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Continuous and discrete tight frames of orthogonal polynomials for a radially symmetric weight

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ABSTRACT

This paper considers tight frame decompositions of the Hilbert space $\mathcal{P}_n$ of orthogonal polynomials of degree n for a radially symmetric weight on $\mathbb{R}^d$, e.g., the multivariate Gegenbauer and Hermite polynomials. We explicitly construct a single zonal polynomial $p \in \mathcal{P}_n$ with property that each $f \in \mathcal{P}_n$ can be reconstructed as a sum of its projections onto the orbit of p under $\mathrm{SO}(d)$ (symmetries of the weight), and hence of its projections onto the zonal polynomials p_ξ obtained from p by moving its pole to $\xi \in S := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. Furthermore, discrete versions of these integral decompositions also hold where $\mathrm{SO}(d)$ is replaced by a suitable finite subgroup, and S by a suitable finite subset. One consequence of our decomposition is a simple closed form for the reproducing kernel for $\mathcal{P}_n$.

Key Words: multivariate orthogonal polynomials, Jacobi polynomials, Gegenbauer polynomials, ultraspherical polynomials, Legendre polynomials, Hermite polynomials, Laguerre polynomials, harmonic functions, spherical harmonics, zonal harmonics, quadrature for trigonometric polynomials, cubature on the sphere, representation theory, radial functions, ridge functions, zonal functions, tight frames

AMS (MOS) Subject Classifications: primary 33C45, 33D50, secondary 06B15, 42C15

1. Introduction

We construct orthogonal–type expansions (tight frames) with the simplest possible form for the space $\mathcal{P}_n$ of orthogonal polynomials of degree n for a general radially symmetric weight function on $\mathbb{R}^d$. By way of motivation, first consider the Legendre polynomials (constant weight) on the unit disc $\mathbb{D} := \{x \in \mathbb{R}^2 : |x| \leq 1\}$ given by the inner product

$$\langle f, g\rangle := \int_{\mathbb{D}} fg = \int_0^{2\pi} \int_0^1 (fg)(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

These polynomials, and bases for them, are often referred to as Zernike polynomials.

Let U_n be the n–th *Chebyshev polynomial of the second kind*, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through the angle θ, which acts on functions $p : \mathbb{R}^2 \to \mathbb{R}$ via $R_\theta p := p \circ R_\theta^{-1}$. Then the polynomial p given by

$$p(x, y) := \frac{1}{\sqrt{\pi}} U_n(x), \tag{1.1}$$

and each of its rotations $R_\theta p$ have unit norm and belong to $\mathcal{P}_n$. In [W08] it was shown that each Legendre polynomial of degree n on the unit disc can be expressed as a sum of its projections onto these "simple" polynomials as follows

$$f = \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} p\rangle R^j_{\frac{2\pi}{k}} p = \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta p\rangle R_\theta p\, d\theta, \qquad \forall f \in \mathcal{P}_n, \tag{1.2}$$

where $k \geq n+1$ with k not even if $k \leq 2n$. A special case of this is the orthogonal expansion of [LS75]

$$f = \sum_{j=0}^{n} \langle f, R_{\frac{j\pi}{n+1}} p\rangle R_{\frac{j\pi}{n+1}} p, \qquad \forall f \in \mathcal{P}_n. \tag{1.3}$$

The calculations leading to (1.2) exploited the identification of $\mathbb{R}^2$ with $\mathbb{C}$ and so it is not immediately clear how these expansions generalise to $d > 2$. The generalisation of (1.2) which is presented here has the following key features:

- The "simple" function p is replaced by a *zonal function*.
- The group of rotations $\langle R_{\frac{2\pi}{k}}\rangle$ is replaced by an appropriate *subgroup of* $\mathrm{SO}(d)$.

We consider two possible interpretations for $R_\theta p$ above

- The group element $g = R_\theta \in \mathrm{SO}(2)$ acting on p.
- The ridge polynomial p in the direction $e_1 := (1, 0)$ with its direction changed to $\xi \in \mathbb{R}^2$, $\xi := R_\theta e_1$, i.e.,

$$R_\theta p = p_\xi := \frac{1}{\sqrt{\pi}} U_n(\langle\cdot, \xi\rangle).$$

The integral in (1.2) can then be expressed as

$$\dim(\mathcal{P}_n) \int_{g\in \mathrm{SO}(2)} \langle f, gp\rangle gp \, d\mu_2(g), \qquad \frac{\dim(\mathcal{P}_n)}{\mathrm{area}(S)} \int_S \langle f, p_\xi\rangle p_\xi \, d\xi$$

respectively, where μ_d is the normalised Haar measure on $\mathrm{SO}(d)$, and $\mathrm{area}(S)$ is the area (length) of the circle $S := \{x \in \mathbb{R}^2 : |x| = 1\}$.

The main result of this paper is for any $d \geq 2$ the explicit construction of a zonal polynomial $p \in \mathcal{P}_n$ for which

$$\begin{aligned} f &= \dim(\mathcal{P}_n) \int_{g\in \mathrm{SO}(d)} \langle f, gp\rangle gp \, d\mu(g) = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{g\in G} \langle f, gp\rangle gp \\ &= \frac{\dim(\mathcal{P}_n)}{\mathrm{area}(S)} \int_S \langle f, p_\xi\rangle p_\xi \, d\xi = \frac{\dim(\mathcal{P}_n)}{|V|} \sum_{\xi\in V} \langle f, p_\xi\rangle p_\xi, \qquad \forall f \in \mathcal{P}_n, \end{aligned} \tag{1.4}$$

where G is suitable finite subgroup of $\mathrm{SO}(d)$, and V is suitable finite subset of the unit sphere S. The integrals in (1.4) are known as continuous (tight) frame decompositions (cf [C03:§5.8]), and the finite sums as tight frame decompositions.

Our approach to (1.4) is to first prove the continuous versions, which with hindsight are the most natural decompositions as they inherit all the symmetries of the space $\mathcal{P}_n$. We then obtain the discrete decompositions from these. This is in contrast with [W08], where the discrete versions were proved first, and then the continuous versions were obtained from these by using a quadrature rule.

The rest of the paper is set out as follows. In the next section we give some of the basic definitions and facts that our results are based on. This includes the definition of $\mathcal{P}_n$ and a discussion of zonal and harmonic functions. Following that, we give the decomposition of $\mathcal{P}_n$ into absolutely irreducible $\mathrm{SO}(d)$–invariant subspaces. This is used in the following sections to obtain the continuous and discrete frame decompositions of (1.4). One nice application of our continuous tight frame decomposition is a simple closed form for the reproducing kernel of $\mathcal{P}_n$.

2. Basic definitions and results

Our results hold for the orthogonal polynomials given by an appropriate radially symmetric measure μ (cf [X05]). However, for simplicity of exposition, we consider the case where μ is given by Lebesgue integration with a radial weight function w on the ball

$$B = B_R := \{x \in \mathbb{R}^d : |x| := \sqrt{x_1^2 \cdots + x_d^2} < R\}$$

of radius $0 < R \leq \infty$. This covers all the cases of known interest.

