MATHS 730 FC Lecture Notes March 5, 2014

1 Introduction

Definition. If A, B are sets and there exists a bijection $A \to B$, they have the same **cardinality**, which we write as $|A|, \#A$. If there exists one-to-one map $A \to B$, then $|A| \leq |B|$

Remark 1.1. $\lim_{n\to\infty} x_n = 0$ means that (x_n) diverges to ∞: i.e. $\forall c > 0, \exists N$ such that $x_n > c, \forall n > N$.

Definition. Defined the **extended real numbers** to be $\overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{-\infty, \infty\}$.

Remark 1.2. We have that $-\infty < x < \infty, \forall x \in \mathbb{R}$.

Recall. Each non-empty set $A \subset \mathbb{R}$ is either
(a) Bounded above, in which it has a supremum $\sup(A)$,
(b) A is unbounded above.
In case (b) we will say supremum (in the extended sense) of A is ∞ and write $\sup(A) = \infty$.

Remark 1.3. Every non-empty subset of $\overline{\mathbb{R}}$ has a supremum in $\overline{\mathbb{R}}$. Every increasing sequence (x_n) in $\overline{\mathbb{R}}$ has a limit in $\overline{\mathbb{R}}$.

Definition. If (x_n) is a sequence of extended real numbers, we define the **limit superior** and the **limit inferior** of this sequence by

$$\overline{\lim} x_n = \limsup x_n = \inf_m \left(\sup_{n\geq m} x_n \right) = \lim_{m\to\infty} \sup_{n\geq m} x_n \in [-\infty, \infty],$$
$$\underline{\lim} x_n = \liminf x_n = \sup_m \left(\inf_{n\geq m} x_n \right) = \lim_{m\to\infty} \inf_{n\geq m} x_n \in [-\infty, \infty].$$

It is easy to check that

$$\liminf x_n \leq \limsup x_n,$$
$$\liminf(-x_n) = -\limsup(x_n).$$

If the limit inferior and the limit superior are equal, then their value if called the **limit** of the sequence.

Theorem 1.1. Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Then exists subsequences of (x_n) converging to $m = \liminf x_n, M = \limsup x_n$ (in the extedned sense). Further, if (x_{n_k}) is a convergent subsequence,

$$m \leq \lim_{k\to\infty} x_{n_k} \leq M.$$

Corollary 1.1. x_n converges if and only if $\limsup x_n = \liminf x_n$, in which case $x_n \to \limsup x_n$.

2 Measurable Functions

Definition 2.1. Let X be a non-empty set, A collection $\mathscr{S}$ of subsets of X is said to be a **σ-algebra** (or a **σ-field**) if:
(i) $\emptyset, X$ belong to $\mathscr{S}$.
(ii) If E belongs to $\mathscr{S}$, then the complement $\mathscr{C}(A) = \overline{A} = X \backslash E \in \mathscr{S}$.
(iii) If $(E_j) \in \mathscr{S}, \forall j \in \mathbb{N}$, then

$$\bigcup_{j=1}^{\infty} E_j \in \mathscr{S}.$$

Remark 2.1. $\mathscr{S}$ is closed under countable intersection, this is easy to check using De Morgan.

Example 2.1.

1. $\mathscr{S} = \{\emptyset, X\}$ is the trivial σ-algebra .
2. $\mathscr{S} = \mathscr{P}(X)$ power set $:=$ collection of all subsets of X.
3. Let X be uncountable, $\mathscr{S} := \{Y \subset X : Y$ or X is countable $\}$.
4. If $\{\mathscr{S}_j\}$ are σ-algebra of subsets of X. Then $\cap_j \mathscr{S}_j$ is σ-algebra .

Definition. Let A be a nonempty collection of subsets of X. We observe that there is a smallest σ-algebra of subsets of X containing A. To see this, observe that the family of all subsets of X is a σ-algebra containing A and the intersection of all the σ-algebra containing A is also a σ-algebra containing A. This smallest σ-algebra is called the **σ-algebra generated by A**, we denote it $\sigma(A)$.

Definition. Let (X, ρ) be a metric space (or topological space). Let A be the collection of all open subsets of X. Then σ-algebra generated by A is called the **Borel σ-algebra** and is denoted $\mathscr{B}(X)$.

Remark 2.2. Clearly the Borel σ-algebra is generated by all closed sets

Example 2.2. Let $X = \mathbb{R}$. The Borel σ-algebra is generated by all open intervals (a, b) in $\mathbb{R}$.

Remark 2.3. Extended Borel σ-algebra is generated by all open intervals in $\overline{\mathbb{R}}$, i.e.

$$(a, b), (a, \infty), (a, \infty], (-\infty, b), [-\infty, b), [-\infty, \infty].$$

Definition. A function $f : X \to Y$ between measurable spaces $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ is **measurable** (**$(\mathscr{S}, \mathscr{T})$-measurable**) if

$$f^{-1}(A) \in \mathscr{S}, \quad \forall A \in \mathscr{T}$$

Definition. (Working Definition)
Let $(X, \mathscr{S})$ be a measurable space. Then $f : X \to \mathbb{R}$ or $\overline{\mathbb{R}}$ is **$\mathscr{S}$-measurable** is $\{x \in X : f(x) > \alpha\} \in \mathscr{S}, \forall \alpha \in \mathbb{R}$.

Lemma 2.1. The following statements are equivalent for a function f on X to $\mathbb{R}$:
(a) For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to $\mathscr{S}$.
(b) For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X : f(x) \leq \alpha\}$ belongs to $\mathscr{S}$.
(c) For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X : f(x) \geq \alpha\}$ belongs to $\mathscr{S}$.
(d) For every $\alpha \in \mathbb{R}$, the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to $\mathscr{S}$.
(e) f is $\mathscr{S}$-measurable.

