X International Meeting on Approximation
of the University of Jaén
Úbeda, June 27--July 2, 2009

Finite tight frames in Approximation Theory

Shayne Waldron

Department of Mathematics, University of Auckland, New Zealand
waldron@math.auckland.ac.nz

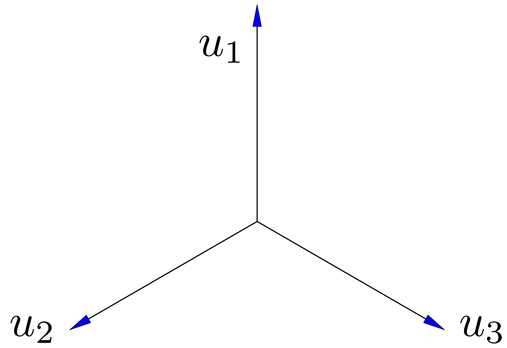
ABSTRACT

Over the last two decades, the use of frames for the construction of wavelets has been highly successful. Similar applications of finite tight frames are now emerging in areas as diverse as signal processing, quantum information theory and orthogonal polynomials.

I outline how my interest in finite tight frames originated from questions about the eigenstructure of the Bernstein operator and multivariate Jacobi polynomials. I then give a brief outline of current research in finite frame theory, including: the special geometry of finite tight frames, the use of equiangular tight frames for signal reconstruction when there are erasures, and Zauner's conjecture for rank one quantum measurements.

A question

Let u_1, u_2, u_3 be three equally spaced unit vectors in $\mathbb{R}^2$.



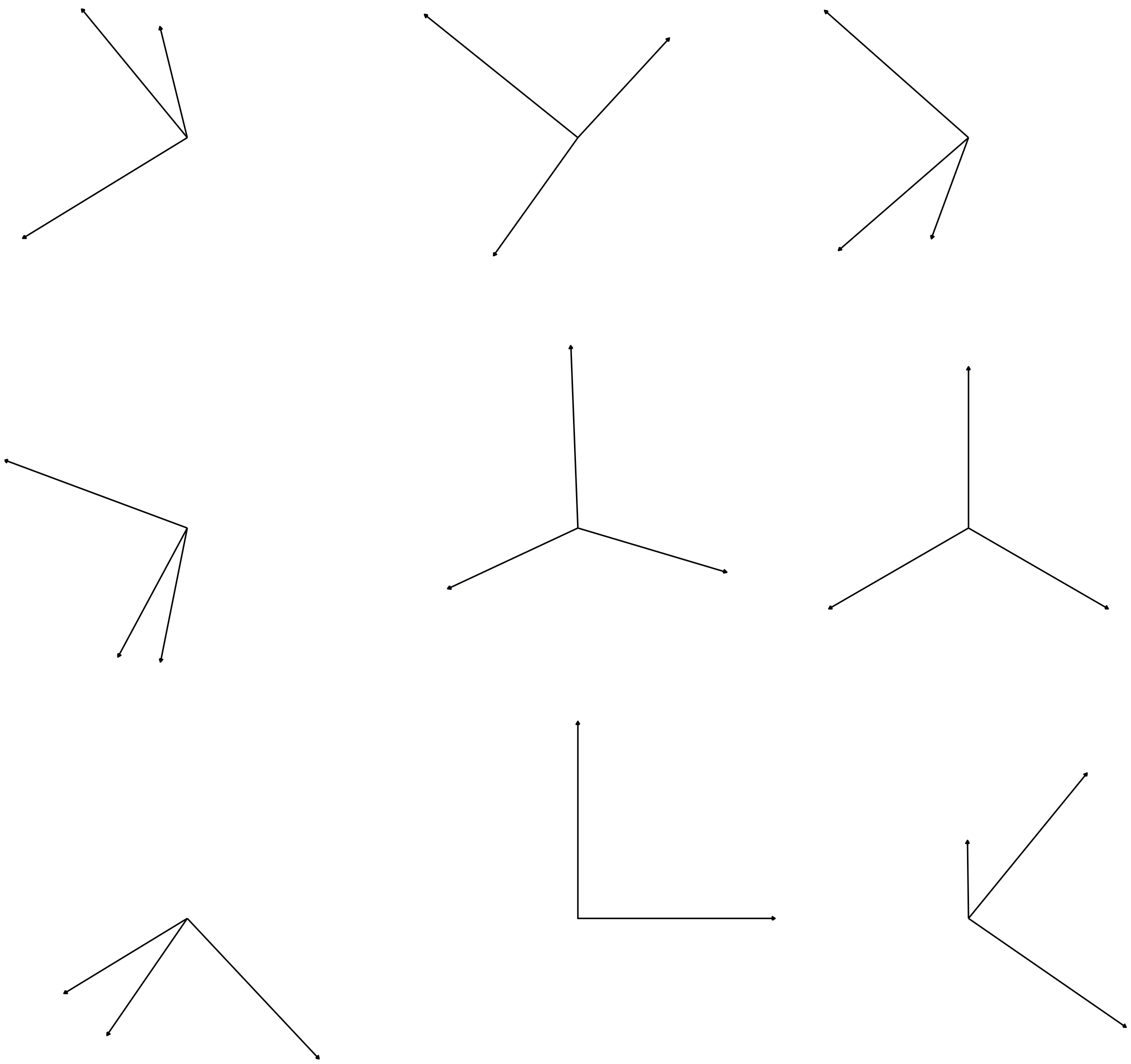
For a given nonzero vector $f \in \mathbb{R}^2$, what is the sum of its orthogonal projections onto these vectors?

