$, Schur's Lemma implies that b must be a (unit) scalar matrix cI, and hence $a = cS^{-\beta}\Omega^\alpha \in \hat{H}$. □

The order of $SL_2(\mathbb{Z}_d)$ is known to be (see [Gun62] Theorem 3, Chapter I)

$$|\,SL_2(\mathbb{Z}_d)| = d^3 \prod_{p|d}\Big(1-\frac{1}{p^2}\Big), \qquad (p \text{ the prime factors of } d).$$

Hence, by Lemma 3.2, the number of Clifford operations is

$$|\mathrm{PC}(d)| = \Big|\frac{\mathrm{C}(d)}{[I]}\Big| = \Big|\frac{\hat{H}}{[I]}\Big|\,\Big|\frac{\mathrm{C}(d)}{\hat{H}}\Big| = d^2|\,SL_2(\mathbb{Z}_d)| = d^5\prod_{p|d}\Big(1-\frac{1}{p^2}\Big).$$

Example 3.1 *From Lemma 3.1, we have the following $\psi_a \in SL_2(\mathbb{Z}_d)$,*

$$\psi_F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \psi_{P_\sigma} = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix}. \tag{3.22}$$

Example 3.2 *If $h = cS^a\Omega^b \in \hat{H}$, then $\psi_h = I$ and $z_h(j,k) = \omega^{bj-ak}$ (a character), since (2.2) gives*

$$hU_{(j,k)}h^{-1} = S^a\Omega^b S^j\Omega^k\Omega^{-b}S^{-a} = S^a(\omega^{bj}S^j\Omega^b)\Omega^{k-b}S^{-a} = \omega^{bj-ak}S^j\Omega^k = \omega^{jb-ak}U_{(j,k)}.$$

A function z_a satisfying (3.16) is called a **second degree character** of $\mathbb{Z}_n^2$ associated to the bicharacter

$$B : \mathbb{Z}_n^2 \times \mathbb{Z}_n^2 \to \mathbb{T}, \qquad B(p,q) := \omega^{p^T\sigma_A q},$$

given by σ_A (see [Rei89]). A continuous map $B : G \times G \to \mathbb{T}$ is a **bicharacter** of a locally compact abelian group G if for any fixed choice of one argument the resulting function $G \to \mathbb{T}$ is a character. All the second degree characters associated to a given bicharacter can be obtained from one by multiplying it by the characters.

Using a variation of the above argument, in [FHK+08] it is shown that if

$$\hat{H} \to \hat{H} : cU_\lambda \mapsto cz(\lambda)U_{A\lambda}, \qquad c \in \mathbb{T}, \quad z : \mathbb{Z}_d \to \mathbb{T}, \quad A \in GL_2(\mathbb{Z}_d), \tag{3.23}$$

is an automorphism of $\hat{H}$, then $A \in SL_2(\mathbb{Z}_d)$ and z is a second degree character (given by σ_A). For $a \in \mathrm{C}(d)$, the map $cU_\lambda \mapsto a(cU_\lambda)a^{-1} = cz_a(\lambda)U_{A\lambda}$, $A = \psi_a$, is an automorphism of $\hat{H}$. The elements of $\mathrm{C}(d)$ are termed **metaplectic** operations, and they are said to "intertwine the automorphisms of $\hat{H}$".

4 Generators for the Clifford group

It turns out that the matrices ψ_F and ψ_R of (3.22) generate $SL_2(\mathbb{Z}_d)$, and hence we obtain the following generators for the Clifford group.

Theorem 4.1 *(Clifford group generators) The homomorphism*

$$\psi : \mathrm{C}(d) \to SL_2(\mathbb{Z}_d) : a \mapsto \psi_a$$

maps F and R to generators for $SL_2(\mathbb{Z}_d)$, and hence is onto. Therefore $\mathrm{C}(d)$ is generated by the unitary scalar matrices, and S, Ω, F, R.

Proof: By Lemma 3.2, the kernel of $a \mapsto \psi_a$ is $\hat{H}$. Since $\hat{H}$ is generated by the unitary scalar matrices and S, Ω, it suffices to show that $SL_2(\mathbb{Z}_d)$ is generated by

$$\psi_F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{4.24}$$

It is well known that these matrices generate $SL_2(\mathbb{Z})$. Since the map of taking the entries of $A \in SL_2(\mathbb{Z})$ modulo d is a homomorphism onto $SL_2(\mathbb{Z}_d)$, they generate $SL_2(\mathbb{Z}_d)$. □

We call the subgroup of the Clifford group $\mathrm{C}(d)$ generated by F, R (and the scalars) the **symplectic unitaries**

$$\mathrm{C}_{\mathrm{Sp}}(d) := \langle F, R, [I] \rangle,$$

and the elements of $\mathrm{C}_{\mathrm{Sp}}(d)/[I]$ the **symplectic operations**. We will show that this is equivalent to the definition of [AYAZ13] that an element of the Clifford group is a symplectic unitary if it has an Appleby index of the form $[A, 0]$. Elements of the Heisenberg group $\hat{H}$ (or $\hat{H}/[I]$) are referred to as **Heisenberg operations**, (**Weyl**) **displacements** or **time–frequency shifts**. It follows from Theorem 4.1, that

Every Clifford operation is the product of a symplectic operation and a displacement.

For d even, an elementary calculation gives

$$\Omega^{\frac{d}{2}} = R^d, \quad S^{\frac{d}{2}} = F^{-1}\Omega^{\frac{d}{2}}F = F^{-1}R^dF, \quad S^{\frac{d}{2}}\Omega^{\frac{d}{2}} = F^{-1}R^dFR^d. \tag{4.25}$$

Thus there are nontrivial symplectic operations which are also displacements.[2] It turns out that (4.25) are the only cases (see Corollary 7.1). This makes the description of the Clifford group more technical for d even (here R has order $2d$).

5 Indexing the Clifford operations

We now show each Clifford operation is uniquely determined by the pair (ψ_a, z_a). Define the semidirect product $SL_2(\mathbb{Z}_d) \ltimes \mathbb{T}^{\mathbb{Z}_d^2}$ via the multiplication

$$(A, z_A)(B, z_B) := (AB, (z_A \circ B)z_B), \tag{5.26}$$

where functions $\mathbb{Z}_d^2 \to \mathbb{T}$ are multiplied pointwise.

Corollary 5.1 *With the multiplication (5.26), the map*

$$\mathrm{C}(d) \to SL_2(\mathbb{Z}_d) \ltimes \mathbb{T}^{\mathbb{Z}_d^2} : a \mapsto (\psi_a, z_a) \tag{5.27}$$

is a homomorphism with kernel $[I]$. Thus every Clifford operation $[a] \in \mathrm{C}(d)/[I]$ has a unique index (ψ_a, z_a), and these satisfy

$$\psi_{ab} = \psi_a\psi_b, \qquad z_{ab} = (z_a \circ \psi_b)z_b, \tag{5.28}$$

$$\psi_{a^*} = \psi_{a^{-1}} = \psi_a^{-1}, \qquad z_{a^*} = z_{a^{-1}} = \overline{z_a} \circ \psi_{a^*}, \tag{5.29}$$

$$\psi_{\overline{a}} = J\psi_a J, \qquad z_{\overline{a}} = \overline{z_a} \circ J, \qquad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.30}$$

Further, if $\psi_a = \psi_b$, then z_a/z_b is a character.

[2]Subgroups of the symplectic unitaries are sometimes said to be *displacement free*.

Proof: It is easy to check $SL_2(\mathbb{Z}_d) \ltimes \mathbb{T}^{\mathbb{Z}_d^2}$ is a group with the multiplication (5.26), identity $(I, 1)$, and inverse $(A, z_A)^{-1} = (A^{-1}, z_A^{-1} \circ A^{-1})$. By (3.20), we have

$$\psi_{ab} = \psi_a \psi_b, \qquad z_{ab} = (z_a \circ \psi_b) z_b,$$

i.e., the map $a \mapsto (\psi_a, z_a)$ is a homomorphism. Thus (5.28) holds, as does (5.29) by the calculation $(\psi_{a^{-1}}, z_{a^{-1}}) = (\psi_a, z_a)^{-1} = (\psi_a^{-1}, z_a^{-1} \circ \psi_a^{-1})$. and (5.30), since $\overline{U_\lambda} = U_{J\lambda}$ gives

$$\overline{a} U_\lambda \overline{a}^{-1} = \overline{a U_{J\lambda} a^{-1}} = \overline{z_a(\lambda) U_{AJ\lambda}} = \overline{z_a}(\lambda) U_{JAJ\lambda}.$$

Now suppose that a is in the kernel, i.e., $\psi_a = I$, $z_a = 1$. By Lemma 3.2, we have $a = cS^j\Omega^k \in \hat{H}$. Using (2.2), we therefore obtain (see Example 3.2)

$$a S^{p_1} \Omega^{p_2} a^{-1} = S^j \Omega^k S^{p_1} \Omega^{p_2} \Omega^{-k} S^{-j} = \omega^{kp_1 - jp_2} S^{p_1} \Omega^{p_2},$$

so that $z_a(p) = \omega^{kp_1 - jp_2} = 1$, $\forall p \in \mathbb{Z}_d^2$. Thus $j = k = 0$ and $a = cI \in [I]$, as supposed.

For $\psi_a = \psi_b = A$, it follows from (3.16) or (3.18) that z_a / z_b is a character. □

Example 5.1 *For $a = RF$, from (3.22) and Lemma 3.1, we calculate*

$$\psi_{RF} = \psi_R \psi_F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$z_{RF} = (z_R \circ \psi_F) z_F \quad \Longrightarrow \quad z_{RF}(j,k) = \mu^{(-k)(-k+d)} \omega^{-jk} = \mu^{k(k+d)+2jk}.$$

We call the subgroup of $SL_2(\mathbb{Z}_d) \ltimes \mathbb{T}^{\mathbb{Z}_d^2}$ given by

$$\mathrm{Ind}(d) := \{(\psi_a, z_a) : a \in \mathrm{C}(d)\}$$

the **index group** of the Clifford operations, and the **index map** is the isomorphism

$$\mathrm{C}(d)/[I] \to \mathrm{Ind}(d) : [a] \mapsto (\psi_a, z_a). \tag{5.31}$$

In view of the multiplication

$$z_{ah} = (z_a \circ \psi_h) z_h = z_a z_h, \qquad h \in \hat{H},$$

Example 3.2 (that all characters of $\mathbb{Z}_d^2$ have the form z_h, $h \in \hat{H}$), and the fact that $a \mapsto \psi_a$ is onto $SL_2(\mathbb{Z}_d)$ (Theorem 4.1), the elements of $\mathrm{Ind}(d)$ consist of all pairs (A, z), where $A \in SL_2(\mathbb{Z}_d)$ and z is a second degree character given by σ_A. In other words:

Corollary 5.2 *The automorphisms of the Heisenberg group of the form (3.23) are given by conjugation by Clifford operations.*

Proof: All the possible automorphisms of this type have z a second degree character given by σ_A, where $A \in SL_2(\mathbb{Z}_d)$, i.e., $(A, z) \in \mathrm{Ind}(d)$. If a is a Clifford operation with this index, then conjugation by a gives such an automorphism of $\hat{H}$. □

6 Appleby indexing

If d is odd, then $-\mu = \omega^{\frac{d+1}{2}}$, and it follows from (3.16) that

$$z_a(p) = (-\mu)^{p^T \sigma_A p}\, \hat{z}_a(p), \qquad \forall p \in \mathbb{Z}_d^2, \tag{6.32}$$

where $A = \psi_a$, and $\hat{z}_a$ is a character. If d is even, then the factor $(-\mu)^{p^T \sigma_A p}$ above is not well defined. To obtain an analogue of (6.32), it is necessary to "lift" A to a $B \in SL_2(\mathbb{Z}_{2d}^2)$. This "doubling" works, but the corresponding (Appleby) index $[B, \chi]$ is not unique. We now give the details as described by [App05], using Corollary 5.1 to streamline the proof.