2.1. Orthogonal polynomials for a radially symmetric weight

Throughout, fix $d \geq 2$ and let $\mathbb{B}$ be the unit ball, and S the unit sphere in $\mathbb{R}^d$, i.e.,

$$\mathbb{B} := B_1 = \{x \in \mathbb{R}^d : |x| < 1\}, \qquad S := \{x \in \mathbb{R}^d : |x| = 1\}.$$

Let $w : [0, R) \to \mathbb{R}$, $0 < R \leq \infty$ be a positive function for which

$$\langle f, g \rangle = \langle f, g \rangle_w := \int_{B_R} f(x) g(x)\, w(|x|)\, dx \tag{2.1}$$

defines an inner product on Π_n (the polynomials on $\mathbb{R}^d$ of degree $\leq n$).

By analogy with the univariate case, the space $\mathcal{P}_n = \mathcal{P}_n^w$ of orthogonal polynomials of degree n corresponding to the weight $w(|\cdot|)$ on B_R is given by

$$\mathcal{P}_n := \{f \in \Pi_n : \langle f, g \rangle_w = 0, \forall g \in \Pi_{n-1}\}.$$

This is a Hilbert space with the inner product given by (2.1), and has dimension

$$\dim(\mathcal{P}_n) = \binom{n+d-1}{d-1}.$$

The orthogonal polynomials on $\mathbb{B} = B_1$ corresponding to the weight

$$w(r) := (1 - r^2)^\alpha r^{2\beta}, \qquad \alpha > -1,\ \beta > -\frac{d}{2}, \tag{2.2}$$

will be called the **generalised Gegenbauer polynomials**, **Gegenbauer polynomials** (when $\beta = 0$), and **Legendre polynomials** (when $\alpha = \beta = 0$). Those on $\mathbb{R}^d = B_\infty$ corresponding to the weight

$$w(r) = r^{2\beta} e^{-r^2}, \qquad \beta \geq 0, \tag{2.3}$$

are called **generalised Hermite polynomials**, and **Hermite polynomials** ($\beta = 0$).

The orthogonal group $\mathrm{O}(d)$ acts naturally on functions f defined on the ball B_R via

$$gf := f \circ g^{-1}, \qquad g \in \mathrm{O}(d).$$

It fixes $w(|\cdot|)$, so that $g \in \mathrm{O}(d)$ is a symmetry of our (radially symmetric) inner product, i.e.,

$$\langle f_1, f_2 \rangle = \langle gf_1, gf_2 \rangle, \qquad \forall g \in \mathrm{O}(d). \tag{2.4}$$

2.2. Zonal functions

A function defined on the sphere S is said to be *zonal* (on the sphere S) with pole $\xi \in \mathbb{R}^d$, $|\xi| = 1$ if it is invariant under the action of the subgroup of $\mathrm{O}(d)$ which fixes ξ, i.e., is constant on the intersection of S with any hyperplane in $\mathbb{R}^d$ which is orthogonal to the vector ξ. We extend this notion as follows.

Definition. *We say a function f defined on a $\mathrm{O}(d)$–invariant subset of $\mathbb{R}^d$ (such as S or B_R) is* **zonal** *with pole $\xi \in S$ if it is invariant under the action of the subgroup of $\mathrm{O}(d)$ which fixes ξ.*

A function f is zonal if and only if it can be written in the form

$$f(x) = g(\langle x, \xi \rangle, |x|).$$

Compare this with the corresponding conditions for being a ridge and radial function, i.e.,

$$f(x) = g(\langle x, \xi \rangle) \quad \text{(ridge function with direction } \xi), \qquad f(x) = g(|x|) \quad \text{(radial function).}$$

Thus a zonal function on B_R is a generalisation of a ridge function and of a radial function (and a zonal function on S is the restriction of a ridge function).

If f_η is zonal with pole η, then we can move the pole of f_η to be ξ by applying any $g \in \mathrm{O}(d)$ with $\xi = g\eta$. We use the notation f_ξ to denote the corresponding (well defined) zonal function

$$f_\xi := g f_\eta, \qquad g \in \mathrm{O}(d), \quad \xi = g\eta. \tag{2.5}$$

2.3. Harmonic functions

A function f (defined on B_R) is *harmonic* if it satisfies Laplace's equation, i.e.,

$$\Delta f = 0, \qquad \Delta := D_1^2 + \cdots D_d^2.$$

Let $\mathcal{H}_n$ be the space of homogeneous harmonic polynomials of degree n. The map of restriction of a function to the sphere $f \mapsto f|_S$ applied to $\mathcal{H}_n$ has trivial kernel, so that

$$\dim(\mathcal{H}_n) = \dim(\mathcal{H}_n(S)), \qquad \mathcal{H}_n(S) := \{f|_S : f \in \mathcal{H}_n\}.$$

The spaces $\mathcal{H}_n$ and $\mathcal{H}_n(S)$ are called (*solid* and *surface*) *spherical harmonics* of degree n, are invariant under the action of $\mathrm{O}(d)$, and have dimension

$$\begin{aligned} \dim(\mathcal{H}_n) = \dim(\mathcal{H}_n(S)) &= \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} \\ &= \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2} \quad (n \geq 1). \end{aligned} \tag{2.6}$$

Spherical harmonics of different degrees are orthogonal to each other with respect to the inner product

$$\langle f, g \rangle_S := \int_S f(\omega) g(\omega)\, d\omega, \tag{2.7}$$

where $d\omega$ denotes Lebesgue measure on the sphere. We will refer to the "area" of the sphere S (whatever the dimension d is), and denote it by

$$\operatorname{area}(S) := \int_S 1\, d\omega = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{2.8}$$

2.4. Zonal harmonics

The *Zonal harmonic* of degree k with pole $\xi \in S$ is the unique function $Z = Z_\xi^{(k)} \in \mathcal{H}_k$ which represents point evaluation at ξ with respect to the inner product (2.7) on $\mathcal{H}_k$, i.e., with the property

$$f(\xi) = \langle f, Z_\xi^{(k)} \rangle_S = \int_S f Z_\xi^{(k)}, \qquad \forall f \in \mathcal{H}_k. \tag{2.9}$$

We recall some basic facts about zonal harmonics (cf [SW71] and[ABR92]).