(a) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = 0 \quad (\text{since } u_1 + u_2 = u_3 = 0).$$

(b) $$\sum_{j=1}^{3} \langle f, u_j \rangle u_j = \frac{3}{2} f, \qquad \forall f \in \mathbb{R}^2.$$

Frames in finite dimensional spaces

The following sets of vectors $\{v_j\}_{j=1}^3$ form tight frames for $\mathbb{R}^2$



i.e., give decompositions of the form

$$f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{R}^2.$$

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).

The start of a (long) story

The **Bernstein operator** $B_n : C([0,1]) \to \Pi_n$

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

has the diagonal form

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)}\, p_k^{(n)}\, \mu_k^{(n)}(f),$$

where the eigenvalues $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0$ are

$$\lambda_k^{(n)} := 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

and the corresponding eigenfunctions have the form

$$p_k^{(n)}(x) = x^k - \frac{k}{2} x^{k-1} + \text{lower order terms}.$$

The limiting eigenfunctions

The Bernstein operator converges as $n \to \infty$

$$\begin{array}{ccccccc} B_n f &=& \displaystyle\sum_{k=0}^{n} & \lambda_k^{(n)} & p_k^{(n)} & \mu_k^{(n)}(f) \\ \downarrow & & & \downarrow & \downarrow & \downarrow \\ f &=& \displaystyle\sum_{k=0}^{\infty} & 1 \ \cdot & p_k^* \ \cdot & \mu_k^*(f), \end{array}$$

where the "limit" eigenfunctions

$$p_k^*(x) = \frac{k!(k-2)!}{(2k-2)!} x(x-1) P_{k-2}^{(1,1)}(2x-1), \quad k \geq 2$$

are related to the Jacobi polynomials.

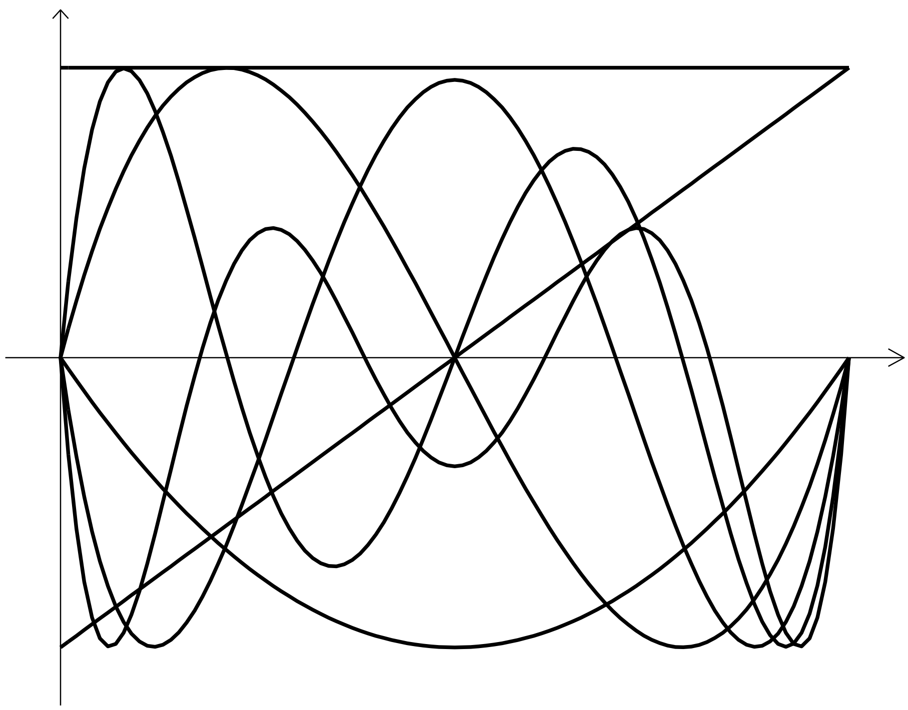


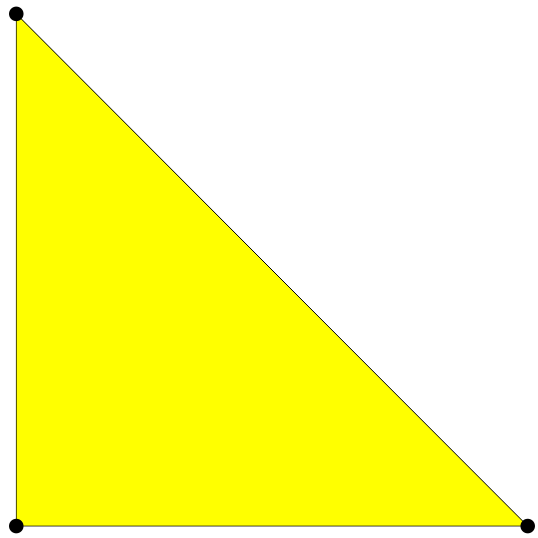
Fig. The first few limit eigenfunctions p_k^*.