Define displacement operators by

$$\hat{D}_p := (-\mu)^{p_1 p_2} S^{p_1} \Omega^{p_2}, \qquad p \in \mathbb{Z}^2. \tag{6.33}$$

These satisfy $\det(\hat{D}_p) = 1$,

$$\hat{D}_p^{-1} = \hat{D}_{-p}, \qquad \hat{D}_p \hat{D}_q = (-\mu)^{\langle\!\langle p,q \rangle\!\rangle} \hat{D}_{p+q} = \omega^{\langle\!\langle p,q \rangle\!\rangle} \hat{D}_q \hat{D}_p, \tag{6.34}$$

and

$$\hat{D}_{p+dq} = \begin{cases} \hat{D}_p, & d \text{ odd}; \\ (-1)^{\langle\!\langle p,q \rangle\!\rangle} \hat{D}_p, & d \text{ even}, \end{cases} \tag{6.35}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the *symplectic form*

$$\langle\!\langle p, q \rangle\!\rangle := p_2 q_1 - p_1 q_2 = p^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} q,$$

which has the property

$$\langle\!\langle Ap, Aq \rangle\!\rangle = \det(A) \langle\!\langle p, q \rangle\!\rangle, \qquad \forall p, q. \tag{6.36}$$

It follows from (6.35) that $\hat{D}_p$ depends only on $p \bmod d'$, where

$$d' := \begin{cases} d, & d \text{ odd}; \\ 2d, & d \text{ even}. \end{cases}$$

The appearance of d' in the description of $\mathrm{C}(d)$ is due to the fact that R has order d'. The *overlaps* $\chi_p^{\Pi} := \mathrm{trace}(\Pi \hat{D}_p)$, $p \in \mathbb{Z}_{d'}^2$, for a SIC fiducial $\Pi = vv^*$ play a crucial role in describing the Galois symmetries of a SIC.

We now generalise (6.32), to show that for each $[a] \in \mathrm{C}(d)/[I]$ there exists a $B \in SL_2(\mathbb{Z}_{d'})$ and $\chi \in \mathbb{Z}_d^2$, such that

$$a \hat{D}_p a^{-1} = \omega^{\langle\!\langle \chi, Bp \rangle\!\rangle} \hat{D}_{Bp}, \qquad \forall p \in \mathbb{Z}_{d'}^2.$$

Here $\langle\!\langle \chi, Bp \rangle\!\rangle$ is interpreted as $\langle\!\langle \chi, Ap \rangle\!\rangle$, $A := B \bmod d$, when d is even. We will write the pair (B, χ) as $[B, \chi]$, and call it an **Appleby index**.

Theorem 6.1 *Define the semidirect product $SL_2(\mathbb{Z}_{d'}) \ltimes \mathbb{Z}_d^2$ via the multiplication*

$$[B_1, \chi_1][B_2, \chi_2] := [B_1 B_2, \chi_1 + A_1 \chi_2], \quad A_1 := B_1 \bmod d. \tag{6.37}$$

Then there is a unique surjective homomorphism onto the Clifford operations

$$f : SL_2(\mathbb{Z}_{d'}) \ltimes \mathbb{Z}_d^2 \to \mathrm{C}(d)/[I], \tag{6.38}$$

with the property that for $[a] = f([B, \chi])$

$$a \hat{D}_p a^{-1} = \omega^{\langle\!\langle \chi, Bp \rangle\!\rangle} \hat{D}_{Bp}, \qquad \forall p \in \mathbb{Z}_{d'}^2, \tag{6.39}$$

i.e.,

$$A := \psi_a = B \bmod d, \qquad z_a(p) = \omega^{\langle\!\langle \chi, Ap \rangle\!\rangle} (-\mu)^{p^T \sigma_B p}, \quad \forall p \in \mathbb{Z}_d^2. \tag{6.40}$$

This f is an isomorphism for d odd (i.e., $d' = d$), and for d even it has kernel

$$\ker f = \Big\{ \Big[\begin{pmatrix} 1 + rd & sd \\ td & 1 + rd \end{pmatrix}, \begin{pmatrix} s\frac{d}{2} \\ t\frac{d}{2} \end{pmatrix} \Big] : r, s, t \in \{0, 1\} \Big\}. \tag{6.41}$$

Proof: If $a \in \mathrm{C}(d)$ satisfies (6.39), then (6.40) follows. Here $p^T \sigma_B p$ is calculated modulo $2d$, and its value only depends on $p \bmod d$. In view of the isomorphism (5.31), f is uniquely defined, and it suffices to show that

$$\theta : SL_2(\mathbb{Z}_{d'}) \ltimes \mathbb{Z}_d^2 \to \mathrm{Ind}(d) : [B, \chi] \mapsto (A, z_a),$$

given by (6.40) is a surjective homomorphism.

We first show it is a homomorphism (as a map to $SL_2(\mathbb{Z}_d) \ltimes \mathbb{T}^{\mathbb{Z}_d^2}$). Now

$$\theta\big([B_1, \chi_1][B_2, \chi_2]\big) = \theta\big([B_1 B_2, \chi_1 + A_1 \chi_2]\big) = (A_1 A_2, z_{a_1 a_2}),$$

$$A_j := B_j \bmod d, \qquad z_{a_1 a_2}(p) := \omega^{\langle\!\langle \chi_1 + A_1 \chi_2, A_1 A_2 p \rangle\!\rangle} (-\mu)^{p^T \sigma_{B_1 B_2} p},$$

and

$$\theta\big([B_1, \chi_1]\big)\theta\big([B_2, \chi_2]\big) = (A_1, z_{a_1})(A_2, z_{a_2}) = (A_1 A_2, (z_{a_1} \circ A_2) z_{a_2}),$$

$$\big((z_{a_1} \circ A_2) z_{a_2}\big)(p) = \omega^{\langle\!\langle \chi_1, A_1 A_2 p \rangle\!\rangle} (-\mu)^{(B_2 p)^T \sigma_{B_1} B_2 p} \omega^{\langle\!\langle \chi_2, A_2 p \rangle\!\rangle} (-\mu)^{p^T \sigma_{B_2} p},$$

so that θ is a homomorphism provided that

$$\langle\!\langle \chi_1 + A_1 \chi_2, A_1 A_2 p \rangle\!\rangle = \langle\!\langle \chi_1, A_1 A_2 p \rangle\!\rangle + \langle\!\langle \chi_2, A_2 p \rangle\!\rangle,$$

$$p^T \sigma_{B_1 B_2} p = (B_2 p)^T \sigma_{B_1} B_2 p + p^T \sigma_{B_2} p.$$

The first follows since (6.36) gives

$$\langle\!\langle \chi_2, A_2 p \rangle\!\rangle = \langle\!\langle A_1 \chi_2, A_1 A_2 p \rangle\!\rangle,$$

and the second follows by the identity

$$\sigma_{B_1 B_2} = B_2^T \sigma_{B_1} B_2 + \det(B_1) \sigma_{B_2}.$$

We calculate (as in Examples 3.1 and 3.2)

$$\theta\big([\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},0]\big) = (\psi_F, z_F), \qquad \theta\big([\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},0]\big) = (\psi_R, z_R), \tag{6.42}$$

and

$$\theta\big([I,\begin{pmatrix}\alpha\\ \beta\end{pmatrix}]\big) = (z_{S^\alpha\Omega^\beta}, \psi_{S^\alpha\Omega^\beta}), \tag{6.43}$$

so that θ maps generators for $SL_2(\mathbb{Z}_{d'}) \ltimes \mathbb{Z}_d^2$ to generators for $\mathrm{Ind}(d)$, and hence is a surjective homomorphism.

Finally, we determine $\ker f = \ker\theta$. By (6.40), we have $[B,\chi] \in \ker f$ if

$$A := \psi_a = B \bmod d = I, \qquad z_a(p) = \omega^{\langle\!\langle \chi, Ap\rangle\!\rangle}(-\mu)^{p^T\sigma_B p} = 1, \quad \forall p.$$

For d odd, $d' = d$, and so $B = I$ and $z_a(p) = \omega^{\chi_2 p_1 - \chi_1 p_2} = 1$, $\forall p$. Thus $[B,\chi] = [I,0]$, and f is an isomorphism. For d even, $B \bmod d = I$ gives

$$B = \begin{pmatrix}1+rd & sd\\ td & 1+ud\end{pmatrix}, \qquad r,s,t,u \in \{0,1\},$$

and the condition $\det(B) = 1$ gives

$$\det(B) = (1+rd)(1+ud) - std^2 \equiv 1 + (r+u)d \bmod d' \quad \Longrightarrow \quad r = u,$$

so that

$$B = \begin{pmatrix}1+rd & sd\\ td & 1+rd\end{pmatrix}, \quad \sigma_B = \begin{pmatrix}td(1+rd) & tdsd\\ sdtd & sd(1+rd)\end{pmatrix} \equiv \begin{pmatrix}td & 0\\ 0 & sd\end{pmatrix} \bmod d'.$$

Hence $z_a(p) = \omega^{\langle\!\langle \chi, p\rangle\!\rangle}(-\mu)^{p^T\sigma_B p} = \omega^{\chi_2 p_1 - \chi_1 p_2}(-\mu)^{tdp_1^2 + sdp_2^2} = 1$, which gives

$$\omega^{\chi_1 p_2 - \chi_2 p_1} = (-1)^{tp_1^2 + sp_2^2} = (-1)^{tp_1 + sp_2}\omega^{\frac{d}{2}(tp_1+sp_2)}, \quad \forall p.$$

Thus, $\chi_1 = \frac{d}{2}s$, $\chi_2 = -\frac{d}{2}t = \frac{d}{2}t$, and we obtain (6.41). □

Each Clifford operation has an Appleby index $[B,\chi] \in SL_2(\mathbb{Z}_{d'}) \times \mathbb{Z}_d^2$.

