On the Structure of Kergin Interpolation for Points in General Position

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Abstract

For $n+1$ points in $\mathbb{R}^d$, in general position, the Kergin polynomial interpolant of C^n functions may be extended to an interpolant of C^{d-1} functions. This results in an explicit set of reduced Kergin functionals naturally stratified by their dependence on certain directional derivatives of order k, $0 \le k \le d-1$. We show that the polynomials dual to the functionals depending on derivatives of order k are multi-ridge functions of $d-k$ variables and moreover, that the polynomials dual to the purely interpolating functionals ($k=0$) are always harmonic.

1 Introduction

Kergin interpolation is a natural multivariate extension of the Newton form of classical Hermite polynomial interpolation, introduced by Kergin in [4], and since then much studied (see, e.g., [6], [7], [1], [2]). It is most easily defined as follows.

Suppose that Θ is a sequence of $n+1$ (possibly coincident) points in $\mathbb{R}^d$

$$\Theta := [\theta_0, \theta_1, \ldots, \theta_n].$$

Then the **Kergin interpolant** of $f \in C^n(\mathbb{R}^d)$ is the polynomial $K_\Theta f$ of degree n, defined by

$$K_\Theta f(x) := \sum_{k=0}^{n} \int_{[\theta_0,\ldots,\theta_k]} d^k f(x-\theta_0, x-\theta_1, \ldots, x-\theta_{k-1}). \tag{1.1}$$

Here $d^k f$ denotes the k–th total derivative of f, and the linear functional

$$f \mapsto \int_{[\theta_0,\ldots,\theta_k]} f := \frac{1}{k!\,\mathrm{vol}_k(S)} \int_S f \circ A, \tag{1.2}$$

where S is any k–simplex in $\mathbb{R}^s$ with (k–dimensional) volume $\mathrm{vol}_k(S)$, and $A : \mathbb{R}^s \to \mathbb{R}^d$ is any affine map taking the $k+1$ vertices of S onto the points $\theta_0, \ldots, \theta_k$. The change of variables formula shows that (1.2) does not depend on the particular choice of S and A.

In the univariate case (1.1) reduces to the Newton form of the Hermite interpolant of f at Θ, via the Hermite–Genocchi formula for the divided difference of f at $\theta_0, \ldots, \theta_k$, i.e.,

$$[\theta_0, \ldots, \theta_k]f = \int_{[\theta_0,\ldots,\theta_k]} D^k f.$$

Often it is convenient to write $K_\Theta f$ in terms of directional derivatives

$$K_\Theta f(x) = \sum_{k=0}^{n} \int_{[\theta_0,\ldots,\theta_k]} D_{x-\theta_0} D_{x-\theta_1} \cdots D_{x-\theta_{k-1}} f,$$

and to make a particular choice of S and A in (1.2), such as

$$\int_{[\theta_0,\ldots,\theta_k]} f = \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1.$$

Amongst others, Kergin interpolation enjoys the following desirable properties:

<u>Interpolation at Θ</u>: $K_\Theta f(\theta_j) = f(\theta_j)$, $0 \le j \le n$.

<u>Symmetry in Θ</u>: K_Θ does not depend on the ordering of the points in Θ.

<u>Error Formula</u>: For each $f \in C^{n+1}(\mathbb{R}^d)$,

$$f(x) - K_\Theta f(x) = \int_{[\theta_0,\ldots,\theta_n]} D_{x-\theta_0} D_{x-\theta_1} \cdots D_{x-\theta_n} f.$$

In particular, K_Θ is a linear projector onto $\Pi_n(\mathbb{R}^d)$, the polynomials on $\mathbb{R}^d$ of degree $\le n$.

<u>Continuity and Coalescence</u>: For each $f \in C^n(\mathbb{R}^d)$, the map

$$\mathbb{R}^{d\times(n+1)} \to \Pi_n(\mathbb{R}^d) : \Theta \mapsto K_\Theta f$$

is continuous, and if $\theta_0 = \cdots = \theta_n = a$, then $K_\Theta f$ is the degree n Taylor polynomial of f centred at a.

