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Errors of Linear Interpolation on a Triangle

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We attempt to determine best possible bounds for the errors in function value and derivative when a function is approximated by linear interpolation between values at the vertices of a triangle of known shape and size. We consider both L_2 and L_∞ bounds, in terms of L_2 and L_∞ measures of smoothness of the function.

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1 Introduction

The questions addressed here are at first sight simple ones, and date back at least to 1975 [8]: when a smooth function of two variables is approximated on a triangle by linear interpolation between its values at the vertices (as it might be in one element of a finite-element decomposition), how tight a bound can we find for the resulting errors in (a) the value of the function and (b) its gradient, in terms of the size of its second derivatives and the shape and size of the triangle?

It turns out that we can, if we define our terms carefully, write down a precise bound for (a) in very simple form. We have had less success, however, with (b). We are able to obtain a simple bound in this case, but it is far from the best possible.

2 The one-dimensional analogue

The corresponding questions in one dimension are easily answered.

Consider a line segment, say the interval $[0, a]$. Let u denote the smooth underlying function, U its linear interpolate

$$U(x) := xu(a) + (a - x)u(0),$$

and e the interpolation error

$$e := u - U,$$

so that $e(0) = e(a) = 0$.

We shall look here at L_∞ and L_2 error bounds:

- $\|e\|_\infty$ and $\|e'\|_\infty$ in terms of $\|u''\|_\infty$ and
- $\|e\|_2$ and $\|e'\|_2$ in terms of $\|u''\|_2$.

[Since U is linear, we have $\|e''\|_\infty = \|u''\|_\infty$ and $\|e''\|_2 = \|u''\|_2$.]

2.1 L_∞ bounds

- We can start from the Green's formula

$$e(x) = -\int_0^a g(x,y)e''(y)\mathrm{d}y, \tag{2.1}$$

and its derivative

$$e'(x) = -\int_0^a g_x(x,y)e''(y)\mathrm{d}y, \tag{2.2}$$

where $g(x,y)$ is the Green's function

$$g(x,y) = \begin{cases} \dfrac{(a-x)y}{a}, & y \le x, \\ \dfrac{x(a-y)}{a}, & y \ge x, \end{cases} \qquad \text{with} \quad g_x(x,y) = \begin{cases} -\dfrac{y}{a}, & y \le x, \\ \dfrac{(a-y)}{a}, & y \ge x. \end{cases}$$

- It follows at once from (2.1) that

$$|e(x)| \le \|e''\|_\infty \int_0^a |g(x,y)|\,\mathrm{d}y$$
$$= \|u''\|_\infty \frac{x(a-x)}{2} \tag{2.3}$$

so that

$$\|e\|_\infty \le \|u''\|_\infty \frac{a^2}{8} \tag{2.4}$$

(the bound on $|e(x)|$ attaining its maximum where $x = \frac{1}{2}a$).

- It follows likewise from (2.2) that

$$|e'(x)| \le \|e''\|_\infty \int_0^a |g_x(x,y)|\,\mathrm{d}y$$
$$= \|u''\|_\infty \frac{a^2 - 2x(a-x)}{2a} \tag{2.5}$$

so that

$$\|e'\|_\infty \le \|u''\|_\infty \frac{a}{2} \tag{2.6}$$

(the bound on $|e'(x)|$ attaining its maximum where $x = 0$ or $x = a$).

2.1.1 Related elementary inequalities

For possible future reference, we display here the corresponding bounds on functions not vanishing at both ends of the interval.

- If $f(0) = 0$ and $f'(a) = 0$ then we have the Green's formula

$$f(x) = -\int_0^x y f''(y)\mathrm{d}y - \int_x^a x f''(y)\mathrm{d}y.$$

It follows that

$$|f(x)| \le \|f''\|_\infty \frac{x(2a-x)}{2}, \tag{2.7}$$
$$\|f\|_\infty \le \|f''\|_\infty \frac{a^2}{2}. \tag{2.8}$$