Remark 2.4. Sets such as

$$\begin{aligned}
&\{x : f(x) = a\} = f^{-1}(\{a\}) \\
&\{x : a \leq f(x) \leq b\} \\
&\{x : f(x) = \infty\} \\
&\{x : f(x) \text{ is finite }\}
\end{aligned}$$

are all $\mathscr{S}$-measurable if f is $\mathscr{S}$-measurable.

Example 2.3. the characteristic function $f : \chi_E$ is $\mathscr{S}$-measurable if and only if $E \in \mathscr{S}$.

Example 2.4. Continuous functions are Borel measurable, i.e. if $(X, \mathscr{S})$ combines $\mathscr{B}(X)$ and $(Y, \mathscr{T})$ is $\mathscr{B}(Y)$, then $f : X \to Y$ measurable is continuous.

Lemma 2.2. Let $(X, \mathscr{S})$ be a measurable space, $E \in \mathscr{S}$. Then $f \to \overline{\mathbb{R}}$ is $\mathscr{S}$-measurable on X if and only if it is $\mathscr{S}$-measurable on E and $X \backslash E$.

Theorem 2.1. Every reasonable combination of measurable functions is measurable.
Let $(X, \mathscr{S})$ be a measurable space, $E \in \mathscr{S}$ and f, g, f_n, g_n be $\mathscr{S}$-measurable and $X \to \overline{\mathbb{R}}$ (or $\mathbb{R}$) on E. Then the following are $\mathscr{S}$-measurable on F.
1. $\alpha f, \frac{1}{f}, |f|^{\beta}, \quad \alpha \in \overline{\mathbb{R}}, \beta \in \mathbb{R}$.
2. $f + g, fg$.
3. $\sup f_n, \inf f_n, \limsup f_n, \liminf f_n$.
4. f^-, f^+.

3 Measure

Definition 3.1. Let $(X, \mathscr{S})$ be a measurable space. A function $\mu : X \to \overline{\mathbb{R}}$ is called a **measure**:
(i) $\mu(\emptyset) = 0$,
(ii) $\mu(E) \geq 0$ for all $E \in \mathscr{S}$,
(iii) μ is **countably additive** in the sense that if (E_n) is any disjoint sequence of sets in XS, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

We call $(X, \mathscr{S}, \mu)$ a measure space.

Definition. Let $X = \mathbb{N} = \{1, 2, 3, \ldots\}$ and let $\mathscr{S}$ be the σ-algebra of all subsets of $\mathbb{N}$. If $E \in \mathscr{S}$, defined

$$\mu(E) := \begin{cases} |E|, & \text{if } E \text{ if finite} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

Then μ is a measure and is called the **counting measure on** $\mathbb{N}$. Note that μ is not finite, but it is σ-finite. Also, integrals for this measure are sums.

Definition. Let (x_n) be a sequence in X. (p_n) be a sequence in $[0, \infty)$. Define μ on $(X, \mathscr{S})$ by

$$\mu(E) := \sum_{\{n : x_n \in E\}} p_n,$$

then this is the **discrete measure**

Definition. Special case of discrete measure: Let X be any nonempty set, let $\mathscr{S}$ be the σ-algebra of all subsets of X and let p be a fixed element of X. Let μ be defined for $E \in \mathscr{S}$ by

$$\begin{aligned} \mu(E) &= 0, & \text{if } p \notin E, \\ &= 1, & \text{if } p \in E. \end{aligned}$$

then μ is a finite measure and is called the **unit measure concentrated at p** or **Durac-delta measure**.

Definition. There exists a measure space $(\mathbb{R}, \mathscr{L}, m)$ with $\mathscr{L} \supset \mathscr{B}(\mathbb{R})$ such that m is translation invariant and $m(I) = b - a$ where I is the interval with endpoints a and b. This measure is unique and is usually called **Lebesgue** (or **Borel**) **measure**. It is not a finite measure, but it is σ-finite.

Lemma 3.1. $\mathscr{L}$ cannot be all of $\mathscr{P}(\mathbb{R})$.

Definition. If $X = \mathbb{R}, \mathscr{S} = \mathscr{B}$, and f is a continuous monotone increasing function, then there exists a unique measure λ_f defined on $\mathscr{B}$ such that if $E = (a, b)$, then $\lambda_f(E) = f(b) - f(a)$. This measure λ_f is called the **Borel-Stieltjes measure generated by f**.

Remark 3.1. If $E \subseteq F, E, F, \in \mathscr{S}$, then
(a) $\mu(E) \leq \mu(F)$,
(b) If $\mu(F) < \infty$, then $\mu(E) = \mu(F) - \mu(F \backslash E)$.

Definition. Let $(X, \mathscr{S}, \mu)$ be a measure space. We say $E \subset X$ is a **null set** or a **set of measure zero** if $E \subset F \in \mathscr{S}, \mu(F) = 0$. These sets can be ignored for integration.

Definition. We say a certain proposition holds **μ-almost everywhere**, almost everywhere, or for almost all, etc, if there exists a subset $F \in \mathscr{S}$ with $\mu(F) = 0$ such that the proposition holds on the complement of F.

Example 3.1. For Lebesgue measure m on $\mathbb{R}$, $m(\{a\}) = m([a,b]) - m((a,b]) = 0$ so countable subsets of $\mathbb{R}$ have Lebesgue measure zero.

Definition. We say that two functions f, g are **equal μ-almost everywhere** or that they are **equal for μ-almost all** x in case $f(x) = f(x)$ when $x \notin N$, for some $N \in \mathscr{S}$ with $\mu(N) = 0$. In this case we will often write

$$f = g, \quad \mu\text{-a.e.}$$

Definition. We say that a sequence (f_n) of functions on X **converges μ-almost everywhere** (or **converges for μ-almost all x**) if there exists a set $N \in \mathscr{S}$ with $\mu(N) = 0$ such that $f(x) = \lim f_n(x)$ for $x \notin N$. In this case we often write

$$f = \lim f_n, \quad \mu\text{-a.e.}$$

Definition. If $\{E_n\}$ be a sequence of subsets of X we say $\{E_n\}$ is **expanding** (or **increasing**) if

$$E_{n+1} \supset E_n,$$

and is **contracting** (or **decreasing**) if

$$E_{n+1} \subset E_n.$$

Lemma 3.2. Let $(X, \mathscr{S}, \mu)$ be a measure space, let (E_n) be a sequence in $\mathscr{S}$, then
(a) If (E_n) is expanding, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

without restriction.
(b) If (E_n) is decreasing and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Example 3.2. The condition $\mu(E_1) < \infty$ is necessary.
Take $E_n = [n, \infty) \in \mathbb{R}$ with Lebesgue measure, (E_n) contracting with $m(E_n) = \infty$.