Rolle's Property: For *any* homogeneous constant coefficient partial differential operator $q(D)$ of degree $k \le n$,

$$\int_{[\theta_0,\ldots,\theta_k]} q(D)(f - K_\Theta f) = 0. \tag{1.3}$$

PDE Faithfulness: For *any* homogeneous constant coefficient partial differential operator $q(D)$ of degree $k \le n$,

$$q(D)f = 0 \Rightarrow q(D)(K_\Theta f) = 0.$$

The last property involves functions $f : \mathbb{R}^d \to \mathbb{R}$ which are constant on the translates of some affine subspace V of $\mathbb{R}^d$ of codimension m. Since the collection of translates (also called cosets) $\mathbb{R}^d/V := \{x + V : x \in \mathbb{R}^d\}$ partitions $\mathbb{R}^d$,

f is constant on the translates of V

$\iff$ for every affine map $A : \mathbb{R}^d \to \mathbb{R}^s$ with kernel a translate of V, $f = g \circ A$

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where in each case $g : \mathrm{ran}(A) \to \mathbb{R}$ is (well) defined on the range of A by $g(Ax) := f(x)$. We will refer to such an f as a m–**ridge or multi–ridge function**. If $m = 1$, and A is taken to be a nonzero linear functional $\lambda : x \mapsto \langle \lambda^*, x\rangle$, $\lambda^* \in \mathbb{R}^d$, then we obtain $f = g \circ \lambda$, which is commonly referred as a ridge function, or plane wave.

Ridge Friendliness: If $f = g \circ A$ is a multi–ridge function, where $A : \mathbb{R}^d \to \mathbb{R}^s$ is an affine map, and $g \in C^n(\mathrm{ran}(A))$, then

$$K_\Theta f = (K_{A\Theta} g) \circ A.$$

Because of the possibility of coalescence to a single point (Taylor interpolation), the Kergin interpolant is defined in general only for C^n functions. However, in the univariate case there is no smoothness requirement for *distinct* interpolation points, i.e., one has Lagrange interpolation to functions from $C(\mathbb{R})$.

This phenomenon extends to higher dimensional Kergin interpolants for points in general position in $\mathbb{R}^d$, which can be defined for functions of smoothness $f \in C^{d-1}(\mathbb{R}^d)$. This extension can be described by a reduced set of linear functionals (interpolation conditions) that depend only on certain directional derivatives up to order $d-1$, and their corresponding dual polynomials from $\Pi_n(\mathbb{R}^d)$. A complete description of these reduced functionals is given in Theorem 2.3. It is natural to stratify them by the order of the derivative which they depend on, an "ascending" scale of complexity. In Theorem 2.4, we show the corresponding system of dual polynomials has a "descending" scale of complexity; precisely, that the dual polynomials to functionals involving derivatives of order $k = 0, \ldots, d-1$, are $(d-k)$–ridge functions, i.e., functions of only $d-k$ variables. Finally, in Theorem 3.3 we show the surprising fact that the dual polynomials for the $n+1$ point evaluation functionals are always harmonic.

The constant coefficient differential operators $q(D)$ on $\mathbb{R}^d$ used below are homogeneous of degree $r \geq 0$, and are described in terms of the (homogeneous) polynomials $q \in \Pi^0_r(\mathbb{R}^d)$ that correspond to them.

Lemma 1.1. *Suppose that $A : \mathbb{R}^d \to \mathbb{R}^s$ is an affine map, with linear part $L := A - A(0)$. If f is a multi–ridge function of the form $f = g \circ A$, $g \in C^r(\operatorname{ran}(A))$, then*

$$\int_\Theta q(D)f = \int_{A\Theta} (q \circ L^*)(D)g, \qquad \forall q \in \Pi^0_r(\mathbb{R}^d), \tag{1.4}$$

where $L^* : \mathbb{R}^s \to \mathbb{R}^d$ is the adjoint of L.