The zonal harmonic Z (defined on S, $\mathbb{B}$ or $\mathbb{R}^d$) is the unique function in $\mathcal{H}_k$ (up to a scalar) which is zonal with pole ξ. The norm of Z is

$$\|Z_\xi^{(k)}\|_S^2 = \langle Z_\xi^{(k)}, Z_\xi^{(k)} \rangle_S = Z_\xi^{(k)}(\xi) = \frac{\dim(\mathcal{H}_k)}{\operatorname{area}(S)},$$

and the zonal harmonics of a given degree can be obtained from a single one via rotations

$$Z_{g\xi}^{(k)} = g Z_\xi^{(k)} = Z_\xi^{(k)} \circ g^{-1}, \qquad \forall g \in \mathrm{O}(d). \tag{2.10}$$

There is an explicit formula for Z in terms of ultraspherical polynomials

$$\begin{aligned}
\frac{Z_\xi^{(k)}(x)}{\|Z_\xi^{(k)}\|_S} &= \frac{(d+2k-2)}{\sqrt{\operatorname{area}(S)}\sqrt{\dim(\mathcal{H}_k)}} \sum_{j=0}^{[k/2]} (-1)^j \frac{d(d+2)\cdots(d+2k-2j-4)}{2^j j!(k-2j)!} \langle x, \xi \rangle^{k-2j} |x|^{2j} \\
&= \sqrt{\frac{\dim(\mathcal{H}_k)}{\operatorname{area}(S)}} \frac{k!}{(d-2)_k} |x|^k C_k^{\frac{d-2}{2}} \Big(\frac{\langle x, \xi \rangle}{|x|} \Big) \qquad (d > 2) \\
&= \sqrt{\frac{\dim(\mathcal{H}_k)}{\operatorname{area}(S)}} \frac{k!}{(\frac{d-1}{2})_k} |x|^k P_k^{(\frac{d-3}{2}, \frac{d-3}{2})} \Big(\frac{\langle x, \xi \rangle}{|x|} \Big).
\end{aligned} \tag{2.11}$$

Here $(a)_n := a(a+1)\cdots(a+n-1)$ denotes the *Pochhammer symbol.*

2.5. Angular and radial parts

We will repeatedly use the fact that if a function f defined on B_R can be factored into an angular and radial part

$$f(x) = \Theta\big(\frac{x}{|x|}\big) \mathcal{R}(|x|),$$

then (by Fubini's theorem) it can be integrated

$$\int_{B_R} f(x)\, dx = \int_S \int_0^R \Theta(\omega) \mathcal{R}(r)\, r^{d-1} dr\, d\omega = \Big(\int_S \Theta(\omega)\, d\omega \Big) \Big(\int_0^R \mathcal{R}(r)\, r^{d-1} dr \Big).$$

2.6. The $\mathrm{O}(d)$–invariant subspaces of $\mathcal{P}_n$

Our construction is based on the decomposition of $\mathcal{P}_n$ into $\mathrm{O}(d)$–invariant subspaces. This can be deduced from the orthonormal basis given in [X05:Th. 4.7] by observing each polynomial in that basis belongs to some V_j. Since our description differs slightly (we use orthogonal polynomials with argument $|\cdot|^2$ instead of even orthogonal polynomials with argument $|\cdot|$), for completeness and motivation, we provide an indicative proof.

Recall that each homogeneous polynomial p of degree n can be written uniquely in the form

$$p(x) = \sum_{0\le j\le \frac{n}{2}} |x|^{2j} p_{n-2j}(x) = \sum_{j=0}^{[\frac{n}{2}]} |x|^{2j} p_{n-2j}(x), \tag{2.12}$$

where $p_{n-2j} \in \mathcal{H}_{n-2j}$ (see [SW71:Th.2.1]).

We use the *Jacobi polynomials* $P_n^{(\alpha,\beta)}$, which have the normalisation

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n n!} x^n + \text{lower order terms}, \tag{2.13}$$

and the *generalised Laguerre polynomials* L_n^α which are given by

$$L_n^\alpha(x) := \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}), \qquad \int_0^\infty (L_n^\alpha(x))^2\, x^\alpha e^{-x}\, dx = \frac{\Gamma(\alpha+n+1)}{n!}.$$

Lemma 2.14. *For $0 \le j \le \frac{n}{2}$, let $P_j = P_j^{n,d,w}$ be an orthogonal polynomial of degree j for the univariate weight*

$$t \mapsto t^{n-2j+\frac{d-2}{2}} w(\sqrt{t}) \tag{2.15}$$

on $[0,R^2)$. Then $\mathcal{P}_n$ can be written as an orthogonal direct sum of $\mathrm{O}(d)$–invariant subspaces

$$\mathcal{P}_n = \bigoplus_{0\le j\le\frac{n}{2}} V_j, \qquad V_j := \mathcal{H}_{n-2j} P_j(|\cdot|^2). \tag{2.16}$$

The inner product on V_j is given by

$$\langle h_1 P_j(|\cdot|^2), h_2 P_j(|\cdot|^2)\rangle = \langle h_1, h_2\rangle_S \|P_j\|_w^2, \qquad \forall h_1, h_2 \in \mathcal{H}_{n-2j}, \tag{2.17}$$

where

$$\|P_j\|_w^2 := \frac{1}{2}\int_0^\infty |P_j(t)|^2\, t^{n-2j+\frac{d-2}{2}} w(\sqrt{t})\, dt. \tag{2.18}$$

In particular, for the generalised Gegenbauer weight (2.2), we may take

$$\begin{aligned} P_j &= P_j^{(\alpha, n-2j+\frac{d-2}{2}+\beta)}(2(\cdot)-1), \\ \|P_j\|_w^2 &= \frac{1}{2}\frac{\Gamma(j+\alpha+1)\Gamma(n-j+\frac{d}{2}+\beta)}{(\alpha+n+\frac{d}{2}+\beta) j!\,\Gamma(\alpha+n-j+\frac{d}{2}+\beta)}, \end{aligned} \tag{2.19}$$

and for the generalised Hermite weight (2.3), we may take

$$P_j = L_j^{n-2j+\frac{d-2}{2}+\beta}, \qquad \|P_j\|_w^2 = \frac{1}{2}\frac{\Gamma(n-j+\frac{d}{2}+\beta)}{j!}. \tag{2.20}$$

Proof: First we show that $V_j \subset \mathcal{P}_n$. Let $h_{n-2j} \in \mathcal{H}_{n-2j}$ and P_j be a univariate polynomial of degree j. In view of (2.12), the polynomial $h_{n-2j}P_j(|\cdot|^2)$ is in $\mathcal{P}_n$ if and only if it is orthogonal to all polynomials of the form $Q_\ell(|\cdot|^2)p_{k-2\ell}$, where Q_ℓ is a univariate polynomial of degree ℓ and $p_{k-2\ell} \in \mathcal{H}_{k-2\ell}$, $0 \le k < n$, $0 \le \ell \le \frac{k}{2}$, i.e.,

$$\begin{aligned}&\langle h_{n-2j}P_j(|\cdot|^2), Q_\ell(|\cdot|^2)p_{k-2\ell}\rangle \\ &\quad = \Big(\int_S h_{n-2j}(\omega)p_{k-2\ell}(\omega)\,d\omega\Big)\Big(\int_0^R P_j(r^2)Q_\ell(r^2)r^{n-2j+k-2\ell+d-2}w(r)\,rdr\Big)\end{aligned} \tag{2.21}$$

is zero. The orthogonality of spherical harmonics of different degrees with respect to (2.7) implies the above inner product vanishes except for when $n-2j = k-2\ell$, in which case $\ell = j - \frac{1}{2}(n-k) < j$, and the change of variables $t = r^2$ allows the second factor to be written

$$\int_0^R P_j(r^2)Q_\ell(r^2)r^{n-2j+n-2j+d-2}w(r)\,rdr = \int_0^{R^2} P_j(t)Q_\ell(t)\,t^{n-2j+\frac{d-2}{2}}w(\sqrt{t})\,\frac{dt}{2}.$$

By our choice for P_j this integral is zero, and so we conclude that $h_{n-2j}P_j(|\cdot|^2) \in V_j$. A similar calculation shows that the V_j are orthogonal to each other.