- This is unique for d odd.
- There are eight choices (each differing by an element of $\ker f$) for d even.
- Appleby indices for F, R, $S^\alpha\Omega^\beta$ are given by (6.42) and (6.43).

Example 6.1 *By (3.11), the index for the permutation matrix P_σ, $\sigma \in \mathbb{Z}_d^*$, is*

$$(\psi_{P_\sigma}, z_{P_\sigma}) = (P_\sigma, 1),$$

and it has an Appleby index $[P_\sigma, 0]$, where $\sigma \in \mathbb{Z}_{d'}$, i.e., σ^{-1} is calculated modulo $2d$ when d is even (see Proposition 8.1). For example, when $d = 8$ ($d' = 16$), the permutation matrix $P_3 = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$ has eight Appleby indices $[B,\chi]$

$$\Big[\begin{pmatrix}1+8r & 8s\\ 8t & 1+8r\end{pmatrix}, \begin{pmatrix}4s\\ 4t\end{pmatrix}\Big]\Big[\begin{pmatrix}3 & 0\\ 0 & 11\end{pmatrix}, 0\Big] = \Big[\begin{pmatrix}3+8r & 8s\\ 8t & 11+8r\end{pmatrix}, \begin{pmatrix}4s\\ 4t\end{pmatrix}\Big],\ r,s,t \in \{0,1\}.$$

Two of these have $\chi = 0$ (and B diagonal).

a	ψ_a	$z_a(j,k)$	$[B,\chi]$
$S^\alpha\Omega^\beta$	I	$\omega^{\beta j-\alpha k}$	$[I,\begin{pmatrix}\alpha\\ \beta\end{pmatrix}]$
F	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	ω^{-jk}	$[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},0]$
R	$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	$\mu^{j(j+d)}$	$[\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},0]$
P_σ	$\begin{pmatrix}\sigma & 0\\ 0 & \sigma^{-1}\end{pmatrix}$	1	$[\begin{pmatrix}\sigma & 0\\ 0 & \sigma^{-1}\end{pmatrix},0]$

Table 1: The index (ψ_a, z_a) and an Appleby index $[B,\chi]$ for various Clifford operations.

7 Symplectic unitaries

By (4.25), for d even, there are nontrivial *symplectic unitaries* (those generated by F, R and the scalars) which are in the Heisenberg group. We now characterise these.

Let m_d be the surjective homomorphism

$$m_d : SL_2(\mathbb{Z}_{d'}) \to SL_2(\mathbb{Z}_d) : B \mapsto A := B \pmod d,$$

which is the identity for d odd, and for d even has kernel (see Theorem 6.1)

$$K := \left\{ \begin{pmatrix} 1+rd & sd \\ td & 1+rd \end{pmatrix} : r,s,t \in \{0,1\} \right\}, \qquad |K| = 8. \tag{7.44}$$

Corollary 7.1 *A matrix $a \in \mathrm{C}(d)$ is a symplectic unitary if and only if it has an Appleby index of the form $[B,0]$. Indeed, the map*

$$\alpha : SL_2(\mathbb{Z}_{d'}) \to \mathrm{C}_{\mathrm{Sp}}(d)/[I] : B \mapsto f([B,0]) \tag{7.45}$$

is a surjective homomorphism, which is an isomorphism for d odd. When d is even, $\ker\alpha = \{I,(d+1)I\}$, and hence the only nontrivial Heisenberg operations which are symplectic are given by

$$S^{\frac{d}{2}},\ \Omega^{\frac{d}{2}},\ S^{\frac{d}{2}}\Omega^{\frac{d}{2}} \qquad (d \text{ even}).$$

Proof: By (6.37), we have

$$[B_1B_2,0] = [B_1,0][B_2,0],$$

and so α is a homomorphism. It is onto, since by (6.42), its image contains

$$\alpha(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}) = [F], \qquad \alpha(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}) = [R], \tag{7.46}$$

which are generators for $\mathrm{C}_{\mathrm{Sp}}(d)/[I]$. Since ψ has kernel $\hat{H}$ (Lemma 3.2), it induces a well defined homomorphism $\hat{\psi} : \mathrm{C}_{\mathrm{Sp}}(d)/[I] \to SL_2(\mathbb{Z}_d)$, with

$$\hat{\psi}([F]) = \psi(F) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \hat{\psi}([R]) = \psi(R) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}. \tag{7.47}$$

By (7.46) and (7.47), we conclude that

$$m_d = \hat{\psi} \circ \alpha,$$

since it holds for the generators (4.24) of $SL_2(\mathbb{Z}_{d'})$. The kernel of $\hat{\psi}$ consists of the symplectic operations which are also Heisenberg operations, i.e.,

$$\ker \hat{\psi} = \mathrm{C}_{\mathrm{Sp}}(d)/[I] \cap \hat{H}/[I].$$

For d odd, m_d is an isomorphism, so that $\ker \hat{\psi} = \{[I]\}$. For d even,

$$m_d = \hat{\psi} \circ \alpha \quad \Longrightarrow \quad |\ker \hat{\psi}|\,|\ker \alpha| = |\ker m_d| = |K| = 8,$$

and (6.35) gives

$$\hat{D}_{(d+1)Ip} = (-1)^{\langle\langle p,p\rangle\rangle} \hat{D}_p = \hat{D}_{Ip} \quad \Longrightarrow \quad (d+1)I \in \ker \alpha \quad \Longrightarrow \quad |\ker \alpha| \geq 2.$$

In view of (4.25), we must have

$$\ker \alpha = \{I, (d+1)I\}, \qquad \ker \hat{\psi} = \{[I], [S^{\frac{d}{2}}], [\Omega^{\frac{d}{2}}], [S^{\frac{d}{2}}\Omega^{\frac{d}{2}}]\},$$

as claimed. □

Thus, each symplectic operation $[a]$ has an Appleby index of the form $[B, 0]$, and

- This is unique for d odd.
- There are two choices ($[B, 0]$ and $[(d+1)B, 0]$) for d even.

We call $B \in SL_2(\mathbb{Z}_{d'})$ a **symplectic index** for $[a] \in \mathrm{C}_{\mathrm{Sp}}(d)/[I]$.

The following commutative diagram summarises Corollary 7.1.

$$\begin{array}{ccc} SL_2(\mathbb{Z}_{d'}) & \xrightarrow{\quad\alpha\quad} & \mathrm{C}_{\mathrm{Sp}}(d)/[I] \\ & \searrow^{m_d} & \downarrow \hat{\psi} \\ & & SL_2(\mathbb{Z}_d) \end{array} \tag{7.48}$$

In particular, we have the following 1–1 indexing of the symplectic operations

$$\frac{\mathrm{C}_{\mathrm{Sp}}(d)}{[I]} \cong \begin{cases} SL_2(\mathbb{Z}_d), & d \text{ odd}; \\ \frac{SL_2(\mathbb{Z}_{2d})}{\langle (d+1)I \rangle}, & d \text{ even}. \end{cases}$$

The matrices in $SL_2(\mathbb{Z}_{d'})$ are said to be *symplectic*. If a is a symplectic unitary, with symplectic index B, then (6.39) gives

$$a\hat{D}_p a^{-1} = \hat{D}_{Bp}, \qquad \forall p (\in \mathbb{Z}_{d'}),$$

i.e., the conjugation action of a on the displacement $\hat{D}_p$ is given by multiplication of p by the symplectic matrix B. This is the origin of the term *symplectic unitary*.

The group $\mathrm{C}_{\mathrm{Sp}}(d)$ of symplectic unitaries is *not irreducible* for $d > 2$, since its centre contains the nondiagonal matrix

$$P_{-1} = F^2.$$

Calculations in $\mathrm{C}_{\mathrm{Sp}}(d)/[I]$ can be done in the finite group generated by F and R.

8 Permutation matrices

Here we show that the permutation matrices are a subgroup of the symplectic unitaries (as we define them), i.e., each permutation matrix is word in F, R and the scalar matrices.

Proposition 8.1 *The permutation matrices P_b, $b \in \mathbb{Z}_d^*$, are symplectic. Indeed, with $1 \le b < d$, we have*

$$P_b = (c_{b,d})^{-1} R^{b^{-1}} F R^b F R^{b^{-1}} F, \tag{8.49}$$

where b^{-1} is the inverse of b in $\mathbb{Z}_{d'}^*$, and $c_{b,d} = c_{b^{-1},d}$ is the Gauss sum

$$c_{b,d} := \frac{1}{\sqrt{d}} \sum_{j \in \mathbb{Z}_d} \mu^{bj(j+d)} = \frac{1}{2\sqrt{d}} G\big(b(d+1), 2d\big).$$

Proof: Let $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}_{d'})$. Then $\sigma_B = 0$, and so (3.11) gives

$$A := \psi_{P_b} = B \bmod d, \qquad z_{P_b}(p) = 1 = \omega^{\langle\langle 0, Ap \rangle\rangle} (-\mu)^{p^T \sigma_B p}, \quad \forall p \in \mathbb{Z}_d^2.$$

By Theorem 6.1, this implies that $[B, 0]$ is an Appleby index for P_b , which is therefore a symplectic unitary, with symplectic index B. Now B can be factored

$$B = \begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -b^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -b^{-1} \end{pmatrix}. \tag{8.50}$$

In view of (6.42), a symplectic index for $R^b F$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and so applying the homomorphism α of Corollary 7.1 to (8.50) gives (8.49), for some scalar $c_{b,d}$, to be determined. From (3.6) and (3.7), we have

$$(R^b F)_{jk} = \frac{1}{\sqrt{d}} \mu^{bj(j+d)+2jk}. \tag{8.51}$$

Hence, equating the $(0,0)$–entries of $c_{b,d} P_b (R^{b^{-1}} F)^{-1} = R^b F R^{b^{-1}} F$, gives

$$\frac{1}{\sqrt{d}} c_{b,d} = \sum_{j \in \mathbb{Z}_d} (R^b F)_{0j} (R^{b^{-1}} F)_{j0} = \frac{1}{d} \sum_{j \in \mathbb{Z}_d} \mu^{b^{-1} j(j+d)}.$$

We recall that $\mu^{j(j+d)}$ depends only on j modulo d, and $\mu^{jd} = \mu^{dj^2}$, so that

$$c_{b,d} = \frac{1}{2} \frac{1}{\sqrt{d}} \sum_{j=0}^{2d-1} \mu^{b^{-1} j(j+d)} = \frac{1}{2\sqrt{d}} \sum_{j=0}^{2d-1} \mu^{b^{-1}(d+1)j^2} = \frac{1}{2\sqrt{d}} G\big(b^{-1}(d+1), 2d\big).$$