4 The Integral

Definition 4.1. A real valued function is **simple** if it has only a finite number of values.

Remark 4.1. A simple measurable function u can be represented in the form

$$u = \sum_{k=1}^{n} a_k \chi_{A_k},$$

where $a_k \in \mathbb{R}$ and

$$\bigcup_{k=1}^{n} A_k = E, \quad A_k \subset E \;\; \forall k.$$

Definition. Among these representations for u there is a unique **standard representation** characterized by the fact that the a_k are distinct and the A_k are disjoint nonempty subsets of E and are such that $E = \cup_{k=1}^{n} A_k$.

Theorem 4.1. (**Approximation by simple functions**)
Let $(X, \mathscr{S})$ be a measurable space. If f is non-negative $\mathscr{S}$-measurable. Then there is an increasing sequence of non-negative simple measurable function u_n with

$$\lim_{n\to\infty} u_n = f \quad \text{(pointwise)}$$

Definition. For a measurable space $(X, \mathscr{S}, \mu)$, let $M^+(X, \mathscr{S})$ be the set of non-negative $\mathscr{S}$-measurable functions on X.

Definition. If $f \in M^+(X, \mathscr{S})$ is simple, then we define its **integral** to be

$$\int f, \int_X f \,\mathrm{d}\mu, \int_X f(x) \,\mathrm{d}\mu(x) := \sum_{j=1}^{n} \alpha_j \mu(A_j)$$

where

$$f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

is the standard form of f.
With the convention $0 \times \infty := 0$, that is, if $\alpha_k = 0, \mu(A_k) = \infty$, then $\alpha_j \mu(A_k) = 0$.

Lemma 4.1. If $u, v \in M^+(X, \mathscr{S})$ are simple, $0 \le c < \infty$, then
(a) $\int_X cu \,\mathrm{d}\mu = c \int_X u \,\mathrm{d}\mu$, $\int_X (u+v) \,\mathrm{d}\mu = \int_X u \,\mathrm{d}\mu + \int_X v \,\mathrm{d}\mu$.
(b) If define λ by

$$\lambda(E) := \int u \chi_E \,\mathrm{d}\mu, \;\; E \in \mathscr{S},$$

then λ is a measure on $\mathscr{S}$.

Definition. Let $f \in M^+(X, \mathscr{S})$, define the integral of f over X to be

$$\int_X f \,\mathrm{d}\mu := \sup \left\{ \int u \,\mathrm{d}\mu : u \in M^+(X, \mathscr{S}), 0 \le u \le f, u \text{ simple} \right\}$$

is well defined (may be ∞) and ≥ 0.

Definition. $E \in \mathscr{S}$, then for $f \in M^+(X, \mathscr{S})$

$$\int_E f \, \mathrm{d}\mu := \int_X f \chi_E \, \mathrm{d}\mu$$

Lemma 4.2. (**Monotonicity of the integral**)
If f and g belong to $M^+(X, \mathscr{S}), E, F \in \mathscr{S}$, then
(a) If $f \leq g$ on E, then

$$\int_E f \, \mathrm{d}\mu \leq \int_E g \, \mathrm{d}\mu.$$

(b) If $E \subseteq F$, then

$$\int_E f \, \mathrm{d}\mu \leq \int_F f \, \mathrm{d}\mu.$$

Theorem 4.2. MONOTONE CONVERGENCE THEOREM)
If (f_n) is a monotone increasing sequence of functions in $M^+(X, \mathscr{S})$, then

$$\int_X \left(\lim_{n \to \infty} f_n \right) \mathrm{d}\mu = \lim \int_X f_n \, \mathrm{d}\mu.$$

Corollary 4.1. If (f_n) is a monotone increasing sequence of functions in $M^+(X, \mathscr{S})$ which converges μ-almost everywhere on X to a function f in M^+, then

$$\int f \, \mathrm{d}\mu = \lim \int f_n \, \mathrm{d}\mu.$$

Corollary 4.2. Let (g_n) be a sequence in M^+, then

$$\int \left(\sum_{n=1}^{\infty} g_n \right) \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int g_n \, \mathrm{d}\mu \right).$$

Remark 4.2. It should be observed that it is not being assumed that either side of the equation is finite. Indeed, the sequence $(\int f_n \, \mathrm{d}\mu)$ is a monotone increasing sequence of extended real numbers and so always has a limit in $\overline{\mathbb{R}}$, but perhaps not in $\mathbb{R}$.

Recall. For Riemann integrals the Monotone Convergence Theorem fails

$$f_n \coloneqq \chi_{\{q_1,\ldots,q_n\}}, \;\; q_n \in \mathbb{Q}$$

is increasing with $\lim f_n = \chi_{\mathbb{Q}}$, $\lim_{n\to\infty} \int f_n(x)\,\mathrm{d}x = 0$, but $f = \chi_{\mathbb{Q}}$ is not Riemann integrable.

Theorem 4.3. (**Linearity of Integral**)
Let $f, g \in M^+(X, \mathscr{S})$ and $\alpha, \beta \in [0, \infty)$. Then

$$\int_X (\alpha f + \beta g) \; \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu + \beta \int_X g \, \mathrm{d}\mu.$$

Lemma 4.3. Fatou's Lemma
If (f_n) belongs to $M^+(X, \mathscr{S})$, then

$$\int_X (\liminf f_n) \; \mathrm{d}\mu \leq \liminf \int_X f_n \, \mathrm{d}\mu.$$

Remark 4.3. We do not need to worry about whether functions are increasing or not.