- If $f(0) \ne 0$, and/or $f(a) \ne 0$, then it follows from (2.3) that

$$|f(x)| \le \frac{(a-x)\,|f(0)| + x\,|f(a)|}{a} + \|f''\|_\infty \frac{x(a-x)}{2} \tag{2.9}$$

and from (2.6) that

$$\|f'\|_\infty \le \frac{|f(a) - f(0)|}{a} + \|f''\|_\infty \frac{a}{2}. \tag{2.10}$$

2.2 L_2 bounds

2.2.1 Using Schwarz's inequality

- If we apply the Schwarz inequality to the Green's formula (2.1), we get

$$|e(x)| \le \|e''\|_2 \sqrt{\int_0^a g(x,y)^2 \mathrm{d}y}$$
$$= \|u''\|_2 \frac{x(a-x)}{\sqrt{3a}} \tag{2.11}$$

so that (squaring (2.11) and integrating)

$$\|e\|_2 \le \|u''\|_2 \frac{a^2}{3\sqrt{10}}. \tag{2.12}$$

- Treating formula (2.2) in the same way, we get

$$|e'(x)| \le \|e''\|_2 \sqrt{\int_0^a g_x(x,y)^2 \mathrm{d}y}$$
$$= \|u''\|_2 \sqrt{\frac{a^2 - 3x(a-x)}{3a}} \tag{2.13}$$

so that

$$\|e'\|_2 \le \|u''\|_2 \frac{a}{\sqrt{6}}. \tag{2.14}$$

2.2.2 Using the calculus of variations

A more direct approach is to solve the variational problems

$$\max_e! \, \|e\|_2 : e \in \mathcal{H}^2_{(0)}[0,a], \|e''\|_2 \le 1 \tag{2.15}$$

and

$$\max_e! \, \|e'\|_2 : e \in \mathcal{H}^2_{(0)}[0,a], \|e''\|_2 \le 1, \tag{2.16}$$

$\mathcal{H}^2_{(0)}[0,a]$ denoting those functions in $\mathcal{H}^2[0,a]$ that vanish at 0 and a.

- Using the calculus of variations with a Lagrange multiplier (here taken as $(-\lambda^{-4})$ merely to give a nice-looking result), we find that the solution e of the first variational problem (2.15) must satisfy

$$\exists \lambda : \int_0^a e(y)v(y)\mathrm{d}y - \lambda^{-4} \int_0^a e''(y)v''(y)\mathrm{d}y = 0, \; \forall v \in \mathcal{H}^2_{(0)}[0,a]. \tag{2.17}$$

  Integrating the second term of (2.17) twice by parts, we find that this requires that

$$e^{(4)}(y) = \lambda^4 e(y), \; e(0) = e(a) = e''(0) = e''(a) = 0. \tag{2.18}$$

The non-trivial solutions of (2.18) have the form $e(y) = C \sin \lambda y$ where $\lambda > 0$ is a real integer multiple of π/a. For these solutions we have $\|e\|_2 \,/\, \|e''\|_2 = a^2/n^2\pi^2$, so that the upper bound on the function error, found by taking $n = 1$, is expressed by

$$\|e\|_2 \leq \|u''\|_2 \frac{a^2}{\pi^2}. \tag{2.19}$$

- The solution e of the second variational problem (2.16) similarly satisfies

$$\exists \lambda : \int_0^a e'(y)v'(y)\mathrm{d}y - \lambda^{-2} \int_0^a e''(y)v''(y)\mathrm{d}y = 0, \;\; \forall v \in \mathcal{H}^2_{(0)}[0,a]. \tag{2.20}$$

  Integrating the first term of (2.20) once by parts and the second term twice, we find that

$$e^{(4)}(y) + \lambda^2 e''(y), \;\; e(0) = e(a) = e''(0) = e''(a) = 0. \tag{2.21}$$

  The non-trivial solutions of (2.21) again have the form $e(y) = C \sin \lambda y$, for the same values of λ as before, and we have $\|e'\|_2 \,/\, \|e''\|_2 = |a/n\pi|$, so that the upper bound on the gradient error, again found by taking $n = 1$, is expressed by

$$\|e'\|_2 \leq \|u''\|_2 \frac{a}{\pi}. \tag{2.22}$$

The variational bounds (2.19) and (2.22) (which are the best possible) are each slightly stronger than the corresponding Schwarz-inequality bounds (2.12) and (2.13).