Evaluating the $(0,0)$–entries of (8.49), using (8.51), gives

$$c_{b,d} = \frac{1}{d\sqrt{d}} \sum_{j \in \mathbb{Z}_d} \sum_{k \in \mathbb{Z}_d} \mu^{bj(j+d)+2jk+b^{-1}k(k+d)} = c_{b^{-1},d}.$$

□

If the b^{-1} is computed as the inverse in $\mathbb{Z}_d^*$ for d even, then the formula (8.49) only gives P_b up to multiplication by the symplectic Heisenberg operations (4.25). The permutation matrices $\{P_\sigma\}_{\sigma\in\mathbb{Z}_d^*}$ are a subgroup of the symplectic unitaries, since the map $\mathbb{Z}_d^* \to \mathrm{C}_{\mathrm{Sp}}(d) : \sigma \to P_\sigma$ is a group homomorphism, by the calculation

$$(P_{\sigma_1\sigma_2})_{jk} = \sum_r (P_{\sigma_1})_{jr}(P_{\sigma_2})_{rk} = \delta_{j,\sigma_1 r}\delta_{r,\sigma_2 k} = \sigma_{j,\sigma_1\sigma_2 k} = (P_{\sigma_1\sigma_2})_{jk}.$$

The formulas for evaluating Gauss sums imply that $c_{b,d}$ is an 8–th root of unity, e.g., if b has odd order, then $c_{b,d} = (\sqrt{i})^{1-d}$.

Example 8.1 *When $b = 1$, (8.49) gives*

$$(RF)^3 = c_{1,d}P_1 = e^{-\frac{2\pi i}{8}(d-1)}I.$$

Thus $[RF]$ is a symplectic operation of order three.

The symplectic unitary of order three (as a matrix), given by

$$Z := \zeta^{d-1}RF, \qquad \zeta := e^{\frac{2\pi i}{24}}, \qquad \psi_Z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

is called the **Zauner matrix**. This matrix plays a central role in the construction of SICs, since the majority of the known SICs are can be obtained as eigenvectors of Z. The Zauner matrix satisfies

$$R^{-1}ZR = \overline{Z}^2, \qquad R^{-1}Z^2R = \overline{Z}, \tag{8.52}$$

$$Z(S^j\Omega^k)Z^{-1} = \mu^{k(k-2j+d)}S^{-k}\Omega^{j-k}, \tag{8.53}$$

and by (5.28), (5.30)

$$\psi_Z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \psi_{Z^2} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \psi_{\overline{Z}} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \psi_{\overline{Z}^2} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

In view of its definition, Z can substitute for either F or R as the generators for the Clifford group given by Theorem 4.1. Since the antilinear map of entrywise complex conjugation

$$C : \mathbb{C}^d \to \mathbb{C}^d : z \mapsto \overline{z}$$

maps SIC fiducials to SIC fiducials (as does the Clifford group), it is natural to consider the extended Clifford group $\mathrm{EC}(d)$, which is generated by C and the Clifford group. Thus we have the following cute corollary of Theorem 4.1.

Corollary 8.1 *The extended Clifford group is generated by $\hat{H}$, and*

$$C \text{ (order 2)}, \qquad Z \text{ (order 3)}, \qquad F \text{ (order 4)}.$$

In addition to being symplectic and of order three, Z has *Clifford trace*

$$\mathrm{trace}(\psi_Z) = -1 \in \mathbb{Z}_d.$$

We now give a complete characterisation of all such symplectic unitaries.

9 The symplectic unitaries of order 3

The **Clifford trace** is the map

$$\mathrm{tr_C} : \mathrm{C}(d) \to \mathbb{Z}_d : a \mapsto \mathrm{trace}(\psi_a).$$

Since $a \mapsto \psi_a$ is a homomorphism with kernel $\hat{H}$ (Lemma 3.2), this satisfies

$$\mathrm{tr_C}(ab) = \mathrm{tr_C}(ba), \qquad \forall a, b \in \mathrm{C}(d), \tag{9.54}$$

$$\mathrm{tr_C}(ah) = \mathrm{tr_C}(a), \qquad \forall a \in \mathrm{C}(d),\ \forall h \in \hat{H}. \tag{9.55}$$

In particular, the Clifford trace of any conjugate of Z or $Z^{-1} = Z^2$ is -1, e.g.,

$$\mathrm{tr_C}(gZg^{-1}) = \mathrm{tr_C}(Zg^{-1}g) = \mathrm{tr_C}(Z) = \mathrm{trace}(\psi_Z) = -1,$$

and the Clifford trace is well defined on the Clifford operations, i.e.,

$$\mathrm{tr_C}([a]) := \mathrm{tr_C}(a), \qquad \forall [a] \in \mathrm{PC}(d).$$

The order of a Clifford operation is related to its Clifford trace, since

$$A^2 = \mathrm{trace}(A)A - I, \qquad \forall A \in SL_2(\mathbb{Z}_d). \tag{9.56}$$

Lemma 9.1 *A nonidentity extended Clifford operation $[a] \in \mathrm{EC}(d)/[I]$ with index (A, z_a) and Clifford trace $t = \mathrm{trace}(A)$ has order 3 if and only if*

$$(t^2 - 1)A = (t + 1)I, \qquad z_a((t + 1)Ap) = \omega^{(t+1)p^T M_A p}, \quad \forall p \in \mathbb{Z}_d^2, \tag{9.57}$$

where $M_A = \begin{pmatrix} \gamma(\alpha^3 + 2\alpha^2\delta + \alpha\delta^2 - 2\alpha - \delta) & \beta\gamma(\alpha + \delta - 1)(\alpha + \delta + 1) \\ \beta\gamma(\alpha + \delta - 1)(\alpha + \delta + 1) & \beta(\delta^3 + 2\alpha\delta^2 + \alpha^2\delta - 2\delta - \alpha) \end{pmatrix}$, $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Proof: Since a product of three antiunitaries is a unitary matrix, we have $a \in \mathrm{C}(d)$. In view of the isomorphism (5.31), $[a]$ has order 3 if and only if

$$(A, z_a)^3 = (A^3, (z_a \circ A^2)(z_a \circ A)z_a) = (I, 1).$$

From (9.56), we obtain

$$A^3 = A(tA - I) = t(tA - I) - A = (t^2 - 1)A - tI,$$

so that the condition $A^3 = I$ can be written as the first condition of (9.57).

We now consider the condition $(z_a \circ A^2)(z_a \circ A)z_a = 1$. By (3.16), we calculate

$$\begin{aligned} z_a(p)z_a(Ap)z_a(A^2p) &= \omega^{-p^T\sigma_A(Ap)} z_a(p + Ap)z_a(A^2p) \\ &= \omega^{-p^T\sigma_A(Ap)}\omega^{-(p+Ap)^T\sigma_A(A^2p)} z_a(p + Ap + A^2p) \\ &= \omega^{-p^T(\sigma_A A + \sigma_A A^2 + A^T\sigma_A A^2)p} z_a(p + Ap + A^2p). \end{aligned}$$

By (9.56), we have

$$I + A + A^2 = I + A + tA - I = (1 + t)A.$$

Using $\det(A) = \alpha\delta - \beta\gamma = 1$, a calculation gives

$$\begin{aligned} &\sigma_A A + \sigma_A A^2 + A^T\sigma_A A^2 \\ &\quad = (\alpha + \delta + 1)\begin{pmatrix} \gamma(\alpha^3 + 2\alpha^2\delta + \alpha\delta^2 - 2\alpha - \delta) & \beta\gamma(\alpha + \delta - 1)(\alpha + \delta + 1) \\ \beta\gamma(\alpha + \delta - 1)(\alpha + \delta + 1) & \beta(\delta^3 + 2\alpha\delta^2 + \alpha^2\delta - 2\delta - \alpha) \end{pmatrix}. \end{aligned}$$

Thus we may rewrite the condition $(z_a \circ A^2)(z_a \circ A)z_a = 1$, to obtain the result. □

Example 9.1 *Since $z_a(0) = 1$, $\forall a \in \mathrm{EC}(d)$, and $\mathrm{tr}_{\mathrm{C}}(I) = 2 = -1$ if and only if $d = 3$, we have that if $a \in \mathrm{C}(d)$ has Clifford trace -1 and $d \neq 3$, then $[a]$ has order 3.*

By taking the trace of the condition $(t^2 - 1)A = (t + 1)I$, we have that the Clifford trace t of a Clifford operation of order 3 satisfies

$$(t - 2)(t + 1)^2 = 0. \tag{9.58}$$

For d a prime, the Clifford operators of order 3 must have Clifford trace -1.

Proposition 9.1 *Suppose that $d \neq 3$ and $a \in \mathrm{C}(d)$. Then*

1. *If a has Clifford trace -1, then $[a]$ has order 3.*
2. *If d is prime, then $[a]$ has order 3 if and only if a has Clifford trace -1.*

Proof: Since we have already proved 1, it suffices to prove for $d \neq 3$ prime and $[a]$ of order 3 that the Clifford trace $t = \mathrm{tr}_{\mathrm{C}}(a)$ is -1. We recall that t is a root of (9.58).

If $t \neq -1$, then $t + 1$ is a unit (all nonzero elements of $\mathbb{Z}_d$ are units for d prime), so that $t = 2$. But, if $t = 2$, then (9.57) gives $3A = 3I$, and hence $A = I$ ($3 \in \mathbb{Z}_d^*$ for $d \neq 3$ prime), so that $a \in \hat{H}$ (by Lemma 3.2). Since $(S^j \Omega^k)^3 = \omega^3 S^{3j} \Omega^{3k}$ and S, Ω have order d, the order of $[a]$ cannot be 3 (since 3 does not divide d). Thus $t = \mathrm{tr}_{\mathrm{C}}(a) - 1$ (when $[a]$ has order 3 and $d \neq 3$ is prime). □

A Clifford operation of order 3 is said to be **canonical order** 3 if it has Clifford trace -1 (see [App05]), e.g., the Zauner matrix Z and M_1 are canonical order 3.

Example 9.2 *It follows from (9.54) and (9.55) that left or right multiplication of a canonical order 3 Clifford operation by a displacement operation gives another canonical order 3 operation, e.g., $[h_1 Z h_2]$ is canonical order 3 for any $h_1, h_2 \in \hat{H}$.*

There are Clifford operations of order 3 with Clifford trace 2.