Remark 4.4. Discrete version:

$$f_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Then we have

$$\liminf a_n + \liminf b_n \leq \liminf (a_n + b_n) \, .$$

Remark 4.5. The inequality can be strict in Fatou's Lemma

Example 4.1. Suppose $f_n \in M^+(X, \mathscr{S}), f_n \to f, f_n \leq f$ then $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$.

Theorem 4.4. Let $f \in M^+(X, \mathscr{S})$ and

$$\begin{aligned}\nu(E) &:= \int_E f \,\mathrm{d}\mu \\ &= \int_X f\chi_E \,\mathrm{d}\mu,\end{aligned}$$

then ν is the measure on $(X, \mathscr{S})$.

Theorem 4.5. Suppose that f belongs to $M^+(X, \mathscr{S})$. Then $f(x) = 0$ μ-almost everywhere on X if and only if

$$\int_X f \,\mathrm{d}\mu = 0.$$

Theorem 4.6. Let $f, g \in M^+(X, \mathscr{S})$, then
(i) If $f \leq g$ μ-a.e. on X then

$$\int_X f \,\mathrm{d}\mu \leq \int_X g \,\mathrm{d}\mu.$$

(ii) If $f = g$ μ-a.e. on X then

$$\int_X f \,\mathrm{d}\mu = \int_X g \,\mathrm{d}\mu.$$

5 Integrable Functions

Definition 5.1. The collection $L = L_1(X, \mathscr{S}, \mu)$ of **integrable** (or **summable**) **functions** consists of all (extended) real-valued $\mathscr{S}$-measurable functions f defined on X, such that both the positive and negative parts f^+, f^-, of f have finite integrals with respect to μ. In this case, we define the **integral of f over X with respect to** μ to be

$$\int_X f \,\mathrm{d}\mu = \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu.$$

If E belongs to $\mathscr{S}$, we define

$$\int_E f \,\mathrm{d}\mu = \int_E f^+ \,\mathrm{d}\mu - \int_E f^- \,\mathrm{d}\mu.$$

Recall.

$$f^+ = \max\{f(x), 0\}, \quad f^- = \max\{-f(x), 0\}, \quad f = f^+ - f^-, \quad |f| = f^+ + f^-$$

Theorem 5.1. Let f be measurable, then $f \in L_1(X, \mathscr{S}, \mu)$ if and only if $|f| \in L_1(X, \mathscr{S}, \mu)$. i.e. $|f|$ has finite integral. Moreover,

$$\left|\int_X f \,\mathrm{d}\mu\right| \leq \int_X |f| \,\mathrm{d}\mu.$$

Lemma 5.1. If $f \in L(X, \mathscr{S}, \mu)$, then
(i) f is finite-valued a.e.
(ii) If

$$\tilde{f}(x) := \begin{cases} f(x), & f(x) \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{f} \in L(X, \mathscr{S}, \mu)$ and

$$\int_X \tilde{f} \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Theorem 5.2. Linearity of integral
If $f, g \in L(X, \mathscr{S}, \mu) 1$ and $\alpha \in \mathbb{R}$, then $\alpha f, f+g \in L_1(X, \mathscr{S}, \mu)$ and

$$\int_X \alpha f \, \mathrm{d}\mu = a \int_X f \, \mathrm{d}\mu, \quad \int_X (f+g) \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

Theorem 5.3. (i) If g is measurable, $f \in L(X, \mathscr{S}, \mu)$ and $g = f$ μ-a.e. then $g \in L(X, \mathscr{S}, \mu)$ and

$$\int_X g \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu.$$

(ii) If $f, g \in L(X, \mathscr{S}, \mu), f \leq g$ μ-a.e. on X then

$$\int_X f \, \mathrm{d}\mu \leq \int_X g \, \mathrm{d}\mu.$$

Theorem 5.4. LEBESGUE DOMINATED CONVERGENCE THEOREM
Suppose $f_n \in L(X, \mathscr{S}, \mu), \forall n$ and $f_n \to f$ μ-a.e. on X.
If there exists a $g \in L(X, \mathscr{S}, \mu)$ (dominating function) such that $|f_n| \leq g$ for all n, then $f \in L(X, \mathscr{S}, \mu)$ and

$$\int_X f \, \mathrm{d}\mu = \int_X \lim_{n \to \infty} f_n \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu.$$

6 The Lebesgue Spaces $\mathbf{L_p}$

Definition 6.1. If V is a real linear (= vector) space, then a real-valued function $\|\cdot\|$ on V is said to be a **norm** for V in case it satisfies
(i) $\|v\| \geq 0$ for all $v \in V$;
(ii) $\|v\| = 0$ if and only if $v = 0$;
(iii) $\|\alpha v\| = |\alpha| \, \|v\|$ for all $v \in V$ and real α;
(iv) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.
If condition (ii) is dropped, the function $\|\cdot\|$ is said to be a **semi-norm** or a **pseudo-norm** for V. A **normed linear space** is a linear space V together with a norm for V.

Definition. We say $\mathscr{S}$-measurable functions f and g are μ**-equivalent** if $f = g$ μ-a.e. on X.

Remark 6.1. This is an equivalence relation

Definition. We denote the equivalence classes for this relation by

$$\begin{aligned}[f] &:= \{g : f = g \ \mu\text{-a.e. }\} \\ &= \{f + g : g = 0 \ \mu\text{-a.e. }\}\end{aligned}$$

Remark 6.2. On these equivalence classes we can define $+, \cdot$.

$$[f] + [g] = [f + g]\,, \quad \alpha\,[f] = [\alpha f]\,.$$

Ultimately, this gives a vector space with zero $[0] = [f + (-f)]$.