Proof. We compute

$$\begin{aligned} D_u f(x) &:= \lim_{t\to 0} \frac{g(A(x+tu)) - g(A(x))}{t} \\ &= \lim_{t\to 0} \frac{g(Ax + tLu) - g(A(x))}{t} \\ &= (D_{Lu} g)(Ax), \end{aligned}$$

and so (by iterating)

$$D_{u_1} \cdots D_{u_r} f = (D_{Lu_1} \cdots D_{Lu_r} g) \circ A, \qquad \forall u_1, \ldots, u_r \in \mathbb{R}^d.$$

Hence it follows from (1.2) that

$$\int_\Theta D_{u_1} \cdots D_{u_r} f = \int_\Theta (D_{Lu_1} \cdots D_{Lu_r} g) \circ A = \int_{A\Theta} D_{Lu_1} \cdots D_{Lu_r} g. \tag{1.5}$$

Since the differential operator $q(D) = D_{u_1} \cdots D_{u_r}$ corresponds to the homogeneous polynomial $q(x) := (u_1^T x) \cdots (u_r^T x)$, and $D_{Lu_1} \cdots D_{Lu_r}$ to

$$((Lu_1)^T x) \cdots ((Lu_r)^T x) = (u_1^T L^* x) \cdots (u_r^T L^* x) = q(L^* x) = (q \circ L^*)(x),$$

(1.5) can be expressed in the form (1.4). □

2 The Ridge Structure of the Kergin Dual Polynomials

Definition 2.1. *Suppose that V is a linear subspace of $\mathbb{R}^d$, with orthogonal complement $V^\perp$. Let $\Pi_k(V)$ denote the subspace of $\Pi_k(\mathbb{R}^d)$ consisting of multi–ridge polynomials which vary in the V directions, i.e., which are constant on translates of $V^\perp$, and $\Pi_k^0(V)$ those which are in addition homogeneous of degree k.*

If V has dimension m, then the natural identification with polynomials on $\mathbb{R}^m$ gives

$$\dim(\Pi_k(V)) = \binom{k+m}{m}, \qquad \dim(\Pi_k^0(V)) = \binom{k+m-1}{m-1}.$$

It follows from the Rolle's property and the error formula that K_Θ is the unique linear projector of $C^n(\mathbb{R}^d)$ onto $\Pi_n(\mathbb{R}^d)$ for which $\lambda(K_\Theta f) = \lambda(f)$, $\forall \lambda \in \Lambda$, where

$$\Lambda := \sum_{j=1}^{n+1} \sum_{\substack{\Psi\subset\Theta \\ \#\Psi=j}} \mathrm{Span}\{f \mapsto \int_\Psi q(D)f : q \in \Pi_{j-1}^0(\mathbb{R}^d)\}.$$

Here $\Psi \subset \Theta$, $\#\Psi = j$ indicates that Ψ is a subsequence of Θ of cardinality j. Henceforth we will refer to Λ as the (space of) interpolation conditions of K_Θ.

Definition 2.2. *For Ψ a sequence of $k+1$ affinely independent points in $\mathbb{R}^d$, let Λ_Ψ^i be the space of (continuous) linear functionals $C^i(\mathrm{conv}(\Psi)) \to \mathbb{R}$*

$$\Lambda_\Psi^i := \{f \mapsto \int_\Psi q(D)f : q \in \Pi_i^0(\mathrm{aff}(\Psi)^\perp)\}, \qquad i = 0, 1, \ldots. \tag{2.1}$$

The $q(D)$ occuring above consist of all i–th order derivatives normal to $\mathrm{aff}(\Psi)$, i.e., are spanned by $D_{\mathrm{n}_1} D_{\mathrm{n}_2} \cdots D_{\mathrm{n}_i}$, where the $\mathrm{n}_1, \mathrm{n}_2, \ldots, \mathrm{n}_i$ are directions orthogonal to $\mathrm{aff}(\Psi)$.

The following decomposition of the interpolation conditions Λ was given in [8].

Theorem 2.3. **(Reduced functionals)** *Suppose the points in Θ are in general position. Then the space of interpolation conditions for K_Θ can be written as a direct sum*

$$\Lambda = \bigoplus_{j=1}^{d} \Lambda_{j-1}, \qquad \Lambda_{j-1} := \bigoplus_{\substack{\Psi\subset\Theta \\ \#\Psi=j}} \Lambda_\Psi, \tag{2.2}$$

where the linear functionals

$$\Lambda_\Psi := \Lambda_\Psi^{j-1} = \{f \mapsto \int_\Psi q(D)f : q \in \Pi_{j-1}^0(\operatorname{aff}(\Psi)^\perp)\}$$

are continuous on $C^{d-1}(\mathbb{R}^d)$.