Since space $\mathcal{H}_{n-2j}$ and the polynomial $P_j(|\cdot|^2)$ are $\mathrm{O}(d)$–invariant, so is the subspace $V_j := \mathcal{H}_{n-2j}P_j(|\cdot|^2)$.

For the generalised Gegenbauer weight, we have

$$t^{n-2j+\frac{d-2}{2}}w(\sqrt{t}) = t^{n-2j+\frac{d-2}{2}}(1-t)^\alpha t^\beta,$$

so that $P_j = P_j^{(\alpha, n-2j+\frac{d-2}{2}+\beta)}(2(\cdot)-1)$, and for the generalised Hermite weight, we have

$$t^{n-2j+\frac{d-2}{2}}w(\sqrt{t}) = t^{n-2j+\frac{d-2}{2}}t^\beta e^{-t},$$

so that $P_j = L_j^{n-2j+\frac{d-2}{2}+\beta}$. The calculation of (2.19) and (2.20) are straightforward. □

The dimension of the V_j is given by $\dim(V_j) = \dim(\mathcal{H}_{n-2j})$, and the formula (2.6).

3. Continuous tight frames for $\mathcal{P}_n$

The continuous tight frame decomposition given here can be viewed as an example of the continuous analogue of [VW05:Th. 6.18]. Let

$$\mathcal{P}_n = \bigoplus_{0 \le j \le \frac{n}{2}} V_j, \qquad \dim(V_j) = \dim(\mathcal{H}_{n-2j}) \tag{3.1}$$

be the decomposition $\mathcal{P}_n$ into $\mathrm{O}(d)$–invariant subspaces given by (2.16). The representation of $\mathrm{SO}(d)$ on the harmonic polynomials $\mathcal{H}_k$ is *absolutely irreducible* (see [FH87:Th. I.5]), i.e., if p is any nonzero polynomial in the complex vector space $\mathcal{H}_k$ then $\{gp\}_{g\in\mathrm{SO}(d)}$ spans $\mathcal{H}_k$. In view of (2.17), it follows that (3.1) is an orthogonal decomposition of $\mathcal{P}_n$ into absolutely irreducible $\mathrm{SO}(d)$–invariant subspaces V_j.

Further, none of the V_j are $\mathbb{C}\mathrm{SO}(d)$–isomorphic to each other (cf [FH87:Prop. I.10]), where we recall that $\mathrm{SO}(d)$–invariant subspaces V_j and V_k are $\mathbb{C}\mathrm{SO}(d)$–isomorphic if there is a $\mathbb{C}$-vector space isomorphism $S: V_j \to \mathcal{V}_k$ for which

$$gS(f) = S(gf), \qquad \forall f \in V_j, \quad \forall g \in \mathrm{SO}(d).$$

This is easily seen for $d \geq 3$ where (2.6) implies that the V_j have different dimensions, and for $d = 2$ from following the explicit description of V_j

$$V_j = \mathrm{span}\{(x,y) \mapsto \Re((a+ib)(x+iy)^{n-2j})P_j(x^2+y^2) : a,b \in \mathbb{R}\}.$$

Our main result uses Schur's Lemma (see, e.g., [JL93:9.1]) in the following form. If $S : V_j \to V_k$ is a $\mathbb{C}\mathrm{SO}(d)$–homomorphism (between absolutely irreducible $\mathrm{SO}(d)$–invariant subpaces), then either $S = 0$ or it is a $\mathbb{C}\mathrm{SO}(d)$–isomorphism of the form $Sf = cf$, $\forall f \in V_j$ where $c \in \mathbb{C}$is a fixed scalar.

Henceforth, let the orthogonal polynomials P_j for the weight (2.15) be normalised so that the coefficient of $(\cdot)^j$ is positive, i.e.,

$$\frac{P_j(|x|^2)}{\|P_j\|_w} = \sum_{k=0}^{j} p_k^j |x|^{2k} = p_j^j |x|^{2j} + \text{lower order terms}, \qquad p_j^j > 0. \tag{3.2}$$

Theorem 3.3. *Let $p \in \mathcal{P}_n$ be any unit norm polynomial of the form*

$$p = \sum_{0 \leq j \leq \frac{n}{2}} \sqrt{\frac{\dim(\mathcal{H}_{n-2j})}{\dim(\mathcal{P}_n)}} p_j, \qquad p_j \in V_j, \quad \|p_j\| = 1. \tag{3.4}$$

Then $\{gp\}_{g\in\mathrm{SO}(d)}$ is an isometric continuous tight frame for $\mathcal{P}_n$, i.e.,

$$f = \dim(\mathcal{P}_n) \int_{\mathrm{SO}(d)} \langle f, gp \rangle gp \, d\mu(g), \qquad \forall f \in \mathcal{P}_n, \tag{3.5}$$

where μ denotes the normalised Haar measure on $\mathrm{SO}(d)$, and these are all such $p \in \mathcal{P}_n$. Moreover, p can be chosen to be a zonal function, in which case

$$f = \frac{\dim(\mathcal{P}_n)}{\mathrm{area}(S)} \int_S \langle f, p_\xi \rangle p_\xi \, d\xi, \qquad \forall f \in \mathcal{P}_n. \tag{3.6}$$

There are a finite number of such $p = p_\xi$ with a given pole ξ. Amongst these, we call

$$p_\xi := \sqrt{\frac{\mathrm{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0 \leq j \leq \frac{n}{2}} Z_\xi^{(n-2j)} \frac{P_j(|\cdot|^2)}{\|P_j\|_w}. \tag{3.7}$$

the **canonical** *choice of* p*. This uniquely maximises the value* $p_\xi(r\xi)$*, where* $r > 0$ *is any number with*

$$r > \max\{t \geq 0 : t^2 \text{ is a root of } P_j, 0 \leq j \leq \frac{n}{2}\}.$$

Proof: Let $G := \mathrm{SO}(d)$. Choose $v_j \in V_j$, $v_k \in V_k$, and define $S : V_j \to V_k$ by

$$Sf = \int_G \langle f, gv_j\rangle gv_k \, d\mu(g), \qquad \forall f \in V_j.$$

Then S is a $\mathbb{C}G$–isomorphism, since by (2.4) it follows that S commutes with every $g \in G$

$$gS(f) = g\int_G \langle f, hv_j\rangle hv_k \, d\mu(h) = \int_G \langle gf, ghv_j\rangle ghv_k \, d\mu(h) = S(gf).$$