Example 9.3 *If 3 divides d, then the symplectic unitary $R^{\frac{d'}{3}}$ (and its inverse) has order 3 and Clifford trace*

$$\mathrm{tr}_{\mathrm{C}}(R^{\frac{d'}{3}}) = \mathrm{trace}\Big(\begin{pmatrix} 1 & 0 \\ \frac{d'}{3} & 1 \end{pmatrix}\Big) = 2,$$

as do the Weyl displacement operators $S^{\frac{d}{3}}, \Omega^{\frac{d}{3}}, S^{\frac{d}{3}} \Omega^{\frac{d}{3}}$.

There are Clifford operations of order 3 with Clifford trace $t \neq -1, 2$, i.e., for which (9.58) holds with $t - 2$ and $t + 1$ not units in $\mathbb{Z}_d$.

Example 9.4 *For $d = 10$, $SL_2(\mathbb{Z}_{10})$ has a single conjugacy class of elements of order 3 and trace 4 and 7. These have representatives*

$$A = \begin{pmatrix} 3 & 2 \\ 6 & 1 \end{pmatrix} \quad \textit{(trace 4)}, \qquad B = \begin{pmatrix} 6 & 5 \\ 5 & 1 \end{pmatrix} \quad \textit{(trace 7)}.$$

These can be lifted to symplectic indices which give symplectic unitaries of order 3 and Clifford trace 4 and 7, e.g., $a = R^{10} F R^8 F^{-1} R^6$, $b = R^{10} F R^5 F^{-1} R^{15}$.

The main technical result of the paper is the following lemma. This is essentially a proof the Conjecture 4 of [Fla06] on the number of conjugacy classes (d is replaced by $2d$ for d even), which was proved for d prime.

Lemma 9.2 *Suppose that $d \geq 2$, and let*

$$z := \psi_Z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad z^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \tag{9.59}$$

$$m_1 := \begin{pmatrix} 1 & 3 \\ \frac{d-3}{3} & -2 \end{pmatrix}, \quad d \equiv 3 \bmod 9, \tag{9.60}$$

$$m_2 := \begin{pmatrix} 1 & 3 \\ \frac{2d-3}{3} & -2 \end{pmatrix}, \quad d \equiv 6 \bmod 9. \tag{9.61}$$

Then the conjugacy classes of elements of order 3 and trace -1 in $SL_2(\mathbb{Z}_d)$ have representatives

$$\{z\}, \quad d \not\equiv 0 \bmod 3, \tag{9.62}$$

$$\{z, z^2\}, \quad d \equiv 0 \bmod 9 \text{ or } d = 3, \tag{9.63}$$

$$\{z, z^2, m_1\}, \quad d \equiv 3 \bmod 9, \ d \neq 3, \tag{9.64}$$

$$\{z, z^2, m_2\}, \quad d \equiv 6 \bmod 9. \tag{9.65}$$

By the Chinese remainder theorem, it is sufficient to prove this for d a prime power. The proof is given in the appendix. It is elementary, but long, since each case involves the solution of a binary quadratic equation.

We also need the following technical lemmas.

Lemma 9.3 *Let $\varphi : G \to H$ be a homomorphism of G onto H, with $|\ker \varphi| = 2^k$. If $h \in H$ has order 3, then there is an element $g \in G$ of order 3 with $\varphi(g) = h$.*

Proof: By the first isomorphism theorem for groups, we may assume that $H = G/K$, where $K = \ker \varphi$. Suppose that $h = aK \in G/K$ has order 3, i.e., $a^3 = x \in K$, where $a \notin K$. By Bézout's identity (the Euclidean algorithm) choose integers α, β with $1 = -3\alpha + 2^k\beta$. Let $g = ax^\alpha \in \langle a \rangle$. Then $\varphi(g) = ax^\alpha K = aK$, and

$$g^3 = (ax^\alpha)^3 = a^3 x^{3\alpha} = x^{3\alpha+1} = x^{2^k\beta} = 1.$$

□

Lemma 9.4 *For d even, $SL_2(\mathbb{Z}_{2d})$ has no elements of of order 3 and trace $d-1$.*

Proof: If $A \in SL_2(\mathbb{Z}_{2d})$ has order 3, and $t = \text{trace}(A)$, then, by (9.56), we have

$$A^3 = A(tA - I) = t(tA - I) - A = (t^2 - 1)A - tI = I \quad \Longrightarrow \quad (t^2 - 1)A = (t+1)I.$$

For $t = d - 1$, this gives $(d^2 - 2d)A = 0 = dI \pmod{2d}$, which not possible. □

We now characterise all symplectic unitaries of canonical order 3.

Theorem 9.1 *(Characterisation) The symplectic operations of canonical order* 3 *are conjugate in* $\mathrm{C_{Sp}}(d)/[I]$ *to* $[a]$*, where* $a \in \mathrm{C_{Sp}}(d)$ *is one of the following*

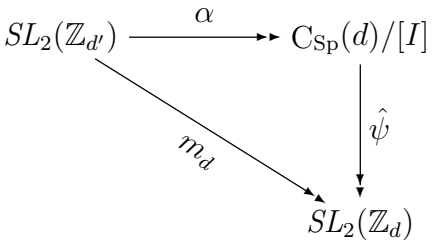
$$\begin{aligned} \{Z\}, &\quad d \not\equiv 0 \bmod 3, \\ \{Z, Z^2\}, &\quad d \equiv 0 \bmod 9 \text{ or } d = 3, \\ \{Z, Z^2, W_1\}, &\quad d \equiv 3 \bmod 9, d \neq 3, \\ \{Z, Z^2, W_2\}, &\quad d \equiv 6 \bmod 9, \end{aligned}$$

where

$$\begin{aligned} Z &:= e^{\frac{2\pi i}{24}(d-1)} RF^{-1}, \\ W_a &:= (-1)^{d-1} R^{\frac{2d}{3}a} F^{-1} R^3 F R, \end{aligned} \tag{9.66}$$

have order 3 *in* $\mathrm{C_{Sp}}(d)$*.*

Proof: The key idea is to apply the fact that group homomorphisms map conjugacy classes to conjugacy classes to the commutative diagram (7.48) of §7, i.e.,



We observe that

- Since the kernel of $\hat{\Psi}$ has order 1 or 4 (d odd or even), the conjugacy classes of elements of order 3 and Clifford trace -1 in $\mathrm{C_{Sp}}(d)/[I]$ map onto the conjugacy classes of elements of order 3 and trace -1 in $SL_2(\mathbb{Z}_d)$ (by Lemma 9.3).
- Since the kernel of α has 1 or 2 (d odd or even), each conjugacy class of an element of order 3 and Clifford trace -1 in $\mathrm{C_{Sp}}(d)/[I]$ is the image under α of the conjugacy class of an element of order 3 in $SL_2(\mathbb{Z}_{d'})$ (by Lemma 9.3) and of trace -1 (by Lemma 9.4).

Thus the conjugacy classes of elements of order 3 and trace -1 in $SL_2(\mathbb{Z}_{d'})$ map onto the conjugacy classes of elements of canonical order 3 in $\mathrm{C_{Sp}}(d)/[I]$, which in turn map onto the conjugacy classes of elements of order 3 and trace -1 in $SL_2(\mathbb{Z}_d)$. A count of the conjugacy classes in $SL_2(\mathbb{Z}_{d'})$ and $SL_2(\mathbb{Z}_d)$ (for d even) shows that these maps are 1–1, i.e., representatives of the conjugacy classes of elements of order 3 and trace -1 in $SL_2(\mathbb{Z}_{d'})$ give symplectic indices for representatives of the conjugacy classes of the symplectic operations of canonical order 3.

We now use Lemma 9.2 to calculate these symplectic indices (and show the injectivity asserted above) for the various cases.

For $d \not\equiv 0 \pmod 3$, we have $2d \not\equiv 0 \pmod 3$, and so there is a single conjugacy class with symplectic index z.

For $d \equiv 0 \pmod 9$, $d \neq 3$, we have $2d \equiv 0 \pmod 9$, and so there is are two conjugacy classes given by the symplectic indices z, z^2. For $d = 3$, we have $d' = d$, and there are two conjugacy classes given by the symplectic indices z, z^2.

For $d \equiv 3 \pmod 9$, $d \neq 3$, we have $2d \equiv 6 \pmod 9$, so that there are three conjugacy classes given by the symplectic indices z, z^2, and

$$\begin{pmatrix} 1 & 3 \\ \frac{d-3}{3} & -2 \end{pmatrix} \in SL_2(\mathbb{Z}_d) \quad (d \text{ odd}), \qquad \begin{pmatrix} 1 & 3 \\ \frac{2(2d)-3}{3} & -2 \end{pmatrix} \in SL_2(\mathbb{Z}_{2d}) \quad (d \text{ even}).$$

The second formula gives the first for d odd, and so works in both cases.

For $d \equiv 6 \pmod 9$, we have $2d \equiv 3 \pmod 9$, so that there are three conjugacy classes given by the symplectic indices z, z^2, and

$$\begin{pmatrix} 1 & 3 \\ \frac{2d-3}{3} & -2 \end{pmatrix} \in SL_2(\mathbb{Z}_d) \quad (d \text{ odd}), \qquad \begin{pmatrix} 1 & 3 \\ \frac{2d-3}{3} & -2 \end{pmatrix} \in SL_2(\mathbb{Z}_{2d}) \quad (d \text{ even}).$$

In the last two cases, the third conjugacy class is given by the symplectic indices m_1 and m_2 (respectively), where

$$m_j := \begin{pmatrix} 1 & 3 \\ \frac{4dj-3}{3} & -2 \end{pmatrix} \in SL_2(\mathbb{Z}_{d'}).$$

and m_j^2 is conjugate to m_j (since otherwise there would be four conjugacy classes). For convenience of presentation, we take the representative with symplectic index

$$w_j := m_j^2 = \begin{pmatrix} -2 & -3 \\ 1 + \frac{2d}{3} j & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\frac{2d}{3} j} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

By taking the symplectic operations corresponding to the representatives z, z^2, w_1, w_2 in the above conjugacy classes, i.e., Z, Z^2, W_1, W_2, we obtain representatives for the conjugacy classes of canonical order 3 symplectic operations. The normalisation of W_a in its definition (9.66) ensures that it has order 3. □

From the above proof, we have:

The conjugacy classes of order 3 *and trace* -1 *elements in* $SL(\mathbb{Z}_{d'})$ *are in* 1–1 *correspondence with the conjugacy classes of canonical order* 3 *symplectic operations.*

The the canonical order 3 symplectic operations Z (the Zauner matrix) and M_1 appear as symmetries of the Scott–Grassl SICs [SG10], where they are denoted by the symplectic indices F_z and F_a, i.e.,

$$B_z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = (d+1)F_z, \qquad B_{m_1} = \begin{pmatrix} d+1 & 3 \\ \frac{d-3}{3} & d-2 \end{pmatrix} = (d+1)F_a.$$

As yet, no numerical SIC fiducials have been found that are eigenvectors of M_2. We propose the name a **ghost SIC** for such SIC.