This was proved by using the identity

$$\int_{[\Theta,v,w]} D_{v-w} f = \int_{[\Theta,v]} f - \int_{[\Theta,w]} f,$$

to reduce the order of the derivatives occuring in the interpolation conditions as much as possible, then doing a dimension count of the conditions so obtained. This result implies K_Θ has a continuous extension to $C^{d-1}(\mathbb{R}^d) \to \Pi_n(\mathbb{R}^d)$.

Theorem 2.4. **(Dual polynomials)** *Let* P_Ψ^Θ *be the subspaces of* $\Pi_n(\mathbb{R}^d)$ *dual to those of* (2.2)*, i.e., satisfying* $\dim(\Lambda_\Psi|_{P_\Psi^\Theta}) = \dim(P_\Psi^\Theta)$*, and* $\Lambda_{\Psi_1}(P_{\Psi_2}^\Theta) = 0$*,* $\Psi_1 \neq \Psi_2$*, which implies*

$$\Pi_n(\mathbb{R}^d) = \bigoplus_{j=1}^{d} \bigoplus_{\substack{\Psi\subset\Theta \\ \#\Psi=j}} P_\Psi^\Theta. \tag{2.3}$$

Then P_Ψ^Θ*,* $\#\Psi = j$*, consists of* $(d-j+1)$*–ridge functions in the directions* $\operatorname{aff}(\Psi)^\perp$*, i.e.,*

$$P_\Psi^\Theta \subset \Pi_n(\operatorname{aff}(\Psi)^\perp).$$

Proof. Fix $\Psi_0 \subset \Theta$, $\#\Psi_0 = k+1$, and consider the space of functionals Λ_{Ψ_0}. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection onto $V := \operatorname{aff}(\Psi_0)^\perp$. Then, by the ridge–friendliness of Kergin interpolation,

$$K_\Theta(g \circ A) = (K_{A\Theta} g) \circ A, \qquad \forall g \in C^{d-1}(V). \tag{2.4}$$

Here $K_{A\Theta} g$ is the Kergin interpolant for the sequence of "projected" points $A\Theta \subset V$, which is defined for $g \in C^{d-1}(V)$ via (1.1). The $k+1$ points of Ψ_0 project onto a single point, say ψ_0, and the remaining projected points are distinct (since Θ is in general position).

By Lemma 1.1, and $A = A^*$ (orthogonal projections are self adjoint), the linear functionals in Λ_Ψ restricted to multi–ridge functions $f = g \circ A$ have the form

$$f \mapsto \int_\Psi q(D)f = \int_{A\Psi} (q \circ A)(D) g, \qquad q \in \Pi_{j-1}^0(\operatorname{aff}(\Psi)^\perp). \tag{2.5}$$

Let $\Lambda^{|}_{\Psi}$ denote the corresponding space of projected interpolation conditions

$$C^{d-1}(V) \to \mathbb{R} : g \mapsto \int_{A\Psi} (q \circ A)(D)g, \qquad q \in \Pi^0_{j-1}(\mathrm{aff}(\Psi)^{\perp}).$$

Then, by (2.4), the interpolation conditions of $K_{A\Theta}$ are $\Lambda^{|} := \sum_{\Psi} \Lambda^{|}_{\Psi}$, and we now show

$$\Lambda^{|} = \Lambda^{|}_{\Psi_0} \bigoplus \sum_{\substack{\Psi\subset\Theta \\ \Psi\neq\Psi_0}} \Lambda^{|}_{\Psi}, \qquad \dim(\Lambda^{|}_{\Psi_0}|_{\Pi_n(\mathbb{R}^d)|_V}) = \dim(\Lambda_{\Psi_0}). \tag{2.6}$$

Firstly, by (2.5), the space $\Lambda^{|}_{\Psi_0}$ consists of the linear functionals

$$g \mapsto \frac{1}{k!} q(D)g(\psi_0), \qquad q \in \Pi^0_k(V),$$

and so has

$$\dim(\Lambda^{|}_{\Psi_0}|_{\Pi_n(\mathbb{R}^d)|_V}) = \dim(\Pi^0_k(V)) = \dim(\Lambda_{\Psi_0}).$$

