Technical Report
23 November 2010

Frames for vector spaces and affine spaces

Shayne Waldron

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand
e–mail: waldron@math.auckland.ac.nz

ABSTRACT

A finite frame for a finite dimensional Hilbert space is simply a spanning sequence. We show that the linear functionals given by the dual frame vectors *do not* depend on the inner product, and thus it is possible to extend the frame expansion (and other elements of frame theory) to any finite spanning sequence for a vector space. The corresponding coordinate functionals generalise the dual basis (the case when the vectors are linearly independent), and are characterised by the fact that the associated Gramian matrix is an *orthogonal projection*. Existing generalisations of the frame expansion to Banach spaces involve an analogue of the frame bounds and frame operator.

The potential applications of our results are considerable. Whenever there is a natural spanning set for a vector space, computations can be done directly with it, in an efficient and stable way. We illustrate this with a diverse range of examples, including multivariate spline spaces, generalised barycentric coordinates, and vector spaces over the rationals, such as the cyclotomic fields.

Key Words: finite frames, vector spaces over the rationals, least squares (minimum norm) solutions, affine spaces, barycentric coordinates, multivariate splines, cyclotomic fields

AMS (MOS) Subject Classifications: primary 15A03, 15A21, 41A45, 42C15, secondary 12Y05, 15B10, 41A15, 52B11, 65F25,

1. Introduction

Vectors $(f_j)_{j=1}^n$ in a Hilbert space $\mathcal{H}$ are a *frame* for $\mathcal{H}$ if for some $A, B > 0$

$$A\|f\|^2 \le \sum_j |\langle f, f_j \rangle|^2 \le B\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

This is equivalent to (f_j) spanning the finite dimensional space $\mathcal{H}$. The well established theory of frames (cf. [C03], [W11]), then gives the dual frame expansion

$$f = \sum_j \langle f, \tilde{f}_j \rangle f_j = \sum_j \langle f, f_j \rangle \tilde{f}_j, \qquad \forall f \in \mathcal{H}, \tag{1.1}$$

where the *dual frame* vectors $(\tilde{f}_j)$ are characterised by the fact that the coefficients $(\langle f, \tilde{f}_j \rangle)$ in the first expansion for f above have the minimal ℓ_2–norm amongst all possible choices. Such expansions, which generalise orthogonal and biorthogonal expansions, have many applications, e.g., in wavelets and signal analysis (see [K94] and [CK07]).

A careful examination of the above characterisation of the dual frame vectors, shows that the linear functionals $c_j : f \mapsto \langle f, f_j \rangle$ (giving the coefficients with minimal ℓ_2–norm) *do not* depend on the inner product – though, when there is one, the Riesz representation conveniently allows them to be expressed via the inner product with a dual frame vector.

This paper outlines the consequences of this simple observation: the extension of the dual frame expansion, and other elements of frame theory, to finite spanning sequences for a vector space X. Key features of our "canonical expansion" include

- It is easy to calculate.
- The corresponding coordinates have nice properties, which characterise them.
- It depends continuously on the frame vectors.
- It transforms naturally under linear maps, and preserves symmetries.
- The dimension of the vector space does not need to be known.
- There is an analogue for affine spaces, i.e., generalised barycentric coordinates.

The development is elementary, requiring no knowledge of frame theory. However, we do indicate the corresponding notions in frame theory, when they exist. In these terms, our principal result (Theorem 4.9) is

- If $(f_j)_{j=1}^n$ spans a vector space X, then there is a unique inner product on X for which it is a *normalised tight frame*, i.e., (1.1) holds with $\tilde{f}_j = f_j$.

If the success of finite frames in applications (cf. [CK07]) is anything to go by, then the possible applications of these results are considerable. Whenever there is a natural spanning set/sequence for a vector space or affine space, computations can be done directly with it, in an efficient and stable way. This avoids the need to obtain a basis by thinning or applying Gram–Schmidt to an ad hoc ordering of the spanning set, which may destroy the inherent geometry. Spanning sets also have natural advantages over bases is when the dimension of the space is difficult to determine, e.g., multivariate spline spaces. A number of indicative examples are given, including the cyclotomic fields as vector spaces over the rationals (where the roots of unity form a natural spanning set). By way of contrast, the generalisations of the frame expansion to Banach spaces initiated by [G91] typically involve an analogue of the frame bounds and frame operator – something which can completely be dispensed with in the finite dimensional case (where convergence is not an issue).

The rest of the paper is set out as follows. Next we give the basic linear algebra of a spanning sequence $\Phi = (f_j)$. In particular, we show that the linear dependencies of Φ can be described in a canonical way, via an *associated projection* matrix P_Φ (Lemma 3.3). We define the *canonical* dual functionals $c^\Phi = (c_j)$ for Φ and characterise them (Section 4), and then describe their coordinate like properties (Section 5). We conclude with a parallel development for affine spanning sequences for affine spaces, i.e., generalised barycentric coordinates.

2. Duality

Throughout, let $\mathbb{F}$ denote a subfield of $\mathbb{C}$, e.g., the rationals $\mathbb{Q}$ or the reals $\mathbb{R}$, and X be a d–dimensional vector space over $\mathbb{F}$. The $\mathbb{F}$–vector space of all linear functionals $\lambda : X \to \mathbb{F}$ is called the (algebraic) dual X and is denoted by X'. If $(f_j)_{j=1}^d$ is a basis for X, then the unique coefficients $\lambda_j(f)$ for which $f = \sum_j \lambda_j(f) f_j$ define linear functionals λ_j which are a basis for X', so that $\dim(X) = \dim(X')$. These satisfy $\lambda_j(f_k) = \delta_{jk}$, and are called the *dual basis*. The *bidual* $X'' = (X')'$ is canonically isomorphic to X via the map

$$X \to X'' : x \mapsto \hat{x}, \qquad \hat{x}(\lambda) := \lambda(x), \qquad \forall \lambda \in X'. \tag{2.1}$$

Let e_j denote the j–th standard basis vector for $\mathbb{F}^J$.

For a finite sequence $\Phi = (f_j)_{j\in J}$ in X we define the **synthesis map**

$$V = [f_j]_{j\in J} : \mathbb{F}^J \to X : a \mapsto \sum_{j\in J} a_j f_j.$$

As the matrix notation suggests, this maps the coefficients (a_j) to the linear combination $\sum_j a_j f_j$ of the "columns" $f_j = Ve_j$ of V. Thus the sequence Φ spans X if and only if V is onto X. Similarly, for a finite sequence $\Psi = (\lambda_j)_{j\in J}$ of linear functionals on X, the **analysis map** is

$$\Lambda : X \to \mathbb{F}^J, \qquad \Lambda(f) := (\lambda_j(f))_{j\in J},$$

and we write $\Lambda = (\lambda_j)_{j\in J}$ for short (the same notation is used for the sequence Ψ). The duality (2.1) gives that Λ is 1–1 if and only if (λ_j) spans X'.

If X has an inner product, then each $\lambda \in X'$ has a Riesz representation $\lambda(x) = \langle x, f\rangle$ for some $f \in X$. For this reason, the product of the analysis map and synthesis map of finite sequences $\Psi = (\lambda_j)_{j\in J}$ and $\Phi = (f_k)_{k\in K}$ in X' and X

$$\Lambda V = (\lambda_j)_{j\in J}[f_k]_{k\in K} = [\lambda_j(f_k)]_{j\in J,k\in K}$$

is often called the **Gramian** matrix of the sequences. Let I_X denote the identity on X.

Our generalisation of (1.1) is an expansion of the following type.

Proposition 2.2 (Dual sequences). *Let $(f_j)_{j\in J}$ and $(\lambda_j)_{j\in J}$ be spanning sequences for a vector space X and its algebraic dual X', respectively, with Gramian*

$$G := [\lambda_j(f_k)]_{j,k\in J} = \Lambda V, \qquad V := [f_j]_{j\in J} : \mathbb{F}^J \to X, \quad \Lambda = (\lambda_j)_{j\in J} : X \mapsto \mathbb{F}^J.$$

Then the following are equivalent
(a) $V\Lambda = I_X$.
(b) $f = \sum_j \lambda_j(f) f_j$, $\forall f \in X$.
(c) $\lambda = \sum_j \lambda(f_j)\lambda_j$, $\forall \lambda \in X'$.
(d) $G^2 = G$.
(e) $V = VG$.
(f) $\Lambda = G\Lambda$.

If $(f_j)_{j\in J}$ and $(\lambda_j)_{j\in J}$ are spanning sequences for X and X' (with the same index set) satisfying any of the conditions of Proposition 2.2, then (by analogy with the case of bases) we say that they are **dual**. If (f_j) is not a basis, then there are many sequences of functionals which are dual to it. In Section 4, we show that there is a *canonical* choice if $\mathbb{F}$ is closed under complex conjugation, and give various characterisations for it. For example, it is the unique choice for which the Gramian is an orthogonal projection matrix.