3 The problem in two dimensions

We now try to go as far as we can in extending these results to a general triangle in two dimensions.

- Let Δ denote the triangle ABC, having sides $BC = a$, $CA = b$ and $AB = c$, and angles α at A, β at B and γ at C. Let $|\Delta|$ denote its area

$$|\Delta| := \iint_\Delta \mathrm{d}x \; \mathrm{d}y.$$

  Let O denote the centre of the circumscribed circle and R its radius:

$$\begin{aligned} R &= \frac{abc}{\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)}} \\ &= \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}} \\ &= \frac{abc}{4\,|\Delta|} = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} \\ &\quad [\geq \tfrac{1}{2}\max\{a,b,c\}]. \end{aligned} \tag{3.1}$$

3.1 L_2 bounds

3.1.1 Using the calculus of variations

The variational formulation extends directly to the triangle without much difficulty, provided that we are satisfied with numerical bounds.

- Let u denote the underlying function, let U denote the linear interpolating function (of the form $U(x,y) = lx + my + n$ with coefficients l, m, n chosen so that $U(x,y) = u(x,y)$ at each vertex of Δ), and let e denote the interpolation error $e := u - U$. Say that we measure the function error by

$$\|e\|_2 := \sqrt{\iint_\Delta e(x,y)^2 \, dx \, dy} \tag{3.2}$$

and the gradient error by

$$\|\nabla e\|_2 := \sqrt{\iint_\Delta e_x^2 + e_y^2 \, dx \, dy}, \tag{3.3}$$

while we measure the second derivative by

$$|e|_2 = |u|_2 := \sqrt{\iint_\Delta u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \, dx \, dy}. \tag{3.4}$$

(Note that each of these measures (3.2), (3.3) and (3.4) is invariant under rotation of the axes, (3.4) being similar to the seminorm that is minimised in the definition of thin-plate splines.)

The error bounds are then

$$\|e\|_2 \le M_2 \, |u|_2 , \quad \|\nabla e\|_2 \le M_2' \, |u|_2 , \tag{3.5}$$

where

$$M_2 := \sup_{e \in \mathcal{H}^2_{(0)}(\Delta)} \frac{\|e\|_2}{|e|_2}, \quad M_2' := \sup_{e \in \mathcal{H}^2_{(0)}(\Delta)} \frac{\|\nabla e\|_2}{|e|_2}, \tag{3.6}$$

$\mathcal{H}^2_{(0)}(\Delta)$ denoting the space of those functions in $\mathcal{H}^2(\Delta)$ that vanish at the vertices of Δ.

- The extrema (3.6) may be computed by solving the partial differential equations

$$\nabla^4 e(x,y) = \lambda^4 e(x,y), \ (x,y) \in \Delta,$$
$$e_{nn} = 0, \quad (2\nabla^2 e)_n = e_{nnn}, \ (x,y) \in \partial\Delta,$$
$$\nabla^4 e(x,y) + \lambda^2 \nabla^2 e(x,y) = 0, \ (x,y) \in \Delta,$$
$$e_{nn} = 0, \quad (2\nabla^2 e + \lambda^2 e)_n = e_{nnn}, \ (x,y) \in \partial\Delta,$$

with $e = 0$ at the vertices of Δ (e_n denoting the normal derivative on the boundary).

It is, however, more natural to attack the variational problem directly. Let $\{\phi_j\}$ denote a set of basis functions, all vanishing at the three vertices of Δ. We assemble symmetric matrices Φ_0, Φ_1 and Φ_2 with elements

$$(\Phi_0)_{ij} = \iint_\Delta \phi_i \phi_j \, dx \, dy,$$
$$(\Phi_1)_{ij} = \iint_\Delta \phi_{ix}\phi_{jx} + \phi_{iy}\phi_{jy} \, dx \, dy,$$
$$(\Phi_2)_{ij} = \iint_\Delta \phi_{ixx}\phi_{jxx} + 2\phi_{ixy}\phi_{jxy} + \phi_{iyy}\phi_{jyy} \, dx \, dy,$$

so that $\|e\|_2^2 = z^T\Phi_0 z$, $\|\nabla e\|_2^2 = z^T\Phi_1 z$ and $|e|_2^2 = z^T\Phi_2 z$ when $e = \sum_j z_j\phi_j$. We then solve the generalised eigenvalue problems

$$\Phi_2 z_0 = \lambda_0^4 \Phi_0 z_0, \quad \Phi_2 z_1 = \lambda_1^2 \Phi_1 z_1. \tag{3.7}$$