Let L be any orthogonal projection onto a 1–dimensional subspace of V which maps only the points Ψ_0 onto $L\psi_0 = L(A\psi_0)$. Since Θ is in general position all but a finite number of the possible L will have this property. By (1.4) and the Hermite–Genocchi formula, a linear functional from $\Lambda^{|}_{\Psi}$ applied to a ridge function of the form $f = g \circ L$ is a multiple of the divided difference of the univariate function $g \in C^{d-1}(\mathrm{ran}(L))$ at the points $L\Psi$. Hence, if

$$\lambda^{|} \in \Lambda^{|}_{\Psi_0} \bigcap \sum_{\substack{\Psi\subset\Theta \\ \Psi\neq\Psi_0}} \Lambda^{|}_{\Psi},$$

then $\lambda^{|}(f) = 0$, since the only $L\Psi$, $\Psi \neq \Psi_0$ that involve the point $L\psi_0$ $k+1$ times are those with $\Psi \supset \Psi_0$. By the fundamentality of such ridge functions $f = g \circ L$, we have $\lambda^{|} = 0$.

Let Q be the subspace of $\Pi_n(\mathbb{R}^d)|_V$ dual to the first summand of (2.6), i.e., satisfying

$$\dim(\Lambda^{|}_{\Psi_0}|_Q) = \dim(\Lambda_{\Psi_0}), \qquad \Lambda^{|}_{\Psi}(Q) = 0, \quad \forall \Psi \neq \Psi_0.$$

Then $P^{\Theta}_{\Psi_0} = Q \circ A \subset \Pi_n(\mathrm{aff}(\Psi_0)^{\perp})$, since

$$\dim(\Lambda_{\Psi_0}|_{Q\circ A}) = \dim(\Lambda^{|}_{\Psi_0}|_Q) = \dim(\Lambda_{\Psi_0}) \quad \forall \Psi \neq \Psi_0$$

and

$$\Lambda_\Psi(Q \circ A) = \Lambda^|_\Psi(Q) = 0, \quad \forall \Psi \neq \Psi_0.$$

□

Remark. Kergin interpolation can also be defined for points in $\mathbb{C}^d$, via an extension of the functional (1.2), cf. [1]. This inherits all the properties of real Kergin interpolation, in a natural way, and a simple check shows the dual polynomials for Kergin interpolation to points in general position in $\mathbb{C}^d$ have the multi–ridge structure of Theorem 2.4.

Example. Consider the interpolation conditions Λ_{Ψ_0}, where $\#\Psi_0 = d$. This space is 1–dimensional, with a basis given by

$$\lambda : f \mapsto \int_{\Psi_0} D_{\mathrm{n}}^{d-1} f,$$

where n is a unit normal to the hyperplane $\mathrm{aff}(\Psi_0)$. Let $p = p_\lambda \in P_{\Psi_0}^\Theta$ be the polynomial dual to λ, then by our proof, this is the ridge function

$$p(x) = g(\langle \mathrm{n}, x\rangle), \qquad g := (\cdot - \psi_0)^{d-1} \prod_{\theta \in \Theta \setminus \Psi_0} (\cdot - \langle \mathrm{n}, \theta\rangle), \quad \psi_0 := \langle \mathrm{n}, \Psi_0\rangle.$$

where g is the Hermite interpolant to the data

$$g(\langle \mathrm{n}, \theta\rangle) = 0, \quad \theta \in \Theta \setminus \Psi_0, \qquad (n - d \text{ conditions})$$

and

$$g(\psi_0) = Dg(\psi_0) = \cdots = D^{d-2}g(\psi_0) = 0, \quad \frac{D^{d-1}g(\psi_0)}{(d-1)!} = 1 \qquad (d \text{ conditions}).$$

3 Harmonicity of the dual polynomials for point evaluation

Let Δ denote the Laplace operator in $\mathbb{R}^d$. It is well known that the map

$$\Delta|_{\Pi_n(\mathbb{R}^d)} : \Pi_n(\mathbb{R}^d) \to \Pi_{n-2}(\mathbb{R}^d) : f \mapsto \Delta f$$

is onto, and so its kernel $\mathbb{H}_n(\mathbb{R}^d)$, the harmonic polynomials of degree $\leq n$ in $\mathbb{R}^d$, has

$$\dim(\mathbb{H}_n(\mathbb{R}^d)) = \binom{n+d}{d} - \binom{n-2+d}{d}.$$

Since $\dim(\mathbb{H}_n(\mathbb{R}^d)) < \dim(\Pi_n(\mathbb{R}^d)) = \dim\Lambda$, $n \geq 2$, the Kergin interpolation conditions Λ of (2.2) are not linearly independent over $\mathbb{H}_n(\mathbb{R}^d)$, i.e., as functionals restricted to $\mathbb{H}_n(\mathbb{R}^d)$. We now describe these dependencies, which also hold for all harmonic functions. These then imply that the dual polynomials to point evaluations are harmonic for $d > 1$.