3. Linear dependencies and the associated projection

From now on, we assume that the field $\mathbb{F} \subset \mathbb{C}$ is *closed under complex conjugation*. This ensures that if A is a matrix with entries in $\mathbb{F}$, then so is its Hermitian transpose A^*. In particular, the inner product between vectors with entries from $\mathbb{F}$ is in $\mathbb{F}$.

The linear combinations of a finite spanning sequence $\Phi = (f_j)_{j\in J}$ for a vector space X can be conveniently described as elements of the range of the synthesis map

$$V = [f_j]_{j\in J} : \mathbb{F}^J \to X : a \mapsto \sum_j a_j f_j.$$

The subspace of $\mathbb{F}^J$ consisting of all **(linear) dependencies** is $\operatorname{dep}(\Phi) := \ker(V)$, i.e.,

$$a = (a_j)_{j\in J} \in \operatorname{dep}(\Phi) \quad \iff \quad \sum_j a_j f_j = 0.$$

Now let $\mathbb{C}^J$ (and hence $\mathbb{F}^J$) have the Euclidean inner product $\langle x, y\rangle = \sum_j x_j \overline{y_j}$, and

$$\operatorname{dep}(\Phi)^\perp := \{x \in \mathbb{F}^J : \langle x, a\rangle = 0, \forall a \in \operatorname{dep}(\Phi)\}.$$

For $\mathbb{F}$ the real or complex field, we have the familar orthogonal decomposition

$$\mathbb{F}^J = \operatorname{dep}(\Phi)^\perp \oplus \operatorname{dep}(\Phi).$$

It turns out that this also holds for any subfield $\mathbb{F}$ of $\mathbb{C}$ which is closed under conjugation. As this is not obvious or well known, even for $\mathbb{F} = \mathbb{Q}$ (cf. [ER49]), we now give the details.

Lemma 3.1 (Orthogonal projections in $\mathbb{F}^J$). *Suppose that $\overline{\mathbb{F}} = \mathbb{F}$, and let $\mathcal{W}$ be a subspace of $\mathbb{F}^J$. Then there is the orthogonal direct sum decomposition*

$$\mathbb{F}^J = \mathcal{W}^\perp \oplus \mathcal{W}, \qquad \mathcal{W}^\perp := \{x \in \mathbb{F}^J : \langle x, a\rangle = 0, \forall a \in \mathcal{W}\},$$

where the matrices $P, Q \in \mathbb{F}^{J\times J}$ giving the projections onto the components $\mathcal{W}^\perp$ and $\mathcal{W}$ are complementary orthogonal projection matrices.

Proof: Clearly, we have a direct sum $\mathcal{W}\oplus\mathcal{W}^\perp$ (where $\mathcal{W}$ and $\mathcal{W}^\perp$ are orthogonal), and so it suffices to show this subspace is all of $\mathbb{F}^J$, and that one of P or $Q = I - P$ is an orthogonal projection matrix with entries in $\mathbb{F}$. We do this by constructing Q explicitly.

Let (v_k) be a basis for $\mathcal{W}$. Observe that for $v \in \mathbb{F}^J$ nonzero, the matrix giving the orthogonal projection of $\mathbb{C}^J$ onto $\mathrm{span}_{\mathbb{C}}\{v\}$ is $\frac{1}{\langle v,v\rangle} vv^* \in \mathbb{F}^{J\times J}$ (since the field $\mathbb{F}$ is closed under conjugation). Thus applying Gram–Schmidt (without normalisation) to (v_k) gives an orthogonal basis (w_k) for $\mathcal{W} \subset \mathbb{F}^J$. Let $Q := \sum_k \frac{1}{\langle w_k,w_k\rangle} w_k w_k^* \in \mathbb{F}^{J\times J}$, which is an orthogonal projection matrix, i.e., $Q^2 = Q$ and $Q^* = Q$, with

$$Qa = \sum_k \frac{\langle a, w_k\rangle}{\langle w_k, w_k\rangle} w_k \in \mathcal{W}, \quad \forall a \in \mathbb{F}^J, \qquad Qa = a, \quad \forall a \in \mathcal{W}.$$

Let $P := I - Q \in \mathbb{F}^{J\times J}$ be the complementary orthogonal projection matrix. Then

$$a = Pa + Qa, \qquad \forall a \in \mathbb{F}^J.$$

which gives $\mathbb{F}^J = W^\perp \oplus \mathcal{W}$, provided that $Pa \in \mathcal{W}^\perp$. Since (w_k) is a basis for $\mathcal{W}$, and

$$\langle Pa, w_k\rangle = \langle (I-Q)a, Qw_k\rangle = \langle (Q-Q^2)a, w_k\rangle = \langle 0, w_k\rangle = 0, \qquad \forall k,$$

it follows that $Pa \in \mathcal{W}^\perp$, $\forall a \in \mathbb{F}^J$, which completes the proof. □

For $\Phi = (f_j)_{j\in J}$ in X and $\mathcal{W} = \mathrm{dep}(\Phi)$, the **associated projection** $P_\Phi \in \mathbb{F}^{J\times J}$ is the matrix P of Lemma 3.1 (and the linear map $\mathbb{F}^J \to \mathbb{F}^J$ that it gives), i.e., the orthogonal projection onto $\mathrm{dep}(\Phi)^\perp$, which is characterised by

$$\mathrm{ran}(P_\Phi) = \mathrm{dep}(\Phi)^\perp, \qquad \ker(P_\Phi) = \mathrm{ran}(I - P_\Phi) = \mathrm{dep}(\Phi).$$

The kernel of the synthesis map $V = [f_j]_{j\in J}$ is $\mathrm{dep}(\Phi) = \mathrm{ran}(I - P_\Phi)$, and so it can be factored

$$V = VP_\Phi. \tag{3.2}$$

We say spanning sequences $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ for vector spaces X and Y are **similar** if there is an invertible linear map $Q : X \to Y$ with $g_j = Qf_j$, $\forall j$.

Lemma 3.3 (Linear dependencies). *Let $\Phi = (f_j)_{j\in J}$ and $\Psi = (g_j)_{j\in J}$ be spanning sequences for the $\mathbb{F}$–vector spaces X and Y. Then the following are equivalent*
(a) *Φ and Ψ are similar, i.e., there is a invertible linear map $Q : f_j \mapsto g_j$.*
(b) $\operatorname{dep}(\Phi) = \operatorname{dep}(\Psi)$ *(the dependencies are equal).*
(c) $P_\Phi = P_\Psi$ *(the associated projections are equal).*

Proof: (a)$\Longrightarrow$(b). Suppose there is an invertible $Q : X \to Y$ with $g_j = Qf_j$, $\forall j$. Then $[g_j] = [Qf_j] = Q[f_j]$, so that $\operatorname{dep}(\Psi) = \ker(Q[f_j]) = \ker([f_j]) = \operatorname{dep}(\Phi)$.

(b)$\Longrightarrow$(c). This follows since P_Φ depends only on $\mathcal{W} = \operatorname{dep}(\Phi)$.

(c)$\Longrightarrow$(a). Suppose that $P_\Phi = P_\Psi$. Let $V = [f_j]_{j\in J}$ and $W = [g_j]_{j\in J}$. Since $V = VP_\Phi$ and $\operatorname{ran}(P_\Phi) = \ker(V)^\perp$, the restriction map $V| = V|_{\operatorname{ran}(P_\Phi)} : \operatorname{ran}(P_\Phi) \to X$ is a bijection, as is $W| = W|_{\operatorname{ran}(P_\Psi)} = W|_{\operatorname{ran}(P_\Phi)}$. Thus $Q := W|(V|)^{-1} : X \to Y$ is a bijection. Since $f_j = Ve_j = V(P_\Phi e_j)$, we have

$$Qf_j = W(V|)^{-1}V(P_\Phi e_j) = WP_\Phi e_j = WP_\Psi e_j = We_j = g_j,$$

so that Φ and Ψ are similar. $\square$

More generally, since a linear map $L : X \to Y$ preserves linear combinations, we have

$$\operatorname{dep}(L\Phi) \supset \operatorname{dep}(\Phi) \quad \Longleftrightarrow \quad P_{L\Phi}P_\Phi = P_{L\Phi},$$

with equality if and only if L is 1–1, i.e., Φ and $L\Phi$ are similar.

Example 1. The sequence $(P_\Phi e_j)$ of the columns of the matrix P_Φ (which span $\operatorname{dep}(\Phi)^\perp$) is similar to Φ, since the synthesis map of these columns is P_Φ, which has kernel $\operatorname{dep}(\Phi)$. This gives a *canonical* representative of the similarity class of the spanning sequence Φ. In particular, there is a vector space isomorphism (cf. Theorem 4.9)

$$L = L_\Phi : X \mapsto \operatorname{ran}(P_\Phi) : f_j \mapsto P_\Phi e_j, \quad \forall j. \tag{3.4}$$

When Φ is a basis $\operatorname{ran}(P_\Phi) = \mathbb{F}^J$ and $Lf_j = e_j$, the standard basis for $\mathbb{F}^J$.