But none of the V_j are $\mathbb{C}G$–isomorphic, so Schur's Lemma gives

$$\int_G \langle f, gv_j\rangle gv_k \, d\mu(g) = 0, \qquad \forall f \in V_j, \quad j \neq k, \tag{3.8}$$

$$\int_G \langle f, gv_j\rangle gv_j \, d\mu(g) = cf, \qquad \forall f \in V_j, \tag{3.9}$$

where $c \in \mathbb{C}$ Now

$$\mathrm{trace}(\langle \cdot, gv_j\rangle gv_j) = \mathrm{trace}(gv_j(gv_j)^*) = \mathrm{trace}(gv_jv_j^*g^*) = \mathrm{trace}(v_j^*v_j) = \|v_j\|^2,$$

so taking the trace of (any matrix representing) the operators $V_j \to V_j$ defined by (3.9) gives

$$\begin{aligned} c\dim(V_j) &= \mathrm{trace}(c\,\mathrm{Id}_{V_j}) = \mathrm{trace}\int_G \langle \cdot, gv_j\rangle gv_j \, d\mu(g) = \int_G \mathrm{trace}(\langle \cdot, gv_j\rangle gv_j) \, d\mu(g) \\ &= \int_G \|v_j\|^2 \, d\mu(g) = \|v_j\|^2\mu(G) = \|v_j\|^2, \end{aligned}$$

and so (3.8) becomes

$$\int_G \langle f, gv_j\rangle gv_j \, d\mu(g) = \frac{\|v_j\|^2}{\dim(V_j)} f, \qquad \forall f \in V_j. \tag{3.10}$$

Let $p \in \mathcal{P}_n$ be a polynomial of the form

$$p = \sum_{0 \leq j \leq \frac{n}{2}} c_jv_j, \qquad v_j \in V_j, \ \|v_j\| = 1, \quad c_j \in \mathbb{C}$$

It suffices to prove (3.5) for $f \in V_j$, since both sides of (3.5) are linear functions of f. For $f \in V_j$, (3.8) and (3.10) give

$$\int_G \langle f, gp\rangle gp \, d\mu(g) = \sum_{0 \leq \ell \leq \frac{n}{2}} \sum_{0 \leq k \leq \frac{n}{2}} \overline{c_\ell}c_k \int_G \langle f, gv_\ell\rangle gv_k \, d\mu(g) = \frac{|c_j|^2}{\dim(V_j)} f.$$

This will be a scalar multiple λf of f and give a p of unit norm if and only if

$$\frac{|c_j|^2}{\dim(V_j)} = \lambda, \quad \|p\|^2 = \sum_j |c_j|^2 = 1 \quad \iff \quad \lambda = \frac{1}{\sum_j \dim(V_j)}, \quad |c_j| = \sqrt{\frac{\dim(V_j)}{\dim(\mathcal{P}_n)}}.$$

Thus all polynomials p for which (3.5) holds are given by (3.4).

All the choices for p which are zonal with pole ξ are obtained by taking the harmonic factor of the V_j component to be a scalar multiple of the zonal harmonic $Z_\xi^{(n-2j)}$, i.e.,

$$p_\xi = \sqrt{\frac{\text{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0 \le j \le \frac{n}{2}} s_j Z_\xi^{(n-2j)} \frac{P_j(|\cdot|^2)}{\|P_j\|_w}, \qquad s_j \in \{-1, 1\}.$$

The value of these p_ξ at $r\xi$ is

$$p_\xi(r\xi) = \sum_{0 \le j \le \frac{n}{2}} s_j c_j, \qquad c_j := \sqrt{\frac{\text{area}(S)}{\dim(\mathcal{P}_n)}} r^{n-2j} Z_\xi^{(n-2j)}(\xi) \frac{P_j(r^2)}{\|P_j\|_w} > 0.$$

This is clearly maximised by the choice $s_j = 1$, $\forall j$, which gives (3.7).

Finally, suppose p is zonal, so that p_ξ is zonal with pole $\xi \in S$. Let G_ξ be the subgroup of $\mathrm{SO}(d)$ which fixes ξ, i.e.,

$$G_\xi := \{g \in \mathrm{SO}(d) : g\xi = \xi\} \approx \mathrm{SO}(d-1).$$

with normalised Haar measure denoted by ν. Then by [SD80:Th. III.3.2] the integral of (3.5) can be computed

$$\begin{aligned} \int_{\mathrm{SO}(d)} \langle f, gp \rangle gp \, d\mu(g) &= \frac{1}{\text{area}(S)} \int_S \int_{\mathrm{SO}(d-1)} \langle f, gp \rangle gp \, d\nu(g) \, d\xi \\ &= \frac{1}{\text{area}(S)} \int_S \int_{G_\xi} \langle f, p_\xi \rangle p_\xi \, d\nu(g) \, d\xi = \frac{1}{\text{area}(S)} \int_S \langle f, p_\xi \rangle p_\xi \, d\xi, \end{aligned}$$

which gives (3.6). □

For $d > 2$ the decomposition (3.6) is preferable to (3.5), as the underlying manifold S has dimension $d-1$ (and is easily parametrised), whereas $\mathrm{SO}(d)$ has dimension $\frac{1}{2}d(d-1)$.

The canonical choice for $p = p_\xi$ depends continuously on the weight w. By (2.11), it has the structural form

$$p_\xi(x) = \sum_{0 \le k \le j \le \frac{n}{2}} c_{jk} |x|^{2k} \langle x, \xi \rangle^{n-2j}.$$

For the Legendre polynomials we will see that it reduces to a ridge function.

Example 1. Let $\mathcal{P}_4$ be the quartic Legendre polynomials on the unit disc. Write the summands of the canonical choice $p_\xi(x)$ as

$$v_j := \sqrt{\frac{\operatorname{area}(S)}{\dim(\mathcal{P}_n)}} Z_\xi^{(n-2j)}(x) \frac{P_j(|x|^2)}{\|P_j\|_w}.$$

Then by (2.11) and (2.19), one has

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{\pi}}\Big(2|x|^4 - 16|x|^2\langle x,\xi\rangle^2 + 16\langle x,\xi\rangle^4\Big), \\ v_1 &= \frac{1}{\sqrt{\pi}}\Big(-8|x|^4 + 16|x|^2\langle x,\xi\rangle^2 + 6|x|^2 - 12\langle x,\xi\rangle^2\Big), \\ v_2 &= \frac{1}{\sqrt{\pi}}\Big(6|x|^4 - 6|x|^2 + 1\Big), \end{aligned}$$

and so the canonical choice is

$$p_\xi(x) := v_0 + v_1 + v_2 = \frac{1}{\sqrt{\pi}}\Big(16\langle x,\xi\rangle^4 - 12\langle x,\xi\rangle^2 + 1\Big) = \frac{1}{\sqrt{\pi}} U_4(\langle x,\xi\rangle).$$

The formulas for the other zonal polynomials with pole ξ (up to to multiplication by ± 1) are given by $v_0 - v_1 + v_2$ (is a rotation of p_ξ by $\frac{\pi}{2}$), $q_\xi(x) := v_0 + v_1 - v_2$ and $v_0 - v_1 - v_2$ (is a rotation of q_ξ by $\frac{\pi}{2}$).

