Highly symmetric Tight Frames

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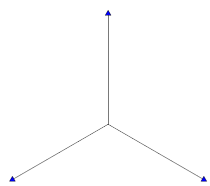
Introduction

The prototypical example of a *tight frame* is the three unit vectors of the Mercedes-Benz frame in $\mathbb{R}^2$,



which provides the generalised orthogonal expansion $x = \frac{2}{3}\sum_{j=1}^{3} \langle x, u_j \rangle \, u_j, \forall x \in \mathbb{R}^2$.

Since tight frames are generalisations of orthonormal bases (they may have redundancy) they may have properties not possible for bases. For instance the Mercedes-Benz frame above has more symmetry than an orthonormal basis. This thesis focuses on the construction of such 'highly symmetric tight frames' from groups such as the symmetry groups of the regular polytopes, and even more generally, from abstract groups.

The formal definition of a tight frame is as follows:

Definition 1. *A sequence of vectors $\Phi = (\phi_i)_{i \in \mathcal{I}}$ in a finite dimensional Hilbert space $\mathcal{H}$ is a* ***frame*** *if it spans $\mathcal{H}$, and it is a* ***tight frame*** *if one has a generalised orthogonal expansion*

$$\exists c \forall u \in \mathcal{H} \quad u = c \sum_{i \in \mathcal{I}} \langle u, \phi_i \rangle \phi_i.$$

Frame theory has recently been used in finite dimensional applications such as signal analysis and quantum measurement theory. There is a well developed theory of wavelet frames for infinite dimensional spaces where wavlets with properties such as smoothness and small support can be constructed.

We will only be dealing with a finite dimensional context and will focus on the class of tight frames known as group frames. These are frames produced as the orbit of a vector under the action of a finite group of linear transformations. What is interesting about group frames is they inherit a structure from the group. In the literature there is a lot of interest in the use of abelian and Heisenberg groups. We will outline the properties of those popular group frames before turning to the symmetry groups of regular polytopes.

If you think of mathematical objects which possess symmetry as beautiful then it is hard to go past the preeminent objects of classical Greek mathematics - the platonic solids.

The sequence of vectors produced by centering the regular polytope at the origin and taking vectors at the vertices is a tight frame. By virtue of its origins in a highly regular shape the polytopal tight frame possesses a natural symmetry.

Motivated by the nice structure of regular real polytopes we began by looking at Coxeter's regular complex polytopes to investigate whether these would produce interesting tight frames in $\mathbb{C}^d$. Given the rising interest of the quantum information community in frames with equal angles between distinct vectors in $\mathbb{C}^d$ the regularity of the polytopes was attractive although this work has not turned up any new maximal equiangular tight frames.

The symmetry groups of the regular complex polytopes are contained within the class of irreducible complex reflection groups that have been fully classified. Investigating frames that came from irreducible complex reflection groups has led to the definition of a new class of frames - highly symmetric tight frames.

The highly symmetric tight frames can be computed from abstract groups and it will be shown that they form a finite class up to unitary equivalence. This is striking as we are not dealing with abelian groups and typically there are uncountably many group frames that arise from non-abelian groups.

This report will lay out a path to arrive at highly symmetric tight frames and consider numerous examples. The next chapter will introduce the language of frame theory and group representations so that chapter 3 can outline two common examples of group frames. In preparation for defining highly symmetric tight frames chapter 4 will define regular complex polytopes and their symmetry groups and contextualise these within the irreducible complex reflection groups. Chapter 5 will define the highly symmetric tight frames and chapter 6 will draw on the highly symmetric tight frames computed from the irreducible complex reflection groups to investigate the connection between these and the well known harmonic and equiangular frames. Finally chapter 7 will define how to construct a highly symmetric tight frame in general from an abstract group.

2 Frame Theory and Group Representation

One class of frames that have interesting structures are the group frames that act on $\mathbb{F}^d$. For a given finite group G these have the form

$$\Phi = (gv)_{g \in G} \qquad \text{where } v \in \mathbb{F}^d.$$

This sequence may include repeated vectors if v is stabilised by numerous elements in the group, in particular we define the stabiliser of v as

$$\mathrm{Stab}(v) = \{g \in G : gv = v\}.$$

The G-frame has the following generalised orthogonal expansion,

$$\forall u \in \mathbb{F}^d \qquad u = \frac{d}{|G|} \frac{1}{\|v\|^2} \sum_{g \in G} \langle u, gv \rangle \, gv.$$

These definitions assume that the group elements are linear transformations acting on $\mathbb{F}^d$. To fully define the notion of a group frame and the action gv will require the use of representation theory.

2.1 Representation Theory

Representation theory allows us to move between abstract groups, defined in terms of the relations on their generators, and a more concrete presentation, which specifies the actual form of the generators. For instance we may want all the generators to be reflections which satisfy the defining relations of the abstract group.

Definition 2. *Given a group G a* ***representation*** *over $\mathbb{F}^d$ is a group homomorphism*

$$\rho : G \to \mathcal{GL}(d, \mathbb{F}),$$

where $\mathcal{GL}(d, \mathbb{F})$ is the general linear group of invertible linear transformations that act on $\mathbb{F}^d$. In particular

$$\begin{aligned} \rho(g)\rho(h) &= \rho(gh) \qquad \forall g, h \in G \\ \rho(g^{-1}) &= (\rho(g))^{-1} \\ \rho(1_G) &= I_d \end{aligned}$$

Throughout this report the field $\mathbb{F}$ will be taken to be $\mathbb{R}$ or $\mathbb{C}$. It is also possible to consider representations over finite fields but some of the standard representation theory results such as Maschke's theorem will require extra conditions.

The group G is said to act on $\mathbb{F}^d$. In particular the action of $g \in G$ on $v \in \mathbb{F}^d$ is defined by $gv := \rho(g)v$. If there is a subspace $U \subset \mathbb{F}^d$ such that

$$\forall u \in U \quad gu \in U$$

then U is called a G-submodule.

Definition 3. *If $\{0\}$ and $\mathbb{F}^d$ are the only G-submodules of G under the representation*

*$\rho : G \to \mathcal{GL}(d, \mathbb{F})$ then G is **irreducible**, otherwise it is reducible.*

In other words G is irreducible if the span of the orbit $\{gv\}_{g \in G}$ equals $\mathbb{F}^d$ for all non-zero v in $\mathbb{F}^d$.

For instance if G is the cyclic group of order 3 then $G = \langle a \rangle = \{1, a, a^2\}$ has the following reducible representation $\rho : G \to \mathcal{GL}(2, \mathbb{C})$ where $\omega = e^{\frac{2\pi i}{3}}$,

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(a) = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(a^2) = \begin{bmatrix} \omega^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case $U = \operatorname{span}\{[1, 0]^T\}$ is a G-submodule.

So far $\rho(g)$ has been described as a linear transformation but it would be more useful if it was also unitary. Given that G will always be taken to be finite it is possible to define an appropriate positive definite hermitian form $\langle \cdot, \cdot \rangle_\rho$ which will make $\rho(g)$ unitary. More specifically, given any positive definite hermitian form $\langle \cdot, \cdot \rangle$ then

$$\langle u, v \rangle_\rho = \frac{1}{|G|} \sum_{g \in G} \langle gu, gv \rangle$$

is a hermitian form. This guarantees that $\langle \cdot, \cdot \rangle_\rho$ has the following properties;

(i). $\langle u, u \rangle_\rho \geq 0 \quad$ for all $u \in \mathbb{F}^d$;

(ii). $\langle gu, gv \rangle_\rho = \langle u, v \rangle_\rho \quad$ (G-invariant);

(iii). $\langle hu, v \rangle_\rho = \langle u, h^{-1}v \rangle_\rho \quad$ hence $h^* = h^{-1}$ with respect to $\langle \cdot, \cdot \rangle_\rho$.

The phrase ‘G is a unitary group in $\mathbb{F}^d$’ should be taken to mean that there is a representation $\rho : G \to \mathcal{GL}(d, \mathbb{F})$ along with an appropriate G-invariant hermitian form $\langle \cdot, \cdot \rangle_\rho$ on $\mathbb{F}^d$ such that $\rho(g)$ is unitary with respect to $\langle \cdot, \cdot \rangle_\rho$ for all $g \in G$.

For the most part the representation theory is hidden in the background so it is important to note that the concept of a reflection group includes the particular representation. For instance given the cyclic group $G = \langle a \rangle$ of order m, let $\rho_1, \rho_2 : G \to \mathcal{GL}(3, \mathbb{C})$ be two representations such that,

$$\rho_1(a) = \begin{bmatrix} \omega & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \rho_2(a) = \begin{bmatrix} \omega & & \\ & \omega & \\ & & 1 \end{bmatrix}, \quad \text{where } \omega = e^{\frac{2\pi i}{m}}.$$

Then $\rho_1(a)$ is a reflection but $\rho_2(a)$ is not. Hence whether something is a reflection group depends on the particular representation.

2.2 Group Orbits and Frames

The properties of a group help us in understanding the collection of vectors in the orbit. In general the orbit of a vector under the action of an arbitrary finite group will not be a tight frame, although in some cases we can guarantee tight frames.

Theorem 1. *The orbit of any non-zero vector under an irreducible group forms a tight frame.*

Having an irreducible group means any non-zero orbit will be a spanning set for the vector space which satisfies the first condition of what it means to be a tight frame. It can also be shown that such an orbit satisfies the condition for having a generalised orthogonal expansion [VW04].

Theorem 1 allows us to generate a large number of tight frames - all you need is an irreducible representation of a group. To be more specific about the sort of frames that this method produces there are two approaches; first you can rely on the properties of the groups and secondly you can look at the action of the group on a vector space. The first approach is used for generating harmonic frames by requiring that the irreducible group is an abelian group. We will use the second approach when it comes to defining a highly symmetric tight frame.

The strength of considering both of these methods is that they can be combined for a more specific tailoring of the tight frame. If you combine the method to produce highly symmetric tight frames (that will be outlined later) along with the condition that the groups used are abelian then you get harmonic highly symmetric tight frame. This thesis will develop on these ideas to explore the variety of group frames that can be produced.

3

Groups and Frames

To understand how the structure of the group can alter the frame this chapter will discuss two examples: harmonic frames that come from abelian groups and equiangular frames that comes from Heisenberg groups. Both of these examples come up a lot in the literature and have many applications. Harmonic frames are used in electrical engineering and maximal equiangular frames are useful for signal processing and quantum information theory.

