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A SIC-POVM with a nonabelian index group

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ABSTRACT

Over the last decade, there has been intensive work on the construction of G–covariant SIC-POVMs, i.e., d^2 equiangular lines in $\mathbb{C}^d$, for the abelian group $G = \mathbb{Z}_d \times \mathbb{Z}_d$. These equiangular tight frames for $\mathbb{C}^d$ with the maximal number of vectors have applications to quantum measurement theory and signal analysis. Here we present the first example of a SIC-POVM which is G–covariant for a *nonabelian* group G. It is in six dimensions, and is constructed from a nice error basis with index group

$$G = \mathbb{Z}_3 \times A_4 = \texttt{SmallGroup}(36, 11),$$

and canonical abstract error group $\texttt{SmallGroup}(216, 42)$.

This interdisciplinary area is heavy with terminology, so we begin with some motivating examples:

Three equiangular lines in $\mathbb{R}^2$

It is easy to construct three (**but not four**) equiangular lines in the plane.

Four equiangular lines in $\mathbb{C}^2$

Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are used to study spin in quantum mechanics.
The four vectors

$$v := \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{\frac{\pi}{4}i}\sqrt{3-\sqrt{3}} \end{pmatrix}, \qquad \sigma_1 v, \quad \sigma_2 v, \quad \sigma_3 v$$

give a set of four equiangular lines in $\mathbb{C}^2$.

Equiangular lines/vectors

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ (it makes a big difference).

We say that unit vectors (f_j) in $\mathbb{F}^d$ (or the lines they represent) are **equiangular** if

$$|\langle f_j, f_k \rangle| = C < 1, \qquad j \neq k.$$

Theorem. *Let (f_j) be n equiangular unit vectors in $\mathbb{F}^d$. Then the orthogonal projections*

$$P_j : f \mapsto \langle f, f_j \rangle f_j, \qquad j = 1, \ldots, n$$

are linearly independent, and hence

$$n \leq \begin{cases} \frac{1}{2} d(d+1), & \mathbb{F}{=}\mathbb{R}; \\ d^2, & \mathbb{F}{=}\mathbb{C} \end{cases}$$

with equality iff $\{P_j\}_{j=1}^n$ is a basis for the Hermitian matrices.

Proof: Recall $C < 1$. Observe

$$\operatorname{trace}(P_j P_k^*) = |\langle f_j, f_k \rangle|^2 = C^2, \qquad j \neq k,$$

so the Frobeneous norm of the linear combination $\sum_j c_j P_j$ is

$$\begin{aligned} \| \sum_j c_j P_j \|_F^2 &= \operatorname{trace}(\sum_j c_j P_j \sum_k \overline{c_k} P_k^*) \\ &= \sum_j \sum_k c_j \overline{c_k} \operatorname{trace}(P_j P_k^*) \\ &= \sum_j \sum_k c_j \overline{c_k} C^2 + \sum_j c_j \overline{c_j} (1 - C^2) \\ &= C^2 |\sum_j c_j|^2 + (1 - C^2) \sum_j |c_j|^2, \end{aligned}$$

which is zero only for the trivial linear combination. $\square$

Lines in $\mathbb{R}^d$

For $d = 2$, the maximal number of lines possible is

$$\frac{1}{2}d(d+1) = 3,$$

our set of three equiangular lines. This bound is known to be reached only for

$$d = 2, 3, 7, 23$$

which is a pretty small list (which hasn't changed for ages).

This is the motivating problem for algebraic graph theory.

Lines in $\mathbb{C}^d$

For $d = 2$, the maximal number of lines possible is

$$d^2 = 4,$$

the Pauli matrix example. This bound is known to be reached only for

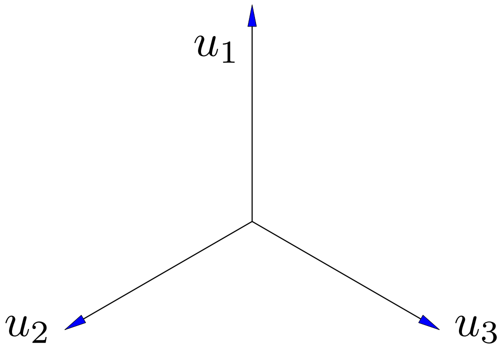
$$d = 2, 3, \ldots, 15, 19, 24, 35, 48,$$

which is changing often ($d = 16, 28$ were just announced).