4. The canonical dual functionals

Recall, the (Moore–Penrose) *pseudoinverse* of a linear map $A : \mathcal{H} \to \mathcal{K}$ between finite dimensional Hilbert spaces, is the unique linear map $A^+ : \mathcal{K} \to \mathcal{H}$ satisfying

$$AA^+A = A, \qquad A^+AA^+ = A^+, \qquad (AA^+)^* = AA^+, \qquad (A^+A)^* = A^+A. \tag{4.1}$$

We can now define the *canonical dual functionals* to a spanning sequence.

Lemma 4.2 (Existence). *Let X be a vector space over a field $\mathbb{F}$, with $\overline{\mathbb{F}} = \mathbb{F}$. Suppose that $\Phi = (f_1, \ldots, f_n)$ are vectors which span X. Then there exist unique coefficients $c^\Phi(f) = (c_j(f))_{j=1}^n \in \mathbb{F}^n$ of minimal ℓ_2–norm for which*

$$f = \sum_{j=1}^n c_j(f) f_j. \tag{4.3}$$

These are given by

$$c^\Phi(f) = (\Lambda V)^+ \Lambda f, \qquad V = [f_1, \ldots, f_n], \tag{4.4}$$

where $\Lambda = (\lambda_k)_{k=1}^m : X \to \mathbb{F}^m$ is any 1–1 linear map, i.e., $\lambda_1, \ldots, \lambda_m$ span X'. Further, the Gramian

$$[c_j(f_k)]_{j,k=1}^n = c^\Phi V = (\Lambda V)^+ \Lambda V = P_\Phi, \tag{4.5}$$

where P_Φ is the orthogonal projection matrix associated with Φ.

Proof: Let $a = c(f) \in \mathbb{F}^n$ be a solution to (4.3), i.e., to $Va = f$. Since the linear functionals $\lambda_1, \ldots, \lambda_m$ span X', the equation $Va = f$ is equivalent to $\lambda_k(Va) = \lambda_k(f)$, $\forall k$, i.e.,

$$\Lambda V a = \Lambda f,$$

where $\Lambda V \in \mathbb{F}^{m \times n}$ and $\Lambda f \in \mathbb{F}^n$. This (possibly) underdetermined linear system has a unique minimal ℓ_2–norm (least squares) solution $a \in \mathbb{C}^n$ given by $a = c(f) = (\Lambda V)^+ \Lambda f$. Since this solution does not depend on the particular choice of Λ, we can take $\lambda_1, \ldots, \lambda_m$ to be a basis for X', so that ΛV is onto. In this case, the explicit formula for the pseudoinverse gives

$$(\Lambda V)^+ = (\Lambda V)^* (\Lambda V (\Lambda V)^*)^{-1} \in \mathbb{F}^{n \times d}, \tag{4.6}$$

and so we conclude that $a \in \mathbb{F}^n$. By (4.4), the $c_j : X \to \mathbb{F}$ are linear functionals.

By the properties (4.1) of the pseudoinverse, the Gramian matrix $G = (\Lambda V)^+ \Lambda V$ is Hermitian, and

$$G^2 = (\Lambda V)^+ (\Lambda V (\Lambda V)^+ \Lambda V) = (\Lambda V)^+ \Lambda V = G.$$

Thus G is an orthogonal projection matrix, and $\ker(G) = \ker(V) = \operatorname{dep}(\Phi) = \ker(P_\Phi)$ gives $G = P_\Phi$. □

The sequence of linear functionals $c^\Phi = c = (c_j)$ defined by (4.4) *do not* depend on the choice of Λ. Indeed, they can be calculated from P_Φ (which depends only on Φ) via

$$c^\Phi(f) = P_\Phi a, \qquad f = Va.$$

We will refer to c^Φ as the **canonical dual functionals** or (**linear**) **coordinates** for the spanning sequence $\Phi = (f_j)$. When Φ is a basis they are simply the dual functionals.

Remark: Minimal ℓ_2–norm solutions, such as $c^\Phi(f)$ above, have been used extensively in numerical linear algebra, where the method is referred to as *total least squares*. It is used in spline theory, e.g., selecting the (natural) cubic interpolating spline which minimises the L_2–norm of the second derivative.

Example 2. Suppose X is a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. If $\Phi = (f_j)$ is finite frame for X, i.e., a spanning sequence for X, then $V = [f_j]$ is onto, so that $\Lambda = V^* : f \mapsto (\langle f, f_j\rangle)$ is 1–1, and so by (4.4), the canonical dual functionals $c = c^\Phi$ are given by

$$c = (V^*V)^+V^* \quad \Longleftrightarrow \quad c_j(f) = \langle f, V(V^*V)^+e_j\rangle \quad (\text{since } (A^+)^* = (A^*)^+).$$

In frame theory the Riesz representers $V(V^*V)^+e_j$ are called the *dual frame* vectors. They are usually denoted $\tilde{f}_j$, and computed via

$$\tilde{f}_j = S^{-1}f_j, \qquad S := VV^*, \tag{4.7}$$

where S is called the *frame operator*. By (4.5) and (4.7), the associated projection is

$$P_\Phi = c^\Phi V = (V^*V)^+V^*V = [\langle f_k, \tilde{f}_j\rangle] = V^*S^{-1}V. \tag{4.8}$$

The projection matrix of (4.8) is the Gramian of the *canonical tight frame* $\Psi = (f_j^{\mathrm{can}})$, i.e.,

$$P_\Phi = [\langle f_k^{\mathrm{can}}, f_j^{\mathrm{can}}\rangle], \qquad f_j^{\mathrm{can}} := S^{-\frac{1}{2}}f_j, \quad S := VV^*.$$

This frame is similar to Φ (by its definition), and has the property that $c_j^\Psi(f) = \langle f, f_j^{\mathrm{can}}\rangle$. In the absense of some canonical identification between X and X' (such as that given by the Riesz representation), there is no analogue of the canonical dual frame for a spanning sequence Φ for a vector space X. That being said, the columns of P_Φ, which are the orthogonal projection of the orthonormal basis (e_j), form a normalised tight frame (for their span) which is similar to Φ via (3.4), and is *unique* (up to a unitary transformation). In this way, each Φ is uniquely associated with a normalised tight frame.

Further, (3.4) induces a unique inner product on X for which Φ is a normalised tight frame, namely

$$\langle f, g\rangle_X := \langle L_\Phi f, L_\Phi g\rangle, \qquad \forall f, g \in X.$$

This formulation of our canonical coordinates is a new result in frame theory:

Theorem 4.9. *Let $\Phi = (f_j)$ be a finite spanning sequence for a vector space X. Then there exists a unique inner product on X for which Φ is a normalised tight frame, namely*

$$\langle f_j, f_k\rangle_X := \langle P_\Phi e_j, P_\Phi e_k\rangle = (P_\Phi)_{kj}. \tag{4.10}$$

Proof: If Φ is a normalised tight frame for $\langle\cdot,\cdot\rangle_X$, then its Gramian is P_Φ, i.e., (4.10) holds. We have already observed that this defines an inner product on X (for which Φ is a normalised tight frame). $\square$

The canonical dual functionals can be characterised in a number of ways:

Theorem 4.11 (Characterisation). *Let X be a vector space over a field $\mathbb{F}$, with $\overline{\mathbb{F}} = \mathbb{F}$. Suppose that $\Phi = (f_j)$ in X and $\Psi = (\lambda_j)$ in X' are dual, with Gramian matrix*

$$G = \Lambda V = [\lambda_j(f_k)], \qquad V := [f_j] : \mathbb{F}^J \to X, \quad \Lambda = (\lambda_j) : X \to \mathbb{F}^J.$$

Then the following are equivalent
(a) *(λ_j) are the canonical dual functionals for (f_j).*
(b) *$(\hat{f}_j)$ are the canonical dual functionals for (λ_j).*
(c) *For all $f \in X$, the $a = (a_j)$ with $f = \sum_j a_j f_j$ of minimal ℓ_2–norm is $a = (\lambda_j(f))$.*
(d) *For all $\lambda \in X'$, the $b = (b_j)$ with $\lambda = \sum_j b_j \lambda_j$ of minimal ℓ_2–norm is $b = (\lambda(f_j))$.*
(e) *$G = G^*$, i.e., G is an orthogonal projection.*
(f) *$G = P_\Phi$.*
(g) *$G^T = P_\Psi$.*
(h) *$P_\Psi = P_\Phi^T$.*
(i) *$\operatorname{dep}(\Psi) = \overline{\operatorname{dep}(\Phi)}$.*

Proof: By Proposition 2.2 and (3.2), Φ and Ψ being dual is equivalent to

$$V P_\Phi \Lambda = V\Lambda = I_X.$$

Thus Λ is a right inverse of VP_Φ. Since $VP_\Phi c^\Phi = I_X$ and VP_Φ is 1–1 on $\operatorname{ran}(P_\Phi)$, Λ must have the form

$$\Lambda = c^\Phi + (I - P_\Phi)R, \qquad R : X \to \mathbb{F}^J, \tag{4.12}$$

and by (4.5)

$$G = \Lambda V = c^\Phi V + (I - P_\Phi)RV = P_\Phi + (I - P_\Phi)RVP_\Phi. \tag{4.13}$$

(a)$\Longleftrightarrow$(c), (b)$\Longleftrightarrow$(d). By definition.