Jacobi polynomials on a simplex

Let $T = \mathrm{conv}(V)$ be a simplex in $\mathbb{R}^d$ with $d+1$ vertices V, with corresponding barycentric coordinates $\xi = (\xi_v)_{v\in V}$, and define the Jacobi inner product

$$\langle f, g\rangle_\nu := \int_T f g\, \xi^{\nu-1}, \qquad \nu = (\nu_v)_{v\in V} > 0.$$

e.g., for $d=2$, $T = \mathrm{conv}\{e_1, e_2, 0\}$, $\nu - 1 = (\alpha, \beta, \gamma)$



$$\xi_{e_1}(x,y) = x$$

$$\xi_{e_2}(x,y) = y$$

$$\xi_0(x,y) = 1 - x - y$$

$$\langle f, g\rangle_\nu = \int_0^1 \int_0^{1-x} f(x,y) g(x,y)\, x^\alpha y^\beta (1-x-y)^\gamma \, dy\, dx$$

The **Jacobi polynomials** of degree k are

$$\mathcal{P}_k^\nu := \{ f \in \Pi_k : \langle f, p\rangle_\nu = 0, \forall p \in \Pi_{k-1} \}.$$

This space has

$$\dim(\mathcal{P}_k^\nu) = \binom{k+d-1}{d-1}.$$

Each polynomial in $\mathcal{P}_k^\nu$ is uniquely determined by its leading term, e.g., for ξ_0^2 + lower order terms, the leading term is

$$\{(1-x-y)^2\}_\downarrow = x^2 - 2xy + y^2.$$

Orthogonal and biorthogonal systems

We describe the known representations for $\mathcal{P}_k^\nu$ in terms of the leading terms (for the case $d = 2$, $k = 2$).

Biorthogonal system (Appell 1920's): partial symmetries

$$x^2, \quad xy, \quad y^2.$$

Orthogonal system (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \quad x^2 - y^2, \quad x^2 - y^2 - 4xy.$$

For the *three* dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the *six* polynomials

$$x^2, \quad xy, \quad y^2, \quad x(1-x-y), \quad y(1-x-y), \quad (1-x-y)^2.$$

Let

$$\Phi := \{p_{\xi^\alpha} = \xi^\alpha + \text{l.o.t} \in \mathcal{P}_2 : |\alpha| = 2\}$$

be these six functions. Then Φ is a frame for $\mathcal{P}_2^\nu$ (i.e., it spans) but it is *not* tight. We would like to find contants $c_\alpha > 0$ with

$$f = \sum_{|\alpha|=2} c_\alpha \langle f, p_{\xi^\alpha} \rangle p_{\xi^\alpha} = \sum_{|\alpha|=2} \langle f, \tilde{p}_{\xi^\alpha} \rangle \tilde{p}_{\xi^\alpha}, \qquad \forall f \in \mathcal{P}_2^\nu,$$

where $\tilde{p}_{\xi^\alpha} := \sqrt{c_\alpha} p_{\xi^\alpha}$.

A tight frame for the Jacobi polynomials

Let ϕ_α^ν be the orthogonal projection of

$$\xi^\alpha/(\nu)_\alpha, \qquad |\alpha| = n$$

onto $\mathcal{P}_n^\nu$, which is given by

$$\begin{aligned}\phi_\alpha^\nu &:= \frac{(-1)^n}{(n+|\nu|-1)_n} F_A\biggl(\begin{matrix}|\alpha|+|\nu|-1, -\alpha \\ \nu\end{matrix}; \xi\biggr) \\ &= \frac{(-1)^n}{(n+|\nu|-1)_n} \sum_{\beta\le\alpha} \frac{(n+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!},\end{aligned}$$

with F_A the Lauricella function of type A.

Theorem [WXR]. *The Jacobi polynomials on a simplex have the tight frame representation*

$$f = (|\nu|)_{2n} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi_\alpha^\nu\rangle_\nu \phi_\alpha^\nu, \qquad \forall f \in \mathcal{P}_n^\nu,$$

where the normalisation is $\langle 1, 1\rangle_\nu = 1$.

Remark. *It can be shown that the polynomials*

$$p_\alpha^\nu := (\nu)_\alpha \phi_\alpha^\nu = \xi^\alpha + \textit{lower order terms}, \qquad |\alpha| = k$$

have a limit p_α^* *as* $\nu \to 0^+$*, and that* p_α^* *is a limit eigenfunction of the Bernstein operator* B_n *on the simplex* T.