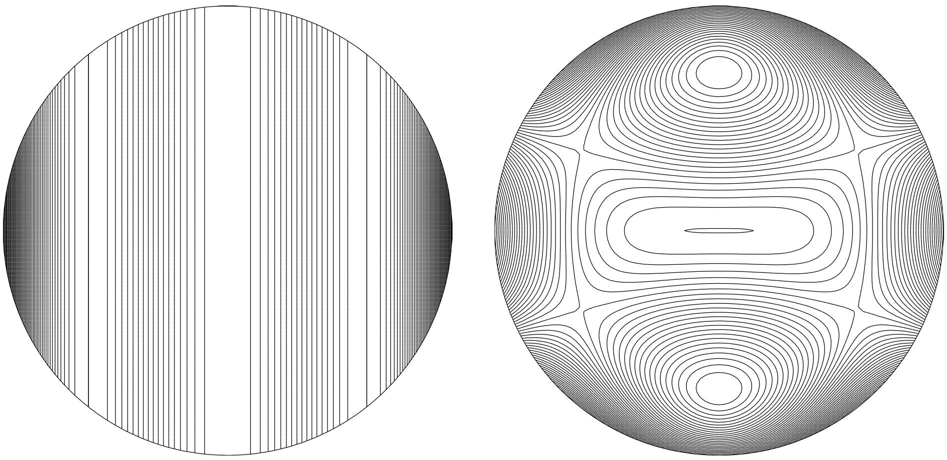

Fig 1. Contour plots of the quartic Legendre polynomials p_ξ and q_ξ from Ex. 1 for $\xi = (1,0)$. Clearly the canonical choice p_ξ is a ridge function.

The following result (cf [BS06]) allows us to determine when p is a ridge function.

Lemma 3.11. *A homogeneous ridge polynomial can be expressed in terms of the Zonal harmonics as follows*

$$\langle x,\xi\rangle^n = \frac{n!\operatorname{area}(S)}{2^n(\frac{d}{2})_n} \sum_{0\le j\le \frac{n}{2}} \frac{(n-j+\frac{d}{2})_j}{j!} |x|^{2j} Z_\xi^{(n-2j)}(x),$$

where $n \geq 1$ and $|\xi| = 1$.

For p to be a ridge function, each of its homogeneous terms must be ridge functions. In particular its leading term $p_\uparrow(x)$, which is given by

$$p_\uparrow(x) := \sqrt{\frac{\operatorname{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0\leq j\leq \frac{n}{2}} Z_\xi^{(n-2j)}(x) p_j^j |x|^{2j},$$

must be a scalar multiple of $\langle x, \xi\rangle^n$. In view of Lemma 3.11 and the fact the polynomials $\{|\cdot|^{2j} Z_\xi^{n-2j}\}_{0\leq j\leq \frac{n}{2}}$ are linearly independent, this is equivalent to the p_j^j of (3.2) satifying

$$p_j^j = c\frac{(n-j+\frac{d}{2})_j}{j!}, \qquad 0 \leq j \leq \frac{n}{2}, \tag{3.12}$$

for some constant $c > 0$.

For the Gegenbauer polynomials, (2.13) and (2.19) give

$$p_j^j = \sqrt{\frac{2(\alpha+n+\frac{d}{2})j!\Gamma(\alpha+n-j+\frac{d}{2})}{\Gamma(j+\alpha+1)\Gamma(n-j+\frac{d}{2})}} \frac{(\alpha+n-j+\frac{d}{2})_j}{j!}, \tag{3.13}$$

and so it follows the canonical choice for p is a ridge function if (and only if) $\alpha = 0$.

In this way, we recover the result of Petrushev [P99] that the decomposition (3.6) holds for the Legendre polynomials.

Corollary 3.14 (Legendre polynomials [P99]). *For the constant weight 1 on the unit ball, the canonical choice for p in Theorem 3.3 is the ridge polynomial given by*

$$p_\xi(x) = \frac{\sqrt{2n+d}}{\sqrt{\operatorname{area}(S)}\sqrt{\dim(\mathcal{P}_n)}} C_n^{d/2}(\langle x, \xi\rangle). \tag{3.15}$$

Proof: Let $p = p_\xi$ be the canonical choice (3.7) for $w = 1$ on $\mathbb{B}$. By (3.13)

$$p_j^j = \sqrt{2n+d}\frac{(n-j+\frac{d}{2})_j}{j!},$$

and so by Lemma 3.11 the leading term of p (term of highest degree of the decomposition of p into its homogeneous components) is given by

$$\begin{aligned} p_\uparrow(x) &= \sqrt{\frac{\operatorname{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0\leq j\leq \frac{n}{2}} Z_\xi^{(n-2j)}(x)\sqrt{2n+d}\frac{(n-j+\frac{d}{2})_j}{j!}|x|^{2j} \\ &= \sqrt{\frac{\operatorname{area}(S)}{\dim(\mathcal{P}_n)}}\sqrt{2n+d}\frac{1}{\operatorname{area}(S)}\frac{2^n(\frac{d}{2})_n}{n!}\langle x, \xi\rangle^n, \end{aligned}$$

which is a ridge function. It is well known (cf [DX01:Prop. 6.1.13]) that $C_n^{d/2}(\langle\cdot,\xi\rangle)$, $\xi\in S$ is a Legendre polynomial. Since orthogonal polynomials of degree n are uniquely determined by their leading terms, and the ultraspherical polynomials have the normalisation

$$C_n^{d/2}(\langle x,\xi\rangle)=\frac{(\frac{d}{2})_n 2^n}{n!}\langle x,\xi\rangle^n+\text{lower order terms}$$

we therefore conclude that

$$p_\xi(x)=\frac{\sqrt{2n+d}}{\sqrt{\operatorname{area}(S)}\sqrt{\dim(\mathcal{P}_n)}}\Big(\frac{2^n(\frac{d}{2})_n}{n!}\langle x,\xi\rangle^n+\cdots\Big)=\frac{\sqrt{2n+d}}{\sqrt{\operatorname{area}(S)}\sqrt{\dim(\mathcal{P}_n)}}C_n^{d/2}(\langle x,\xi\rangle).$$

□

Example 2. For the Legendre polynomials on the disc, (3.15) gives

$$p_\xi(x)=\frac{\sqrt{2n+2}}{\sqrt{2\pi}\sqrt{n+1}}C_n^1(\langle x,\xi\rangle)=\frac{1}{\sqrt{\pi}}U_n(\langle x,\xi\rangle),$$

and both (3.5) and (3.6) reduce to

$$f=\frac{n+1}{2\pi}\int_0^{2\pi}\langle f,R_\theta p_\xi\rangle R_\theta p_\xi\,d\theta,$$

which is the integral formula of (1.2).

For the Legendre polynomials on the unit ball in $\mathbb{R}^3$, (3.15) gives

$$p_\xi(x)=\frac{\sqrt{2n+3}}{\sqrt{4\pi}\sqrt{\binom{n+2}{2}}}C_n^{\frac{3}{2}}(\langle x,\xi\rangle).$$

The integral in (3.5) is over the manifold SO(3) of dimension 3, and the integral in (3.6) is over S which has dimension 2.