3.1 Abelian Groups and Harmonic Frames

The original conception of harmonic frames was in terms of cyclic groups. In the context of cyclic groups if $\Phi = (\phi_i)_{i\in\mathcal{I}}$ is a cyclic-harmonic tight frame of N vectors in $\mathbb{C}^d$ then the frame matrix $V = \begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_N\end{bmatrix}$ is a $d \times N$ matrix made up of d rows from the

$N \times N$ discrete Fourier transform matrix

$$F = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & \omega_1 & \omega_1^2 & \cdots & \omega_1^{N-1} \\ 1 & \omega_2 & \omega_2^2 & \cdots & \omega_2^{N-1} \\ \vdots & & & \ddots & \vdots \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \end{bmatrix}$$

where ω_i are distinct N-th roots of unity.

To define Φ in terms of group orbits consider the unitary transformation

$$U : \mathbb{C}^d \to \mathbb{C}^d, \qquad U(x) = \sum_{i \in \mathcal{I}} \langle x, \phi_i \rangle \, \phi_{i+1}$$

where ϕ_{N+1} is defined to be ϕ_1. Then $U^N = I$ and $U^k \neq I$ for $1 \leq k \leq N-1$ and $G = \langle U \rangle$ is a unitary representation of the cyclic group of order N. Let $v = [1, \cdots, 1]^T = \phi_1$ and

$$\Phi = \{gv : g \in G\} = \{U^k \phi_1 : 1 \leq k \leq N\}$$

then Φ is a G-frame where G is a cyclic group.

Alternatively we can say that the cyclic-harmonic frame of N vectors is obtained by taking d rows of the character table of the cyclic group of order N. In particular the Fourier transform matrix is the character table for a cyclic group of order N.

The concept of a character comes from representation theory. In particular if $\rho : G \to \mathcal{GL}(d, \mathbb{C})$ is a representation then $\chi : G \to \mathbb{C}$ is the character of the representation defined by $\chi(g) := trace(\rho(g))$. Each row in the character table corresponds to a different character and hence different representation of the group G.

The role of cyclic groups in creating harmonic frames can be replaced with the more general notion of an abelian group.

Definition 4. *A* ***harmonic tight frame*** *of N vectors in $\mathbb{C}^d$ is a G-frame, where G is abelian, obtained by taking d rows of the character table of G.*

To consider how many harmonic frames there are we need a notion of unitary equivalence.

Definition 5. *Two frames $\{\phi_i\}_{i\in\mathcal{I}}$ and $\{\psi_i\}_{i\in\mathcal{I}}$ are* ***unitarily equivalent*** *if there exists a unitary transformation U such that*

$$\forall i \in \mathcal{I} \quad U\phi_i = \psi_i.$$

What is special about harmonic frames is that there are only finitely many harmonic frames of N vectors in $\mathbb{C}^d$ up to unitary equivalence. This follows from the fact that there are only finitely many abelian groups of order N and a finite number of ways of selecting d rows from their character tables.

Generally there will be infinitely many G-frames of N vectors when G is not abelian. Consider the smallest nonabelian group, the dihedral group D_3, acting on $\mathbb{R}^2$ where the generating elements of D_3 are a reflection in the x-axis and a rotation of $\frac{2\pi}{3}$. In each of

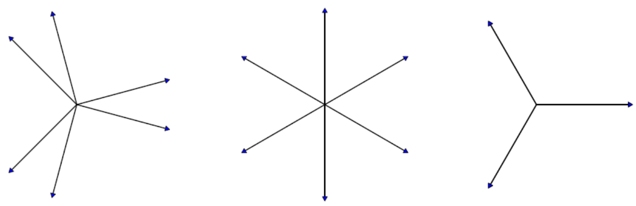


Figure 3.1: Orbits under D_3

the three frames pictured there are different angles between pairs of distinct vectors. As unitary transformations preserve angles there is no unitary transformations between the pictured frames and hence they are not unitarily equivalent. In fact, there are uncountably many unitarily inequivalent tight frames, these correspond to D_3-frames generated from the uncountably many initial vectors $v = (\cos(\theta), \sin(\theta))$ where $0 \leq \theta \leq \frac{\pi}{6}$.

One of the frames above looks like it only has 3 vectors while the others have 6; this is because of the issue of repeated vectors turning up in the sequence of a frame. Sometimes we will exclude repetition of vectors and other times we will consider a frame as a sequence of vectors which may include repetition. This is a small technical difference between considering a frame as a set or as a sequence of vectors.

3.2 Equiangular Frames and the Heisenberg Group

There is a lot of interest in tight frames where the angle between any distinct pair of vectors is constant. In particular if $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ is a tight frame of N unit vectors in $\mathbb{C}^d$ then

$$\mathrm{max}_{i\neq j}|\,\langle\phi_i,\phi_j\rangle\,| \geq \sqrt{\frac{N-d}{d(N-1}},$$

and this bound is only achieved if the vectors are equiangular.

Lemma 2. *An equiangular tight frame of N vectors in $\mathbb{C}^d$ can only exist if $N \leq d^2$.*

This means that if Φ is an equiangular tight frame that has the maximum number of vectors then for all $i \neq j$, $|\,\langle\phi_i,\phi_j\rangle\,| = \sqrt{\frac{1}{d+1}}$.

In 1999, Gerhard Zauner conjectured that there exists d^2 equiangular lines in $\mathbb{C}^d$ for all values of d [Zau99]. There has been a lot of interest in whether it is possible to have a frame of maximal size. For instance these maximal equiangular frames offer a robustness to erasure that is useful for signals processing. Despite an increasing body of numerical and analytical solutions in small dimensions it remains an open problem.

The most promising direction for a general analytic result involves the Heisenberg group of $d \times d$ matrices with entries from $\mathbb{C}$. In particular the eigenvectors of specific elements in the normaliser of the Heisenberg group have so far been good candidates as an initial vector for the orbit. So far this has been a case by case approach that has not been generalised. The most recent computer survey verified that such maximal equiangular frames exist in dimension d for $d \leq 67$ and with some uncertainty around $d = 66$ [GS09].

Definition 6. *The **Heisenberg group** in d-dimensions, H_d, is generated by a shift matrix S and a diagonal matrix Ω,*

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \omega^{d-1} \end{bmatrix},$$

where $\omega = e^{\frac{2i\pi}{d}}$ *is the primitive* d*-th root of unity.*

In particular $\Omega S = \omega S \Omega$ and so the elements in H_d commute up to scalar multiplication.

If $\Phi = (hv)_{h \in H}$ is a tight frame obtained from a Heisenberg group H in d-dimensions then the d vectors $\omega^r S^i \Omega^j$ are scalar multiples of each other for fixed i, j and $1 \leq r \leq d$.

Example: The orbit of $[1, -1, 0]$ under the Heisenberg group H_3 produces an equiangular tight frame of 9 vectors in $\mathbb{C}^3$ (ignoring scalar multiples of vectors).

Unlike the harmonic frames which had a clear link to abelian groups the connection for equiangular frames is more questionable. So far the Heisenberg groups have been useful but the results have resisted being generalised. There is no definitive statement that can be made about how the structure of a group influences the structure of the resulting frame but there are a lot of interesting relationships that can be developed in the future.

4

Polytopes and Reflection Groups

There is a difference between a polytope in $\mathbb{R}^d$ and $\mathbb{C}^d$. Similarly there is a contrast between real and complex reflections, in part because of the existence of more roots of unity in $\mathbb{C}$. The definitions of polytopes and reflections are interrelated but which is first will determine how this relationship is expressed. If you start with a structural definition of a polytope as a combinatorial object and consider the symmetries of the polytope then you can determine features of a complex reflection. By contrast if you develop the theory of complex reflections and consider the possible arrangements of their mirrors then you are narrowing down what sort of structure a complex polytope will be.

We will begin by presenting the definitions of regular complex polytopes and complex reflections. From there we will give two alternative accounts of the symmetry groups of regular complex polytopes; firstly beginning with the structure of a polytope we will construct the reflections and secondly by considering the reflections and abstract groups we will provide some restrictions for the possible structures of complex polytopes.

4.1 Defining a Complex Polytope

Euclid defined the terms 'point', 'line' and 'straight' but then proceeded to talk about polyhera without ever saying what a polyhedron actually is. Geometers continue to discuss polyhedral properties without defining polyhedra - the original sin of the theory of polyhedra as Branko Grunbaum says.

We will be looking at d-dimensional polyhedra, called polytopes, in complex space. A polytope is made up of elements of increasing dimension which includes the 0-dimensional vertices all the way up to the d-dimensional polytope itself. We will refer to an ***i-face*** $= f_i$ as an i-dimensional element of the polytope. For instance a 0-face $= f_0$ is a vertex, a 1-face $= f_1$ is an edge and the d-face $= f_d$ is the full polytope. For formal completeness we also include an empty face which is called the (-1)-face.

An i-face f_i is ***incident*** with a j-face f_j if one of them is contained within the other. For instance if v is a vertex on the edge e, then the 0-face v is incident with the 1-face e and we will denote this by saying $v \subset e$. Furthermore f_i and f_j are successively incident if $f_i \subset f_j$ and $j = i + 1$. For all i-faces f_i where $0 \leq i \leq d-1$ we have that $f_{-1} \subset f_i \subset f_d$. From this is it follows that the incidence relation $\subset$ is a partial ordering on the face elements.

In defining a polytope we want to capture the intuition that, like a polyhedron, a polytope is a connected shape where at least two i-faces are required to carve out an (i+1)-face. For instance we want to rule out a situation where one vertex with a loop attached defines a face and so we say that a single 1-face is not enough to define a 2-face.

The definition of an abstract real polytope as used by Peter McMullen and Egon Schulte [MS02] is as follows;

Definition 7. *A* ***real polytope*** *in $\mathbb{R}^d$ is a collection of i-faces, for $-1 \leq i \leq d$, that satisfy the following conditions;*

I. *If f_{i-1} and f_{i+1} are incident faces then there exists* ***exactly two*** *intermediary faces, f_i, f_i', such that $f_{i-1} \subset f_i$, $f_i' \subset f_{i+1}$;*

II. *If f_i, f_j are two faces then they are connected by some sequence of successively incident faces $f_{a_1}, f_{a_2}, ..., f_{a_n}$ such that $f_{a_1} = f_i$ and $f_{a_n} = f_j$.*

Given a polytope such as the cube in $\mathbb{R}^3$ there are exactly two vertices incident with the empty face and a given edge, this is an instance of the first property of a real polytope. However in $\mathbb{C}^d$ we want to allow more than two vertices on an edge. Part of the motivation for this is that the 1-dimensional complex space is isomorphic as a vector space to a 2-dimensional real space and so the structure of a 1-dimensional complex edge is naturally going to look different.