A nice example

Since the group of symmetries of the triangle (the dihedral group $G = D_3 \approx S_3$) induces an irreducible representation on the quadratic Legendre polynomials $\mathcal{P}_2$ on the triangle, we can construct a single polynomial

$$f = (2\sqrt{5}-5\sqrt{2})\left(\xi_v^2+\xi_w^2+\xi_u^2-\frac{1}{2}\right)+15\sqrt{2}\left(\xi_v^2-\frac{4}{5}\xi_v+\frac{1}{10}\right) \in \mathcal{P}_2$$

whose orbit under G consists of *three* polynomials which form an orthonormal basis for $\mathcal{P}_2$.

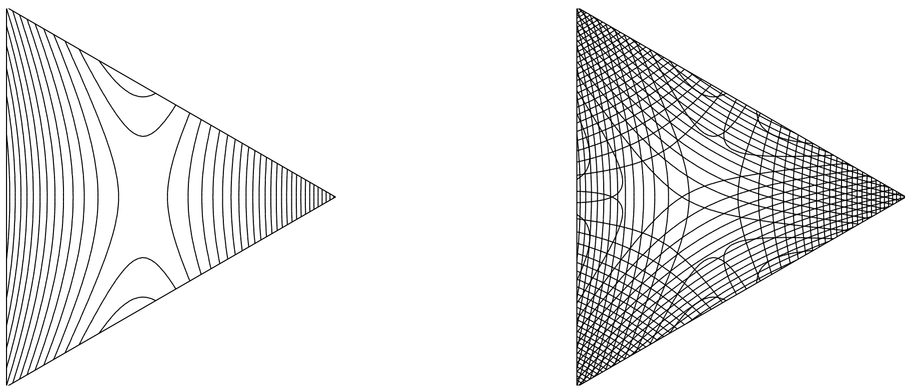


Fig. 1. Contour plots of f and those of its orbit showing the triangular symmetry.

Orthogonal polynomials on the disc

Let $\mathcal{P}_n = \mathcal{P}_n^w$ be the $n+1$ dimensional space of orthogonal polynomials of degree n on the unit disc

$$D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

given by the radially symmetric inner product

$$\langle f, g \rangle := \int_D f g\, w = \int_0^{2\pi} \int_0^1 (fg)(r\cos\theta, r\sin\theta)\, w(r)\, r dr\, d\theta.$$

These polynomials have long been used to analyse the optical properties of a circular lens, and to reconstruct images from Radon projections, etc.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through the angle θ, i.e.,

$$R_\theta(x,y) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

Let the group of rotations of the disc (which are symmetries of the weight)

$$\mathrm{SO}(2) = \{R_\theta : 0 \leq \theta < 2\pi\}$$

act on functions defined on the disc in the natural way, i.e.,

$$R_\theta f := f \circ R_\theta^{-1}.$$

The Logan Shepp polynomials

[Logan, Shepp 1975] showed the **Legendre polynomials** on the disc (constant weight $w = 1$) have an orthonormal basis given by the $n+1$ polynomials

$$p_j(x,y) := \frac{1}{\sqrt{\pi}} U_n\big(x \cos \frac{j\pi}{n+1} + y \sin \frac{j\pi}{n+1}\big), \qquad j = 0, \ldots, n,$$

where U_n is the n–th *Chebyshev polynomial of the second kind.*

This says that an orthonormal basis can be constructed from a single simple polynomial p_0 (a ridge function obtained from a univariate polynomial) by rotating it through the angles

$$\frac{j\pi}{n+1}, \qquad 0 \le j \le n.$$

It turns out, that for *any* weight w such an orthogonal expansion always exists, though the 'simple' polynomial p_0 is not in general a ridge function, but a *zonal function.*

Moreover, such an expansion reflects the rotational symmetry of the weight in a deeper way, e.g., for the Legendre polynomials there exists the tight frame decompositions

$$\begin{aligned} f &= \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} p_0 \rangle R^j_{\frac{2\pi}{k}} p_0 \\ &= \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta p_0 \rangle R_\theta p_0 \, d\theta, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

where $k \ge n+1$ with k not even if $k \le 2n$.

Zonal functions

A function f on the ball or $\mathbb{R}^d$ is **zonal** if it can be written in the form

$$f(x) = g(\langle x, \xi \rangle, |x|).$$

Compare this with

$$f(x) = g(\langle x, \xi \rangle) \quad \text{(ridge function with direction } \xi\text{)},$$
$$f(x) = g(|x|) \quad \text{(radial function)}.$$