(a)$\Longrightarrow$(b). Suppose $\Psi = (\lambda_j) = (c_j^\Phi)$. Since $\Psi = (c_j^\Phi)$ spans X' and $(\hat{f}_j)$ spans X'', we can use Lemma 4.2 to calculate the canonical dual functionals for $\Psi = (\lambda_j)$ as

$$c^\Psi = (LW)^+ L, \qquad W := [c_j^\Phi] : \mathbb{F}^J \to X', \quad L = (\hat{f}_j) : X' \to \mathbb{F}^J.$$

Since $LW = [\hat{f}_j(c_k^\Phi)] = [c_k^\Phi(f_j)] = P_\Phi^T$, a projection matrix, we have $(LW)^+ = LW$, and so

$$c_j^\Psi(\lambda) = (P_\Phi^T L\lambda)_j = \sum_k (P_\Phi^T)_{jk}(L\lambda)_k = \sum_k c_k^\Phi(f_j)\hat{f}_k(\lambda) = \sum_k c_k^\Phi(f_j)\lambda(f_k)$$
$$= \lambda\Big(\sum_k c_k^\Phi(f_j) f_k\Big) = \lambda(f_j) = \hat{f}_j(\lambda), \quad \forall \lambda \in X' \quad \Longrightarrow \quad c_j^\Psi = \hat{f}_j.$$

(b)$\Longrightarrow$(g). Since $(\hat{f}_j)$ are the canonical dual functionals for Ψ, (4.5) gives

$$P_\Psi = [\hat{f}_j(\lambda_k)] = [\lambda_k(f_j)] = G^T.$$

(g)$\Longrightarrow$(e). Since $P_\Psi = P_\Psi^*$, we have $G = P_\Psi^T = (P_\Psi^*)^T = (P_\Psi^T)^* = G^*$.

(e)$\Longrightarrow$(f). Since $P_\Phi = P_\Phi^*$ and $V = VP_\Phi$, by (4.13), we have $G = G^*$ if and only if

$$(I - P_\Phi)RV = ((I - P_\Phi)RVP_\Phi)^* = P_\Phi(RV)^*(I - P_\Phi). \tag{4.14}$$

The operator above maps into $\operatorname{ran}(I - P_\Phi) \cap \operatorname{ran}(P_\Phi) = 0$, and so is zero, giving $G = P_\Phi$.

(f)$\Longrightarrow$(h). Suppose $G = P_\Phi$. Then (4.5) gives $\lambda_j(f_k) = c_j^\Phi(f_k)$, $\forall j,k$. Since (f_k) spans X, this gives $\lambda_j = c_j^\Phi$, $\forall j$. By (a)$\Longrightarrow$(b), we have $c_j^\Psi = \hat{f}_j$, and hence

$$P_\Psi = [c_j^\Psi(\lambda_k)] = [\hat{f}_j(c_k^\Phi)] = [c_k^\Phi(f_j)] = P_\Phi^T.$$

(h)$\Longrightarrow$(i). Suppose $P_\Psi = P_\Phi^T$. Then $\operatorname{dep}(\Psi) = \ker(P_\Psi) = \ker(P_\Phi^T)$ gives

$$a \in \operatorname{dep}(\Psi) \quad \Longleftrightarrow \quad P_\Phi^T a = 0 \quad \Longleftrightarrow \quad \overline{P_\Phi^T a} = P_\Phi^* \overline{a} = P_\Phi \overline{a} = 0 \quad \Longleftrightarrow \quad \overline{a} \in \operatorname{dep}(\Phi).$$

(i)$\Longrightarrow$(a). In view of (4.12) it suffices to show $(I - P_\Phi)R = 0$, where P_Φ is thought of as a map $\mathbb{F}^J \to \mathbb{F}^J$. Let $a \in \operatorname{ran}(I - P_\Phi) = \ker(P_\Phi) = \operatorname{dep}(\Phi)$. Then

$$a \in \operatorname{dep}(\Phi) = \ker(P_\Phi) \quad \Longrightarrow \quad a^* c^\Phi V = a^* P_\Phi = (P_\Phi a)^* = 0 \quad \Longrightarrow \quad a^* c^\Phi = 0,$$

$$\overline{a} \in \overline{\operatorname{dep}(\Phi)} = \operatorname{dep}(\Psi) \quad \Longrightarrow \quad \sum_j \overline{a_j} \lambda_j = \overline{a}^T \Lambda = 0 \quad \Longrightarrow \quad a^* \Lambda = 0.$$

Thus (4.12) gives

$$\operatorname{ran}((I - P_\Phi)R) \perp a, \qquad \forall a \in \operatorname{ran}(I - P_\Phi),$$

and so $(I - P_\Phi)R = 0$, since $\mathbb{F}^J = \operatorname{ran}(I - P_\Phi) \oplus \operatorname{ran}(P_\Phi)$ (orthogonal direct sum). $\square$

Example 3. Suppose that $\Phi = (f_j)_{j=1}^n$ has just one dependency: $f_1 + f_2 + \cdots + f_n = 0$. Then $a = (1, 1, \ldots, 1)$ spans $\operatorname{dep}(\Phi)$, so that $P_\Phi = I - Q_\Phi = I - \frac{1}{n} a^* a$, $d = \dim(X) = n - 1$, i.e.,

$$c_j(f_k) = \begin{cases} 1 - \frac{1}{d+1}, & j = k; \\ \frac{-1}{d+1}, & j \neq k. \end{cases}$$

Example 4. By Theorem 4.9, each spanning sequence Φ is associated with the unique normalised tight frame with Gramian P_Φ. Via this identification we can extend elements of frame theory to spanning sequences. For example, we say Φ is **equiangular** if the tight frame given by the columns of $P_\Phi = [p_{jk}]$ is, i.e., P_Φ has constant diagonal and $|p_{jk}| = C$, $j \neq k$. Equiangular sequences of vectors are useful in many applications (cf. [SH03], [HP04], [R05]). An analytic construction of d^2 equiangular vectors (or lines) in $\mathbb{C}^d$ (for a general d) is an important problem in a number of areas (cf. [RBSC04], [A05]). The current approach to this (successful for small values of d) is to obtain Φ as the orbit of a single vector $v \in \mathbb{C}^d$ under the action of a discrete Heisenberg group. Given the above observations, a general construction might come as a spanning sequence of d^2 vectors for a d–dimensional space (which is not presented as $\mathbb{C}^d$), as was done in Example 3.

Example 5. Let ω be the n–th root of unity $\omega = e^{2\pi i/n}$. The **cyclotomic field** $\mathbb{Q}[\omega]$ is a $\mathbb{Q}$–vector space of dimension $d = \varphi(n)$, where φ is the *Euler Phi* function. A natural spanning sequence for this space is given by the n–th roots themselves $\Phi = (1, \omega, \omega^2, \cdots, \omega^{n-1})$. In the coordinates for Φ, multiplication by ω is given by the *cyclic forward shift* operator S, i.e., $c^\Phi(\omega f) = Sc^\Phi(f)$, and so P_Φ is a *circulant matrix.* Circulant matrices are (unitarily) diagonalised by the Fourier transform matrix (characters χ_j of $\mathbb{Z}_n$), and so (with a little calculation [CW10]) one obtains

$$P_\Phi = \sum_{j \in \mathbb{Z}_n^*} \frac{1}{n} \chi_j \chi_j^*, \qquad \chi_j = (1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j})^T, \tag{4.15}$$

where $\mathbb{Z}_n^*$ is the (multiplicative) group of units in $\mathbb{Z}_n$, and (by Galois theory) $nP_\Phi \in \mathbb{Z}^{n \times n}$. Thus the canonical coordinates are given by

$$c_j(\omega^k) = \nu_n(j-k) = \frac{1}{n} \sum_{\ell \in \mathbb{Z}_n^*} \omega^{\ell(j-k)}.$$

The formula (4.15) makes a direct connection between $\dim_{\mathbb{Q}}(\mathbb{Q}[\omega]) = \varphi(n)$ and $\mathbb{Z}_n^*$, or, equivalently, the primitive n–th roots $\{\omega^j : j \in \mathbb{Z}_n^*\}$. The primitive n–th roots are a natural basis for $\mathbb{Q}[\omega]$ when n is prime (and square free [T10]), but not in general, e.g., for $n = 4$ the primitive roots $\{-i, i\}$ are linearly dependent over $\mathbb{Q}$. When the primitive roots are not a basis, bases with additional properties can be constructed in a noncanonical way. Most prominently used are the *integral bases* (each element of the ring of integers has its coefficients in $\mathbb{Z}$), and *power bases* (these have the form $\{1, z, z^2, \cdots, , z^{d-1}\}$).