Definition 8. *A* ***complex polytope*** *in $\mathbb{C}^d$ is a collection of i-faces, for $-1 \leq i \leq d$, that satisfy the following conditions;*

I. If f_{i-1} and f_{i+1} are incident faces then there exists ***at least two*** *intermediary faces, f_i, f_i', such that $f_{i-1} \subset f_i$, $f_i' \subset f_{i+1}$;*

II. If f_i, f_j are two faces then they are connected by some sequence of successively incident faces $f_{a_1}, f_{a_2}, ..., f_{a_n}$ such that $f_{a_1} = f_i$ and $f_{a_n} = f_j$.

To continue the analogy of real and complex space it is worth considering the difference between real and complex reflections.

4.2 Defining a Complex Reflection

In a real space a reflection is an involution but this is not the defining characteristic. To develop a complex reflection as an analogy of a real reflection we need to be clear about which properties we want to preserve.

A **real reflection** in $\mathbb{R}^d$ is a linear transformation that satisfies the following conditions;

(i). There is a hyperplane, H, which is fixed by the reflection;

(ii). The reflection exchanges the two pieces of the complement of H, namely it is of order two.

The first condition is maintained for a complex reflection - a hyperplane is fixed. To see why the second condition is different consider reflections acting on the one-dimensional real and complex spaces. In both cases the reflection will fix a zero-dimensional space, namely the origin. The real reflection acts on $0^{\perp} = \mathbb{R}/\{0\}$ by multiplying by -1 while the complex reflection can multiply x by any root of unity and still satisfy the condition of being a linear transformation that fixes a hyperplane.

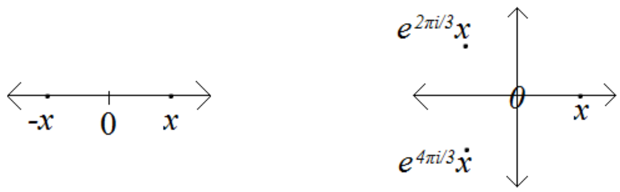


Figure 4.1: Reflection in $\mathbb{R}^1$ and $\mathbb{C}^1$.

In $\mathbb{R}$ the complement of 0 is split into two pieces and so it makes sense to talk of a reflection as an operation which exchanges these two halves. However in $\mathbb{C}$ the complement of 0 does not have the same structure to it. Instead of thinking of the reflection in $\mathbb{R}^1$ as something that switches two pieces we can think of it as an operation that multiplies everything by -1. In $\mathbb{R}$ the only non-trivial root of unity is -1 however in $\mathbb{C}$ there are roots of unity of order n, for all $n \geq 2$.

The important feature of what makes something a reflection is best seen in terms of a fixed mirror, or hyperplane.

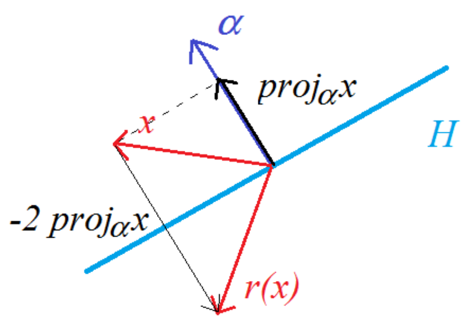
Definition 9. *A **complex reflection** in $\mathbb{C}^d$ is a linear transformation $r : \mathbb{C}^d \to \mathbb{C}^d$ that leaves a hyperplane H pointwise invariant.*

An appropriate change of basis or adoption of a different hermitian form will mean that a reflection of finite order can be made into a unitary transformation and so we will refer to such reflections as unitary transformations from now on.

4.2.1 Reflections and Roots

The modern treatment of reflections and complex reflections expresses unitary transformations in terms of root systems. In particular a root of the reflection r of order m is an eigenvector α such that $r(\alpha) = \xi\alpha$ where ξ is an m-th root of unity.

In $\mathbb{R}^d$ a reflection r of order 2 fixes a hyperplane H with normal vector α which will be shown to be an eigenvector of the transformation. The reflection r then has the following representation, where $\xi = -1$ is the root of unity of order 2, the same order as the



$$\begin{aligned} r(x) &= x - 2\mathrm{proj}_\alpha x \\ &= x - 2\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \\ &= x + (\xi - 1)\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \end{aligned}$$

reflection r. From this it follows that for all $v \in H$, $\mathrm{proj}_\alpha v = 0$ and hence $r(v) = v$ so the hyperplane is pointwise fixed. Furthermore $r(\alpha) = -\alpha = \xi\alpha$.

In $\mathbb{C}^d$ the notion of a reflection is more general but the matrix representation of the linear transformation is still diagonalizable. There is a more general concept of a reflection where it is non diagonalizable but given that $\mathrm{char}\mathbb{C} = 0$ and that we will only be considering reflections of finite order, all of the reflections in this thesis are diagonalizable [Kan01].

Similar to the case in $\mathbb{R}^d$ a reflection $r : \mathbb{C}^d \to \mathbb{C}^d$ of order m has the following representation,

$$r(x) = x + (\xi - 1)\frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha,$$

where $\xi^m = 1$ and α is the eigenvector corresponding to the eigenvalue ξ, namely $r(\alpha) = \xi\alpha$. Let H be the hyperplane that is pointwise fixed by r, then $\alpha \notin H$ and by Maschke's theorem $\mathbb{C}^d = H \oplus \mathbb{C}\alpha$. Moreover for all $v \in H$,

$$\langle v, \alpha\rangle = \langle r(v), r(\alpha)\rangle = \langle v, \xi\alpha\rangle = \overline{\xi}\,\langle v, \alpha\rangle\,,$$

hence $\langle v, \alpha\rangle = 0$ so the root vector is orthogonal to the hyperplane H.

If r is a reflection of order m and $\xi^m = 1$ is an eigenvalue for r then r^k is a reflection for all $1 \leq k \leq m-1$ where

$$r^k(x) = x + (1 + \xi + \cdots \xi^{k-1})(\xi - 1)\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

If r is a reflection with hyperplane H and $g \in \mathcal{GL}(d, \mathbb{F})$ then the conjugate transformation grg^{-1} is also a reflection with hyperplane gH.

4.3 Going From Polytopes to Reflections

Beginning with the structure of a polytope we need to isolate a special subset of the i-faces with which to investigate the symmetries.

Definition 10. *A **flag**, F, of a polytope in $\mathbb{C}^d$ is a sequence of successively incident i-faces, f_i for $-1 \leq i \leq d$ where f_{-1} is the empty face which is incident with all faces.*

$$F = (f_{-1}, f_0, f_1, ..., f_d) \textit{ where } f_i \subset f_{i+1} \textit{ for } -1 \leq i \leq d-1.$$

The symmetry group of a polytope consists of all the linear transformations that map flags to flags.

Definition 11. *A polytope $\mathcal{P}$ is **regular** if the symmetry group, $Sym(\mathcal{P})$, is transitive on the flags. In particular given two flags $F = (f_{-1}, f_0, ..., f_d)$ and $F' = (f_{-1}, f_0', ..., f_d')$ then there exists some unitary transformation $g \in Sym(\mathcal{P})$ such that $g\ f_i = f_i'$ for $0 \leq i \leq d$.*

To motivate the construction of reflections in the symmetry group of a complex polytope we will begin by considering the symmetry group of the cube in $\mathbb{R}^3$.

4.3.1 Symmetries of the Real Cube

A flag for the cube in $\mathbb{R}^3$ will be made up of any mutually incident vertex, edge and face (along with the cube itself and the empty face). The sequences $F = (f_{-1}, f_0, f_1, f_2, f_3)$ and $F' = (f_{-1}, f_0, f_1', f_2, f_3)$ are both flags for the cube.

The symmetry group of the cube has to be transitive on its flags. The most basic example of two different flags is the one pictured above where they only differ in one place. Hence

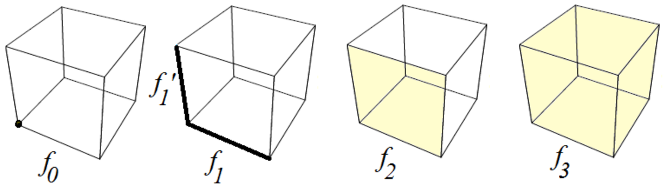


Figure 4.2: Two flags for the cube.

we will define the linear transformations in terms of how they act on a pair of flags such as F and F'.

In general, given the flag $F = (f_{-1}, f_0, ..., f_3)$ of the cube consider the sequence of points $C = (0_0, 0_1, 0_2, 0_3)$ such that 0_i is the point at the centre of the face f_i. Define the transformation R_j such that it only change the j-th flag element and hence only fixes O_i for all $i \neq j$.

For instance consider the flags F and F' pictured above that only differ in the f_1 position. Define the linear transformation R_1 which acts as follows

$$f_1 \mapsto f_1' \quad f_i \mapsto f_i, i \neq 1, \qquad \text{which is the same as} \qquad F \mapsto F'.$$

As R_1 fixes the three centers O_0, O_2 and O_3 which are not collinear based on the structure of the polytope then R_1 fixes a plane in $\mathbb{R}^3$.

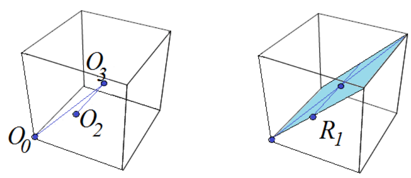


Figure 4.3: Mirror for R_1.

By requiring that R_1 maps the flag F to the flag F', it follows that R_1 is a linear transformation that fixes a plane in $\mathbb{R}^3$, hence R_1 is a reflection. So by beginning with the structural definition of a polytope we single out the fact that the generating linear transformations, which we call reflections, fix a hyperplane.

The symmetries of the cube are reflections and rotations. It is well known that a rotation by θ can be achieved by the product of two reflections whose mirrors are at an angle of $\theta/2$. Hence the symmetry group of the cube can be generated by reflections.