For a sequence of points Ψ in $\mathbb{R}^d$, denote by $\Delta_\Psi^\perp$ the Laplacian in the directions orthogonal to $\operatorname{aff}(\Psi)$, i.e.,

$$\Delta_\Psi^\perp = q_\Psi^\perp(D) := D_{\mathrm{n}_1}^2 + D_{\mathrm{n}_2}^2 + \cdots + D_{\mathrm{n}_{d-k}}^2, \qquad q_\Psi^\perp \in \Pi_2^0(\operatorname{aff}(\Psi)^\perp),$$

where $\mathrm{n}_1, \ldots, \mathrm{n}_{d-k}$ is any orthonormal basis for $\operatorname{aff}(\Psi)^\perp$. Denote by $\operatorname{conv}(\Psi)$ the convex hull of the points in Ψ.

Lemma 3.1. *Suppose that Ψ is a sequence of $k+1$ affinely independent points in $\mathbb{R}^d$, and let $\lambda \in \Lambda_\Psi^i$, $i \geq 2$, be the linear functional given by*

$$\lambda(f) := \int_\Psi \Delta_\Psi^\perp p(D) f, \qquad p \in \Pi_{i-2}^0(\operatorname{aff}(\Psi)^\perp). \tag{3.1}$$

Then

$$\lambda|_{\mathrm{H}} \in \sum_{\substack{\tilde\Psi \subset \Psi \\ \#\tilde\Psi = k}} \Lambda_{\tilde\Psi}^{i-1}|_{\mathrm{H}}, \qquad \mathbb{H} := \{f \in C^i(\operatorname{conv}(\Psi)) : \Delta f = 0\}. \tag{3.2}$$

Proof. Let Δ_Ψ denote the Laplacian in directions parallel to $\operatorname{aff}(\Psi)$, i.e., split $\Delta = \Delta_\Psi + \Delta_\Psi^\perp$. Suppose that f is harmonic. Then

$$\Delta p(D) f = p(D)\Delta f = 0 \Rightarrow \Delta_\Psi^\perp p(D) f = -\Delta_\Psi p(D) f,$$

and so, by (1.2),

$$\lambda(f) = \int_\Psi \Delta_\Psi^\perp p(D) f = -\int_\Psi \Delta_\Psi p(D) f = -\frac{1}{k!\operatorname{vol}_k(\operatorname{conv}(\Psi))} \int_{\operatorname{conv}(\Psi)} \Delta_\Psi p(D) f.$$

Now, by Green's theorem,

$$\int_{\operatorname{conv}(\Psi)} \Delta_\Psi g = \int_{\partial(\operatorname{conv}(\Psi))} D_{\mathrm{n}} g = \sum_{\substack{\tilde\Psi \subset \Psi \\ \#\tilde\Psi = k}} \int_{\operatorname{conv}(\tilde\Psi)} D_{\mathrm{n}} g, \qquad \forall g \in C^2(\operatorname{conv}(\Psi)),$$

where n is the outward normal to the boundary of the k–simplex $\operatorname{conv}(\Psi)$, which consists of the faces $\operatorname{conv}(\tilde\Psi)$. Hence, using (1.2), $\lambda|_{\mathrm{H}}$ can be expressed as a linear combination of functionals

$$\lambda(f) = \sum_{\substack{\tilde\Psi \subset \Psi \\ \#\tilde\Psi = k}} c_{\tilde\Psi} \int_{\tilde\Psi} D_{\mathrm{n}} p(D) f, \qquad c_{\tilde\Psi} := -\frac{\operatorname{vol}_{k-1}(\operatorname{conv}(\tilde\Psi))}{k\operatorname{vol}_k(\operatorname{conv}(\Psi))},$$

each of which belongs to $\Lambda^{i-1}_{\tilde\Psi}$, since $p \in \Pi^0_{i-2}(\mathrm{aff}(\Psi)^\perp) \Rightarrow p \in \Pi^0_{i-2}(\mathrm{aff}(\tilde\Psi)^\perp)$.
□

There are also linear dependencies, of a different type, between the functionals Λ^1_{Ψ} over harmonic functions.