The existence of d^2 equiangular lines in $\mathbb{C}^d$ is called **Zauner's conjecture**.

A question

Let u_1, u_2, u_3 be three equally spaced unit vectors in $\mathbb{R}^2$.



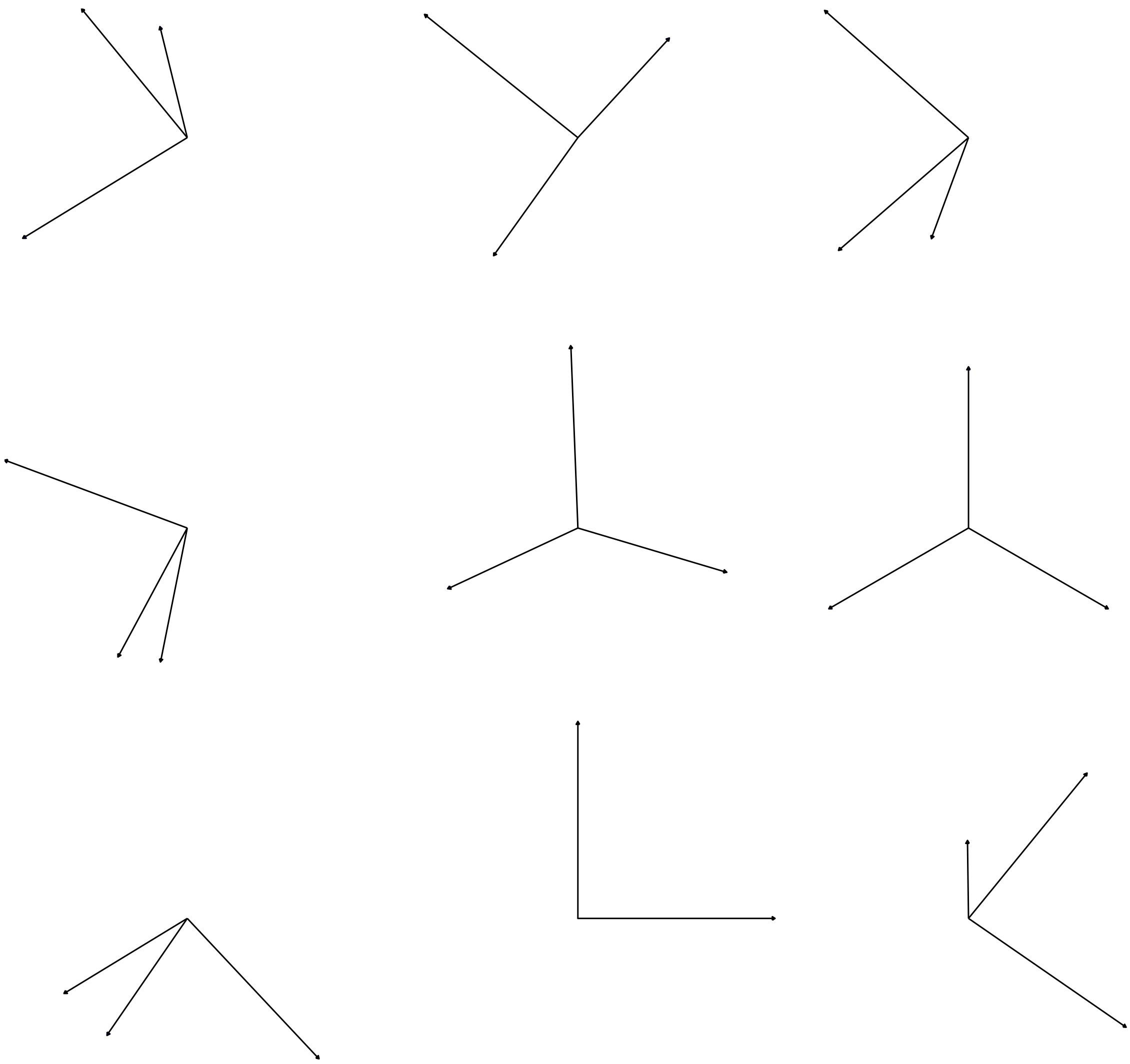
For a given nonzero vector $f \in \mathbb{R}^2$, what is the sum of its orthogonal projections onto these vectors?