Definition. On the μ-equivalence classes we can define a **norm** for $1 \leq p < \infty$ by

$$\|[f]\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p},$$

this normed is well-defined with values in $[0, \infty]$.

Definition. Key for L_p spaces
We define $L_p(\mu)$ to be the set of all μ-equivalence classes of $\mathscr{S}$-measurable functions for which

$$\|[f]\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} < \infty$$

Claim. $L_p(X, \mathscr{S}, \mu)$ is a vector space. Check $L_p(\mu)$ is a vector space:

Recall. If $A, B \geq 0, q \leq p < \infty$ then

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q},$$

where q is the **conjugate exponent** to p. i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 6.1. HÖLDER'S INEQUALITY
Fix $(X, \mathscr{S}, \mu)$. Let p and q be conjugate exponents $(1 \leq p < \infty)$. If f, g are measurable, then $fg \in L_1$ and

$$\int_X |fg| \, \mathrm{d}\mu \leq \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} \left(\int_X |g|^q \, \mathrm{d}\mu \right)^{1/q}$$

i.e. $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Remark 6.3. We may have $\|fg\|_1, \|f\|_p, \|g\|_q$ be ∞.

Corollary 6.1. $f \in L_p, g \in L_q$ then $fg \in L_1$.

Theorem 6.2. CAUCHY-BUNYAKOVSKIĬ-SCHWARZ INEQUALITY
$f, g \in L_2$ then $fg \in L_1$ and

$$\langle f, g \rangle = \left| \int_X fg \, \mathrm{d}\mu \right| \leq \int_X |fg| \, \mathrm{d}\mu \leq \|f\|_2 \|g\|_2 .$$

Definition. $L_p(\mu) :=$ μ-equivalences of $\mathscr{S}$-measurable functions f with

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} < \infty.$$

Theorem 6.3. MINKOWSKI'S INEQUALITY
If f and g belong to $L_p, p \geq 1$, then $f + g$ belongs to L_p and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

Claim. $L_p(\mu)$ is a Banach space.

7 Modes of Convergence

Definition. The sequence (f_n) **converges uniformly** to f if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that if $n \geq N$ and $x \in X$, then $|f_n(x) - f(x)| < \varepsilon$. i.e. $\|f_n - f\|_\infty < \varepsilon, \forall n > N$.

Definition. The sequence (f_n) **converges pointwise** to f if $\forall x \in X, \varepsilon > 0, \exists N := N_\varepsilon(x) \in \mathbb{N}$, such that if $n \geq N_\varepsilon(x)$, then $|f_n(x) - f(x)| < \varepsilon$.
That is, we need different N for different x, clearly uniform convergence implies pointwise convergece. The two are equivalent if we have continuous functions on compact set.

Example 7.1. Let $f_n = \chi_{[n,n+1]}$, then f is continuous pointwise but not uniformly.

Remark 7.1. The usual interpretation of convergence is pointwise.

Definition. The sequence (f_n) **converges (pointwise)** μ **almost everywhere** to f if there exists a set $E \in \mathscr{S}$ with $\mu(X \backslash E) = 0$ such that $(f_n|_E)$ converges pointwise to $f|_E$.

Example 7.2. Let $f_n = n\chi_{\mathbb{Q}} = 0, \mu$-a.e. but converging at the point of rationals.

7.1 Convergence in $\mathbf{L_p}$

Definition. A sequence (f_n) in $L_p = L_p(X, \mathscr{S}, \mu)$ **converges in $\mathbf{L_p}$** to $f \in L_p$, if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that if $n \geq N$, then

$$\|f_n - f\|_p = \left\{ \int |f_n - f|^p \, \mathrm{d}\mu \right\}^{1/p} < \varepsilon.$$

In this case, we sometimes say that the sequence (f_n) **converges to f in mean (of order p)**.

Example 7.3. Define with Lebesgue measure

$$f_n = \frac{1}{n^{1/p}} \chi_{[0,n]},$$

then $f_n \to f = 0$ uniformly, but not in L_p since

$$\|f_n - f\|_p = \left(\int_0^n \left(\frac{1}{n^{1/p}} \right)^p \mathrm{d}x \right)^{1/p} = 1 \nrightarrow 0.$$

Remark 7.2. Uniform convergence does not imply convergence in L_p unless measure is finite. e.g. $\mu(X) < \infty$.

Definition. A sequence (f_n) in L_p is said to be **Cauchy in $\mathbf{L_p}$**, if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that if $m, n \geq N$, then

$$\|f_m - f_n\|_p = \left\{ \int |f_m - f_n|^p \, \mathrm{d}\mu \right\}^{1/p} < \varepsilon.$$

Theorem 7.1. Suppose that $\mu(X) < +\infty$ and that (f_n) is a sequence in L_p which converges uniformly on X to f. Then f belongs to L_p and the sequence (f_n) converges in L_p to f.

Example 7.4. Let $f_n = n^{1/p} \chi_{\left[\frac{1}{n}, \frac{2}{n}\right]}$, $f_n : [0, 2] \to \mathbb{R}$, then $f_n \to 0$ pointwise but not in L_p.

7.2 Convergence in Measure

Definition 7.1. A sequence (f_n) of measurable real-valued functions is said to **converge in measure** to a measurable real-valued function f in case

$$\lim_{n \to \infty} \mu \left(\{ x \in X : |f_n(x) - f(x)| \geq \varepsilon \} \right) = 0$$

for each $\varepsilon > 0$. The sequence (f_n) is said to be **Cauchy in measure** in case

$$\lim_{m,n \to \infty} \mu \left(\{ x \in X : |f_n(x) - f_n(x)| \geq \varepsilon \} \right) = 0$$

for each $\varepsilon > 0$.

Remark 7.3. Clearly, uniform convergence implies convergence in measure.

Example 7.5. Let $f_n = \chi_{[n,n+1]}$. Then $f_n \to 0$ pointwise but it does not converge in measure.