Lemma 3.2. *Suppose that Ψ is a sequence of $d+1$ affinely independent points in $\mathbb{R}^d$. Then, for $f \in C^1(\mathrm{conv}(\Psi))$ harmonic,*

$$\sum_{\substack{\tilde\Psi\subset\Psi\\ \#\tilde\Psi=d}} a_{\tilde\Psi}\int_{\tilde\Psi} D_{\mathrm{n}} f = 0, \qquad a_{\tilde\Psi} := (d-1)!\,vol_{d-1}(\mathrm{conv}(\tilde\Psi)),$$

where n *denotes the outward normal to* $\mathrm{conv}(\tilde\Psi)$ *as a face of the simplex* $\mathrm{conv}(\Psi)$.

Proof. By Green's Theorem,

$$\sum_{\substack{\tilde\Psi\subset\Psi\\ \#\tilde\Psi=d}} a_{\tilde\Psi}\int_{\tilde\Psi} D_{\mathrm{n}} f = \sum_{\substack{\tilde\Psi\subset\Psi\\ \#\tilde\Psi=d}} \int_{\mathrm{conv}(\tilde\Psi)} D_{\mathrm{n}} f = \int_{\partial(\mathrm{conv}(\Psi))} D_{\mathrm{n}} f = \int_{\mathrm{conv}(\Psi)} \Delta f = 0.$$

□

In [2], using the identification of $\mathbb{R}^2$ with $\mathbb{C}$, it was shown that the dual polynomials to point evaluation for Kergin interpolation to points in general position in $\mathbb{R}^2$ are given by the real part of dual polynomials for univariate complex Lagrange interpolation to the points in $\mathbb{C}$, and so are harmonic. We now show this harmonicity extends to higher dimensions (including the odd dimensions!).

Theorem 3.3. (Harmonicity). *For Kergin interpolation to Θ in general position in $\mathbb{R}^d$, $d \ge 2$, the dual polynomials to point evaluation are harmonic, i.e., $\Delta P^{\Theta}_{[\psi]} = 0$, $\forall \psi \in \Theta$.*

Proof. We begin by giving an alternative proof of the bivariate result, which is easily generalised to cover the so called scale of mean value interpolants (see, e.g., [8]).

In the bivariate case, the reduced interpolation conditions Λ of (2.2) consist of $n+1$ point evaluations Λ_0, and $\binom{n+1}{2}$ first derivative functionals Λ_1. By Lemma 3.2, for each triple of points $\Psi := [\psi_1, \psi_2, \psi_3] \subset \Theta$, the associated first derivative functionals have a linear dependency over harmonic functions given by

$$\|\psi_2-\psi_1\|\int_{[\psi_1,\psi_2]} D_{\mathrm{n}} f + \|\psi_3-\psi_2\|\int_{[\psi_2,\psi_3]} D_{\mathrm{n}} f + \|\psi_1-\psi_3\|\int_{[\psi_3,\psi_1]} D_{\mathrm{n}} f = 0.$$

Since each pair of points is in a triple containing a given point θ_0, this implies the n first derivative functionals that involve θ_0 and another point span Λ_1 over $\mathbb{H}_n(\mathbb{R}^2)$. But,

$$\dim(\Lambda|_{\mathbb{H}_n(\mathbb{R}^2)}) = \dim(\mathbb{H}_n(\mathbb{R}^2)) = 2n+1 = (n+1)+n = \dim(\Lambda_0)+n,$$

and so these n functionals must be a basis for Λ_1 over $\mathbb{H}_n(\mathbb{R}^2)$, which when appended by the $(n+1)$ point evaluations Λ_0 forms a dual basis for $\mathbb{H}_n(\mathbb{R}^2)$. Let $h \in \mathbb{H}_n(\mathbb{R}^2)$ be the harmonic polynomial which is 1 at a point ψ of Θ, 0 at the others, and is annihilated by the n first derivative functionals above, and hence all of Λ_1. Then this h satisfies the conditions that characterise the Kergin dual polynomial to point evaluation at ψ, which is therefore harmonic.