The projection $Q_\Phi = I - P_\Phi$ onto $\mathrm{dep}(\Phi)$ is circulant, and so there is a spanning set for $\mathrm{dep}(\Phi)$ which has *cyclic symmetry*, i.e., $\{S^j a_\Phi : 0 \leq j < n\}$ where S is the cyclic forward shift operator and

$$a_\Phi = n(e_1 - P_\Phi e_1) = ne_1 - \sum_{\ell \in \mathbb{Z}_n^*} \chi_\ell \in \mathbb{Z}^n.$$

There is a large body of research on the dependencies (over $\mathbb{Z}$) of the n–th roots of unity, largely concerned with finding *vanishing sums* with minimal numbers of terms (cf. [M65], [CJ76], [LL00], [S08]). To the best of our knowledge the spanning sequence $\{S^j a_\Phi\}$ is new.

As an example, when $n = 6$, $\mathbb{Z}_6^* = \{1, 5\}$ and the coordinates for $\Phi = \{1, \omega, \ldots, \omega^5\}$, $\omega := e^{\frac{2\pi i}{6}}$ are given by

$$P_\Phi = \frac{1}{6}(\chi_1 \chi_1^* + \chi_5 \chi_5^*) = \frac{1}{6} \begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \end{pmatrix}.$$

Thus $a_\Phi = (4,-1,1,2,1,-1)^T$, i.e., the dependencies between the roots can be expressed as

$$4\omega^j - \omega^{j+1} + \omega^{j+2} + 2\omega^{j+3} + \omega^{j+4} - \omega^{j+5} = 0, \qquad 0 \le j < 6.$$

Example 6. We can define *matrices* with respect to spanning sequences in the usual way. The (**canonical**) **matrix** representing a linear map $L: X \to Y$ with respect to spanning sequences $\Phi = (f_j)_{j=1}^n$ and $\Psi = (g_k)_{k=1}^m$ for X and Y is the $A = A_L \in \mathbb{F}^{m\times n}$ given by

$$j\text{–th column of } A = Ae_j = c^\Psi(Lf_j).$$

i.e., $c^\Psi(Lf) = A(c^\Phi(f))$, $\forall f \in X$. The map $L \mapsto A_L$ is linear, and L can be recovered from $A = A_L$ via

$$L = WAc^\Phi, \qquad W = [g_k].$$

Computations with such matrices are *stable*, since A_L depends *continuously* on Φ and Ψ.

5. Properties of the linear coordinates

We now show that the *linear* coordinates c^Φ for a sequence of vectors Φ transform naturally under linear maps and symmetries. This, together with the fact that they depend continuously on Φ makes them ideal for applications such as multivariate spline spaces. First we consider the *coordinate like* properties of c^Φ which generalise the biorthogonality of a basis and its dual basis, and follow from the fact that P_Φ is a projection matrix.

We write the sequence obtained by removing the vector f_j from $\Phi = (f_1, \ldots, f_n)$ as

$$\Phi \setminus f_j := (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n).$$

Proposition 5.1 (Properties of c^Φ). *Let $c^\Phi = (c_j)$ be the coordinates (canonical dual functionals) for a finite spanning sequence $\Phi = (f_j)$ for a vector space X. Then*
(a) $0 \le |c_j(f_k)| \le 1$, $c_j(f_j) \ge 0$, *and* $c_j(f_k) = \overline{c_k(f_j)}$.
(b) $|c_j(f_k)| = 1$ *if and only if* $k = j$ *and* $f_j \notin \operatorname{span}(\Phi \setminus f_j)$, *in which case* $c_j(f_j) = 1$ *and* $c_j = 0$ *on* $\operatorname{span}(\Phi \setminus f_j)$.
(c) $c_j(f_j) = 0$ *if and only if* $f_j = 0$.
(d) $c_j = \alpha c_k$, $\alpha \in \mathbb{F}$ *if and only if* $f_j = \overline{\alpha} f_k$.
(e) $\sum_j c_j(f_j) = d = \dim(X)$.

Proof: Recall that $P = [p_{jk}] = P_\Phi = [c_j(f_k)]$ is an orthogonal projection matrix.
(a) Since $P = P^*$, we have $c_j(f_k) = p_{jk} = \overline{p_{kj}} = \overline{c_k(f_j)}$, and $\|Px\| \le \|x\|$, $\forall x$, gives

$$0 \le |c_j(f_k)| = |p_{jk}| \le \|Pe_k\| = \Big(\sum_j |p_{jk}|^2\Big)^{1/2} \le \|e_k\| = 1.$$

Since P is positive semidefinite, we have $c_j(f_j) = e_j^* P e_j \ge 0$.

(b) We have $|c_j(f_k)|=1$ in the above if and only if $Pe_k=e_k$ and $|p_{jk}|=1$, i.e., $j=k$, $c_j(f_j)=1$ and $c_j(f_\ell)=0$, $\ell\neq j$. The condition $Pe_j=e_j$ is equivalent to the j–th column of P not being in the span of the others, and since the columns of P and the vectors of Φ have the same linear dependencies, this is the same as $f_j\not\in \operatorname{span}(\Phi\setminus f_j)$.

(c) Suppose that $c_j(f_j)=0$. Then for any $t\in\mathbb{R}$, we may write

$$f_j=\sum_{k\neq j} c_k(f_j)f_k = tf_j+(1-t)\sum_{k\neq j} c_k(f_j)f_k.$$

The coefficients in the first linear combination above have minimal ℓ_2–norm, so that

$$t^2+\{(1-t)^2-1\}\sum_{k\neq j}|c_k(f_j)|^2 = t^2\Big\{1+\sum_{k\neq j}|c_k(f_j)|^2\Big\}-2t\sum_{k\neq j}|c_k(f_j)|^2\geq 0,$$

and so (by taking $t\to 0$) we conclude that $\sum_{k\neq j}|c_k(f_j)|^2=0$, i.e., $c_k(f_j)=0$, $\forall k$. Thus $f_j=\sum_k c_k(f_j)f_k=0$. The converse is immediate.

(d) Suppose that $c_j=\alpha c_k$. Then $c_j(f_\ell)=\alpha c_k(f_\ell)$, $\forall\ell$, i.e., $e_j^*P=\alpha e_k^*P$, which gives

$$Pe_j=\overline{\alpha}Pe_k \quad\Longrightarrow\quad f_j=\sum_\ell c_\ell(f_j)f_\ell=\sum_\ell \overline{\alpha}c_\ell(f_k)f_\ell=\overline{\alpha}f_k.$$

Conversely, if $f_j=\overline{\alpha}f_k$, i.e., $c_\ell(f_j)=c_\ell(\overline{\alpha}f_k)=\overline{\alpha}c_\ell(f_k)$, $\forall\ell$, then the second part of (a) gives

$$c_j(f_\ell)=\overline{c_\ell(f_j)}=\alpha\overline{c_\ell(f_k)}=\alpha c_k(f_\ell),\quad \forall\ell \quad\Longrightarrow\quad c_j=\alpha c_k.$$

(e) For *any* dual sequences, we may use Proposition 2.2 to calculate

$$\sum_j \lambda_j(f_j)=\operatorname{trace}(G)=\operatorname{trace}(\Lambda V)=\operatorname{trace}(V\Lambda)=\operatorname{trace}(I_X)=\dim(X)=d,$$

or take the trace of P (an orthogonal projection matrix of rank d). □

There are similar properties for the sum of the squares of the coordinates:

Proposition 5.2. *Let $c^\Phi=(c_j)$ be the coordinates (canonical dual functionals) for a finite spanning sequence $\Phi=(f_j)$ for a vector space X, and*

$$C_\Phi(f):=\sum_j|c_j(f)|^2.$$

Then
(a) $C_\Phi(f)\geq 0$ *with equality if and only if* $f=0$.
(b) $C_\Phi(f_j)\leq 1$ *with equality if and only if* $f_j\not\in\operatorname{span}(\Phi\setminus f_j)$.
(c) $\sum_j C_\Phi(f_j)=\dim(X)=d$.