As the cube is in 3-dimensions there are 3 generating reflections which are pictured below. In each case R_j permutes two f_j faces as seen in yellow.

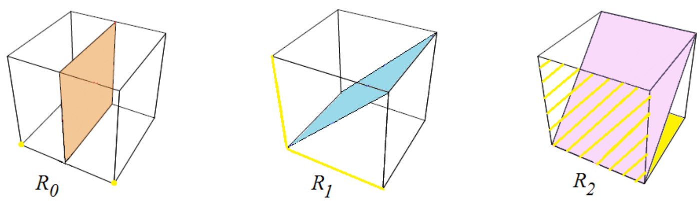
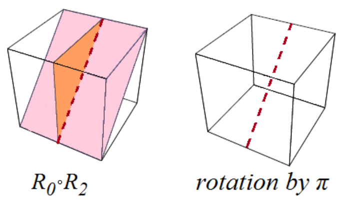


Figure 4.4: Generating reflections for the symmetry group.

Using those three generating symmetries we can produce all the symmetries of the cube. For instance given that the mirrors of R_0 and R_2 are at an angle of $\frac{\pi}{2}$ the composition $R_0 \circ R_2$ produces a rotation of π about their intersection.



From the definition of a real polytope there is only one flag which will differ from F in only one place. Hence a reflection can only interchange the two flags and will only have order two.

Let $\mathcal{P}$ be a complex polytope with three flags F_1, F_2, F_3 which only differ in the i-th place. This is possible because of the definition of a complex polytope that requires at least two i-faces between any fixed incident pair of faces f_{i-1}, f_{i+1}. So there can be 3 flags that only differ in the i-th place.

There will exist a linear transformation in the generators of the symmetry group of $\mathcal{P}$ with order 3 as it will interchange the 3 flags. Hence we end up with complex reflections that have order greater than 2.

4.3.2 Symmetries of a Regular Complex Polytope

A regular complex polytope in $\mathbb{C}^d$ requires d generators for the symmetry group. Consider the flag

$$F = (f_{-1}, f_0, ..., f_d)$$

and the corresponding sequence of centres

$$C = (0_0, ..., 0_d).$$

The complex reflection R_i, for $0 \leq i \leq d-1$, will be the unitary transformation which fixes the d points 0_j for $j \neq i$ and the hyperplane defined by them. The order of R_i is determined by the structure of the polytope.

As f_{i-1} and f_{i+1} are in the flag F then they are incident. By definition of a complex polytope there exists at least two intermediary i-faces. If there are n intermediary i-faces then there are n flags, including F, which differ from each other in the i-th place, namely $F_k = (f_{-1}, ..., f_{i-1}, f_i^{(k)}, f_{i+1}, ..., f_d)$.

Let $F_n = F$ and $R_i^k(F) = F_k$ for $1 \leq k \leq n$ then R_i has order n.

4.4 Going From Reflections to Polytopes

If we start with reflections we need to consider how the arrangements of multiple mirrors can create structures in space by starting with one vector or point and reflecting it repeatedly. In the following picture two reflections spaced $\frac{\pi}{4}$ apart generate the vertices of a square as an orbit under one vertex that lies in the mirror of one of the reflections.

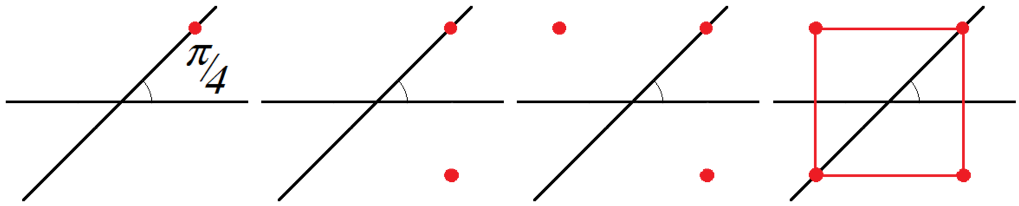


Figure 4.5: Generating a Square from Two Reflections and One Vertex

Given the two reflections that produced the square we can consider how the reflections act on the structure of the square. The blue reflection permutes two edges around a vertex while the yellow reflection permutes two vertices around an edge. It should be noted that the edges were not created under the action, only the collection of vertices. For the tight frames we are only interested in the collection of vertices but to see the relationship between reflections and polytopes it is useful to include the edges for now.

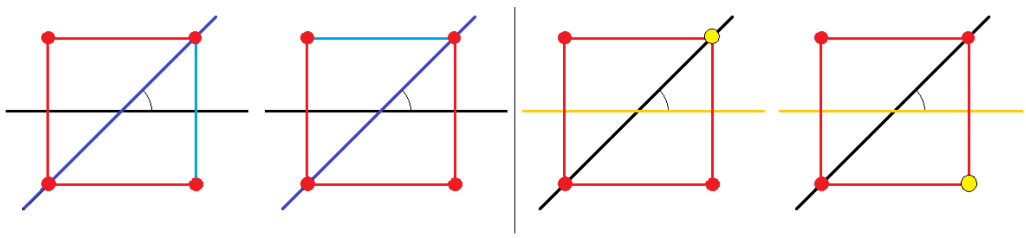


Figure 4.6: Reflections acting on the Square

All of that information is captured in the Schläfli symbol $2\{4\}2$ where the first 2 denotes the number of vertices on an edge, the 4 refers to the angle between the two generating reflections and the last two is the number of edges through each vertex. A real polygon will always have the form $2\{q\}2$ but a complex polygon may look like $p_1\{q\}p_2$.

Reflections in a real space can only permute 2 vertices around a given edge or 2 edges around a given vertex. However a complex reflection can permute p_1 vertices around a given edge or p_2 edges around a vertex. Hence the Schläfli symbol captures some of the structure of the complex polygon.

The complex square 3{2}2 has three vertices on each edge corresponding to the third roots of unity $\omega = e^{\frac{2\pi i}{3}}$. There are two edges through each vertex and adjacent edges are at an angle of $\frac{\pi}{2}$. In the following pictures the vertical edges are shown in orange and the horizontal edges in blue.

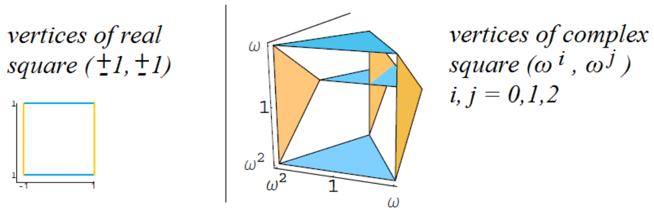


Figure 4.7: Complex Cube

As a complex polygon in $\mathbb{C}^2$ should be projected into $\mathbb{R}^4$ the following diagram is only a loose schematic. The complex reflection $x = 0$ is of order 3 and will act like the real reflection $x = 0$ in that it will both permute the vertices on a horizontal edge and permute the vertical edges.

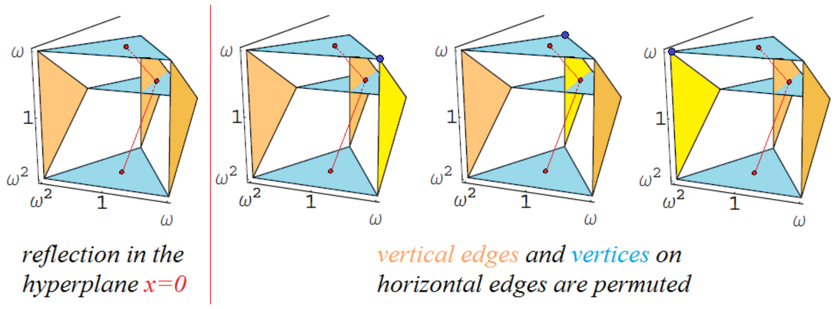


Figure 4.8: Reflection in the Complex Square

4.4.1 Schläfli Symbol

The Schläfli symbol for the complex polygon $p_1\{q\}p_2$ indicates the possible local structure of the polytope. In particular by specifying the number and arrangement of i-faces. However not all combinations are possible. There are only three infinite families of complex polygons and then a finite number of exceptions. These restrictions are derived from the considerations of the placement of mirrors. Consider the abstract symmetry group for

$p_1\{q\}p_2$ which has the following presentation [Cox91],

$$R_1^{p_1} = R_2^{p_2} = 1 \quad \underbrace{R_0R_1R_0...}_{q} = \underbrace{R_1R_0R_1...}_{q}.$$

The relationships between these reflections are determined by considering the angles in a spherical triangle. Hence we can derive the conditions

$$p_i > 1, \quad q > 2 \qquad \frac{1}{p_1} + \frac{2}{q} + \frac{1}{p_2} > 1.$$

From these conditions we see that there are very few possible complex polygons $p_1\{q\}p_2$. The three infinite families are the real polygons $2\{q\}2$, the complex cube $p\{4\}2$ and its dual the complex cross $2\{4\}p$. Along with these there are 31 regular non-starry complex polygons although some of them share the same symmetry group.

More generally the Schläfli symbol of a complex polytope $\mathcal{P}$ in $\mathbb{C}^d$ is $p_1\{q_1\}p_2\{q_2\}\cdots\{q_{d-1}\}p_d$. In the complex polygon p_2 is the vertex figure - it represents the structure that takes place around a given vertex, in the 2-dimensional case that is just a matter of how many edges go through the vertex. In $\mathbb{C}^d$ the vertex figure is $p_2\{q_2\}\cdots\{q_{d-1}\}p_d$ and the cell figure is $p_1\{q_1\}\cdots p_{d-2}\{q_{d-2}\}_{d-1}\}$.

The symmetry group of the polytope $\mathcal{P}$ is generated by d reflections $R_1, \cdots, R_d$ which satisfy the following relationships,

$$R_i^{p_i} = 1 \quad \underbrace{R_iR_{i+1}R_i...}_{q_i} = \underbrace{R_{i+1}R_iR_{i+1}...}_{q_i} \quad R_iR_j = R_jR_i, j > i+1.$$

For this to be a polytope the symmetry group has to be finite and the values in the Schläfli symbol have to satisfy the condition of the polygons,

$$p_i > 1, \quad q_i > 2 \qquad \frac{1}{p_i} + \frac{2}{q_i} + \frac{1}{p_{i+1}} > 1.$$

In dimension $d > 4$ there are only three polytopes; the simplex $\alpha_d = 2\{3\}2...2\{3\}2$, the generalized cross $\beta_d^p = 2\{3\}2...2\{3\}2\{4\}p$ and the generalized cube $\gamma_d^p = p\{4\}2\{3\}2...2\{3\}2$.