Now the case $d \geq 3$. The idea is the same, except we use the linear dependencies of Lemma 3.1. Let us count how many different dependencies this gives. Since Θ is in general position, for $2 \leq k < d$, each of the $\binom{n+1}{k+1}$ subsequences Ψ of $k+1$ points is affinely independent, and the subspace $\Lambda_\Psi^{\mathrm{dep}}$ of $\Lambda_\Psi = \Lambda_\Psi^k$ of functionals of the form (3.1) has

$$\dim(\Lambda_\Psi^{\mathrm{dep}}) = \dim(\Pi_{k-2}^0(\mathrm{aff}(\Psi)^\perp)) = \binom{k-2+(d-k-1)}{d-k-1} = \binom{d-3}{d-k-1}.$$

This gives a total number of dependencies

$$\sum_{k=2}^{d-1} \binom{d-3}{d-k-1}\binom{n+1}{k+1} = \binom{n-2+d}{d}, \tag{3.3}$$

as the above sum represents the number of ways of choosing d objects from $n-2+d$ objects after having split them into two groups, one of size $d-3$ and the other of size $n+1$. Let Λ_Ψ^* be a complement of $\Lambda_\Psi^{\mathrm{dep}}$ in Λ_Ψ, i.e., write $\Lambda_\Psi = \Lambda_\Psi^* \oplus \Lambda_\Psi^{\mathrm{dep}}$. Then, by (3.2),

$$\Lambda_\Psi|_{\mathrm{H}} = \Lambda_\Psi^*|_{\mathrm{H}} + \Lambda_\Psi^{\mathrm{dep}}|_{\mathrm{H}} \subset \Lambda_\Psi^*|_{\mathrm{H}} + \Lambda_{k-1}|_{\mathrm{H}}, \qquad \mathbb{H} := \{f \in C^{d-1}(\mathrm{conv}(\Theta)) : \Delta f = 0\},$$

and so the space of functionals

$$\Lambda^* := \Lambda_0 \oplus \Lambda_1 \oplus \bigoplus_{j=3}^{d} \bigoplus_{\substack{\Psi\subset\Theta \\ \#\Psi=j}} \Lambda_\Psi^*,$$

spans Λ over $\mathbb{H}$. By (3.3),

$$\dim(\Lambda^*) = \binom{n+d}{d} - \binom{n-2+d}{d} = \dim(\mathbb{H}_n(\mathbb{R}^d)),$$

and so Λ^* is dual to $\mathbb{H}_n(\mathbb{R}^d)$. Let $h \in \mathbb{H}_n(\mathbb{R}^d)$ be the harmonic polynomial which is 1 at a point ψ of Θ, 0 at the others, and is annihilated by the other summands in the definition of Λ^*, and hence by $\Lambda_1, \ldots, \Lambda_{d-1}$. Then this h satisfies the conditions that characterise the Kergin dual polynomial to point evaluation at ψ, which is therefore harmonic. $\square$

Remark. The above result can be generalised to the mean value interpolation of [3]. Briefly, for points in general position (see [8]), there is a family of linear projectors

$$\mathcal{H}_\Theta^{(m)} : C^{n-m-1}(\mathbb{R}^d) \to \Pi_{n-m-1}(\mathbb{R}^d), \quad 0 \leq m \leq d-1,$$

called the scale of mean value interpolations, determined by the interpolation conditions

$$\Lambda = \bigoplus_{j=m+1}^{d} \Lambda_{j-m-1}^{(m)} = \bigoplus_{j=m+1}^{d} \bigoplus_{\substack{\Psi \subset \Theta \\ \#\Psi = j}} \Lambda_\Psi^{j-m-1},$$

which contain Kergin interpolation as the special case $K_\Theta = \mathcal{H}_\Theta^{(0)}$. A modification of the above argument shows that for all mean value interpolation operators, except Hakopian interpolation, when $m = d-1$, the dual polynomials to the functionals $\Lambda_0^{(m)}$ (which involve the function, but no derivatives) are harmonic.

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