8 Decomposition of Measures

Definition. If μ is a measure, $f \in M^+(X, \mathscr{S})$ then

$$\lambda(E) := \int_E f \, \mathrm{d}\mu,$$

defines a measure with $\mu(E) = 0$ implies $\lambda(E) = 0$, and we say λ is **absolutely continuous with respect to** μ, and write $\lambda \ll \mu$.

Definition 8.1. Let $(X, \mathscr{S})$ be a measurable space, then a real-valued function $\lambda : \mathscr{S} \to \mathbb{R}$ is said to be a **charge** in case
(i) $\lambda(\emptyset) = 0$,
(ii) λ is countably additive in the sense that if (E_n) is a disjoint sequence of sets in $\mathscr{S}$, then

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

Example 8.1.

- A finite measure is a charge
- If $f \in L_1(X, \mathscr{S}, \mu)$, $\lambda(E) := \int_E f \, \mathrm{d}\mu$ is a charge.

Definition.

- We say $P \in \mathscr{S}$ is **positive** if $\lambda(P \cap E) \geq 0, \forall E \in \mathscr{S}$.
- We say $N \in \mathscr{S}$ is **negative** if $\lambda(N \cap E) \leq 0, \forall E \in \mathscr{S}$.
- We say $K \in \mathscr{S}$ is **null** if $\lambda(K \cap E) = 0, \forall E \in \mathscr{S}$.

Theorem 8.1. HAHN DECOMPOITION THEOREM
If λ is a charge on $\mathscr{S}$, then there exist set P and N in $\mathscr{S}$ with $X = P \cup N, P \cap N = \emptyset$, and such that P is positive and N is negative with respect to λ.

Example 8.2. If $f \in L_1(X, \mathscr{S}, \mu), \lambda(E) \coloneqq \int_E f \, \mathrm{d}\mu$, then

$$\begin{aligned} P &= \{x : f^+(x) > 0\} \\ N &= \{x : f^-(x) < 0\} \\ K &= \{x : f(x) = 0\} \end{aligned}$$

Definition. A pair P, N of measurable sets satisfying the conclusion of the preceding theorem is said to form a **Hahn decomposition** of X with respect to λ.

Lemma 8.1. If P_1, N_1 and P_2, N_2 are Hahn decomposition for λ, and E belongs to $\mathscr{S}$, then

$$\lambda(E \cap P_1) = \lambda(E \cap P_2), \quad \lambda(E \cap N_1) = \lambda(E \cap N_2).$$

Definition 8.2. Let λ be a charge on $\mathscr{S}$ and let P, N be a Hahn decomposition for λ. The **positive** and the **negative variations** of λ are the finite measures λ^+, λ^- defined on E in $\mathscr{S}$ by

$$\lambda^+(E) \coloneqq \lambda(E \cap P), \quad \lambda^-(E) \coloneqq -\lambda(E \cap N).$$

The **total variation** of λ is the measure $|\lambda|$ defined for E in $\mathscr{S}$ by

$$|\lambda|(E) \coloneqq \lambda^+(E) + \lambda^-(E).$$

Remark 8.1.

$$\lambda(E) = \lambda^+(E) - \lambda^-(E),$$

and $|\lambda|, \lambda^+, \lambda^-$ are finite measures.

Theorem 8.2. If f belongs to $L_1(X, \mathscr{S}, \mu)$ and λ is defined by

$$\lambda(E) = \int_E f \, \mathrm{d}\mu,$$

then λ^+, λ^-, and $|\lambda|$ are given for E in $\mathscr{S}$ by

$$\lambda^+(E) = \int_E f^+ \, \mathrm{d}\mu, \quad \lambda^-(E) = \int_E f^- \, \mathrm{d}\mu, \quad |\lambda|(E) = \int_E |f| \, \mathrm{d}\mu.$$

Definition 8.3. A measure λ on $\mathscr{S}$ is said to be **absolutely continuous** with respect to a measure μ on $\mathscr{S}$ if $E \in \mathscr{S}$ and $\mu(E) = 0$ imply that $\lambda(E) = 0$. In this case we write $\lambda \ll \mu$. A charge λ is **absolutely continuous** with respect to a charge μ in case the toal variation $|\lambda|$ of λ is absolutely continuous with respect to $|\mu|$.

Lemma 8.2. Let λ and μ be finite measure on $\mathscr{S}$. The $\lambda \ll \mu$ if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that $E \in \mathscr{S}$ and $\mu(E) < \delta$ imply that $\lambda(E) > \varepsilon$.

Theorem 8.3. RADON-NIKODÝM THEOREM
Let λ and μ be σ-finite measures defined on $\mathscr{S}$ and suppose that λ is absolutely continuous with respect to μ. Then there exists a function f in $M^+(X, \mathscr{S})$ such that

$$\lambda(E) = \int_E f \, \mathrm{d}\mu, \quad E \in \mathscr{S}.$$

Moreover, the function f is uniquely determined μ-almost everywhere.
Sometimes f is denoted

$$f = \frac{\mathrm{d}\lambda}{\mathrm{d}\mu},$$

which is called the **Radon-Nikodým derivative**.

Remark 8.2. f need not be integrable.

Definition 8.4. Two measures λ, mu on $\mathscr{S}$ are said to be **mutually singular** if there are disjoint sets A, B in $\mathscr{S}$ such that $X = A \cup B$ and $\lambda(A) = \mu(B) = 0$. In this case we write $\lambda \perp \mu$. Although the relation of singularity is symmetric in λ and μ, we shall sometimes say that λ is **singular with respect to** μ.

Theorem 8.4. LEBESGUE DCOMPOSITION THEOREM
Let λ, μ be σ-finite measures, then λ can be uniquely decomposed

$$\lambda = \lambda_1 + \lambda_2,$$

with $\lambda_1 \perp \mu, \lambda_2 \ll \mu$.