There are a finite number of regular complex polytopes that do not fall into those three infinite families and these all occur in dimensions 2,3 and 4.

4.5 Irreducible Complex Reflection Groups

The symmetry groups of the regular complex polytopes are only one example of the class of groups called the irreducible complex reflection groups. These groups were fully classified by Shephard and Todd in 1954 [ST54] and so we will now look at the irreducible complex reflection groups.

There is a sub-categorisation of the irreducible complex reflection groups based on whether they are imprimitive or primitive. The three infinite families of polytopes all fall under the imprimitive groups which is an infinite family of groups. The primitive groups contain 34 exceptional cases in dimensions 2-8 and these include the symmetry groups of all the regular complex polytopes that are not in the infinite families. The primitive groups do not have a general description.

4.5.1 Imprimitive Complex Reflection Groups

Definition 12. *A group G of unitary transformations on $\mathbb{C}^d$ is* ***imprimitive*** *if*

$$\mathbb{C}^d = E_1 \oplus E_2 \oplus ... \oplus E_k$$

where the family $\{E_1, ..., E_k\}$ of non-trivial proper subspaces, which is called the system of imprimitivity, is invariant under G.

As the groups we are looking at are generated by reflections then this gives us some more information about the system of imprimitivity $\{E_1, ..., E_k\}$ [LT09].

Lemma 3. *If G is an imprimitive irreducible complex reflection group in $\mathbb{C}^d$ and $\{E_1, ..., E_k\}$ is a system of imprimitivity for G, then $k = d$ and for all i, $dim(E_i) = 1$.*

Proof. As G is irreducible it has to act transitively on $\{E_1, ..., E_k\}$. Given $1 \leq i \leq k$ there exists a reflection $r \in G$ such that $rE_i \neq E_i$. As r is a reflection then it fixes a hyperplane whose intersection with E_i is equal to $\{0\}$. Hence $\dim(E_i) = 1$ for all i which implies that $k = d$. □

The groups we are looking at are denoted $G(m, p, d)$ where $p|m$ and they act on the space $\mathbb{C}^d$. If $m, d > 1$ and $(m, p, d) \neq (2, 2, 2)$ then $G(m, p, d)$ is an imprimitive irreducible complex reflection group. The group $G(2, 2, 2)$ is more familiarly known as the klein four group and it is not irreducible. While $G(1, 1, d)$ is not irreducible it is interesting because of its connection to the family of simplexes, similarly the cyclic groups $G(m, 1, 1)$ are a familiar group and so they are listed here for completeness.

One presentation of the groups is as follows;

$$\begin{aligned}
G(1,1,d) &= \langle r_1, \cdots r_{d-1} \rangle \approx S_d; \\
G(m,1,1) &= \langle \omega \rangle \quad \text{ where } \omega = e^{\frac{2\pi i}{m}}, m > 1; \\
G(m,1,d) &= \langle t, r_1, \cdots r_{d-1} \rangle ; \\
G(m,m,d) &= \langle s, r_1, \cdots r_{d-1} \rangle ; \\
G(m,p,d) &= \langle s, t^p, r_1, \cdots r_{d-1} \rangle , \quad p \neq 1, m.
\end{aligned}$$

In the presentation above r_i is a reflection of order 2 which interchanges the basis elements e_i and e_{i+1}, t is a reflection of order m such that $e_1 \mapsto \omega e_1$ where $\omega = e^{\frac{2\pi i}{m}}$ and $s = t^{-1} r_1 t$. In particular,

$$r_1 = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad t = \begin{bmatrix} \omega & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad s = \begin{bmatrix} 0 & \overline{\omega} & & & \\ \omega & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

5

Highly Symmetric Tight Frames

5.1 Symmetry Groups of Tight Frames

In the context of a tight frame we are considering a sequence of vectors that are indexed by a finite set $\mathcal{I}$. A symmetry of the frame is a linear transformation mapping frame elements to frame elements which acts like a permutation on the index set. The following definition single out this permutation aspect and uses the permutation to index the linear transformation [VW05].

Definition 13. *Let $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ be the set of N frame elements in $\mathbb{F}^d$. The **symmetry group** of Φ is*

$$Sym(\Phi) := \{\sigma \in S_N | \quad \exists L_\sigma \in \mathcal{GL}(\mathbb{F}, d) \quad L_\sigma \phi_i = \phi_{\sigma i} \quad \forall i \in \mathcal{I}\}$$

As a frame spans $\mathbb{F}^d$ and linear maps are determined by their action on a spanning set then the permutation σ of frame elements determines L_σ. As we are considering the set of frame elements without repetition then for each σ in $Sym(\Phi)$ there is a unique L_σ. The action of the group $Sym(\Phi)$ is defined in terms of the unitary representation

$$\rho : Sym(\Phi) \to \mathcal{GL}(d, \mathbb{F})$$

such that $\rho(\sigma) = L_\sigma$ and the action of $Sym(\Phi)$ on $\mathbb{F}^d$ is, like any representation, $\sigma v := \rho(\sigma)v = L_\sigma v$.

Due to the uniqueness of L_σ the representation ρ is injective and so the representation of $Sym(\Phi)$ is faithfully. If Φ is the set of frame elements without repetition then $|G| = |\Phi|$. As G contains unitary transformations which exchange the frame elements then there is a representation of $Sym(\Phi)$ which includes G, therefore $|Sym(\Phi)| \geq |G| = |\Phi|$. In considering what it means for Φ to possess a high degree of symmetry we want $|Sym(\Phi)| > |\Phi|$.

5.2 Defining Highly Symmetric Tight Frames

A motivating example for the highly symmetric tight frames is the tight frame composed of the vectors that go to the vertices of a platonic solid and more generally the vertices of a regular complex polytope. We want to define a tight frame not in terms of a group but in terms of how a group acts on the vector space. For instance any vector v going from the center of the polytope $\mathcal{P}$ in $\mathbb{F}^d$ to a vertex will lie in the mirrors of $d-1$ reflections. This means the subgroup of the symmetry group of $\mathcal{P}$ generated by those $d-1$ reflections fixes a 1-dimensional space spanned by v. More generally for any vector v in the tight frame there is some subgroup of $Sym(\mathcal{P})$ which fixes the 1-dimensional space spanned by v. Furthermore, as the definition of $Sym(\mathcal{P})$ requires that it is transitive on its flags then it is transitive on the frame elements which represent the vertices of $\mathcal{P}$.

Definition 14. *A finite tight frame Φ of distinct vectors is **highly symmetric** if;*

(i). The action of the symmetry group $Sym(\Phi)$ is irreducible, transitive and faithful;

*(ii). The stabiliser of any vector in Φ is a non-trivial subgroup that fixes a space of dimension **exactly one**.*

We required three things from the action of $Sym(\Phi)$. Firstly to say that the action is irreducible means that for all $v \neq 0 \in \mathbb{F}^d$, $\text{span}\{L_\sigma v : \sigma \in Sym(\Phi)\} = \mathbb{F}^d$. Secondly the transitivity condition requires that for all $i, j \in \mathcal{I}$ there exists $\sigma \in Sym(\Phi)$ such that $L_\sigma \phi_i = \phi_j$ and $L_\sigma \phi_j = \phi_i$. The transitivity condition does not specify how L_σ acts on the rest of the frame. Lastly the faithfulness follows from the fact we are considering a frame of distinct vectors so the representation is injective.

Example: The Mercedes-Benz frame of 3 vectors in $\mathbb{R}^2$ pictured in the introduction is a highly symmetric tight frame. Its symmetry group is the dihedral group of order 6 which is irreducible, transitive and faithful. Also given any of the frame vectors v there is a reflection which fixes the line spanned by v.

Example: The orthonormal basis $\Phi = \{e_1, e_2, ..., e_d\}$ in $\mathbb{C}^d$ is **not** a highly symmetric tight frame. $Sym(\Phi)$ fixes the vector $e_1 + e_2 + \cdots + e_d$ and hence no subgroup can fix the one dimensional space spanned by e_i for any i.

The Mercedes-Benz frame, which is an example of a simplex, and an orthonormal basis are both harmonic frames. This demonstrates that it is possible to choose whether a highly symmetric tight frame is harmonic.

An important feature of the highly symmetric tight frames is that they have a finiteness property like the harmonic frames.

Theorem 4. *There are only finitely many normalised highly symmetric tight frames of N vectors in $\mathbb{F}^d$ up to unitary equivalence.*

Proof. As representations of $Sym(\Phi)$ and its subgroups are what determine Φ by definition then we only need to count the representations and subgroups. If Φ is a normalised highly symmetric tight frame of N vectors then $|Sym(\Phi)| \leq |S_N| = N!$ so there are only finitely many choices for the symmetry group of Φ. By definition of being a highly symmetric tight frame every vector in Φ is fixed by a non-trivial subgroup of some irreducible representation of $Sym(\Phi)$. There are only finitely many irreducible representations of

$Sym(\Phi)$ and hence only finitely many subgroups that can fix a one-dimensional space. Hence there can only be finitely many highly symmetric tight frames. $\square$

6

Computations of Highly Symmetric Tight Frames

Having outlined what defines a highly symmetric tight frame we will consider some examples that arise from calculations done with the irreducible complex reflection groups. The irreducible complex reflection groups provide an example of one sort of irreducible representation where all the generators are represented as reflections.

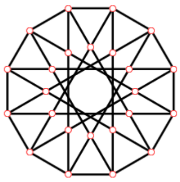
The following examples highlight some of the possible relationships between highly symmetric tight frames and the classes of harmonic and equiangular frames.

6.1 24 Vectors in $\mathbb{C}^2$

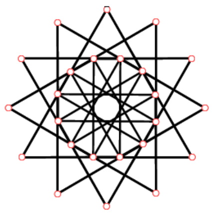
Even within the highly symmetric tight frames obtained from irreducible complex reflection groups there is a notable difference in the frame depending on the different group structures. In particular whether we look at the primitive or the imprimitive groups. This demonstrates the influence of the particular group representation.

In $\mathbb{C}^2$ there are 6 different highly symmetric tight frames of 24 vectors - one comes from the imprimitive group $G(12, 1, 2)$, the others come from the primitive Shephard Todd groups numbers 5, 6, 8 and 12.