4. The reproducing kernel

The **reproducing kernel** for $\mathcal{P}_n$ is the unique function K_n which satisfies

$$f(x)=\langle K_n(x,\cdot),f\rangle=\int_{B_R}K_n(x,y)f(y)\,w(|y|)\,dy,\qquad\forall f\in\mathcal{P}_n.$$

From (3.6), we calculate

$$\begin{aligned}f(x)&=\frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)}\int_S\langle f,p_\xi\rangle p_\xi(x)\,d\xi=\frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)}\int_S\int_{B_R}f(y)p_\xi(y)w(|y|)\,dy\,p_\xi(x)\,d\xi\\&=\int_{B_R}\frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)}\int_S p_\xi(x)p_\xi(y)\,d\xi\,f(y)\,w(|y|)dy,\end{aligned}$$

and hence

$$K_n(x,y) = \frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)} \int_S p_\xi(x) p_\xi(y)\, d\xi. \tag{4.1}$$

In particular, for the Legendre polynomials (3.15) gives the formula of [X07:Prop. 3.1]

$$K_n(x,y) = \frac{2n+d}{(\operatorname{area}(S))^2} \int_S C_n^{d/2}(\langle x,\xi\rangle) C_n^{d/2}(\langle \xi,y\rangle)\, d\xi,$$

which plays a key role in the reconstruction of a function on the ball from its Radon projections. This can also be deduced from [P99], and plays an important role in approximation by neural networks. For the Gegenbauer weight ($\alpha > -\frac{1}{2}$) [X99] (see also [X01]) gives the following formula

$$K_n(x,y) = \frac{(2n+2\alpha+d)\Gamma(\alpha+\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})\operatorname{area}(S)\Gamma(\frac{d}{2})} \int_{-1}^{1} C_n^{\alpha+\frac{d}{2}}(\langle x,y\rangle + t\sqrt{1-|x|^2}\sqrt{1-|y|^2})(1-t^2)^{\alpha-\frac{1}{2}}\, dt.$$

We can simplify our formula for K_n by using the orthogonal decomposition of $\mathcal{P}_n$.

Theorem 4.2. *The reproducing kernel for $\mathcal{P}_n$ is given by the following formulas*

$$K_n(x,y) = \frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)} \int_S p_\xi(x) p_\xi(y)\, d\xi = \sum_{0\le j\le \frac{n}{2}} \frac{P_j(|x|^2)P_j(|y|^2)}{\|P_j\|_w^2} Z^{(n-2j)}(x,y), \tag{4.3}$$

where the polynomial $Z^{(k)}$ is given by

$$Z^{(k)}(x,y) := |x|^k |y|^k Z^{(k)}_{\frac{x}{|x|}}(\frac{y}{|y|}) = |x|^k |y|^k Z^{(k)}_{\frac{y}{|y|}}(\frac{x}{|x|}).$$

Proof: The first formula is (4.1). From the formula (2.11) one obtains

$$Z^{(k)}_\xi(x) = |x|^k Z^{(k)}_{\frac{x}{|x|}}(\xi), \qquad \xi \in S,\ x \in \mathbb{R}^d \setminus \{0\}. \tag{4.4}$$

Substitute (3.7) into (4.1), and use (4.4) and the orthogonality of zonal harmonics of different degrees to obtain

$$\begin{aligned}
K_n(x,y) &= \int_S \Big(\sum_{0\le j\le \frac{n}{2}} Z^{(n-2j)}_\xi(x) \frac{P_j(|x|^2)}{\|P_j\|_w} \Big)\Big(\sum_{0\le k\le \frac{n}{2}} Z^{(n-2k)}_\xi(y) \frac{P_k(|y|^2)}{\|P_k\|_w} \Big)\, d\xi \\
&= \int_S \Big(\sum_{0\le j\le \frac{n}{2}} |x|^{n-2j} Z^{(n-2j)}_{\frac{x}{|x|}}(\xi) \frac{P_j(|x|^2)}{\|P_j\|_w} \Big)\Big(\sum_{0\le k\le \frac{n}{2}} |y|^{n-2k} Z^{(n-2k)}_{\frac{y}{|y|}}(\xi) \frac{P_k(|y|^2)}{\|P_k\|_w} \Big)\, d\xi \\
&= \sum_{0\le j\le \frac{n}{2}} |x|^{n-2j} \frac{P_j(|x|^2)}{\|P_j\|_w} |y|^{n-2j} \frac{P_j(|y|^2)}{\|P_j\|_w} \int_S Z^{(n-2j)}_{\frac{x}{|x|}}(\xi) Z^{(n-2j)}_{\frac{y}{|y|}}(\xi)\, d\xi,
\end{aligned}$$

for x and y nonzero. By the reproducing kernel property (2.9) of zonal harmonics

$$\int_S Z^{(n-2j)}_{\frac{x}{|x|}}(\xi) Z^{(n-2j)}_{\frac{y}{|y|}}(\xi)\, d\xi = Z^{(n-2j)}_{\frac{x}{|x|}}(\frac{y}{|y|}) = Z^{(n-2j)}_{\frac{y}{|y|}}(\frac{x}{|x|}),$$

and so we obtain (4.3). Further, by (2.11) can expand $Z^{(k)}(x,y)$ as

$$Z^{(k)}(x,y) = \frac{(d+2k-2)}{\operatorname{area}(S)} \sum_{j=0}^{[k/2]} (-1)^j \frac{d(d+2)\cdots(d+2k-2j-4)}{2^j j!(k-2j)!} \langle x,y\rangle^{k-2j} |x|^{2j} |y|^{2j},$$

which is a polynomial. Hence (4.3) extends to the case when x or y is zero. $\square$

We observe that the polynomial $(x,y) \mapsto K_n(x,y)$ is of coordinate degree n in x and y, and depends only on $\langle x,y\rangle$, $|x|^2$ and $|y|^2$.

5. Finite tight frames for $\mathcal{P}_n$

For a fixed $x \in \mathbb{R}^d$ and $f \in \mathcal{P}_n$, (3.6) gives

$$f(x) = \frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)} \int_S \langle f, p_\xi\rangle p_\xi(x)\, d\xi.$$

The above integral of the polynomial $\xi \mapsto \langle f, p_\xi\rangle p_\xi(x)$ of degree $2n$ can be replaced by an appropriate quadrature rule (spherical design) to obtain a discrete form of (3.6).

Definition. *A finite subset V of S together with weights $c_\xi \in \mathbb{R}$, $\xi \in V$ is called a* **quadrature** (or **cubature**) **rule** *of* **degree** *k for the sphere if*

$$\frac{1}{\operatorname{area}(S)} \int_S f\, d\xi = \sum_{\xi\in V} c_\xi f(\xi), \qquad \forall f \in \Pi_k.$$

V is termed a **spherical k–design** *if $c_\xi = \frac{1}{|V|}$, $\forall \xi \in V$.*

Here $|V|$ denotes the cardinality of the set V. There is an extensive literature on cubature rules for the sphere, in the first instance see [St71] and [Se01] (equal weights).