9 Generation of Measures

Definition. It is natural to define the length of the half-open interval $(a, b]$ to be the real number $b - a$ and the length of the sets $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$, and $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$, and $(-\infty, +\infty)$ to be the extended real number $+\infty$. We define the length of the union of a finite number of disjoint sets of these forms to be the sum of the corresponding length. Thus the length of

$$\bigcup_{j=1}^{n}(a_j, b_j] \quad \text{is} \quad \sum_{j=1}^{n}(b_j - a_j)$$

provided the intervals do not intersect.

Remark 9.1. It is intuitive to give Lebesgue measure by

$$m^*(E) := \inf\left\{\sum_{j=1}^{\infty} m(I_j) : I_j \text{ is an interval } E \subset \bigcup_{j=1}^{\infty} I_j\right\}.$$

Then m^* is well defined for all $E \subset \mathbb{R}$, however, this is not a measure, namely, it does not necessarily have countable additivity. However, by Carathéodory Extension Theorem we can restrict m^* to a (large) σ-algebra of m^*-measurable sets, then it will be a measure.

Definition 9.1. A family $\mathscr{S}$ of subsets of a set X is said to be an **algebra** or a **field** in case:
(A1) $\emptyset, X \in \mathscr{S}$.
(A2) $E \in \mathscr{S}$ implies $X \backslash E \in \mathscr{S}$.
(A3) If $E_1, \ldots, E_n \in \mathscr{S}$ implies $\bigcup_{j=1}^{n} E_j \in \mathscr{S}$.

Remark 9.2. Clearly, by De Morgan, algebra is closed under finite unions and finite intersections.

Example 9.1. Let J consists of all finite unions of intervals in $\mathbb{R}$, then this is an algebra.

Example 9.2. Let $\mathscr{S}$ be finite disjoint unions of intervals open on the left and closed on the right. This is also an algebra.

Definition 9.2. If $\mathscr{A}$ is an algebra of subsets of a set X, then a **measure** on $\mathscr{A}$ is $\mu_0 : \mathscr{A} \to \overline{\mathbb{R}}$ such that
(MA1) $\mu_0(\emptyset) = 0$,
(MA2) $\mu_0(E) \geq 0$ for all $E \in \mathscr{A}$, and
(MA3) If $(E_n) \in \mathscr{A}$ disjoint and $\bigcup_{n=1}^{\infty} E_n \in \mathscr{A}$, then

$$\mu_0\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

Example 9.3. On J for $A = \bigcup_{j=1}^{\infty} I_j$ (disjoint union of intervals I_j),

$$\mu_0(A) \coloneqq \sum_{j=1}^{n} m(I_j),$$

where $m(I_j)$ is the length of the interval, then this is a measure on the algebra J.

9.1 The Extension of Measures

Definition 9.3. An **outer measure** on X is $\mu^* : \mathscr{P}(X) \to \overline{\mathbb{R}}$ such that
(OM1) $\mu^*(\emptyset) = 0$,
(OM2) $E_1 \subset E_2 \subset X$ implies that $\mu^*(E_1) \leq \mu^*(E_2)$,
(OM3) If (E_n) is an arbitrary sequence of subsets of X, then it satisfy countably subadditivity,

$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Lemma 9.1. The function μ^* defined by

$$\mu^*(B) = \inf \sum_{j=1}^{\infty} \mu(E_j)$$

satisfies the following:
(a) $\mu^*(\emptyset) = 0$.
(b) $\mu^*(B) \geq 0$, for $B \subseteq X$.
(c) If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.
(d) If $B \in \mathscr{A}$, then $\mu^*(B) = \mu(B)$.
(e) If (B_n) is a sequence of subsets of X, then

$$\mu^* \left(\bigcup_{n=1}^{\infty} B_n \right) \leq \sum_{n=1}^{\infty} \mu^*(B_n).$$

This final property is referred to by saying that μ^* is **countably subadditive**.

Lemma 9.2. If μ_0 is a measure on an $\mathscr{A}$ of subsets of X. Then

$$\mu^*(E) := \inf\left\{\sum_{j=1}^{\infty}\mu_0(A_j) : A_j \in \mathscr{A}, E \subset \bigcup_{j=1}^{\infty} A_j\right\}$$

is an outer measure on X.

Definition 9.4. Let μ^* be an outer measure. Then $E \subset X$ is **μ^*-measurable** if

$$\mu^*(Q) = \mu^*(Q \cap E) + \mu^*(Q \backslash E), \quad \forall Q \subset X.$$

Theorem 9.1. CARATHÉODORY EXTENSION THEOREM
IF μ^* is an outer measure on X, $\mathscr{S} :=$ all μ^*-measurable sets then $(X, \mathscr{S})$ is a measurable space, and the restriction of μ^* to $\mathscr{S}$ is a complete measure. Moreover, if (E_n) is a disjoint sequence in $\mathscr{S}$, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty}\mu^*(E_n).$$

Remark 9.3. Complete in this case means all null sets are measurable, that is, if $E \in \mathscr{S}, \mu(E) = 0$, then $F \subset E$ implies $F \in \mathscr{S}$.

9.2 Lebesgue Measure

Definition. Let set $\mathscr{A}$ be all subsets of $\mathbb{R}$ of the form of a finite union of sets of the form

$$(a, b], \quad (-\infty, b], \quad (a, +\infty), \quad (-\infty, +\infty),$$

then $\mathscr{A}$ is an algebra of subsets of $\mathbb{R}$ and the length function $\mu_0 = l$ gives a measure on this algebra $\mathscr{A}$. We call m_0 the **Lebesgue measure** on $\mathscr{A}$.

Definition. The restriction of Lebesgue measure to the Borel sets is called either **Borel** or **Lebesgue measure**.