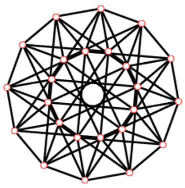
Of the five frames from the primitive groups four of them are complex polygons, pictured below, although none of them are harmonic.



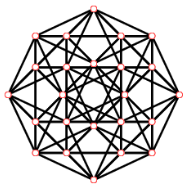
$3\{6\}2 \quad ST(6) = \langle 48, 33\rangle$



$3\{3\}2 \quad ST(6) = \langle 48, 33\rangle$



$3\{4\}3 \quad ST(5) = \langle 72, 25\rangle$



$4\{3\}4 \quad ST(8) = \langle 96, 67\rangle$

By contrast the generalised cross of 24 vectors that comes from the imprimitive group is harmonic. This points to a contrast between the imprimitive and primitive groups which can filter through to the frames themselves.

6.2 The Cross and the Cube

As was seen in the last example the generalized cross that comes from the symmetry group $G(12, 1, 2)$ gave a harmonic frame. We will show that the generalized cross and cube are harmonic frames in all dimensions.

6.2.1 Defining and Picturing the Cross and Cube

The cube and its dual, the octahedron or cross, are familiar polyhedra in $\mathbb{R}^3$ and their symmetry group is $G(2,1,3)$ where the cube has 2^3 vertices and the octahedron has 2×3 vertices.

In a similar pattern $G(m,1,d)$ is the symmetry group for the generalized cube γ_d^m and its dual, the generalized cross β_d^m in d dimensions with m-th roots of unity. These different frames arise from the same symmetry group by using different initial vectors. Given that $G(m,1,d)$ acts on $\mathbb{C}^d$ consider the following vectors,

$$v_k := e_1 + \cdots + e_k, \quad 1 \leq k \leq d.$$

Given v_k it is fixed by the $k!$ permutations of the elements e_1 to e_k that are present in its expression and the $(d-k)!$ permutations of the absent elements. Also multiplying any of the absent $(d-k)$ elements by any of the m-th roots of unity leaves v_k unchanged and hence $|Stab(v_k)| = k!(d-k)!m^{d-k}$.

The order of the symmetry group $G(m,1,d)$ is $m^d d!$ and so the orbit of v_k has size $\binom{d}{k} m^k$. In particular the orbit of v_1 gives the $m \times d$ vertices of the generalized cross and the orbit of v_d gives the m^d vertices of the cube.

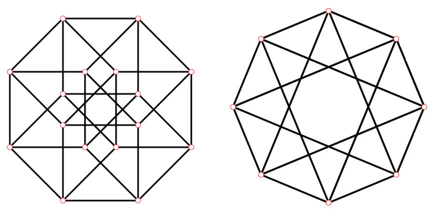


Figure 6.1: The Cube γ_2^4 and Cross β_2^4

6.2.2 Relationship to Harmonic Frames

What is interesting about the generalized cross and cube is that they are both harmonic frames. This means the vertices of the cube, or cross, can be obtained as an orbit under some abelian group. The cube is generated by the group $G_\gamma = \langle q_1, \cdots, q_d \rangle$ where q_j is the reflection defined by,

$$e_j \mapsto \omega e_j \qquad q_j := (r_{j-1}r_{j-2}\cdots r_1)t(r_1 \cdots r_{j-2}r_{j-1}),$$

where r_j and t come from the generators of the imprimitive group that were defined earlier and the elements q_i and q_j commute for all i, j.

An arbitrary vector in γ_d^m looks like $u = [\omega^{a_1}, \omega^{a_2}, \cdots, \omega^{a_d}]$ where ω is an m-th root of unity. Under the action of G_γ

$$u = q_1^{a_1} q_2^{a_2} \cdots q_d^{a_d} v_d,$$

hence all vertices of the generalized cube can be obtained through the action of the abelian group G_γ on v_d.

The cross is generated by the cyclic group $G_\beta = \langle a \rangle$ where the generator is defined in terms of the generating relations of the imprimitive group as

$$a := r_1 r_2 \cdots r_{d-1} t \qquad e_1 \mapsto \omega e_2, \quad e_i \mapsto e_{i+1} \text{ for } i \neq 1.$$

As an arbitrary vertex u in the cross has one non-zero entry ω^i in the j-th position and $a^{d(i-1)} v_1 = [\omega^{i-1}, 0, \cdots, 0]$, $a^{d(i-1)+1} v_1 = [0, \omega^i, \cdots, 0]$ then

$$u = \begin{cases} a^{d(i-1)+j-1} v_1, & j \neq 1 \\ a^{di}, & j = 1 \end{cases}.$$

For $d = 2$ the only highly symmetric tight frames under $G(m, 1, 2)$ are the generalized cross and cube. That is because the only initial vectors of interest are v_1, whose orbit gives the cross, and v_2 whose orbit is the cube. If we consider $d = 3$ there will be three orbits of interest, but only the cross and the cube have been shown to be harmonic.

In the following table $\square$ is used to denote the generalized cube and $\times$ denotes the generalized cross. The calculations find whether there is any subgroup, H, of $G(m,1,3)$ such that the order of H is exactly equal to the number of vectors in the tight frame and that tight frame can be found as an orbit of the same initial vector under the group action of H. If such a subgroup exists the abstract group is listed and it is further classified depending on whether that group is abelian or not.

G(m,p,d)	Abstract Group	Orbit Size	G-Frame	Harmonic G-Frame
(2,1,3)	$\langle 48,48\rangle$	6 $\times$	$\langle 6,1\rangle$	$\langle 6,2\rangle$
		8 $\square$	$\langle 8,3\rangle$	$\langle 8,2\rangle$, $\langle 8,5\rangle$
		12	$\langle 12,3\rangle$	-
(3,1,3)	$\langle 162,10\rangle$	9 $\times$	-	$\langle 9,1\rangle$, $\langle 9,2\rangle$
		27	$\langle 27,4\rangle$, $\langle 27,3\rangle$	-
		27 $\square$	$\langle 27,4\rangle$	$\langle 27,5\rangle$
(4,1,3)	$\langle 384,5557\rangle$	12 $\times$	$\langle 12,1\rangle$	$\langle 12,2\rangle$
		48	$\langle 48,3\rangle$	-
		64 $\square$	$\langle 64,20\rangle$, $\langle 64,85\rangle$	$\langle 64,55\rangle$
(5,1,3)	$\langle 750,26\rangle$	15 $\times$	-	$\langle 15,1\rangle$
		75	$\langle 75,2\rangle$	-
		125 $\square$	-	$\langle 125,5\rangle$
(6,1,3)	$\langle 1296,1827\rangle$	18 $\times$	$\langle 18,3\rangle$	$\langle 18,2\rangle$, $\langle 18,5\rangle$
		108	$\langle 108,21\rangle$, $\langle 108,22\rangle$	-
		216 $\square$	$\langle 216,106\rangle$, $\langle 216,138\rangle$, $\langle 216,139\rangle$	$\langle 216,177\rangle$

These calculations are not sufficient to conclude that the tight frame of 12 vectors, Φ_{12}, obtained from $G(2,1,3)$ is not harmonic. All this shows is that there are no abelian subgroups of $G(2,1,3)$ which can generate Φ_{12}. However the symmetry group of Φ_{12} may be larger than $G(2,1,3)$ and so we can compare $G(2,1,3) = \langle 48,48\rangle$ with the known list of symmetry groups for harmonic tight frames of 12 vectors in dimension 3 [HW06].

The only symmetry groups with order greater than 48 on the list in [HW06] are $\langle 72, 30\rangle$, $\langle 72, 42\rangle$ and $\langle 384, 5557\rangle$. As $48 \nmid 72$ the only potential candidate is $\langle 384, 5557\rangle = G(4, 1, 3)$ which is on the list in reference to the generalized cross β_3^4 which contains different vectors to Φ_{12}. Based on that comparison we can conclude that Φ_{12} is not a harmonic frame.

The comparative analysis above soon runs into problems as Magma is not able to identify groups of order greater than 2000. Let Φ_{27} be the frame of size 27 obtained from $G(3, 1, 3) = \langle 162, 10\rangle$ which is not the cube γ_3^3. In the list of symmetry groups for harmonic frames of size 27 there is an unidentified group of order 4374. Given $162|4374$ we can not definitively conclude that Φ_{27} is not harmonic.

Similarly for the frame of size 48 obtained from $G(4, 1, 3) = \langle 384, 5557\rangle$ we need to consider an unidentified group of order 42576 which is a multiple of 384. This is currently well beyond the bounds of group calculation tools.

The groups $G(m, p, 3)$ for $p \neq 1$ have been left off the table above because $G(m, p, d) \lhd G(m, 1, d)$. Hence any highly symmetric tight frame from $G(m, p, d)$ either already occurs in $G(m, 1, d)$ or it is a subframe of one that does.

Conjecture: *If $m, d > 1$ the only highly symmetric tight frames obtained from $G(m, 1, d)$ that are also harmonic frames are the generalized cube and cross.*

6.3 Heisenberg and the Imprimitive Groups

There is also a connection between the imprimitive group $G(d, 1, d)$ and the Heisenberg group H_d in $\mathbb{C}^d$. In particular $H_d \leq G(d, 1, d)$ where H_d is generated by S and Ω,

$$S = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix} = r_1 \cdots r_d,$$

$$\Omega = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \omega^{d-1} \end{bmatrix} = (r_{d-1} \cdots r_1 t^{d-1})(r_{d-2} \cdots r_1 t^{d-2}) \cdots (r_1 t),$$

where r_i and t are the generators of $G(d, 1, d)$ where $\omega = e^{\frac{2\pi i}{d}}$.

In dimensions $d \geq 4$ H_d does not contain any pseudoreflections but it is a subgroup of a group generated by pseudoreflections.

In 3-dimensions the symmetry group of the Hessian polytope 3{3}3{3}3 (Shephard Todd number 25) is the normaliser of H_3. That is important because a lot of the work on equiangular tight frames has focused on matrices in the normaliser of the Heisenberg group.

Despite this connection between the imprimitive groups, the Hessian and the Heisenberg groups there are not any highly symmetric tight frames which are also maximal equiangular frames. Although almost all the highly symmetric tight frames are not equiangular we can still compare the frame size with the upper bound that is calculated in terms of the number of angles.