Then M_2 and M_2' are derived from the smallest eigenvalues λ_0 and λ_1 as

$$M_2 = \lambda_0^{-2}, \quad M_2' = \lambda_1^{-1}. \tag{3.8}$$

Table 1: Computed L_2 bounds

b	c	$\alpha(°)$	M_2	M_2'	b	c	$\alpha(°)$	M_2	M_2'
1	1	30	.103	.315	1	2	30	.296	1.08
1	1	60	.117	.318	1	2	60	.364	.684
1	1	90	.167	.489	1	2	90	.469	.762
1	1	120	.223	.846	1	2	120	.576	1.28
1	1	150	.262	1.85	1	2	150	.653	2.78

For various triangles defined by the two sides b and c and the included angle α, we have computed the estimates shown in Table 1. This was done using bivariate Bézier polynomial bases and also by using finite-element partitions of Δ into Clough–Tocher C^1 cubic macro-elements [2] and Powell–Sabin C^1 quadratic macro-elements [11]. The three methods gave indistinguishable results, which converged well as the order of the polynomial was raised or the finite-element partition was refined. The value of M_2' for $b = c = 1$, $\alpha = 90°$ agrees with the value computed by Siganevich [13], using a similar finite-element scheme.

3.1.2 Using thin-plate spline theory

The difference between (3.4) and the seminorm minimised by thin-plate splines [6, 9] lies solely in the region of integration. Suppose that we had instead measured the second derivative by the (larger) seminorm, defined by integration over the whole plane,

$$|e|_2 := |u|_2 := \sqrt{\iint_{\mathbf{R}^2} u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \, dx \, dy}, \tag{3.9}$$

associated with the semi-inner product

$$\langle u, v \rangle := \iint_{\mathbf{R}^2} u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy} \, dx \, dy. \tag{3.10}$$

Then the linear functional L_P, defined by

$$L_P(u) := u(P) - \lambda u(A) - \mu u(B) - \nu u(C) \quad (P = \lambda A + \mu B + \nu C,\ \lambda + \mu + \nu = 1),$$

is bounded with respect to $|\cdot|_2$, and has a representing function

$$l_P(Q) := \frac{1}{8\pi}\left(PQ^2 \log PQ - \lambda AQ^2 \log AQ - \mu BQ^2 \log BQ - \nu CQ^2 \log CQ\right) \tag{3.11}$$

such that

$$L_P(u) = \langle u, l_P \rangle \ \forall u$$

If $e(A) = e(B) = e(C) = 0$, then, we have

$$|e(P)| = |L_P(e)| = |\langle e, l_P \rangle| \leq |e|_2 \, |l_P|_2 , \tag{3.12}$$

and

$$\begin{aligned} |l_P|_2^2 &= \langle l_P, l_P \rangle = L_P(l_P) \\ &= l_P(P) - \lambda l_P(A) - \mu l_P(B) - \nu l_P(C) \\ &= \frac{1}{8\pi}\Lambda^T \mathrm{M} \Lambda, \end{aligned} \tag{3.13}$$

where

$$\Lambda := \begin{pmatrix} 1 \\ -\lambda \\ -\mu \\ -\nu \end{pmatrix}, \quad \mathrm{M} := \begin{pmatrix} 0 & AP^2 \log AP & BP^2 \log BP & CP^2 \log CP \\ AP^2 \log AP & 0 & AB^2 \log AB & AC^2 \log AC \\ BP^2 \log BP & AB^2 \log AB & 0 & BC^2 \log BC \\ CP^2 \log CP & AC^2 \log AC & BC^2 \log BC & 0 \end{pmatrix}.$$