Theorem 5.1 (Finite tight frame). *Let $V \subset S$ be a cubature rule of degree $2n$ for the sphere S with weights $(c_\xi)_{\xi\in V}$, and $p = p_\xi$ the canonical choice (3.7). Then we have*

$$f = \dim(\mathcal{P}_n) \sum_{\xi\in V} c_\xi \langle f, p_\xi\rangle p_\xi, \qquad \forall f \in \mathcal{P}_n, \tag{5.2}$$

which for equal weights reduces to

$$f = \frac{\dim(\mathcal{P}_n)}{|V|} \sum_{\xi\in V} \langle f, p_\xi\rangle p_\xi, \qquad \forall f \in \mathcal{P}_n. \tag{5.3}$$

Proof: Let $x \in \mathbb{R}^d$ and $f \in \mathcal{P}_n$. Then (3.6) and the quadrature rule of degree $2n$ give

$$f(x) = \frac{\dim(\mathcal{P}_n)}{\operatorname{area}(S)} \int_S \langle f, p_\xi\rangle p_\xi(x)\, d\xi = \dim(\mathcal{P}_n) \sum_{\xi\in V} c_\xi \langle f, p_\xi\rangle p_\xi(x),$$

which is (5.2). Set $c_\xi = \frac{1}{|V|}$ to obtain (5.3). □

This result uses cubature rules to obtain tight frames of orthogonal polynomials. Usually the opposite relationship is exploited: orthogonal polynomials play a pivotal role in obtaining cubature rules (cf [CMS01]).

Example 1. Consider $d = 2$. Let V_k be any set of k equally spaced points on the cirle S. These give an equal weight quadrature rule of degree $k-1$ for S. Hence for $k \geq 2n+1$ (5.3) holds for $V = V_k$. This also extends to when $k \geq n+1$ and k is odd by the following argument. In this case V_{2k} can be written as the disjoint union $V_k \cup R_\pi V_k$, and so using $R_\pi p_\xi = (-1)^n p_\xi$, we calculate

$$\begin{aligned} f &= \frac{\dim(\mathcal{P}_n)}{2k} \sum_{\xi \in V_{2k}} \langle f, p_\xi \rangle p_\xi = \frac{\dim(\mathcal{P}_n)}{2k} \sum_{\xi \in V_k} (\langle f, R_\pi p_\xi \rangle R_\pi p_\xi + \langle f, R_\pi p_\xi \rangle R_\pi p_\xi) \\ &= \frac{\dim(\mathcal{P}_n)}{2k} \sum_{\xi \in V_k} 2\langle f, R_\pi p_\xi \rangle R_\pi p_\xi = \frac{\dim(\mathcal{P}_n)}{k} \sum_{\xi \in V_k} \langle f, R_\pi p_\xi \rangle R_\pi p_\xi, \qquad \forall f \in \mathcal{P}_n. \end{aligned}$$

In a similar vein, we can obtain discrete versions of (3.5).

Definition. *A finite subgroup G of* $\mathrm{SO}(d)$ *is said to* **generate** *a spherical t–design if the set $V = \{g\eta\}_{g \in G}$ is a spherical t–design for some (and hence every) $\eta \in S$.*

In the literature such groups are said to be *t–homogeneous*, see, e.g., [SHC03] and [HP04].

Corollary 5.4. *If G is a finite subgroup of* $\mathrm{SO}(d)$ *which generates a spherical $2n$–design, and $p = p_\xi$ the canonical choice* (3.7)*. Then we have*

$$f = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{g \in G} \langle f, gp \rangle gp, , \qquad \forall f \in \mathcal{P}_n. \tag{5.5}$$

Proof: Let $V = \{g\xi\}_{g \in G}$ in (5.3), and use (2.5). □

Example 2. Let $d = 2$. Similarily to Example 1, (5.5) holds for $G = \langle R_{\frac{2\pi}{k}} \rangle \subset \mathrm{SO}(2)$, the cyclic group of rotations through multiples of $2\pi/k$ (of order k), where $n+1 \leq k \leq 2n$ and k odd, or $k > 2n$. In particular, this gives the finite tight frame decompositions of [W08] (generalised Gegenbauer polynomials) and [W07] (generalised Hermite polynomials).

A less tractable (but possibly weaker) condition which ensures (5.5) for the canonical choice of p is that the representation of G on $\mathcal{H}_{n-2j}$, $0 \leq j \leq \frac{n}{2}$ is absolutely irreducible. In this case a discrete version of Theorem 3.3 holds.

It also happens that the weakest condition on $\mathrm{SO}(d)$ that could be required for (5.5) to hold, that $\{gp\}_{g \in G}$ spans $\mathcal{P}_n$ for some $p \in \mathcal{P}_n$, does ensure such a decomposition (though not necessarily for p the canonical choice).

Let Π_n^{hom} denote the space of homogeneous polynomials of degree n.

Theorem 5.6. *Let G be a finite subgroup of $\mathrm{SO}(d)$ for which there exists a $q_\uparrow \in \Pi_n^{\mathrm{hom}}$ for which $\{gq_\uparrow\}_{g\in G}$ spans Π_n^{hom}. Then there exists a $p \in \mathcal{P}_n$ for which*

$$f = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{g\in G} \langle f, gp\rangle gp = \dim(\mathcal{P}_n) \int_{\mathrm{SO}(d)} \langle f, gp\rangle gp \, d\mu(g), \qquad \forall f \in \mathcal{P}_n.$$

Proof: Let q be the orthogonal projection of $q_\uparrow$ onto $\mathcal{P}_n$. Since $\{hq_\uparrow\}_{h\in G}$ spans Π_n^{hom}, and each polynomial in $\mathcal{P}_n$ is uniquely determined by its leading term, $\Phi := \{hq\}_{h\in G}$ spans $\mathcal{P}_n$. Following [W08:§2], let $S : \mathcal{P}_n \to \mathcal{P}_n$ be the frame operator for Φ, which is given by

$$Sf := \sum_{h\in G} \langle f, hq\rangle hq, \qquad \forall f \in \mathcal{P}_n,$$

and $p := S^{-\frac{1}{2}} q$, which generates the *canonical dual frame* to Φ, i.e.,

$$f = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{h\in G} \langle f, hp\rangle hp, \qquad \forall f \in \mathcal{P}_n.$$

This implies for all $g \in \mathrm{SO}(d)$ that

$$gf = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{h\in G} \langle gf, ghp\rangle ghp \quad \Longrightarrow \quad f = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{h\in G} \langle f, ghp\rangle ghp, \quad \forall f \in \mathcal{P}_n.$$

Integrate the last equality over the normalised Haar measure, which is right invariant, to obtain

$$\begin{aligned} f &= \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{h\in G} \int_{\mathrm{SO}(d)} \langle f, ghp\rangle ghp \, d\mu(g) = \frac{\dim(\mathcal{P}_n)}{|G|} \sum_{h\in G} \int_{\mathrm{SO}(d)} \langle f, gp\rangle gp \, d\mu(g) \\ &= \dim(\mathcal{P}_n) \int_{\mathrm{SO}(d)} \langle f, gp\rangle gp \, d\mu(g), \qquad \forall f \in \mathcal{P}_n. \end{aligned}$$

□

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