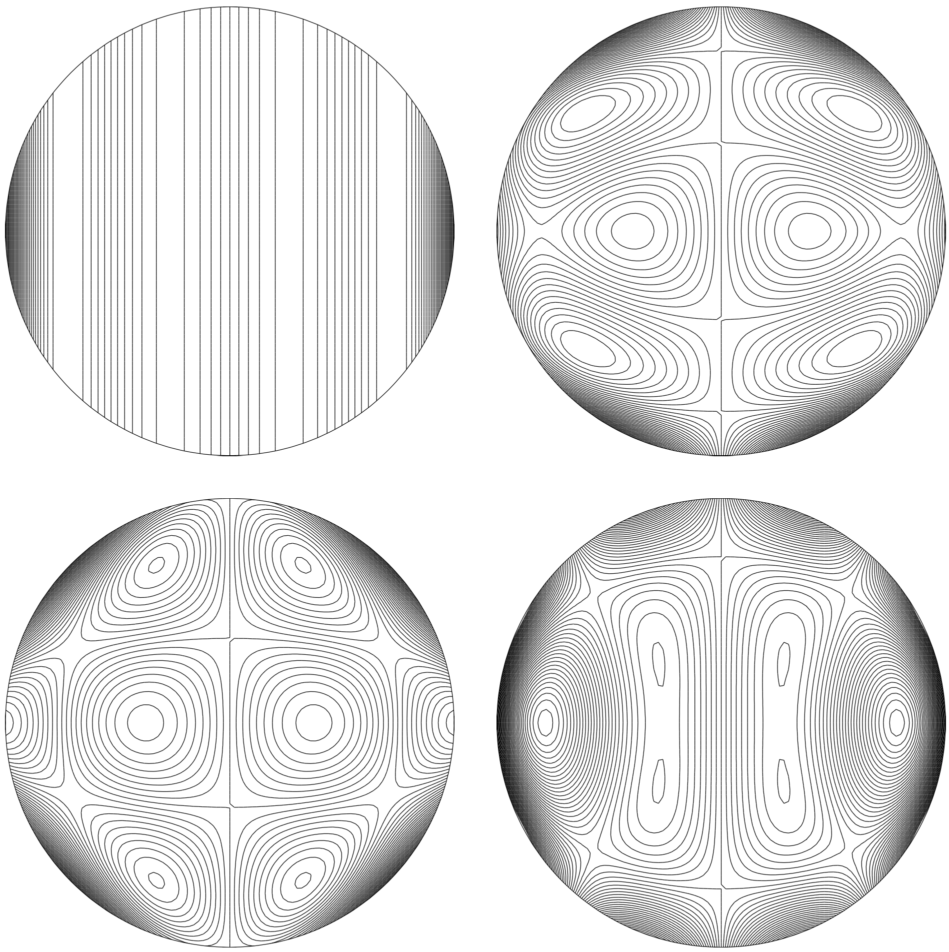


Fig. Contour plots of the zonal polynomials $p_0 \in \mathcal{P}_5$ which give a Logan–Shepp type expansion. The first is the Logan-Shepp polynomial.

Orthogonal polynomials on a ball

Let $\mathcal{P}_n$ be the orthogonal polynomials for a radial weight function w on $\mathbb{R}^d$.

Theorem. *Let $p = p_\xi$ be the zonal function*

$$p_\xi := \sqrt{\frac{\mathrm{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0 \le j \le \frac{n}{2}} Z_\xi^{(n-2j)} \frac{P_j(|\cdot|^2)}{\|P_j\|_w} \in \mathcal{P}_n.$$

Then we have the continuous tight frame expansion

$$\begin{aligned} f &= \dim(\mathcal{P}_n) \int_{\mathrm{SO}(d)} \langle f, gp \rangle gp \, d\mu(g) \\ &= \frac{\dim(\mathcal{P}_n)}{\mathrm{area}(S)} \int_S \langle f, p_\xi \rangle p_\xi \, d\xi, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

where μ denotes the normalised Haar measure on $\mathrm{SO}(d)$*.*

Here $Z_\xi^{(k)}$ is the zonal harmonic of degree k, and P_j is a univariate orthogonal polynomial of degree j.

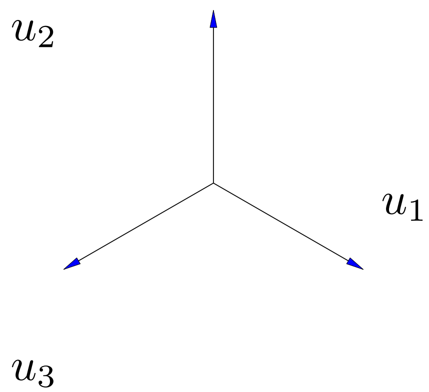
Corollary (Legendre polynomials). *For the weight $w = 1$ on the unit ball p_ξ is is the ridge polynomial given by*

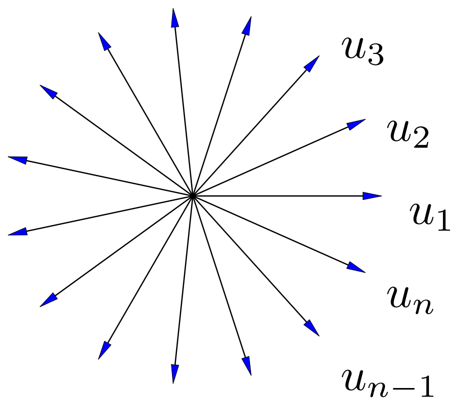
$$p_\xi(x) = \frac{\sqrt{2n+d}}{\sqrt{\mathrm{area}(S)}\sqrt{\dim(\mathcal{P}_n)}} C_n^{d/2}(\langle x, \xi \rangle).$$

Here C_n^λ are Gegenbauer polynomials.