Thus, defining $M_2 := \sup_e \|e\|_2 / |e|_2$, where $\|e\|_2$ is defined by (3.2) but $|e|_2$ by (3.9), we have

$$M_2^2 \leq \iint_\Delta |l_P|_2^2 \, dx \, dy$$

$$= \frac{1}{8\pi}\iint_\Delta \Lambda^T \mathrm{M}\Lambda \; \mathrm{d}x \; \mathrm{d}y$$
$$= -\frac{1}{4\pi}\iint_\Delta (\lambda AP^2 \log AP + \mu BP^2 \log BP + \nu CP^2 \log CP) \; \mathrm{d}x \; \mathrm{d}y +$$
$$+ \frac{|\Delta|}{96\pi}(a^2 \log a + b^2 \log b + c^2 \log c). \tag{3.14}$$

The integral in (3.14) may be evaluated explicitly term by term. For instance, we have

$$\iint_\Delta \lambda AP^2 \log AP \; \mathrm{d}x \; \mathrm{d}y =$$
$$= \frac{|\Delta|}{3600a^4}\{60(b^3 - 3b^2 c\cos\alpha + 3bc^2 - c^3(3 - 2\cos^2\alpha)\cos\alpha)b^3 \log b$$
$$+ 60(c^3 - 3bc^2\cos\alpha + 3b^2c - b^3(3 - 2\cos^2\alpha)\cos\alpha)c^3 \log c$$
$$+ 120b^3c^3\alpha\sin^3\alpha - 60a^2b^2c^2\sin^2\alpha$$
$$- 47(b^6 - 3b^5c\cos\alpha + 4b^3c^3\cos^3\alpha - 3bc^5\cos\alpha + c^6$$
$$+ 3a^2b^2c^2)\}.$$

From (3.14) we have computed the bounds in Table 2. (We cannot use this method to compute bounds for M_2', since the linear functionals L_x and L_y, such that

$$L_x(u) := u_x(P) - \lambda_x u(A) - \mu_x u(B) - \nu_x u(C),$$

are not bounded with respect to $|\cdot|_2$.)

We notice that these bounds are well below those in Table 1. This is to be expected, since the seminorm (3.9) is much stronger than the seminorm (3.4) of Section 3.1.1. We might have been able to come closer by applying this theory in the context of (3.4) — however, we are unable to write down the corresponding representing function $l_P(Q)$ to replace (3.11).

Table 2: Thin-plate-spline estimates for L_2 bounds

b	c	$\alpha(°)$	M_2	b	c	$\alpha(°)$	M_2
1	1	30	.0453	1	2	30	.1067
1	1	60	.0681	1	2	60	.1579
1	1	90	.0844	1	2	90	.1902
1	1	120	.0870	1	2	120	.1926
1	1	150	.0698	1	2	150	.1535

3.2 L_∞ bounds

- L_2 bounds have the practical disadvantage that they indicate only an average error, and can prove misleading if the error has small regions where it takes large values. This is particularly possible in regard to the error in the gradient, which tends to be worst in the neighbourhood of the vertices of Δ.

For the remainder of this work, we shall consider the estimation of M_∞ and M'_∞, defined in terms of L_∞ norms by:

$$M_\infty := \sup_e \frac{\|e\|_\infty}{|e|_\infty}, \quad M'_\infty := \sup_e \frac{\|\nabla e\|_\infty}{|e|_\infty}, \tag{3.15}$$

where

$$\|e\|_\infty := \sup_\Delta |e(x,y)|, \quad \|\nabla e\|_\infty := \sup_\Delta \sqrt{e_x^2 + e_y^2} \tag{3.16}$$

and

$$|e|_\infty = |u|_\infty := \tfrac{1}{2} \sup_{(x,y)\in\Delta} \left(|u_{xx} + u_{yy}| + \sqrt{(u_{xx} - u_{yy})^2 + 4u_{xy}^2}\right) \tag{3.17}$$

(so that we still have measures that are invariant under rotation of the axes — note that (3.17) is the upper bound of the second derivative at any point of Δ and in any direction, and is less than $\sup_\Delta \sqrt{u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2}$).