Proof: Since $C_\Phi(f_k)=\|P_\Phi e_k\|^2$, a careful reading of the proof of Proposition 5.1 gives (a) and (b). Let $\|A\|_F:=\sqrt{\operatorname{trace}(A^*A)}$ be the Frobenius norm. Since P_Φ is an orthogonal projection of rank d, we obtain

$$\sum_j C_\Phi(f_j)=\sum_j\|P_\Phi e_k\|^2=\|P_\Phi\|_F^2=\operatorname{trace}(P_\Phi^*P_\Phi)=\operatorname{trace}(P_\Phi)=d.$$

□

The coordinates transform naturally under the action of a linear transformation:

Proposition 5.3 (Linear maps). *Let $\Phi = (f_1, \ldots, f_n)$ be vectors which span the vector space X, $L : X \to Y$ be an invertible linear map, and $\Psi := L\Phi = (Lf_1, \ldots, Lf_n)$. Then the canonical dual functionals for Φ and Ψ satisfy*

$$c_j^{L\Phi}(Lf) = c_j^{\Phi}(f), \qquad \forall f \in X. \tag{5.4}$$

Proof: Choose Λ as in (4.4), so that

$$c^{\Phi}(f) = (\Lambda V)^{+}\Lambda f, \qquad V = [f_j].$$

Then $\Lambda L^{-1} : Y \to \mathbb{F}^m$ is 1–1, and so, with $W = [Lf_j] = LV$, we have

$$c^{\Psi}(Lf) = (\Lambda L^{-1} W)^{+}\Lambda L^{-1}(Lf) = (\Lambda V)^{+}\Lambda f = c^{\Phi}(f).$$

$\square$

Let $\mathcal{GL}(X)$ be the general linear group of all invertible linear transformations $X \to X$. In [VW10] the **symmetry group** of a finite frame $\Phi = (f_j)_{j\in J}$ for a Hilbert space was defined as the group of permutations on the index set J given by

$$\mathrm{Sym}(\Phi) := \{\sigma \in S_J : \exists L_\sigma \in \mathcal{GL}(X) \text{ with } L_\sigma f_j = f_{\sigma j}, \forall j \in J\}. \tag{5.5}$$

This does not depend on the inner product, and so extends to finite spanning sequences for vector spaces. The symmetry group of similar spanning sequences is the same, and can be computed from P_Φ.

A spanning sequence and its canonical dual functionals have the same symmetries:

Proposition 5.6 (Symmetries). *Let $\Phi = (f_j)$ be a finite spanning sequence for a vector space X. Then Φ and its canonical dual functionals have the same symmetry group, i.e.,*

$$\mathrm{Sym}(\Phi) = \mathrm{Sym}(c^{\Phi}).$$

Proof: In [VW10] is was shown that similar spanning sequences have the same symmetry group, and (by considering the associated tight frame) that a permutation σ is a symmetry of Φ if and only if $P_\sigma^* P_\Phi P_\sigma = P_\Phi$, where P_σ is the permutation matrix given by $P_\sigma e_j = e_{\sigma j}$. Let $\Psi = c^\Phi$, then Theorem 4.11 gives $P_\Phi = P_\Psi^T$, and so we obtain

$$\begin{aligned} \sigma \in \mathrm{Sym}(\Phi) \quad &\iff \quad P_\sigma^* P_\Phi P_\sigma = P_\Phi \quad \iff \quad P_\sigma^* P_\Psi^T P_\sigma = P_\Psi^T \\ &\iff \quad P_\sigma^* P_\Psi P_\sigma = P_\Psi \quad \iff \quad \sigma \in \mathrm{Sym}(\Psi), \end{aligned}$$

since $P_\sigma^T = P_\sigma^*$. $\square$

Example 7. Let $X = \Pi_1'$ be the dual of the (three dimensional) space of linear polynomials on $\mathbb{R}^2$. Consider the spanning sequence for X given by the point evaluations

$$\Phi = \big(\delta_{(0,0)}, \delta_{(1,0)}, \delta_{(0,1)}, \delta_{(a,b)}\big), \qquad \delta_x : f \mapsto f(x).$$

The canonical dual coordinates c^Φ are in $X' = \Pi_1''$. By (2.1) they can be identified with linear polynomials $f_{(0,0)}, \ldots, f_{(a,b)} \in \Pi_1$. By direct computation, using (4.4) and (4.6) with (λ_j) the image of the basis $1, x, y$ for Π_1 under the bidual map $\Pi_1 \to \Pi_1''$, we obtain

$$\begin{aligned} f_{(0,0)}(x,y) &= \frac{(ab-1-a-b^2)x + (ab-1-b-a^2)y + 1 + a^2 + b^2}{2(1+ab+a^2-a+b^2-b)}, \\ f_{(1,0)}(x,y) &= \frac{(2+ab-a+2b^2-2b)x + (a-a^2-2ab)y + ab + a^2 - a}{2(1+ab+a^2-a+b^2-b)}, \\ f_{(0,1)}(x,y) &= \frac{(b-b^2-2ab)x + (2+ab-b+2a^2-2a)y + ab + b^2 - b}{2(1+ab+a^2-a+b^2-b)}, \\ f_{(a,b)}(x,y) &= \frac{(2a+b-1)x + (a+2b-1)y + 1 - a - b}{2(1+ab+a^2-a+b^2-b)}. \end{aligned} \tag{5.7}$$

Note that these polynomials are continuous functions of (a, b).

Example 8. For a given triangulation Δ, the *multivariate spline space* $\mathcal{S}_k^r(\Delta)$ consists of all C^r piecewise polynomial functions of degree $\le k$ on Δ (cf. [LS07]). To compute with these spaces (which give good global approximations based on the local approximation power of polynomials) the prevailing approach is to find a suitable basis (ideally consisting of functions with small support). A major obstacle in the construction of such bases is determining the dimension of $\mathcal{S}_k^r(\Delta)$, which is sensitive to perturbation (cf Ch. 9 of [LS07]). Usually this is done by giving an explicit *determining set* (cf. [AS87]). For example, the triangulations Δ_s and Δ_n (see Fig. 1) have $\dim(\mathcal{S}_2^1(\Delta_s)) = 8$, $\dim\big(\mathcal{S}_2^1(\Delta_n)\big) = 7$, i.e., a perturbation of the interior *singular* vertex of Δ_s causes the dimension to drop by 1. To get around this, singular vertices are either avoided (or carefully accounted for), or a uniform triangulation is used (cf. *box splines* [BHR93]).

Using our results, one could construct a spanning sequence of (minimally supported) splines, and use the canonical dual functionals. Alternatively, one could go down the route of Example 7, and specify the canonical dual functionals first, and then compute the splines that they are the coordinates for. We undertook this calculation in `MAPLE` for the example Δ_n of Fig. 1, for the points $(0,0), (1,0), (1,1), (0,1)$ and a singular vertex $(a,b) \ne (\frac12, \frac12)$.

First we took the 8 linear functionals given by point evaluation at the vertices on the boundary and the midpoints of the boundary edges. The canonical dual spline functions depend continuously on $(a,b) \ne (\frac12, \frac12)$. Their Bernstein–Bézier coefficients are rational functions of (a,b) with a common denominator $(a-\frac12)^2 + (b-\frac12)^2$. As expected, these dual splines do not have a limit as $(a,b) \to (\frac12, \frac12)$, though they do have limits if the path to the limit is restricted to a fixed direction.

Next we took the 13 linear functionals given by the Bernstein-Bézier coefficients of a spline in $\mathcal{S}_2^1(\Delta_n)$. Again the dual splines had rational BB–coefficients (this time with a common denominator being a polynomial in (a,b) of degree 10). Other variations showed similar behaviour. Despite this being quite a restricted example, already features expected for larger triangulations became apparent, e.g., the dual spline to a point evaluation being localised around the point.

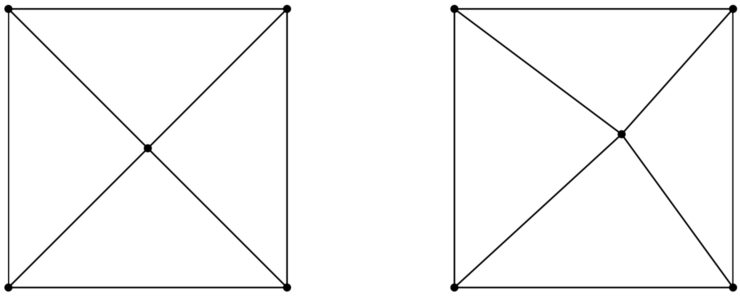

Fig. 1. The triangulations Δ_s (singular vertex) and Δ_n (nonsingular vertex).

6. Generalised barycentric coordinates

We now give the analogue of our results for affine spaces. An *affine space* X is, in effect, a vector space for which there is no distinguished point that plays the role of the origin in a vector space (or, equivalently, the translation of a vector subspace). As such, we can take *affine combinations* of "points" in X, i.e., linear combinations where the sum of the coefficients is 1, and *differences* of points to obtain "vectors" (see [R08] for details).

Let X be an affine space with *affine dimension* $d+1$ (the number of points in affinely independent affine spanning set for X), for short $\operatorname{affdim}(X) = d+1$. A sequence $p_1, \ldots, p_n$ of $n = d+1$ points in X is affinely independent if and only if each point $x \in X$ can be written uniquely as an affine combination of them, i.e.,

$$x = \sum_j \xi_j(x)\, p_j, \qquad \sum_j \xi_j(x) = 1. \tag{6.1}$$

The functions ξ_j, so defined, are called *barycentric coordinates.* They are affine functions that are nonnegative on the simplex given by the convex hull of the points. They satisfy natural symmetry and affine transformation properties. Because of these properties, they are used extensively in CAGD (computer aided geometric design), see, e.g., [B87].