Equal–norm tight frames

Any $n \geq 3$ equally spaced unit vectors $u_1, \ldots, u_n$ in $\mathbb{R}^2$





provide the following tight frame expansion

$$f = \frac{2}{n} \sum_{j=1}^{n} \langle f, u_j \rangle u_j, \qquad \forall f \in \mathbb{R}^2.$$

We say that (f_j) is an **equal norm** tight frame if

$$\|f_j\| = C, \quad \forall j.$$

Ten years ago, it wasn't generally known whether an equal–norm tight frame of $n \geq d$ vectors existed for $\mathbb{R}^d$ (or $\mathbb{C}^d$), $d \geq 3$. At one of the problem sessions at Bommerholz 2000 it was asked what are the best frame bounds for a frame of $n \geq 3$ vectors in $\mathbb{R}^3$.

Independently, a number of people considered this question, e.g., Goyal, et al (signal processing), Zimmermann (in answer to the Bommerholz question), Waldron and Fickus (for the equidistribution of points). The field of construction and application of what are usually called *finite normalised tight frames* was born. Major advocates include Pete Casazza, John Benedetto and Jelena Kovačević.

Harmonic frames

An isometric frame which is generated by an abelian group of symmetries is called an **harmonic frame**.

Example. The character table of the cyclic group of order 3

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad \omega := e^{\frac{2\pi i}{3}},$$

has orthogonal columns, and so the projection of them onto two coordinates gives isometric frames

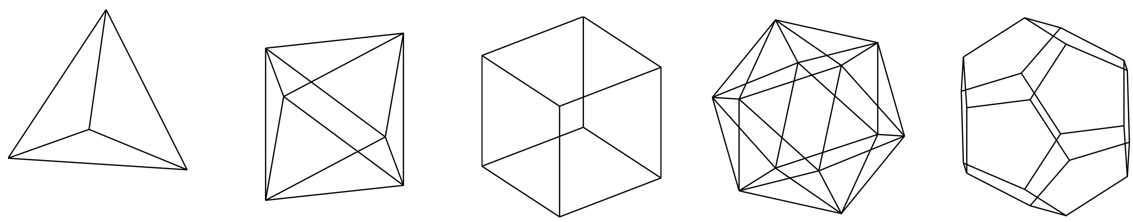
$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix}\right\} \text{ (real)} \qquad \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix}\right\} \text{ (complex)}$$

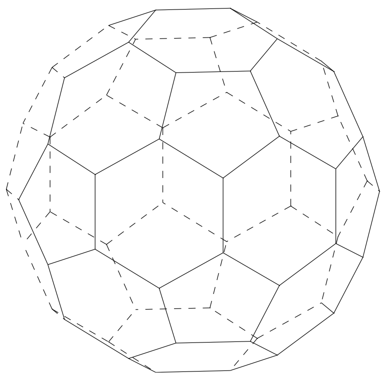
and these are harmonic.

Theorem [VW04]. *All harmonic frames of n vectors can be obtained by taking rows of the character table of an abelian group of order n.*

Well distributed points on the sphere

A number of nice configurations of points on the sphere give isometric tight frames, e.g.,





These turn out to be examples of the orbit of a single vector $v \in \mathbb{C}^d$ under a finite group G of unitary matrices which form an *irreducible representation*, i.e.,

$$\operatorname{span}\{gw : g \in G\} = \mathbb{C}^d, \qquad \forall w \neq 0.$$

Theorem ([VW04]). *If* $\operatorname{span}\{gw\}_{g\in G} = \mathbb{C}^d$ *for some vector* w*, then one can construct a vector* v *for which*

$$Gv := \{gv : g \in G\}$$

is a tight frame for $\mathbb{C}^d$*.*

Heisenberg frames

Let S and Ω be the **shift** and **modulation operators** on $\mathbb{C}^d$

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix}, \ \Omega = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \omega^2 & & & 0 \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & 0 & & & \omega^{d-1} \end{bmatrix}$$

where $\omega := e^{\frac{2\pi i}{d}}$ is a d–th roof of unity. Then we call

$$\{S^j \Omega^k v : 0 \leq j, k \leq d-1\}$$

a **Heisenberg frame** generated by the vector $v \in \mathbb{C}^d$ if it is an equiangular tight frame of $n = d^2$ vectors for $\mathbb{C}^d$.

- Heisenberg frames exist for $d = 2, 3, \ldots, 100$ numerically!
- and for exist for $d = 2, 3, 4, 5, 6, 7, 19$ analytically!