(a) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = 0 \quad (\text{since } u_1 + u_2 = u_3 = 0).$$

(b) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = \frac{3}{2} f, \qquad \forall f \in \mathbb{R}^2.$$

Finite tight frames

The following sets of vectors $\{v_j\}_{j=1}^3$ form tight frames for $\mathbb{R}^2$



i.e., give decompositions of the form

$$f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{R}^2.$$

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).

Tight frames and SIC-POVMs

A sequence of n vectors $\Phi = (f_j)_{j=1}^n$ in the Hilbert space $\mathcal{H} = \mathbb{F}^d$ is a **tight frame** for $\mathcal{H}$ if $\exists C > 0$:

$$f = C \sum_{j=1}^{n} \langle f, f_j \rangle f_j, \qquad \forall f \in \mathcal{H}.$$

If a sequence of d^2 unit vectors (f_j) in $\mathbb{C}^d$ is equiangular, then they are **tight frame**. The orthogonal projections

$$P_j = f_j f_j^* : f \mapsto \langle f, f_j \rangle f_j$$

given by a equiangular tight frame are called a **SIC-POVM** (symmetric informationally complete positive operator valued measure).

Constructing SIC-POVMs

The key feature of the SIC-POVMs (equiangular lines) presented so far, is that they are the (projective) orbit of a group of unitary matrices.

Definition. *Let G be a group of order d^2. Then $d \times d$ unitary matrices $(E_g)_{g\in G}$ are a* **nice (unitary) error basis** *for $M_d(\mathbb{C})$ if*

- *E_1 is a scalar multiple of the identity I.*
- trace$(E_g) = 0$, $g \neq 1$, $g \in G$, *(i.e., they are an error operator basis.)*
- *$E_g E_h = w(g,h) E_{gh}$, $\forall g, h \in G$, where $w(g,h) \in \mathbb{C}$. and G is referred to as the* **index group**.

In the language of group theory, this is equivalent to

$\rho : g \mapsto E_g$ being a unitary irreducible faithful projective representation of G of degree $d = |G|^{\frac{1}{2}}$.

Proposition (Variational characterisation). *If $(E_g)_{g\in G}$ is a nice error basis for $M_d(\mathbb{C})$, then the "orbit" $(E_g v)_{g\in G}$ of $v \in \mathbb{C}^d$ is a SIC-POVM in $\mathbb{C}^d$ if and only if*

$$\sum_{g\in G} |\langle E_g v, v\rangle|^4 = 1 + \frac{d^2-1}{(d+1)^2} = \frac{2d}{d+1}, \qquad \|v\| = 1.$$

For a nice error basis, a vector $v \in \mathbb{C}^d$ giving a SIC-POVM $(E_g v)_{g\in G}$ is called a **generating** or **fiducial** vector.

Heisenberg frames

Let S and Ω be the **shift** and **modulation operators** on $\mathbb{C}^d$

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix}, \ \Omega = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \omega^2 & & & 0 \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & 0 & & & \omega^{d-1} \end{bmatrix}$$

where $\omega := e^{\frac{2\pi i}{d}}$ is a d–th roof of unity. Then

$$\mathbb{Z}_d \times \mathbb{Z}_d : (j,k) \mapsto S^j \Omega^k$$

is a nice error basis (which generalises the Pauli matrices).