If a sequence of points $P = (p_1, \ldots, p_n)$ has affine span X, but they are not affinely independent, then we can define *generalised* barycentric coordinates (ξ_j) as follows:

Lemma 6.2 (Existence). *Let X be an affine space over a field $\mathbb{F}$, with $\overline{\mathbb{F}} = \mathbb{F}$. Suppose that $P = (p_1, \ldots, p_n)$ are points with affine span X. Then there exist unique coefficients $\xi^P(x) = (\xi_j(x))_{j=1}^n \in \mathbb{F}^n$ of minimal ℓ_2–norm for which*

$$x = \sum_{j=1}^n \xi_j(x) p_j, \qquad \sum_{j=1}^n \xi_j(x) = 1. \tag{6.3}$$

These are given by

$$\xi_j(x) = c_j(x - b_P) + \frac{1}{n}, \qquad b_P := \frac{1}{n}\sum_{j=1}^{n} p_j, \tag{6.4}$$

where $c^\Phi = (c_j)$ are the canonical dual functionals for the vectors $\Phi = (f_j)$, $f_j := p_j - b_P$.

Proof: We seek to minimise $\sum_j |\xi_j(x)|^2$ subject to

$$x = \sum_j \xi_j(x) p_j, \qquad \sum_j \xi_j(x) = 1. \tag{6.5}$$

Write $\xi_j(x) = a_j(x) + \frac{1}{n}$. Then $\sum_j \xi_j(x) = 1$ is equivalent to $\sum_j a_j(x) = 0$, and so

$$\sum_j |\xi_j(x)|^2 = \sum_j \Big\{ |a_j(x)|^2 + \frac{1}{n} a_j(x) + \frac{1}{n}\overline{a_j(x)} + \frac{1}{n^2} \Big\} = \sum_j |a_j(x)|^2 + \frac{1}{n}.$$

Since $\sum_j f_j = \sum_j (p_j - b_P) = \sum_j p_j - n b_P = 0$, expanding gives

$$x = \sum_j \xi_j(x) p_j = \sum_j \xi_j(x) f_j + \sum_j \xi_j(x) b_p = \sum_j \{a_j(x) + \frac{1}{n}\} f_j + b_p = \sum_j a_j(x) f_j + b_p.$$

Thus we must minimise $\sum_j |a_j(x)|^2$, subject to the constraints

$$x - b_P = \sum_j a_j(x) f_j, \qquad \sum_j a_j(x) = 0.$$

By Lemma 4.2, the minimiser subject to just the first constraint is $a_j(x) = c_j(x - b_P)$. But $\sum_j f_j = 0$ implies the dependency $\sum_j c_j = 0$ (by Theorem 4.11), and so the second constraint is also satified by this choice for $a_j(x)$. □

The ξ_j defined above are *affine functions*, which we call the **canonical barycentric coordinates** corresponding to the points P. These were introduced in [W09] via Riesz representors, in a setting where the "vectors" in X were endowed with an inner product. There a number of examples and their geometry was considered. In particular, the **region of nonnegativity** for the canonical barycentric coordinates $\xi^P = (\xi_j)$

$$N = N_P := \{x \in X : \xi_j(x) \geq 0, \ \forall j\},$$

was considered. This is a convex polytope with the barycentre b_P as an interior point.

For points $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$, a sequence of functions $\lambda_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, n$ (defined on $\Omega \subset \mathbb{R}^d$ containing P) are called **generalised barycentric coordinates** if (6.1) holds for all $x \in \Omega$ (cf. [FHK06], [LS08]). Thus our canonical barycentric coordinates are generalised barycentric coordinates for the affine space $\mathbb{R}^d$ (typically these are piecewise polynomials or rational functions). From the formula in (6.4), we observe that

- The coordinates of the barycentre b_P are $\xi_j(b_P) = \frac{1}{n}$, $\forall j$.
- The functions ξ_j are constant (equal to $\frac{1}{n}$) if and only if p_j is the barycentre b_P.
- $\xi_j = \xi_k$ if and only if $p_j = p_k$.
- The ξ_j are continuous functions of the points $p_1, \ldots, p_n$ (with affine hull X).

Example 9. Let V be a set of $d+1$ affinely independent points with affine span X, and $\ell = (\ell_v)_{v \in V}$ be the corresponding barycentric coordinates. If $P = (p_j)$ is a sequence of points in V, with each v appearing with multiplicity $m_v \geq 1$, then the canonical barycentric coordinates (ξ_j) for P with $p_j = v$ are equal, and they add to ℓ_v, giving

$$\xi_j = \frac{1}{m_v} \ell_v \quad \text{when} \quad p_j = v. \tag{6.6}$$

We now give the analogues of Propositions 5.1 and 5.2. Denote the affine span (hull) of the points in $P = (p_j)$ by $\mathrm{Aff}(P)$.

Proposition 6.7. *Let X be an affine space, $\xi^P = (\xi_j)$ be the canonical barycentric coordinates for a sequence of points $P = (p_1, \ldots, p_n)$ whose affine span is X. Then*
(a) $0 \leq |\xi_j(p_k)| \leq 1$, $\xi_j(p_j) \geq \frac{1}{n}$ *and* $\xi_j(p_k) = \overline{\xi_k(p_j)}$.
(b) $|\xi_j(p_k)| = 1$ *if and only if* $k = j$ *and* $p_j \notin \mathrm{Aff}(P \setminus p_j)$, *in which case* $\xi_j(p_j) = 1$ *and* $\xi_j = 0$ *on* $\mathrm{Aff}(P \setminus p_j)$.
(c) $\xi_j(p_j) = \frac{1}{n}$ *if and only if* $p_j = b_P$.
(d) $\xi_j = \xi_k$ *if and only if* $p_j = p_k$.
(e) $\sum_j \xi_j(p_j) = d + 1 = \mathrm{affdim}(X)$.

Proof: Most follows directly from Proposition 5.1 and the fact $\xi_j(p_k) = c_j(f_k) + \frac{1}{n}$. Hence we consider only those parts of (a) and (b) requiring further proof.

Since p_k can be written as the affine combination $1p_k + \sum_{j \neq k} 0p_j$, and the canonical barycentric coordinates have minimal ℓ_2–norm amonst all affine combinations giving p_k, we have

$$\sum_j |\xi_j(p_k)|^2 \leq 1^2 + \sum_{j \neq k} 0^2 = 1 \quad \Longrightarrow \quad |\xi_j(p_k)| \leq 1. \tag{6.8}$$

Since $\xi_k(p_k) \geq \frac{1}{n}$, we can have $|\xi_j(p_k)| = 1$ if and only $k = j$ and there is equality in (6.8), i.e., $\xi_\ell(p_j) = 0$, $\ell \neq j$. Thus $\xi_j(p_\ell) = \overline{\xi_\ell(p_j)} = 0$, $\ell \neq j$, and so $\xi_j = 0$ on $\mathrm{Aff}(P \setminus p_j)$.

Conversely, suppose that $p_j \notin \mathrm{Aff}(P \setminus p_j)$. Then the only way p_j can be expressed as an affine combination of the points in P is as $1p_j + \sum_{k \neq j} 0p_k$, so that $\lambda_j(p_j) = 1$. $\square$

Proposition 6.9. *Let X be an affine space, $\xi^P = (\xi_j)_{j=1}^n$ be the canonical barycentric for a finite sequence of points $P = (p_j)$ with affine span X, and*

$$S_P(x) := \sum_j |\xi_j(x)|^2.$$

Then

(a) $S_P(x) \ge \frac{1}{n}$ *with equality if and only if* $x = b_P$ *(the barycentre).*
(b) $S_P(p_j) \le 1$ *with equality if and only if* $p_j \notin \mathrm{Aff}(P \setminus p_j)$.
(c) $\sum_j S_P(p_j) = \mathrm{affdim}(X) = d+1$.

Proof: Let (c_j) be the canonical dual functionals for $\Phi = (f_j)$, $f_j := p_j - b_P$, and recall (from the proof of Lemma 6.2) that $\sum_j c_j = 0$. Thus expanding gives

$$S_P(x) = \sum_j |c_j(x-b_P) + \frac{1}{n}|^2 = \sum_j |c_j(x-b_P)|^2 + \frac{1}{n} = C_\Phi(x-b_P) + \frac{1}{n}.$$

Hence (a) and (c) follow from the corresponding results for C_Φ.