Example. For $d = 2$ a Heisenberg frame is generated by

$$v = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3 + \sqrt{3}} \\ e^{\frac{\pi}{4} i} \sqrt{3 - \sqrt{3}} \end{pmatrix}.$$

Zauner's conjecture

The key to the construction of generating vectors are:

- Simplify the equations (both in number of variables, and algebraic degree), e.g., the Heisenburg group structure reduces the equiangularity condition to

$$|v^* S^j \Omega^k v| = \frac{1}{\sqrt{d+1}}, \qquad \begin{matrix}(j,k) \neq (0,0),\\ (j,k) \in \mathbb{Z}_d^2\end{matrix} \qquad v^*v = 1.$$

- Observe the normaliser of $\langle S, \Omega \rangle$ (in the unitary matrices) maps the set of generating vectors to itself, and that this normaliser is generated by the **Fourier matrix** F and M

$$F_{jk} := \frac{1}{\sqrt{d}} \omega^{jk},$$

$$M_{jk} := \frac{\zeta^{d-1}}{\sqrt{d}} \mu^{j(j+d)+2jk}, \qquad \mu := e^{\frac{2\pi i}{2d}}, \quad \zeta := e^{\frac{2\pi i}{24}}.$$

This mysterious matrix M has order 3, and appears in the latest version of Zauner's conjecture (that there is a equiangular tight frame of $n = d^2$ vectors in $\mathbb{C}^d$, for all $d \geq 2$).

Conjecture. *There is a generating vector $v \in \mathbb{C}^d$, $d \geq 2$, for a Heisenberg frame which is an eigenvector of M, or of a conjugate of M in the normaliser.*

Example $d = 5$

Theorem. *Define a complex number*

$$z := \frac{\sqrt{1+\sqrt{3}}}{2\sqrt{2}} \left(\sqrt{\frac{5-\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{5}} \right).$$

Then the four unit vectors v in the $e^{\frac{2\pi}{3}i}$ eigenspace of A

$$v = v_\beta := \alpha x + \beta y,$$

$$\alpha = \tfrac{1}{2}\sqrt{3-\sqrt{3}} \;= \sqrt{1-|z|^2}, \quad \beta = z, -z, \overline{z}, -\overline{z}$$

each generate a Heisenberg frame for $\mathbb{C}^5$.

Example $d = 7$

Theorem. *For $d = 7$, there are three inequivalent vectors of the form*

$$v = (a, b, b, c, b, c, c)^T, \qquad a \in \mathbb{C}, \quad b, c \in \mathbb{R}$$

which generate a Heisenberg frame for $\mathbb{C}^7$, namely the pair of conjugate solutions given by

$$a = -\frac{\sqrt{8 - 5\sqrt{2}}(2\sqrt{2} + 1 \pm 7i)}{2\sqrt{7}(3\sqrt{2} - 2)},$$

$$b = \frac{\sqrt{8 - 5\sqrt{2}}}{4\sqrt{7}} + \frac{\sqrt[4]{2}}{4}, \quad c = \frac{\sqrt{8 - 5\sqrt{2}}}{4\sqrt{7}} - \frac{\sqrt[4]{2}}{4},$$

and the all real solution given by

$$a = -\frac{\sqrt{3\sqrt{2} - 2}}{2\sqrt{7}},$$

$$b = \frac{\sqrt{6 + 5\sqrt{2}}}{4\sqrt{7}} + \frac{\sqrt{2 - \sqrt{2}}}{4}, \qquad c = \frac{\sqrt{6 + 5\sqrt{2}}}{4\sqrt{7}} - \frac{\sqrt{2 - \sqrt{2}}}{4}.$$

Extra slides

A tight frame

For the weight function $w : [0,1] \to \mathbb{R}^+$ and a fixed n, let

$$P_j \neq 0, \qquad 0 \le j \le \frac{n}{2}$$

be an orthogonal polynomial of degree j for the univariate weight $(1+x)^{n-2j} w(\sqrt{\frac{1+x}{2}})$ on $[-1,1]$, and

$$h_j := \frac{\pi}{2^{n-2j+1}} \int_{-1}^{1} P_j^2(x)(1+x)^{n-2j} w\Big(\sqrt{\frac{1+x}{2}}\Big)\, dx.$$

Theorem [W07]. *Let $v \in \mathcal{P}_n$ be the polynomial with real coefficients defined by*

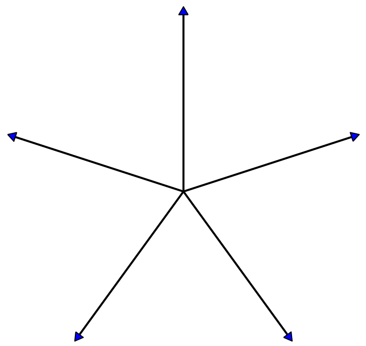
$$v(x,y) := \frac{1}{\sqrt{n+1}} \sum_{0 \le j \le \frac{n}{2}} \frac{2}{1+\delta_{j,\frac{n}{2}}} \frac{1}{\sqrt{h_j}} \operatorname{Re}(\xi_j z^{n-2j}) P_j(2|z|^2 - 1),$$