10 Conjugates of the canonical order 3 symplectic unitaries

We now use Theorem 9.1 to determine when the conjugate of Z or M_1 (or M_2 for that matter) by a symplectic operation is a monomial matrix of the form $R^\alpha P_\sigma$.

Since the permutation matrices in $C(d)$ are symplectic (see §8), there exists a permutation matrix $P_\sigma \in \mathrm{C}(d)$, $\sigma \in \mathbb{Z}_d^*$ of canonical order 3 if and only if

$$P_\sigma^3 = P_{\sigma^3} = I, \qquad \mathrm{tr}_{\mathrm{C}}(P_\sigma) = \sigma + \sigma^{-1} = -1,$$

i.e., the existence of an integer σ (for $d \neq 3$) with

$$\sigma^3 \equiv 1 \bmod d, \qquad \sigma^2 + \sigma + 1 \equiv 0 \bmod d. \tag{10.67}$$

For such a σ, we have $\begin{pmatrix} \sigma & 0 \\ \alpha & \sigma^{-1} \end{pmatrix}^3 = \begin{pmatrix} \sigma^3 & 0 \\ \alpha(1+\sigma+\sigma^2) & \sigma^{-3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that:

If σ satisfies (10.67), then $[R^\alpha P_\sigma]$ is a canonical order 3 symplectic operation.

By the Chinese remainder theorem, it follows (see [App05]) that the condition (10.67) is equivalent to d satisfying:

(i) d has at least one prime divisor $\equiv 1 \bmod 3$.

(ii) d has no prime divisors $\equiv 2 \bmod 3$ (so that d is odd).

(iii) d is not divisible by 9.

The first few such d are

$$d = 7, 13, 19, 21, 31, 37, 39, 43, 49, 57, 61, 67, 73, 79, 91, 93, 97, \ldots$$

By Theorem 9.1, the monomial operation $[R^\alpha P_\sigma]$ is conjugate (via a symplectic operation) to one of $[Z]$, $[Z]^2$, $[M_1] = [W_1]$, $[W_2]$.

- For d not a multiple of 3 ($d \not\equiv 0 \bmod 3$), i.e.,

  $$d = 7, 13, 19, 31, 37, 43, 49, 61, 67, 73, 79, 91, 97, \ldots$$

  there is single conjugacy class, and so all $[R^\alpha P_\sigma]$ are conjugate to $[Z]$.

- For d a multiple of 3, i.e.,

  $$d = 21, 39, 57, 93, 111, 129, 147, 183, 201, 219, 237, \ldots$$

  we have $\frac{d}{3} \equiv 1 \bmod 3$, i.e., $d \equiv 3 \bmod 9$, and so the conjugacy classes are given by $[Z], [Z]^2, [M_1]$.

For a σ satisfying $\sigma^3 = 1$, $1 + \sigma + \sigma^2$, the symplectic index calculations

$$gzg^{-1} = \begin{pmatrix} \sigma & 0 \\ 1 & \sigma^2 \end{pmatrix}, \qquad gz^2g^{-1} = \begin{pmatrix} \sigma^2 & 0 \\ -1 & \sigma \end{pmatrix}, \qquad g := \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

give the following:

For any d, if σ satisfies (10.67), then

1. The monomial operation $[RP_\sigma]$ is a symplectic conjugate of $[Z]$.

2. The monomial operation $[R^{-1}P_\sigma]$ is a symplectic conjugate of $[Z^2]$.

Whenever d is a multiple of 3, i.e., $d \equiv 3 \bmod 9$, it appears (for the d listed above) that $[P_\sigma]$ is always a symplectic conjugate of $[M_1]$.

Example 10.1 *For $d = 21$, no symplectic conjugate of Z is a permutation matrix, but many conjugates are monomial, e.g., the symplectic index calculation*

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 0 \\ 1 & 16 \end{pmatrix},$$

together with Table 1, gives

$$gZg^{-1} = \omega^7 R^{16} P_4, \qquad g := F^{-1}R^{-4}F.$$

11 Appendix

Proof: (of Lemma 9.2) We first show that it suffices to consider the case of d a prime power. Let $d = \prod_{j=1}^m p_j^{r_j}$ be a product of powers of distinct primes. If $a, b \in SL_2(\mathbb{Z}_d)$ are conjugate: $a = gbg^{-1}$, $g \in SL_2(\mathbb{Z}_d)$, then they are conjugate in $SL_2(\mathbb{Z}_{p_j^{r_j}})$, i.e.,

$$ag_j \equiv g_j b \bmod p_j^{r_j}, \qquad 1 \le j \le m, \tag{11.68}$$

where $g_j \equiv g \bmod p_j^{r_j}$ and $g_j \in SL_2(\mathbb{Z}_{p_j^{r_j}})$. Conversely, suppose that a and b are conjugate in $SL_2(\mathbb{Z}_{p_j^{r_j}})$ via $g_j \in SL_2(\mathbb{Z}_{p_j^{r_j}})$, $1 \le j \le m$. Then by the Chinese remainder theorem, there is a $g \in \mathbb{Z}^{2\times 2}$ with $g \equiv g_j \bmod p_j^{r_j}$, so that (11.68) gives

$$ag \equiv gb \bmod d = \prod_{j=1}^m p_j^{r_j}.$$

Similarly, $\det(g) \equiv \det(g_j) \equiv 1 \bmod p_j^{r_j}$, gives $\det(g) \equiv 1 \bmod d$. Thus a and b are conjugate in $SL_2(\mathbb{Z}_d)$.

Hence we assume now that $d = p^r$, p a prime and begin with the easiest case:
Case 1. $p \neq 3$, i.e., d is **not** divisible by 3.

Here we claim that there is one single conjugacy class, i.e., if $a, b \in SL_2(\mathbb{Z}_d)$ with $\mathrm{tr}(a) \equiv \mathrm{tr}(b) \equiv -1 \mod \mathrm{d}$, then a is conjugate to b. Since

$$z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in SL_2(\mathbb{Z}_d)$$

with $\mathrm{tr}(z) \equiv -1 \mod \mathrm{d}$, alternatively we may express this by saying that if $a \in SL_2(\mathbb{Z}_d)$ with $\mathrm{tr}(a) \equiv -1 \mod \mathrm{d}$, then a is conjugate to z, i.e., $\exists g \in SL_2(\mathbb{Z}_d)$ such that

$$a \equiv gzg^{-1} \iff ag \equiv gz \mod \mathrm{d}.$$

It turns out that the existence of such a g reduces to the existence of a solution of an associated Binary Quadratic Equation.

Lemma 11.1 *Suppose that $d \in \mathbb{Z}_{\geq 2}$ (including the divisible by 3 case) and that $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}_d)$ with $\mathrm{tr}(a) \equiv -1 \mod d$. Then there exists $g \in SL_2(\mathbb{Z}_d)$ such that*

$$a \equiv gzg^{-1} \mod d$$

iff

$$g = [x; ax],\ x \in \mathbb{Z}_d^2$$

such that

$$Q_a(x_1, x_2) := \gamma x_1^2 + (\delta - \alpha)x_1x_2 - \beta x_2^2 \equiv +1 \mod d. \tag{11.69}$$

Proof: (of Lemma) Writing $g = [x; y]$ with columns $x, y \in \mathbb{Z}_d^2$,

$$\begin{aligned} gz \equiv ag &\iff [x; y]\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \equiv a[x; y] \\ &\iff [y; -x - y] \equiv [ax; ay] \\ &\iff y \equiv ax \text{ and } ay \equiv -x - y. \end{aligned}$$

But $y \equiv ax$ implies that

$$ay \equiv a^2x \equiv -(a + I_2)x \equiv -ax - x \equiv -y - x.$$

Hence $gz \equiv ag \iff g \equiv [x; Ax]$ for an $x \in \mathbb{Z}_d^2$. Here we have used the fact that, by the modular form of the Cayley-Hamilton Theorem,

$$a^2 - \mathrm{tr}(a)a + \det(a)I_2 \equiv 0 \mod d$$

so that, by our assumptions on a,

$$a^2 \equiv \mathrm{tr}(a)a - \det(a)I_2 \equiv -a - I_2.$$

Now, for g to be in $SL_2(\mathbb{Z}_d)$ we require that $\det(g) \equiv +1 \mod d$. But for $g = [x; ax]$,

$$\begin{aligned} \det(g) &= \det \begin{pmatrix} x_1 & \alpha x_1 + \beta x_2 \\ x_2 & \gamma x_1 + \delta x_2 \end{pmatrix} \\ &= \gamma x_1^2 + (\delta - \alpha)x_1x_2 - \beta x_2^2 =: Q_a(x_1, x_2). \end{aligned}$$

□

We therefore proceed to analyze the solutions of the Binary Quadratic Equation (11.69). We begin with an important subcase.

Lemma 11.2 *Suppose that $p > 2$ is a prime and that $a, b, c \in \mathbb{Z}$ are units modulo p (i.e. not multiples of p). Then*

$$ax^2 + by^2 \equiv c$$

has a solution modulo p^r for all $r \geq 1$.

Proof: (of lemma) We use induction on r. The $r = 1$ case is actually a "well-known" consequence of the Chevalley-Warning Theorem (cf. [IR90]); we give the details for the sake of completeness.

Homogenize the equation to

$$ax^2 + by^2 - cz^2 \equiv 0$$

giving a homogeneous quadratic equation in three variables. The Chevalley-Warning Theorem then implies that there is a non-trivial integer solution (x_0, y_0, z_0) such that

$$ax_0^2 + by_0^2 - cz_0^2 \equiv 0 \mod \mathrm{p}.$$

If $z_0 \not\equiv 0 \mod \mathrm{p}$ it is a unit and we get

$$a(x_0 z_0^{-1})^2 + b(y_0 z_0^{-1})^2 \equiv c \mod \mathrm{p},$$

i.e., $x = x_0 z_0^{-1}$, $y = y_0 z_0^{-1}$ is a sought for solution. Otherwise, if $z_0 \equiv 0 \mod \mathrm{p}$ then we must have $x_0 \not\equiv 0$ and $y_0 \not\equiv 0$ and

$$\begin{aligned} & ax_0^2 + by_0^2 \equiv 0 \mod \mathrm{p} \\ \Longrightarrow \quad & x_0^2 + (ba^{-1})y_0^2 \equiv 0 \\ \Longrightarrow \quad & ba^{-1} \equiv -(x_0 y_0^{-1})^2, \end{aligned}$$

i.e., $-ba^{-1} = t^2$, $t = x_0 y_0^{-1}$. Thus our original equation $ax^2 + by^2 \equiv c$ reduces to

$$\begin{aligned} & x^2 - t^2 y^2 \equiv ca^{-1} \\ \Longleftrightarrow \quad & (x - ty)(x + ty) \equiv ca^{-1} \end{aligned}$$

which is solved, for example, by any solution of the linear system

$$\begin{aligned} x + ty &\equiv ca^{-1} \\ x - ty &\equiv 1, \end{aligned}$$

and in particular by $x = 2^{-1}(ca^{-1} + 1)$, $y = 2^{-1}t^{-1}(ca^{-1} - 1)$.