(b) Follows from (6.8) and the proof of Proposition 6.7. □

The generalised barycentric coordinates map under affine transformations as follows:

Proposition 6.10 (Affine maps). *Let $A : X \to Y$ be an invertible affine map between affine spaces X and Y, and $P = (p_1, \ldots, p_n)$ be points in X with affine span X. Then the canonical barycentric coordinates for P and $Q := AP = (Ap_1, \ldots, Ap_n)$ satisfy*

$$\xi^Q(Ax) = \xi^P(x), \qquad \forall x \in X.$$

Proof: Write $Ax = L(x - b_P) + a$, where L is a linear map (on the vectors in X), and $b_P := \frac{1}{n}\sum_j p_j$ is the barycentre of P. Then the barycentre of Q is

$$b_Q = \frac{1}{n}\sum_j \bigl(L(p_j - b_P) + a\bigr) = L\bigl(\frac{1}{n}\sum_j (p_j - b_P)\bigr) + a,$$

and so $Ax - b_Q = L(x - b_P)$. Let $\Phi = (p_j - b_P)_{j=1}^n$ and $\Psi = (Ap_j - b_Q)_{j=1}^n = L\Phi$. Then using (5.4), we obtain

$$\xi_j^Q(Ax) = c_j^\Psi(Ax - b_Q) + \frac{1}{n} = c_j^{L\Phi}(L(x-b_P)) + \frac{1}{n} = c_j^\Phi(x - b_P) + \frac{1}{n} = \xi_j^P(x).$$

□

By analogy with (5.5), **symmetry group** of a sequence of points $P = (p_j)_{j\in J}$ with affine span X is defined by

$$\mathrm{Sym}(P) := \{\sigma \in S_J : \exists A_\sigma : X \to X \text{ an affine map with } A_\sigma p_j = p_{\sigma j}, \forall j \in J\}.$$

Proposition 6.11 (Symmetries). *Let $P = (p_j)$ be a finite sequence of points with affine span X. Then P and its canonical barycentric coordinates ξ^P have the same symmetries, i.e.,*

$$\mathrm{Sym}(P) = \mathrm{Sym}(\xi^P).$$

Proof: It $\sigma \in \mathrm{Sym}(P)$, then the corresponding affine map A_σ fixes the barycentre b_P, so that $L_\sigma(x - b_P) := A_\sigma x - b_P$ gives a linear map on vectors, and one obtains

$$\mathrm{Sym}(P) = \mathrm{Sym}(\Phi), \qquad \Phi = (f_j), \quad f_j := p_j - b_P.$$

Similarly, $\mathrm{Sym}(\xi^P) = \mathrm{Sym}(c^\Phi)$, and so the result follows from Proposition 5.6. □

Example 10. *Four points in* $\mathbb{R}^2$. In view of Proposition 6.10, we suppose, without loss of generality, that

$$P = (\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}a\\b\end{pmatrix}),$$

where there are no restrictions on (a, b). By direct computation, we find that the canonical barycentric coordinates are given by (5.7), i.e., $\xi_p = f_p$, $\forall p \in P$, where the barycentric coordinates are indexed by the points of P.

Example 11. *Generalised Bernstein polynomials.* Let (ξ_j) be the generalised barycentric coordinates for points $P = (p_1, \ldots, p_n)$ with $\mathrm{Aff}(P) = X$. By the multinomial theorem,

$$(\xi_1 + \cdots + \xi_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} \xi^\alpha = 1, \qquad \binom{k}{\alpha} := \frac{k!}{\alpha!}, \quad \xi^\alpha = \prod_j \xi_j^{\alpha_j},$$

where α is a multi-index, i.e., $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Thus the polynomials

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha : X \to \mathbb{F}, \qquad |\alpha| = k,$$

span the polynomials of degree $\leq k$, and form a partition of unity. They are a basis if and only if $n = \mathrm{affdim}(X)$, in which case they are called the *multivariate Bernstein basis*.

Example 12. *The "dual" affine expansion.* Let $Q = (\xi_j^P)$ be the generalised barycentric coordinates for points $P = (p_1, \ldots, p_n)$ with affine span X. The affine span of Q is $\Pi_1(X)$, the affine space of all affine maps (linear polynomials) $\lambda : X \to \mathbb{F}$. We claim that

$$\xi_j^Q = \hat{p}_j, \qquad \hat{p}_j(\lambda) := \lambda(p_j), \quad \forall \lambda \in \Pi_1(X),$$

i.e., there is the following "dual" of the expansion (6.3)

$$\lambda = \sum_{j=1}^n \hat{p}_j(\lambda) \xi_j^P, \qquad \sum_{j=1}^n \hat{p}_j(\lambda) = 1, \qquad \forall \lambda \in \Pi_1(X). \tag{6.12}$$

We recall that

$$\xi_j^P(x) = c_j^\Phi(x - b_P) + \frac{1}{n}, \qquad \Phi = (f_j), \quad f_j := p_j - b_P, \tag{6.13}$$

and $\sum_j f_j = 0$ implies $\sum_j c_j^\Phi = 0$, so that $b_Q = \frac{1}{n}$. Thus

$$\xi_j^Q(\lambda) = c_j^\Psi(\lambda - \frac{1}{n}) + \frac{1}{n}, \qquad \Psi = (\xi_j^P - \frac{1}{n}).$$

Now (6.13) gives $\xi_j^P - \frac{1}{n} = L c_j^\Phi$, where L is the linear transformation $Lg := g(\cdot - b_P)$. Hence $L^{-1}\Psi = c^\Phi$, and from (5.4) we obtain

$$c_j^{L^{-1}\Psi}(L^{-1}\lambda) = c_j^\Psi(\lambda) = c_j^{c_\Phi}(L^{-1}\lambda) = c_j^{c_\Phi}(\lambda(\cdot + b_P)).$$

Since $c_j^{c^\Phi} = \hat{f}_j$, this gives

$$c_j^\Psi(\lambda - \frac{1}{n}) = (\lambda(\cdot + b_P) - \frac{1}{n})(f_j) = \lambda(f_j + b_P) - \frac{1}{n} \quad \Longrightarrow \quad \xi_j^Q(\lambda) = \lambda(p_j).$$

The second sum in (6.12) follows the theory, or directly: $\sum \hat{p}_j(\lambda) = n\hat{b}_P(\lambda) = n\lambda(b_P) = 1$.

References

[AS87] P. Alfeld and L. L. Schumaker, The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r+1$, *Constr. Approx.* **3** (1987), 189–197.

[A05] D. M. Appleby, Symmetric informationally complete–positive operator valued measures and the extended Clifford group, *J. Math. Phys.* **46** (2005), 1–27.

[B87] C. de Boor, B–form basics, *in* “Geometric Modeling: Algorithms and New Trends” (G. E. Farin Ed.), pp. 131–148, SIAM Publications, Philadelphia, 1987.

[BHR93] C. de Boor, K. Höllig, and S. D. Riemenschneider, “Box Splines”, Springer-Verlag, New York, 1993.

[CK07] A. Chebira and J. Kovačević, Life Beyond Bases: The Advent of Frames (Part I), IEEE Signal Processing Magazine (to appear), 2007.

[CW10] T. Chien and S. Waldron, Tight frames for cyclotomic fields and other rational vector spaces, Preprint, 2010.

[C03] O. Christensen, “An introduction to frames and Riesz bases”, Birkhäuser, Boston, 2003.

[CJ76] J. H. Conway and A. J. Jones, Trigonometric Diophantine equations (On vanishing sums of roots of unity), *Acta Arith.* **30** (1976), 229–240.

[ER49] C. J. Everett and H. J. Ryser, Rational vector spaces I, *Duke Math. J.* **16** (1949), 553–570.

[FHK06] M. S. Floater, K. Hormann, and G. Kós, A general construction of barycentric coordinates over convex polygons, *Adv. Comput. Math.* **24** (2006), 311–331.

[G91] K. Gröchenig, Describing functions: atomic decompositions versus frames, *Monatsh. Math.* **112** (1991), 1–42.

[HP04] R. B. Holmes and V. I. Paulsen, Optimal frames for erasures, *Linear Algebra Appl.* **377** (2004), 31–51.

[K94] G. Kaiser, “A Friendly Guide to Wavelets”, Birkhauser, Boston, 1994.

[LS07] M.-J. Lai and L. L. Schumaker, “Spline functions on triangulations”, Cambridge University Press, Cambridge, 2007.

[LL00] T. Y. Lam and K. H. Leung, On vanishing sums of roots of unity, *J. Algebra* **224** (2000), 91–109.

[LS08] T. Langer and H.-P. Seidel, Higher order barycentric coordinates, *Computer Graphics Forum* **27** (2008), 459–466.

[M65] H. B. Mann, On linear relations between roots of unity, *Mathematika* **12** (1965), 107–117.

[R05] J. M. Renes, Equiangular spherical codes in quantum cryptography, *Quantum Inf. Comput.* **5** (2005), 81–92.

[RBSC04] J. M. Renes, R. Blume–Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, *J. Math. Phys.* **45** (2004), 2171–2180.

[R08] S. Roman, "Advanced linear algebra (Third edition)", Springer, New York, 2008.

[S08] J. P. Steinberger, Minimal vanishing sums of roots of unity with large coefficients, *Proc. London Math. Soc.* **97** (2008), 689–717.

[SH03] T. Strohmer and R. W. Heath, Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmonic Anal.* **14** (2003), 257–275.

[VW10] R. Vale and S. Waldron, The symmetry group of a finite frame, *Linear Algebra Appl.* **433** (2010), 248–262.

[W09] S. Waldron, Generalised affine barycentric coordinates, Submitted, 2010.

[W11] S. Waldron, An Introduction to Finite Tight Frames, Birkhäuser (Boston), 2011.