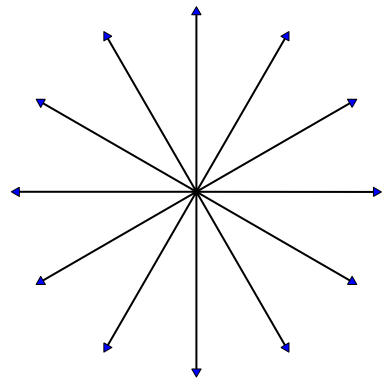
where $z := x + iy$, $\xi_j \in \mathbb{C}$, $|\xi_j| = 1$, with $\xi_{\frac{n}{2}} \in \{-1, 1\}$. Then $\{R^j_{\frac{\pi}{n+1}} v\}_{j=0}^n$ is an orthonormal basis for $\mathcal{P}_n$, and Moreover

$$\begin{aligned} f &= \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} v \rangle R^j_{\frac{2\pi}{k}} v \\ &= \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta v \rangle R_\theta v \, d\theta, \qquad \forall f \in \mathcal{P}_n, \end{aligned}$$

whenever $k \ge n+1$ and k is odd, or $k \ge 2(n+1)$.

Tight frames and equiangularity





A sequence of n vectors $\Phi = (f_j)_{j=1}^n$ in the Hilbert space $\mathcal{H} = \mathbb{F}^d = \mathbb{R}^d, \mathbb{C}^d$ is a **tight frame** for $\mathcal{H}$ if $\exists C > 0$:

$$f = C \sum_{j=1}^{n} \langle f, f_j \rangle f_j, \qquad \forall f \in \mathcal{H}.$$

It is **normalised** if $C = 1$, and it is **equiangular** if $\exists r, c > 0$:

$$\|f_j\|^2 = r, \quad \forall j, \qquad |\langle f_j, f_k \rangle| = c, \quad \forall j \neq k.$$

An **erasure** is the loss of a frame coefficient $\langle f, f_j \rangle$.

A tight frame is optimal for reconstruction from *one* erasure if it has *equal norms*, and from *two* if is *equiangular*.

The Gramian matrix and equations

An equiangular tight frame of n vectors for $\mathbb{F}^d$ is determined (up to unity equivalence) by its Gramian matrix

$$G = \begin{pmatrix} r & cz_{12} & cz_{13} & \cdots & cz_{1n} \\ c\overline{z_{12}} & r & cz_{23} & \cdots & cz_{2n} \\ c\overline{z_{13}} & c\overline{z_{23}} & r & & \\ \vdots & \vdots & & \ddots & \\ c\overline{z_{1n}} & c\overline{z_{2n}} & & & r \end{pmatrix} = rI + c\Sigma, \quad |z_{jk}| = 1,$$

where the $n \times n$ Hermitian matrix Σ (with zero diagonal and off diagonal entries of modulus 1) is called a **signature matrix**.

A set of vectors is a normalised tight frame if and only if its Gramian is a projection matrix.

Theorem. *A signature matrix Σ gives an equiangular tight frame for $\mathbb{F}^d$ if and only if it satisfies the $\frac{1}{2}n(n-1)$ equations*

$$(n-2d)\sqrt{\frac{n-1}{d(n-d)}}z_{jk} = \sum_{i=1}^{j-1} \overline{z_{ij}} z_{ik} + \sum_{i=j+1}^{k-1} z_{ji} z_{ik} + \sum_{i=k+1}^{n} z_{ji} \overline{z_{ki}}.$$

Example. For $d = 2$, $n = 4$ all possible solution are

$$\Sigma = \Lambda \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \pm i & \mp i \\ 1 & \mp i & 0 & \pm i \\ 1 & \pm i & \mp i & 0 \end{pmatrix} \Lambda^*, \qquad \hat{\Sigma} = \begin{pmatrix} 0 & \pm i & \mp i \\ \mp i & 0 & \pm i \\ \pm i & \mp i & 0 \end{pmatrix}$$

$\Lambda = \operatorname{diag}(c_1, \ldots, c_4)$, with $\hat{\Sigma}$ the **reduced signature matrix**.

Graphs and Seidel Matrices

The Seidel matrix of a graph on vertices $1, \ldots, n$ has a -1 in the (j,k)–entry if j and k are adjacent, a 1 if they are not adjacent, and zero diagonals. Thus real signature matrices are in 1–1 correspondence with graphs.

A regular graph with ν vertices and degree k is **strongly regular**, or a srg(ν, k, λ, μ), if there are integers λ, μ such that

- Every two adjacent vertices have λ common neighbours.
- Every two non-adjacent vertices have μ common neighbours.

Theorem. *A signature matrix Σ gives a real equiangular tight frame of n vectors for $\mathbb{R}^d$ if and only if $\hat{\Sigma}$ is the Seidel matrix of a strongly regular graph*

$$\mathrm{srg}(n-1, k, \lambda, \mu), \qquad \mu = \frac{k}{2}, \qquad \lambda = \frac{3k-n}{2}.$$

where

$$d = \frac{1}{2}n - \frac{1}{2}\frac{n(n-2k-2)}{\sqrt{(n-2k)^2 + 8k}}.$$