Continuing by induction, suppose that we have a solution (x_0, y_0) modulo p^r. We will show that then we also have one modulo p^{r+1}. To see this, note first that

$$x_0 + \alpha p^r, \ y_0 + \beta p^r, \quad \alpha, \beta \in \mathbb{Z}$$

are also solutions modulo p^r. We claim that one of these is also a solution modulo p^{r+1}. Substituting into our equation, we have (using the fact that $p^{2r} \equiv 0 \mod \mathrm{p^{r+1}}$)

$$\begin{aligned} & a(x_0 + \alpha p^r))^2 + b(y_0 + \beta p^r)^2 \equiv c \mod \mathrm{p^{r+1}} \\ \Longleftrightarrow \quad & a(x_0^2 + 2x_0\alpha p^r + \alpha^2 p^{2r}) + b(y_0^2 + 2y_0\beta p^r + \beta^2 p^{2r}) \equiv c \mod \mathrm{p^{r+1}} \\ \Longleftrightarrow \quad & ax_0^2 + by_0^2 + p^r(2ax_0\alpha + 2by_0\beta) + 0 \equiv c \mod \mathrm{p^{r+1}} \\ \Longleftrightarrow \quad & (ax_0^2 + by_o^2 - c) + p^r((2ax_0\alpha + 2by_0\beta) + 0 \equiv c \mod \mathrm{p^{r+1}}. \end{aligned}$$

But, by assumption, (x_0, y_0) is a solution modulo p^r and so

$$ax_0^2 + by_0^2 - c = kp^r$$

for some $k \in \mathbb{Z}$.

Consequently, we see that $(x_0 + \alpha p^r, y_0 + \beta p^r)$ is a solution modulo p^{r+1} iff

$$\begin{aligned} & p^r(k + (2ax_0)\alpha + (2by_0)\beta) \equiv 0 \mod \mathrm{p}^{\mathrm{r}+1} \\ \iff \quad & k + (2ax_0)\alpha + (2by_0)\beta) \equiv 0 \mod \mathrm{p}. \end{aligned}$$

But, as c is not a multiple of p by assumption, at least one of x_0, y_0 is also not a multiple of p and as $p > 2$, 2 is a unit, it follows that the linear Diophantine equation $k + (2ax_0)\alpha + (2by_0)\beta) \equiv 0 \mod \mathrm{p}$ for (α, β) has at least one coefficient a unit modulo p^{r+1} and therefore has a solution. □

Lemma 11.3 *Suppose that $p > 2$ is a prime and that for $a, b, c \in \mathbb{Z}$,*

$$Q(x, y) := ax^2 + bxy + cy^2$$

is a Binary Quadratic Form with discriminant

$$\Delta := b^2 - 4ac \not\equiv 0 \mod \mathrm{p}.$$

Let $u \in \mathbb{Z}$ be a unit modulo p. Then for every $r \geq 1$, the equation

$$Q(x, y) \equiv u \mod \mathrm{p}^{\mathrm{r}}$$

has a solution.

Proof: (of Lemma) First suppose that one of $a, c \not\equiv 0 \mod \mathrm{p}$. By symmetry we may assume that it is $a \not\equiv 0 \mod \mathrm{p}$. Then modulo p^r,

$$\begin{aligned} & Q(x, y) \equiv u \\ \iff \quad & 4aQ(x, y) \equiv 4au \quad \text{(as both } a \text{ and 4 are units)} \\ \iff \quad & 4ax^2 + 4abxy + 4acy^2 \equiv 4au \\ \iff \quad & (2ax + by)^2 - \{b^2 - 4ac\}y^2 \equiv 4au \\ \iff \quad & (2ax + by)^2 - \Delta y^2 \equiv 4au. \end{aligned}$$

Setting $x' = 2ax + by$ we have

$$(x')^2 - \Delta y^2 \equiv 4au \mod \mathrm{p}^{\mathrm{r}}$$

and by Lemma 11.2 there is a solution (x_0', y_0) which leads to the solution

$$x = x_0 = 2^{-1}a^{-1}\{x_0' - by_0\}, \; y = y_0$$

to $Q(x, y) \equiv u \mod \mathrm{p}^{\mathrm{r}}$.

If, on the other hand, $a \equiv c \equiv 0 \mod \text{p}$ then as, by assumption, $\Delta \not\equiv 0 \mod \text{p}$, we must have $b \not\equiv 0 \mod \text{p}$. We need in this case to prove our claim by induction on r.

If $r = 1$ then our equation reduces to

$$bxy = u \mod \text{p}$$

which has a solution (among others), $x = 1$, $y = ub^{-1}$, mod p.

Hence suppose that there is a solution (x_0, y_0) modulo p^r. We must show that then there is also a solution modulo p^{r+1}. Indeed, just as in the previous Lemma, we search for such a solution among

$$x = x_0 + \alpha p^r,\ y = y_0 + \beta p^r, \quad \alpha, \beta \in \mathbb{Z}.$$

Then modulo p^{r+1},

$$\begin{aligned} & Q(x_0 + \alpha p^r, y_0 + \beta p^r) \equiv u \\ \iff \quad & (ax_0^2 + bx_0y_0 + cy_0^2 - u) + p^r(2ax_0\alpha + bx_0\beta + by_0\alpha + 2cy_0\beta) \equiv 0 \\ \iff \quad & (ax_0^2 + bx_0y_0 + cy_0^2 - u) + p^r b(x_0\beta + y_0\alpha) \equiv 0 \end{aligned}$$

as $ap^r \equiv cp^r \equiv 0 \mod \text{p}^{\text{r+1}}$ since, by assumption in this case a and c are multiples of p. But as (x_0, y_0) is a solution modulo p^r there must exist a $k \in \mathbb{Z}$ such that

$$ax_0^2 + bx_0y_0 + cy_0^2 - u = kp^r.$$

Therefore, cancelling p^r from both sides, we arrive at the condition

$$k + b(x_0\beta + y_0\alpha) \equiv 0 \mod \text{p}.$$

This linear Diophantine equation has a solution for (α, β) as, in this case, b is a unit and the fact that u is a unit implies that not both x_0 and y_0 can be multiples of p. □

We will use the above to prove

Lemma 11.4 *For* $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}_d)$, $d = p^r$, *p a prime other than 3,* $r \geq 1$, *with* $\text{tr}(a) = \alpha + \delta \equiv -1 \mod \text{d}$, *there is a solution of the quadratic equation (11.69), i.e., of*

$$Q_a(x_1, x_2) = \gamma x_1^2 + (\delta - \alpha)x_1x_2 - \beta x_2^2 \equiv +1 \mod \text{d}.$$

Consequently a is conjugate to z modulo p^r.

Proof: (of lemma) First suppose that $p > 3$. First note that the discriminant of Q_a is

$$\begin{aligned} \Delta &= (\delta - \alpha)^2 + 4\beta\gamma \\ &= (\delta + \alpha)^2 - 4(\alpha\delta - \beta\gamma) \\ &= (\text{tr}(a))^2 - 4\text{det}(a) \\ &= (-1)^2 - 4 \times 1 \\ &= -3 \not\equiv 0 \mod \text{p}. \end{aligned}$$

Since $u = 1$ is a unit, we have from Lemma 11.3 that there exists a solution to $Q_a(x_1, x_2) \equiv +1 \mod \mathrm{p^r}$.

The case $p = 2$, $d = 2^r$ is somewhat different as then 2 is not a unit and Lemma 11.3 is not applicable. Nevertheless, also in this case $Q_a(x_1, x_2) \equiv +1 \mod 2^{\mathrm{r}}$ has a solution.

To see this note that the units modulo 2^r are precisely the odd integers $\leq d-1$. Now, $\mathrm{tr}(a) = \alpha+\delta \equiv -1 \mod 2^{\mathrm{r}}$ implies that $\alpha+\delta$ is odd and hence one of α, δ is odd and the other is even. Consequently the product $\alpha\delta$ us even. Further, $\det(a)\alpha\delta-\beta\gamma \equiv 1 \mod 2^{\mathrm{r}}$ and so $\alpha\delta - \beta\gamma$ is odd with $\alpha\delta$ even. Hence $\beta\gamma$ is odd and indeed *both* β and γ are odd, i.e., are units.

We claim that there is a solution of the form $Q_a(x_1, 1) \equiv +1 \mod 2^{\mathrm{r}}$, i.e., with $x_2 = 1$. In fact, modulo 2^r,

$$\begin{aligned} & Q_a(x_1, 1) \equiv +1 \\ \iff \quad & \gamma x_1^2 + (\delta - \alpha)x_1 - \beta \equiv +1 \\ \iff \quad & x_1^2 + \gamma^{-1}(\delta - \alpha)x_1 \equiv \gamma^{-1}(1 + \beta). \end{aligned}$$

Note that as one of α, δ is odd and the other even, $\delta-\alpha$ is odd and hence a unit. Further γ is odd and hence so is γ^{-1}. Consequently $\gamma^{-1}(\delta - \alpha)$ is *odd*. Moreover, β is odd and so $1 + \beta$ and $\gamma^{-1}(1 + \beta)$ are both *even*.

Consider now the univariate polynomial $P(x) := x^2 + ax$ with $a \in \mathbb{Z}$, odd. We claim that $P(x)$ maps the set of odd integers $\{x \in \mathbb{Z}_d : x \text{ odd}\}$ one-to-one and onto the set of even integers $\{x \in \mathbb{Z}_d : x \text{ even}\}$. Indeed, $P(x) = x^2 + ax = x(x + a)$. Hence if x is odd then $x + a$ is even and $P(x)$ is even. To see that the mapping is one-to-one, suppose that $x, y \in \mathbb{Z}_d$ are both odd. Then modulo $d = 2^r$,

$$\begin{aligned} & P(x) \equiv P(y) \\ \iff \quad & x^2 + ax \equiv y^2 + ay \\ \iff \quad & x^2 - y^2 + a(x - y) \equiv 0 \\ \iff \quad & (x - y)(x + y + a) \equiv 0. \end{aligned}$$

But x, y both odd and a also odd implies that $x + y + a$ is odd and hence a unit modulo 2^r. Consequently we may divide by $x + y + a$ to obtain that $P(x) \equiv P(y)$ iff $x - y \equiv 0$, i.e., $x \equiv y$ and we have shown that the mapping is one-to-one.