3.2.1 Underestimates, using trial functions

We may get underestimates for the L_∞ bounds by substituting any trial function for e. As far as M_∞ is concerned, this is close to the approach used by D'Azevedo and Simpson [4] in seeking an optimal triangulation. They have gone on in like vein, in [5], to seek an optimal triangulation for approximating the gradient. A fundamental difference between their work and ours is that they regard the function as fixed and the shape of triangle as variable — we fix the shape of the triangle and allow the function to vary within certain smoothness constraints.

- The shape of the maximizing functions in Section 3.1.1 suggests that we take as a first attempt the function

  $$e(x,y) = \tfrac{1}{2}(R^2 - (x - x_0)^2 - (y - y_0)^2) \tag{3.18}$$

  where (x_0, y_0) are the coordinates of O. Then $|e|_\infty = 1$.

  The extreme modulus of e is attained at O if $O \in \Delta$ (if Δ is acute-angled, in other words), or otherwise at the midpoint of the longest side, the extreme values being $\frac{1}{2}R^2$ or $\frac{1}{8}\max\{a^2, b^2, c^2\}$ respectively. The extreme magnitude of $|\nabla e|$ is attained at each vertex, and is simply R.

  This yields the lower bounds for M_∞ and M'_∞ shown in Table 3.

Table 3: First underestimates for L_∞ bounds

b	c	$\alpha(^\circ)$	R	M_∞	M'_∞	b	c	$\alpha(^\circ)$	R	M_∞	M'_∞
1	1	30	.518	.134	.518	1	2	30	1.239	.5	1.239
1	1	60	.577	.1667	.577	1	2	60	1.0	.5	1.0
1	1	90	.707	.25	.707	1	2	90	1.118	.625	1.118
1	1	120	1.0	.375	1.0	1	2	120	1.528	.875	1.528
1	1	150	1.932	.4665	1.932	1	2	150	2.909	1.058	2.909

- We shall show presently that the lower bounds for M_∞ in Table 3 are in fact exact values.

  However, it is not difficult to obtain better bounds for M'_∞. By numerically maximizing vertex gradients over the class of all quadratic trial functions with $|e|_\infty = 1$, vanishing at the vertices, we have obtained for these triangles the tighter lower bounds shown in Table 4.

Table 4: Improved underestimates for L_∞ bounds

b	c	$\alpha(^\circ)$	M'_∞	b	c	$\alpha(^\circ)$	M'_∞
1	1	30	.966	1	2	30	1.852
1	1	60	.866	1	2	60	1.823
1	1	90	1.144	1	2	90	2.080
1	1	120	1.5	1	2	120	2.520
1	1	150	2.332	1	2	150	3.687

3.2.2 Overestimates, using the one-dimensional results of Section 2.1

Suppose from now on (for the sake of simplicity) that

$$|e|_\infty = |u|_\infty = 1. \tag{3.19}$$

Then we can prove not only that

$$\left|\mathrm{D}_\theta^2 e\right| \le 1, \ \forall\theta \tag{3.20}$$

but also, more generally, that

$$|\mathrm{D}_\theta(\mathrm{D}_\phi e)| \le 1, \ \forall\theta, \phi, \tag{3.21}$$

where D_θ denotes differentiation with respect to distance along any straight line segment making an angle of θ with the x-axis.

Best possible bounds on the function error are quite easily found.

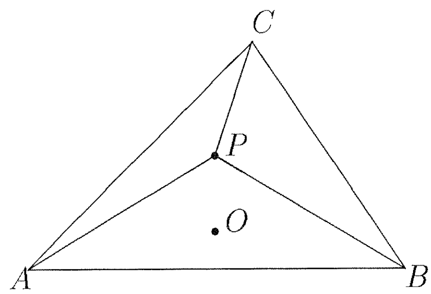

Figure 1:

- If a point P at which $|e(P)| = \|e\|_\infty$ is an interior point of Δ, as in Fig. 1, then $\nabla e(P) = 0$. Since $|e''| \le 1$ along the line segment AP, from (3.20), with $e'(P) = 0$ and $e(A) = 0$, it follows from (2.7) that $|e(P)| \le \frac{1}{2}AP^2$. Hence, considering all three lines AP, BP and CP, we deduce that

$$\|e\|_\infty \le \sup_{P\in\Delta} \tfrac{1}{2}\min\{AP^2, BP^2, CP^2\} \le \begin{cases} \frac{1}{2}R^2, & O \in \Delta, \\ \frac{1}{8}\max\{a^2, b^2, c^2\}, & \text{otherwise.} \end{cases} \tag{3.22}$$

[The last inequality in (3.22) could be strengthened, but is strong enough for our purposes.]