6.4 Angles and Frame Size

Consider the set of moduli of angles in the frame, $A := \{|\langle \phi_i, \phi_j \rangle| : i \neq j \in \mathcal{I}\}$. By counting $|A|$ we can find an upper bound for a frame with that many angles in a given dimension.

Theorem 5. *Let $\Phi = \{\phi_i\}_{i \in \mathcal{I}}$ be a normalised tight frame in $\mathbb{C}^d$ with scalar multiples of frame elements removed. If $A := \{|\langle \phi_i, \phi_j \rangle| : i \neq j \in \mathcal{I}\}$ is the set of moduli of angles in the frame and $s := |A|$ then*

$$|\Phi| \leq \binom{d+s-1}{s}^2$$

If Φ is equiangular then $|A| = 1$ so $|\Phi| \leq d^2$. When more angles are involved we can have larger frames, in fact increasing the dimension while keeping s fixed rapidly increases the upper bound.

In the equiangular case the current work indicates that it would be possible to achieve this upper bound. However in the real case the upper bounds for frame sizes based on angles are scarcely reached. It is not known what the accuracy of the upper bound is from Theorem 1.

The information about $s := |A|$ and the corresponding upper bound on the frame size is recorded in the tables. For instance in Shephard Todd group 13 there are two distinct orbits of size 48 which are referred to as a and b. There are 9 different angles in a and 11 in b which is recorded in the column 'Orbit x: #Moduli'. From that calculation the corresponding upper bounds of 100 for orbit a and 144 for orbit b are obtained and presented in 'Max #Orb'.

None of the frames achieve their upper bound and so they are not maximal. In fact as the dimension increases the upper bound increases much quicker than the frame size.

In the table the symbol $\not\simeq$ indicates that two frames are not unitarily equivalent as they have different angle sets. It is important to note that $\simeq$ merely indicates that the two frames have the same angle set which is a necessary but not sufficient condition for unitary equivalence.

6.5 G-frames of Reflection Type

The information in the tables about G-frames is checking whether the frame can be expressed as a G-frame without repetition of vectors. In particular whether there exists a subgroup H with the same size as the orbit such that the set of frame elements can be obtained as an orbit under H. This is only checking for subgroups of the irreducible complex reflection group and so we will refer to these as G-frames of reflection type.

In dimension 2 all of the highly symmetric tight frames are G-frames of reflection type but this becomes increasingly rare in higher dimensions for the primitive frames.

(When calculation limits of Magma are encountered the missing results are marked *).

Table 6.1: **Primitive Groups in Dimension 2**

ST	Dim	Abstract Group	#Orb	Max #Orb	Orb: #Moduli	G-Frame
4 (P)	2	$\langle 24,3\rangle$	8	9	a: 2	$\langle 8,4\rangle$
5 (P)		$\langle 72,25\rangle$	24	49	a: 6 $\simeq$ b: 6	$\langle 24,3\rangle$, $\langle 24,11\rangle$
6 (P)		$\langle 48,33\rangle$	16	25	a: 4	$\langle 16,13\rangle$
			24	56	a: 7	$\langle 24,3\rangle$
7		$\langle 144,157\rangle$	48	169	a: 12 $\simeq$ b: 12	$\langle 48,47\rangle$, $\langle 48,33\rangle$
			72	196	a: 13	$\langle 72,25\rangle$
8 (P)		$\langle 96,67\rangle$	24	36	a: 5	$\langle 24,3\rangle$, $\langle 24,1\rangle$
9 (P)		$\langle 192,963\rangle$	48	100	a: 9	$\langle 48,4\rangle$, $\langle 48,28\rangle$, $\langle 48,29\rangle$
			96	441	a: 20	$\langle 96,67\rangle$, $\langle 96,74\rangle$
10 (P)		$\langle 288,400\rangle$	72	196	a: 13	$\langle 72,12\rangle$, $\langle 72,25\rangle$
			96	400	a: 19	$\langle 96,54\rangle$, $\langle 96,67\rangle$
11		$\langle 576,5472\rangle$	144	529	a: 22	$\langle 144,69\rangle$, $\langle 144,121\rangle$, $\langle 144,122\rangle$
			192	1156	a: 33	$\langle 192,876\rangle$, $\langle 192,963\rangle$
			288	2704	a: 51	$\langle 288,400\rangle$, $\langle 288,638\rangle$
12		$\langle 48,29\rangle$	24	49	a: 6	$\langle 24,3\rangle$
13		$\langle 96,192\rangle$	48	a:100 b:144	a: 9 $\not\simeq$ b: 11	a:$\langle 48,28\rangle$, a:$\langle 48,29\rangle$, b:$\langle 48,28\rangle$, b:$\langle 48,33\rangle$
14 (P)		$\langle 144,122\rangle$	48	121	a: 10	$\langle 48,26\rangle$, $\langle 48,29\rangle$
			72	289	a: 16	$\langle 72,25\rangle$
15		$\langle 288,903\rangle$	96	400	a: 19	$\langle 96,182\rangle$, $\langle 96,192\rangle$
			144	a:529 b:1024	a: 22 $\not\simeq$ b: 31	a:$\langle 144,121\rangle$, a:$\langle 144,122\rangle$, b:$\langle 144,121\rangle$, b:$\langle 144,157\rangle$
16 (P)		$\langle 600,54\rangle$	120	169	a: 12	$\langle 120,5\rangle$, $\langle 120,15\rangle$
17 (P)		$\langle 1200,483\rangle$	240	289	a: 16	$\langle 240,93\rangle$, $\langle 240,154\rangle$
			600	3136	a: 55	$\langle 600,54\rangle$
18 (P)		$\langle 1800,328\rangle$	360	441	a: 20	$\langle 360,51\rangle$, $\langle 360,89\rangle$
			600	1600	a: 39	$\langle 600,54\rangle$
19		$\langle 3600,*\rangle$	720	1296	a: 35	$\langle 720,420\rangle$, $\langle 720,708\rangle$
			1200	2401	a: 48	$\langle 1200,483\rangle$
			1800	5929	a: 76	$\langle 1800,328\rangle$
20 (P)		$\langle 360,51\rangle$	120	256	a: 15	$\langle 120,5\rangle$
21 (P)		$\langle 720,420\rangle$	240	729	a: 26	$\langle 240,93\rangle$
			360	1521	a: 38	$\langle 360,51\rangle$
22		$\langle 240,93\rangle$	120	529	a: 22	$\langle 120,5\rangle$

Table 6.2: **Primitive Groups in Dimension 3 to 8**

ST	Dim	Order*	#Orb	Max #Orb	Orb: #Moduli	G-Frame
23 (P)	3	120	12	36	a: 2	$\langle 12, 3\rangle$
			20	100	a: 3	-
			30	441	a: 5	-
24		336	42	225	a: 4	$\langle 42, 2\rangle$
			56	441	a: 5	-
25 (P)		648	27	225	a: 4	$\langle 27, 3\rangle$, $\langle 27, 4\rangle$
			72	441	a: 5	-
26 (P)		1296	54	100	a: 3	$\langle 54, 8\rangle$, $\langle 54, 10\rangle$, $\langle 54, 11\rangle$
			72	100	a: 3	-
			216	784	a: 6	$\langle 216, 88\rangle$
27		2160	216	1296	a: 7	-
			270	4356	a: 10	-
			360	8281	a: 12 $\simeq$ b: 12	-
28 (P)	4	1152	24	400	a: 3 $\simeq$ b: 3	$\langle 24, 1\rangle$, $\langle 24, 3\rangle$, $\langle 24, 11\rangle$
			96	14400	a: 7 $\simeq$ b: 7	$\langle 96, 67\rangle$, $\langle 96, 201\rangle$, $\langle 96, 204\rangle$
29		7680	80	400	a: 3	$\langle 80, 30\rangle$
			160	1225	a: 4	-
			320	7056	a: 6 $\simeq$ b: 6	$\langle 320, 1581\rangle$, $\langle 320, 1586\rangle$
			640	132496	a: 11	-
30 (P)		14400	120	3136	a: 5	$\langle 120, 5\rangle$, $\langle 120, 15\rangle$
			600	938961	a: 16	$\langle 600, 54\rangle$
			720	2371600	a: 19	-
			1200	50979600	a: 33	-
31		46080	240	1225	a: 4	-
			1920	81796	a: 10	$\langle 1920, *\rangle$
			3840	1768900	a: 18	-
32 (P)		155520	240	3136	a: 5	-
			2160	313600	a: 13	-
33	5	51840	80	1225	a: 3	-
			270	15876	a: 5	-
			432	44100	a: 6	-
			1080	3312400	a: 12	-
34	6	39191040	*	*	*	*
35		51840	27	3136	a: 3	$\langle 27, 3\rangle$,$\langle 27, 4\rangle$
			72	3136	a: 3	-
			216	627264	a: 7	$\langle 216, 86\rangle$, $\langle 216, 88\rangle$
			720	627264	a: 7	-
36	7	2903040	*	*	*	*
37	8	696729600	*	*	*	*