A nonabelian index group

Define matrices (with 2×2 blocks) by

$$B := \begin{pmatrix} i\sigma_1 & & \\ & i\sigma_2 & \\ & & i\sigma_3 \end{pmatrix}, \qquad S^2 = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} I & & \\ & \omega I & \\ & & \omega^2 I \end{pmatrix}, \qquad \omega := e^{\frac{2\pi i}{3}}.$$

Proposition (Nice error basis). *The unitary 6×6 matrices B, S^2, A generate a group*

$$H := \langle B, S^2, A \rangle \subset SL_6(\mathbb{C}), \qquad |H| = 216 = 6^3$$

which gives a unitary faithful irreducible representation of

$$\texttt{SmallGroup(216,42)},$$

and has centre

$$Z(H) = \langle -\omega I \rangle, \qquad |Z(H)| = 6.$$

In particular, taking a matrix E_g from each coset of

$$G := \frac{H}{Z(H)} = \texttt{SmallGroup}(36, 11)$$

gives a nice error basis (E_g) for $M_6(\mathbb{C})$ with index group G.

A SIC-POVM with nonabelian index group

Theorem (Chien, Waldron 2012). *Let* $(E_g)_{g\in G}$ *be the previous nice error basis with the nonabelian index group* $G :=$ `SmallGroup`$(36, 11)$. *Then the unit vector*

$$v := \begin{pmatrix} \alpha r_0 \\ r_0 \frac{1-i}{\sqrt{2}} \\ r_1\xi_1 \\ \alpha r_1 \xi_1 \frac{1-i}{\sqrt{2}} \\ r_2\xi_2 \\ \alpha r_2 \xi_2 \frac{-1-i}{\sqrt{2}} \end{pmatrix},$$

where

$$\alpha := \frac{\sqrt{2}}{1+\sqrt{3}} = \frac{\sqrt{3-\sqrt{3}}}{\sqrt{3+\sqrt{3}}}, \qquad r_1 := \frac{1}{\sqrt{14}} \frac{\sqrt{7-\sqrt{21}}}{\sqrt{3-\sqrt{3}}},$$

$$r_0 := r_+, \quad r_2 := r_-, \qquad r_\pm := \frac{\sqrt{7+\sqrt{21}\pm\sqrt{14}\sqrt{\sqrt{21}-3}}}{2\sqrt{7}\sqrt{3-\sqrt{3}}},$$

$$\xi_1 = \tau^{50}\sqrt[3]{\beta - i\sqrt{1-\beta^2}}, \quad \xi_2 = \frac{\tau^{31}}{4}\Big(\sqrt{7}-\sqrt{3}-i\sqrt{6+2\sqrt{21}}\Big),$$

$$\beta := -\frac{1}{8}\sqrt{46-6\sqrt{21}+6\sqrt{6\sqrt{21}-18}}, \qquad \tau := e^{\frac{2\pi i}{72}}.$$

gives a G*–covariant SIC-POVM* $(E_g v)_{g\in G}$ *for* $\mathbb{C}^6$.

Brief comments on the result

- SIC-POVMs can easily be constructed numerically using the variational characterisation.
- In all cases, except $d = 3$, there are only finitely many fiducial vectors.
- It seems that $d = 6$ is the first dimension which has a SIC-POVM with a nonabelian index group.
- Finding analytic solutions is very difficult. Usually, the structure of a fiducial vector is guessed. In the Heisenberg case they are eigenvectors of an element of the normaliser of the group generated by $(E_g)_{g \in G}$ (Clifford group).
- Typical tricks include simplifying the equiangularity equations (both in number of variables, and algebraic degree).
- In our example, most of the work involved showing that an analytic solution (obtained using a computer algebra package) was indeed a solution.
- There appear to be many SIC-POVMs with nonabelian index groups for $d = 8$.
- It is hoped that our result might extend to an infinite class of SIC-POVMs.
- The orbit of a fiducial vector under the Heisenberg group gives a tight frame for finite signals which is localised in both space and time, and hence is discrete analogue of a wavelet.