If P is on the boundary of Δ, on the other hand, then we can show by applying (2.4) to the edge on which P lies that

$$\|e\|_\infty \le \tfrac{1}{8}\max\{a^2, b^2, c^2\} \le \tfrac{1}{2}R^2. \tag{3.23}$$

Hence

$$M_\infty \le \begin{cases} \frac{1}{2}R^2, & \Delta \text{ acute-angled,} \\ \frac{1}{8}\max\{a^2, b^2, c^2\}, & \text{otherwise.} \end{cases} \tag{3.24}$$

These bounds (3.24) coincide precisely with the extreme values that were attained in Section 3.2.1 and shown in Table 3, and are therefore best possible.

Finding best possible bounds on the gradient error is quite another matter. Here we describe two attempts.

- First we examine the derivative at A in a direction lying within the angle CAB. In Fig. 2, let D be the point

$$D = \mu B + \nu C$$

(in an obvious geometric notation) where $\mu + \nu = 1$. Then, using (2.3) on the side BC we find that

$$|e(D)| \le \tfrac{1}{2}\mu\nu a^2. \tag{3.25}$$

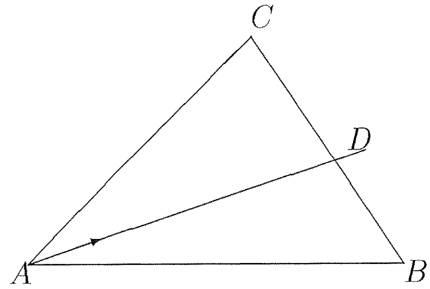

Figure 2:

Applying (2.6) now to the line segment AD, with e'_{AD} denoting differentiation in the direction of AD, we get

$$\begin{aligned}|e'(A)_{AD}| &\leq \frac{|e(D)|}{AD} + \tfrac{1}{2}AD \\ &\leq \tfrac{1}{2}\frac{\mu\nu a^2 + AD^2}{AD} \\ &= \tfrac{1}{2}\frac{\nu b^2 + \mu c^2}{\sqrt{\nu b^2 + \mu c^2 - \mu\nu a^2}}. \end{aligned} \tag{3.26}$$

Maximizing (3.26) over (μ, ν), subject to $\mu \geq 0$, $\nu \geq 0$ and $\mu + \nu = 1$ gives, after a little manipulation,

$$|e'(A)_{AD}| \leq \begin{cases} R, & \beta,\gamma \text{ both acute,} \\ \tfrac{1}{2}\max\{b, c\} & \text{otherwise.} \end{cases} \tag{3.27}$$

In the case where β and γ are both acute, the maximum is attained when the angle $BAD = \frac{1}{2}\pi - \gamma$ and $DAC = \frac{1}{2}\pi - \beta$ — otherwise it is of course attained when D coincides with B or C.

Now, if $e'(B)_{BE}$ is a derivative at B in a direction lying within the angle ABC then, using (3.21), the parallel derivative at A must have a bound

$$|e'(A)_{BE}| \leq |e'(B)_{BE}| + AB. \tag{3.28a}$$

Likewise, the derivative at A in a direction parallel to a line through C in a direction lying within the angle BCA must have a bound

$$|e'(A)_{CF}| \leq |e'(C)_{CF}| + AC. \tag{3.28b}$$

Taking the previous result (3.27) into account, then, we have the following bounds on $|e'(A)|$ in all possible directions, and so on $|\nabla e(A)|$, for triangles of various possible shapes:

shape of triangle	bound on $\lvert\nabla e(A)\rvert$
$\alpha \geq \frac{1}{2}\pi$	$\max\{R, \frac{1}{2}a + \max\{b, c\}\}$
$\beta \geq \frac{1}{2}\pi$	$\max\{\frac{3}{2}b, R + c\}$
$\gamma \geq \frac{1}{2}\pi$	$\max\{\frac{3}{2}c, R + b\}$
other	$R + \max\{b, c\}$

These bounds have the common upper bound $\lvert\nabla e(A)\rvert \leq R + \max\{b, c\}$.