Table 6.3: **Small Imprimitive Groups in Dimension 2**

G(m,p,d)	Abstract Group	#Orb	Max #Orb	Orb x: #Moduli	G-Frame	Harmonic G-Frame
(2,1,2)	$\langle 8,3\rangle$	4	9	a: 2 $\simeq$ b: 2	-	$\langle 4,1\rangle$, $\langle 4,2\rangle$
(3,1,2)	$\langle 18,3\rangle$	6	16	a: 3	$\langle 6,1\rangle$	$\langle 6,2\rangle$
		9	25	a: 4	-	$\langle 9,2\rangle$
(3,3,2)	$\langle 6,1\rangle$	3	9	a: 2	-	$\langle 3,1\rangle$
(4,1,2)	$\langle 32,11\rangle$	8	9	a: 2	$\langle 8,3\rangle$, $\langle 8,4\rangle$	$\langle 8,1\rangle$, $\langle 8,2\rangle$
		16	16	a: 3	$\langle 16,6\rangle$	$\langle 16,2\rangle$
(4,2,2)	$\langle 16,13\rangle$	8	9	a: 2 $\simeq$ b: 2 $\simeq$ c: 2	$\langle 8,3\rangle$, $\langle 8,4\rangle$	$\langle 8,2\rangle$
(4,4,2)	$\langle 8,3\rangle$	4	9	a: 2 $\simeq$ b: 2	-	$\langle 4,1\rangle$, $\langle 4,2\rangle$
(5,1,2)	$\langle 50,3\rangle$	10	49	a: 6	$\langle 10,1\rangle$	$\langle 10,2\rangle$
		25	256	a: 15	-	$\langle 25,2\rangle$
(5,5,2)	$\langle 10,1\rangle$	5	16	a: 3	-	$\langle 5,1\rangle$
(6,1,2)	$\langle 72,30\rangle$	12	16	a: 3	$\langle 12,1\rangle$, $\langle 12,4\rangle$	$\langle 12,2\rangle$, $\langle 12,5\rangle$
		36	56	a: 7	-	$\langle 36,6\rangle$, $\langle 36,14\rangle$
(6,2,2)	$\langle 36,12\rangle$	12	16	a: 3	$\langle 12,4\rangle$	$\langle 12,5\rangle$
		18	25	a: 4 $\simeq$ b: 4	$\langle 18,3\rangle$	$\langle 18,5\rangle$
(6,3,2)	$\langle 24,8\rangle$	12	25	a: 4 $\not\simeq$ b: 3	$\langle 12,1\rangle$, b:$\langle 12,4\rangle$	a:$\langle 12,5\rangle$
(6,6,2)	$\langle 12,4\rangle$	6	9	a: 2 $\simeq$ b: 2	$\langle 6,1\rangle$	$\langle 6,2\rangle$
(7,1,2)	$\langle 98,3\rangle$	14	81	a: 8	$\langle 14,1\rangle$	$\langle 14,2\rangle$
		49	841	a: 28	-	$\langle 49,2\rangle$
(7,7,2)	$\langle 14,1\rangle$	7	25	a: 4	-	$\langle 7,1\rangle$
(8,1,2)	$\langle 128,67\rangle$	16	36	a: 5	$\langle 16,6\rangle$, $\langle 16,7\rangle$, $\langle 16,8\rangle$, $\langle 16,9\rangle$	$\langle 16,1\rangle$, $\langle 16,5\rangle$
		64	324	a: 17	$\langle 64,45\rangle$	$\langle 64,2\rangle$
(8,2,2)	$\langle 64,124\rangle$	16	36	a: 5	$\langle 16,6\rangle$, $\langle 16,7\rangle$, $\langle 16,8\rangle$, $\langle 16,9\rangle$	$\langle 16,5\rangle$
		32	100	a: 9 $\simeq$ b: 9	$\langle 32,11\rangle$, $\langle 32,15\rangle$, $\langle 32,38\rangle$	$\langle 32,3\rangle$
(8,4,2)	$\langle 32,42\rangle$	16	36	a: 5 $\simeq$ b: 5, $\not\simeq$ c: 5	$\langle 16,8\rangle$, $\langle 16,9\rangle$, a,b:$\langle 16,13\rangle$, c:$\langle 16,7\rangle$	a,b:$\langle 16,5\rangle$
(8,8,2)	$\langle 16,7\rangle$	8	16	a: 3 $\simeq$ b: 3	$\langle 8,3\rangle$	$\langle 8,1\rangle$

Table 6.4: **Small Imprimitive Groups in Dimension 3**

G(m,p,d)	Abstract Group	#Orb	Max #Orb	Orb x: #Moduli	G-Frame	Harmonic G-Frame
(2,1,3)	$\langle 48, 48\rangle$	6	36	a: 2	$\langle 6, 1\rangle$	$\langle 6, 2\rangle$
		8	36	a: 2	$\langle 8, 3\rangle$	$\langle 8, 2\rangle$, $\langle 8, 5\rangle$
		12	100	a: 3	$\langle 12, 3\rangle$	-
(2,2,3)	$\langle 24, 12\rangle$	4	9	a: 1	-	$\langle 4, 1\rangle$, $\langle 4, 2\rangle$
		6	36	a: 2	-	-
(3,1,3)	$\langle 162, 10\rangle$	9	36	a: 2	-	$\langle 9, 1\rangle$, $\langle 9, 2\rangle$
		27	a:225, b:100	a: 4 $\not\simeq$ b: 3	$\langle 27, 4\rangle$, b:$\langle 27, 3\rangle$	a:$\langle 27, 5\rangle$
(3,3,3)	$\langle 54, 8\rangle$	9	36	a: 2 $\simeq$ b: 2 $\simeq$ c: 2 $\simeq$ d: 2	-	$\langle 9, 2\rangle$
		32	3025	a: 9 $\simeq$ b: 9	$\langle 32, 11\rangle$, $\langle 32, 15\rangle$, $\langle 32, 38\rangle$	$\langle 32, 3\rangle$
(4,1,3)	$\langle 384, 5557\rangle$	12	36	a: 2	$\langle 12, 1\rangle$	$\langle 12, 2\rangle$
		48	225	a: 4	$\langle 48, 3\rangle$	-
		64	100	a: 3	$\langle 64, 20\rangle$, $\langle 64, 85\rangle$	$\langle 64, 55\rangle$
(4,2,3)	$\langle 192, 944\rangle$	12	36	a: 2	-	-
		32	100	a: 3	$\langle 32, 11\rangle$, $\langle 32, 37\rangle$	$\langle 32, 21\rangle$
		48	225	a: 4	$\langle 48, 3\rangle$	-
(4,4,3)	$\langle 96, 64\rangle$	12	36	a: 2	-	-
		16	36	a: 2	$\langle 16, 6\rangle$	$\langle 16, 2\rangle$
(5,1,3)	$\langle 750, 26\rangle$	15	441	a: 5	-	$\langle 15, 1\rangle$
		75	44100	a: 19	$\langle 75, 2\rangle$	-
		125	314721	a: 32	-	$\langle 125, 5\rangle$
(5,5,3)	$\langle 150, 5\rangle$	15	441	a: 5	-	-
		25	784	a: 6	-	$\langle 25, 2\rangle$
(6,1,3)	$\langle 1296, 1827\rangle$	18	100	a: 3	$\langle 18, 3\rangle$	$\langle 18, 2\rangle$, $\langle 18, 5\rangle$
		108	1296	a: 7	$\langle 108, 21\rangle$, $\langle 108, 22\rangle$	-
		216	8281	a: 12	$\langle 216, 106\rangle$, $\langle 216, 138\rangle$, $\langle 216, 139\rangle$	$\langle 216, 177\rangle$
(6,2,3)	$\langle 648, 266\rangle$	18	100	a: 3	-	-
		108	a:6084, b:1296	a: 11 $\not\simeq$ b: 7	$\langle 108, 22\rangle$	-
(6,3,3)	$\langle 432, 538\rangle$	18	100	a: 3	$\langle 18, 3\rangle$	$\langle 18, 5\rangle$
		72	1296	a: 7 $\simeq$ b: 7 $\simeq$ c: 7	$\langle 72, 29\rangle$, $\langle 72, 30\rangle$, $\langle 72, 42\rangle$, $\langle 72, 47\rangle$	$\langle 72, 50\rangle$
		108	1296	a: 7 $\not\simeq$ b: 7	$\langle 108, 21\rangle$, a:$\langle 108, 32\rangle$, b:$\langle 108, 22\rangle$	a:$\langle 108, 45\rangle$
(6,6,3)	$\langle 216, 95\rangle$	18	100	a: 3	-	-
		36	784	a: 6 $\simeq$ b: 6 $\simeq$ c: 6	$\langle 36, 6\rangle$, $\langle 36, 11\rangle$	$\langle 36, 14\rangle$

7

Constructing Highly Symmetric Tight Frames

To find out whether there are any highly symmetric tight frames of N vectors in dimension d it is possible to use a group calculation program such as Magma where there is a database of abstract groups. The current limitations in Magma mean you cannot work with groups of order greater than 2000.

The following description outlines the algorithm we developed for finding highly symmetric tight frames of N vectors in d dimensions.

Consider all the abstract groups G of order $N \times k$ where $2 \leq k \leq (N-1)(N-2)$ as candidates for the symmetry group of the frame or as subgroups of the symmetry group. From these groups compute all the faithful irreducible representations $\rho : G \to \mathcal{GL}(d, \mathbb{F})$. Some groups will not have any irreducible representations of the desired dimension and it is possible that there are no highly symmetric tight frames of the desired size. For instance there are no highly symmetric tight frames of 5 vectors in 3 dimensions.

Given the various representations $\rho(G)$ which are groups of unitary transformations in a d dimensional space find all the subgroups of $\rho(G)$ that fix a one-dimensional subspace $\text{span}\{v\}$ for some $v \neq 0$. Then $\Phi = \{gv\}_{g\in G}$ is a highly symmetric tight frame of $|G|/|\text{Stab}(v)|$ vectors.

Example Let $G = \langle 18, 3\rangle$ from the small group list in Magma. G has the following presentation,

$$G = \left\langle a, b, c | a^2 = b^3 = c^3 = 1, a^{-1}ca = c^2 \right\rangle .$$

Calculations in Magma reveal that there are six representations of dimension 1 and three of dimension 2. Of the representations of dimension 2 one of them is not faithful and the other two contain the same group elements but merely specifiy different matrices as the generators. Hence we will only consider one representation of interest and let it be $\rho : G \to \mathcal{GL}(2, \mathbb{C})$ where,

$$\rho(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho(b) = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}, \quad \rho(c) = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}.$$

Given a nontrivial subgroup H of $\rho(G)$ we can solve for any v which are fixed by all the generators of H. From this we find two vectors which generate ,

$$\begin{aligned} v_1 &= (1,0) \qquad |\{gv_1\}_{g\in G}| = 6, \\ v_2 &= (1,1) \qquad |\{gv_2\}_{g\in G}| = 9, \end{aligned}$$

where the frame with 6 vectors is a generalized cross and the frame with 9 vectors a generalized cube that were discussed earlier because $\langle 18, 3\rangle = G(3, 1, 2)$. In this example the only highly symmetric tight frames were the ones that came from a representation of the group as a reflection group.

8
Conclusion

The original aim of this research was to clarify the relationship between regular complex polytopes and tight frames. We have shown that the regular complex polytopes are a special case of the finite class of highly symmetric tight frames which can be computed from abstract groups. Many questions arise from this with regards to the relationship between reflection groups and highly symmetric tight frames. In particular whether there is a highly symmetric tight frame that does not come from a reflection group and can every highly symmetric tight frame be interpreted as a semi-regular polytope?

Further, our construction works over any field, even when there is not an inner product. In such a setting one must appeal to the theory of classical groups to interpret the frames as being tight.

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