To see that it is also onto just note that the cardinality of the domain $\#\{x \in \mathbb{Z}_d : x \text{ odd}\} = d/2$ as does the cardinality of the image $\#\{x \in \mathbb{Z}_d : x \text{ even}\} = d/2$.

Thus with $a = \gamma^{-1}(\delta - \alpha)$ it follows that there is an $x_1 \in \mathbb{Z}_d$ such that $P(x_1) = \gamma^{-1}(1 + \beta)$ and we are done. □

In summary, we have shown so far that for any d a product of primes other than 3, i.e., for any d not divisible by 3, every $a \in SL_2(\mathbb{Z}_d)$ with $\mathrm{tr}(a) \equiv -1 \mod \mathrm{d}$, is conjugate to $z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

We now consider the cases when d *is* divisible by 3 beginning with

Case 2. $d = 3$

We claim that there are three conjugacy classes:

1. Those conjugate to $z : C_z := \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\}$
2. Those conjugate to $z^2 : C_{z^2} := \left\{ \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\} = C_z^T$
3. Those conjugate to $I_2 : C_I := \{I_2\}$.

We begin with the lemma for z^2 analogous to Lemma 11.1 for z.

Lemma 11.5 *Suppose that* $d \geq 2$ *and that* $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}_d)$ *with* $\mathrm{tr}(a) \equiv -1 \mod d$. *Then there exists a* $g \in SL_2(\mathbb{Z}_d)$ *such that*

$$a = gz^2g^{-1}$$

iff $g = [Ay; y]$ *for some* $y \in \mathbb{Z}_d^2$ *such that*

$$Q_a(y_1, y_2) = \gamma y_1^2 + (\delta - \alpha)y_1y_2 - \beta y_2^2 \equiv -1 \mod d.$$

Proof: (of lemma) Writing, as before, $g = [x; y]$ with columns $x, y \in \mathbb{Z}_d^2$,

$$\begin{aligned} gz^2 \equiv ag &\iff [x; y]\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \equiv a[x; y] = [ax; ay] \\ &\iff [-x - y; x] \equiv [ax; ay] \\ &\iff x \equiv ay \text{ and } -x - y \equiv ax. \end{aligned}$$

But $x \equiv ay$ implies that

$$\begin{aligned} ax &\equiv a^2y \\ &\equiv -(a + I_2)y \\ &\equiv -ay - y \\ &\equiv -x - y. \end{aligned}$$

Hence $gz^2 \equiv ag$ iff $g = [ay; y]$ for some $y \in \mathbb{Z}_d^2$.

Now, for such g,

$$\begin{aligned} \det(g) &= \det \begin{pmatrix} \alpha y_1 + \beta y_2 & y_1 \\ \gamma y_1 + \delta y_2 & y_2 \end{pmatrix} \\ &= \beta y_2^2 + (\alpha - \delta)y_1y_2 - \gamma y_1^2 \\ &= -Q_a(y_1, y_2). \end{aligned}$$

Hence $\det(g) \equiv +1 \mod d$ iff $Q_a(y_1, y_2) \equiv -1 \mod d$. □

Now if $a \equiv I_2$ then obviously $a \in C_I$. We claim that if $a \not\equiv I_2$ then there is either a solution of

$$Q_a(y_1, y_2) \equiv +1 \mod 3$$

or else of

$$Q_a(y_1, y_2) \equiv -1 \mod 3$$

but not both. To see this note that if β and γ are both $0 \mod 3$ then a is diagonal and it is easily verified that the only diagonal $a \in SL_2(\mathbb{Z}_3)$ with $\mathrm{tr}(a) \equiv -1$ is $a = I_2$. Hence at least one of β, γ are not 0 modulo 3.

Now, if $\gamma \equiv 1$, then $Q_a(y_1, y_2) \equiv +1$ has the solution $y_1 = 1, y_2 = 0$. If $\gamma \equiv -1$, then $Q_a(y_1, y_2) \equiv -1$ has the solution $y_1 = 1, y_2 = 0$. Otherwise, if $\gamma \equiv 0$ and hence $\beta \not\equiv 0$, $Q_a(0, 1) \equiv -1$ if $\beta \equiv +1$ while $Q_a(0, 1) \equiv +1$ if $\beta \equiv -1$.

We now verify that we can not have solutions to *both* $Q_a(y_1, y_2) \equiv \pm 1$, or in other words, z and z^2 are *not* conjugate. Indeed, by Lemma 11.1 z^2 is conjugate to z iff there is a solution of

$$Q_{z^2}(x_1, x_2) \equiv +1 \mod 3.$$

But $Q_{z^2}(x_1, x_2) = -x_1^2 + x_1x_2 - x_2^2$ so, modulo 3,

$$\begin{aligned} & Q_{z^2}(x_1, x_2) \equiv +1 \\ \iff \quad & x_1^2 - x_1x_2 + x_2^2 \equiv -1 \\ \iff \quad & x_1^2 + 2x_1x_2 + x_2^2 \equiv -1 \\ \iff \quad & (x_1 + x_2)^2 \equiv -1. \end{aligned}$$

But -1 is not a perfect square modulo 3 and so this is not possible.

For $d = 3$ there are only 9 different $a \in SL_2(\mathbb{Z}_3)$ with $\mathrm{tr})(a) \equiv -1$ and hence it is a trivial matter to list the conjugacy classes.

Case 3. $d = 3^r$, $r \geq 2$. We claim that here there are two conjugacy classes:

1. Those conjugate to z : $\{a \in SL_2(\mathbb{Z}_d) \,:\, (a \mod 3) \in \mathrm{C_z}\}$
2. Those conjugate to z^2 : $\{a \in SL_2(\mathbb{Z}_d) \,:\, (a \mod 3) \in \mathrm{C_{z^2}}\}$

The class $\{a \in SL_2(\mathbb{Z}_d) \,:\, (a \mod 3) = \mathrm{I_2}\}$ is not present. Indeed, if $a \equiv I_2 \mod 3$, then

$$a = \begin{pmatrix} 1 + 3x & 3y \\ 3z1 + 3w & \end{pmatrix}$$

for some $x, y, z, w \in \mathbb{Z}$.

Then $\mathrm{tr}(a) = 2 + 3(x + w)$ and

$$\det(a) = (1 + 3x)(1 + 3w) - 9yz = 1 + 3(x + w) \mod 9.$$

If $\mathrm{tr}(a) \equiv -1 \mod 3^{\mathrm{r}}$, $r \geq 2$, then $\mathrm{tr}(a) \equiv -1 \mod 9$ and similarly $\det(a) \equiv +1 \mod 9$. Hence we must have

$$\begin{aligned} & 2 + 3(x + w) \equiv -1 \mod 9 \\ \text{and} \quad & 1 + 3(x + w) \equiv +1 \mod 9. \end{aligned}$$

Subtracting the two gives $1 \equiv -2 \mod 9$ which is clearly not possible.

Supposing therefore that $a \not\equiv I_2 \mod 3$, it is easy to check by comparing with the lists of C_z and C_{z^2} (mod 3), that one of $\beta, \gamma \not\equiv 0 \mod 3$. By symmetry we may suppose that it is $\gamma \not\equiv 0$.

We claim that there is either a solution of

$$\begin{aligned} & Q_a(x_1, x_2) \equiv +1 \mod 3^{\mathrm{r}} \\ \text{or} \quad & Q_a(x_1, x_2) \equiv -1 \mod 3^{\mathrm{r}} \end{aligned}$$

but not both.

The $r = 1$ case gives a solution $(x_1, x_2) = (1, 0) \mod 3$ for

$$Q_a(x_1, x_2) \equiv +1 \mod 3 \quad \text{in case } \gamma \equiv +1 \mod 3$$

and

$$Q_a(x_1, x_2) \equiv -1 \mod 3 \quad \text{in case } \gamma \equiv -1 \mod 3.$$

This persists for $r \geq 2$, i.e., there is a solution $(x_1, x_2) = (1, 0) \mod 3$ for

$$Q_a(x_1, x_2) \equiv +1 \mod 3^{\mathrm{r}} \quad \text{in case } \gamma \equiv +1 \mod 3$$

and

$$Q_a(x_1, x_2) \equiv -1 \mod 3^{\mathrm{r}} \quad \text{in case } \gamma \equiv -1 \mod 3.$$

To see this we proceed by induction on r and assume that we have such a solution mod 3^{r}; we will show that there is also one mod $3^{\mathrm{r}+1}$. Indeed,

$$(x_1 + u3^r, v3^r), \ u, v \in \mathbb{Z}$$

are all solutions mod 3^{r}. Then setting $\overline{\gamma} := \gamma \mod 3$, we have

$$\begin{aligned} & Q_a(x_1 + u3^r, v3^r) \equiv \overline{\gamma} \mod 3^{\mathrm{r}+1} \\ \iff \quad & \gamma(x_1 + u3^r)^2 + (\delta - \alpha)(x_1 + u3^r)(v3^r) - \beta(v3^r)^2 \equiv \overline{\gamma} \mod 3^{\mathrm{r}+1} \\ \iff \quad & \gamma(x_1^2 + 2ux_1 3^r) + (\delta - \alpha)(x_1 v 3^r) \equiv \overline{\gamma} \mod 3^{\mathrm{r}+1} \\ \iff \quad & (\gamma x_1^2 - \overline{\gamma}) + 3^r(2\gamma x_1 u + (\delta - \alpha)x_1 v) \equiv 0 \mod 3^{\mathrm{r}+1}. \end{aligned}$$

But, as $(x_1, 0)$ is a solution mod 3^{r} (by assumption),

$$\gamma x_1^2 - \overline{\gamma} = k3^r$$

for some $k \in \mathbb{Z}$.

Thus

$$3^r\{k + (2\gamma x_1)u + ((\delta - \alpha)x_1)v\} \equiv 0 \mod 3^{\mathrm{r}+1}$$

iff

$$k + (2\gamma x_1)u + ((\delta - \alpha)x_1)v \equiv 0 \mod 3$$

iff

$$k + (2\gamma)u + ((\delta - \alpha))v \equiv 0 \mod 3$$

as $x_1 \equiv 1 \mod 3$. But this has the solution

$$\begin{aligned} v &= 0, \\ u &= (2\gamma)^{-1}(-k) \mod 3 \\ &\equiv (-\gamma)^{-1} \mod 3 \\ &\equiv \gamma^{-1}k \mod 3 \\ &\equiv \gamma k \mod 3, \end{aligned}$$

i.e., we have the solution

$$(x_1 + (k\gamma)3^r, 0) \mod 3^{r+1}.$$

The conjugacy classes are distinct as conjugacy modulo 3^{r+1} implies conjugacy modulo 3. □

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