Therefore, under the reasonable assumption that the upper bound of the gradient is highest at one of the vertices, we conclude that

$$\|\nabla e\|_\infty \leq R + \max\{a, b, c\}. \tag{3.29}$$

Table 5: First overestimates for L_∞ bounds M'_∞

b	c	$\alpha(^\circ)$	M'_∞	b	c	$\alpha(^\circ)$	M'_∞
1	1	30	1.518	1	2	30	3.239
1	1	60	1.577	1	2	60	3.0
1	1	90	2.121	1	2	90	3.354
1	1	120	2.732	1	2	120	4.173
1	1	150	3.864	1	2	150	5.819

The upper bounds for M'_∞ resulting from (3.29) are shown in Table 5.

- We can improve on these bounds.

  In Fig. 3, if β and γ are acute, let G be the foot of the perpendicular from A on to BC. Then we can show from (3.26) that

$$\lvert e'(A)_{AG}\rvert \leq R\cos(\beta - \gamma). \tag{3.30}$$

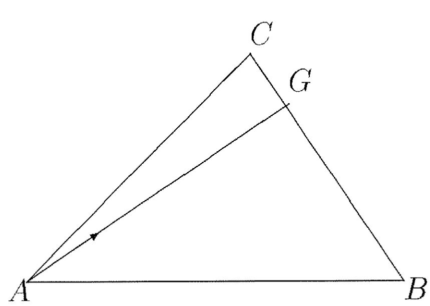

Figure 3:

Now

$$|e'(G)_{BC}| \leq \frac{a^2 - 2bc\cos\beta\cos\gamma}{2a} \leq \tfrac{1}{2}a,$$

using (2.5), so that

$$|e'(A)_{BC}| \leq \frac{a^2 - 2bc\cos\beta\cos\gamma}{2a} + b\sin\gamma, \tag{3.31}$$

using (3.21).

Hence

$$|\nabla e(A)| \leq \sqrt{R^2\cos(\beta-\gamma)^2 + \left\{\frac{a^2 - 2bc\cos\beta\cos\gamma}{2a} + b\sin\gamma\right\}^2} \tag{3.32}$$

If γ is obtuse, then we have

$$\begin{aligned} |e'(A)_{AC}| &\leq \tfrac{1}{2}b, \\ |e'(C)_{BC}| &\leq \tfrac{1}{2}a, \\ |e'(A)_{BC}| &\leq \tfrac{1}{2}a + b \end{aligned}$$

and hence

$$\begin{aligned} |\nabla e(A)| &\leq \frac{\sqrt{|e'(A)_{AC}|^2 + |e'(A)_{BC}|^2 + 2\,|e'(A)_{AC}|\,|e'(A)_{BC}|\cos\gamma}}{\sin\gamma} \\ &\leq \frac{\sqrt{b^2 + (a+2b)^2 + 2b(a+2b)\cos\gamma}}{2\sin\gamma}. \end{aligned} \tag{3.33}$$

Table 6: Better overestimates for L_∞ bounds M'_∞

b	c	$\alpha(^\circ)$	M'_∞	b	c	$\alpha(^\circ)$	M'_∞
1	1	30	1.211	1	2	30	2.063
1	1	60	1.256	1	2	60	2.394
1	1	90	1.581	1	2	90	2.692
1	1	120	1.683	1	2	120	2.828
1	1	150	2.721	1	2	150	4.510

Using (3.32) or (3.33) as appropriate at each vertex, and then taking the largest of the three resulting bounds, gives the improved results in Table 6.

There still remains a considerable gap between Table 6 and Table 4, however, showing that we have not yet found the best possible bounds.

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