Theorem 9.2. There exists a complete measure space $(\mathbb{R}, \mathscr{L}, m)$ with $\mathscr{B} \subset \mathscr{L}$

1. $m(\{x\}) = 0, \quad x \in \mathbb{R}$,
2. for $a < b, m((a,b)) = m((a,b]) = m([a,b)) = m([a,b]) = b-a$,
3. for $a \in \mathbb{R}, m((a,\infty)) = m([a,\infty)) = m((-\infty,a)) = m((-\infty,a]) = m(\mathbb{R}) = \infty$,
4. If f is integrable over interval I with endpoints a and b in $\overline{\mathbb{R}}$, then f is integrable over all intervals with these endpoints and we denoted it by

$$\int_a^b f(x)\,\mathrm{d}x \text{ or } \int_a^b f.$$

Theorem 9.3. Translation invariance
If $E \in \mathscr{L}, x \in \mathbb{R}$ then $E + x \in \mathscr{L}$ and $m(E+x) = m(E)$.

Theorem 9.4. If $E \in \mathscr{L}$, then

1. $m(E) = \inf\{m(U) : U \text{ is open}, E \subset U\}$ (outer regular),
2. $m(E) = \sup\{m(K) : K \text{ is compact}, K \subset E\}$ (inner regular).

Theorem 9.5. LUSIN
Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable which is zero outside a set of finite measure. Then $\forall \varepsilon > 0$, exists g continuous with compact support such that

$$m(\{: f(x) \neq g(x)\}) < \varepsilon, \quad \sup |g| \leq \sup |f| \,.$$

Theorem 9.6. If f is Lebesgue measurable, exists g Borel measurable with $f = g$ m-a.e.

Example 9.4. Lebesgue-Stieltjes measure
Let g be increasing, right continuous, so

$$g(c) = \lim_{h \to 0^+} g(c+h),$$

and define $m_g((a,b]) = g(b) - g(a)$, we have

$$m_g(\{c\} = \text{ jump at } c = g(c) - \lim_{x \to c^-} g(x).$$

Theorem 9.7. RIESZ REPRESENTATION THEOREM
If $\lambda : [a,b] \to \mathbb{R}$ is a bounded positive functional, then exists a (Lebesgue-Stieltjes) measure μ_g (defined on the Borel σ-algebra) with

$$\lambda(f) = \int_a^b f \, \mathrm{d}\mu_g, \quad \forall f \in C\,[a,b]\,, \text{ and}$$

$$\|\lambda\| = \sup_{\|f\|_\infty = 1} \left| \int_a^b f \, \mathrm{d}\mu_g \right| = \mu_g\left([a,b]\right).$$

Remark 9.4. $\lambda \in (C[a,b])^*$, λ is positive if $f \geq 0$, hence $\lambda f \geq 0$. λ looks like

$$\lambda(f) = \sum_{j=1}^{\infty} a_j f(c_j) + \int_a^b f\omega,$$

where $f(c_j)$ is the jump of f at point c_j, ω is the weight function, absolutely continuous with respect to Lebesgue measure m.

10 Product Measure

Definition. Let X and Y be two sets; then the **Cartesian product** $Z = X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Definition 10.1. If $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are measurable spaces, thena set of the form $E \times F$ with $E \in \mathscr{S}$ and $F \in \mathscr{T}$ is called a **measurable rectangle**, or simple a **rectangle**. We shall denote the collection of all finite union of rectangles by $\mathscr{R}$.

$$\mathscr{R} = \mathscr{S} \times \mathscr{T} = \{E \times F : E \in \mathscr{S}, F \in \mathscr{T}\}.$$

Remark 10.1. We wish to define a measure π such that $\pi(E \times F) = \mu(E)\nu(F)$.

Claim. Let $\mathscr{A}_0$ be all finite union of rectangles, we claim that all finite unions of rectangles can be written as a disjoint union of rectangles (proof by picture). So $\mathscr{A}_0$ is closed under finite union. We claim that $\mathscr{A}_0$ is an algebra.

Lemma 10.1. Let $\{D_j \times E_j\}$ be a sequence of disjoint rectangles, and $F_1 \times G_1, \ldots, F_n \times G_n$ be disjoint rectangles with

$$\bigcup_{j=1}^{\infty} D_j \times E_j = \bigcup_{k=1}^{n} F_k \times G_k, \text{ then}$$

$$\sum_{j=1}^{\infty} \mu(D_j)\nu(E_j) = \sum_{k=1}^{n} \mu(F_k)\nu(G_k).$$

Definition. On $\mathscr{A}_0$ define π_0 by

$$\pi_0(A) = \sum_{j=1}^{n} \mu(D_j)\nu(E_j),$$

where A is a disjoint union of rectangles $D_1 \times E_1, \ldots, D_n \times E_n$, by lemma, this is well defined.

Theorem 10.1. π_0 defines a measure on the algebra $\mathscr{A}_0$.

Remark 10.2. There is a naturally associated outer measure (defined on $\mathscr{P}(X \times Y)$)

$$\pi^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \pi_0(R_j) : R_j \text{ is a rectangle } (\in \mathscr{A}_0), E \subset \bigcup_{j=1}^{\infty} R_j \right\}.$$

Bt Carathéodory, there exists σ-algebra $\mathscr{U}$ consisting of the π^*-measurable subsets, for which $\pi^*|_{\mathscr{U}}$ is a complete measure with the properties that $\mathscr{A}_0 \subset \mathscr{U}, \pi^*(A) = \pi_0(A), \forall A \in \mathscr{A}$.

Definition. Let $\mathscr{S} \times \mathscr{T}$ be the σ-algebra generated by $\mathscr{A}_0$ (all rectangles), so $\mathscr{S} \times \mathscr{T} \subset \mathscr{U}$.

Definition. The **product measure** $\pi = \mu \times \nu$ is π^* restricted to $\mathscr{S} \times \mathscr{T}$ (σ-algebra generated by rectangles.)

Definition 10.2. Let $Q \subset \mathscr{S} \times \mathscr{T}, x \in X$, then the **x-section of Q** is the set

$$Q_x := \{y \in Y : (x, y) \in Q\}.$$

Similarly, if $y \in Y$, then the **y-section of Q** is the set

$$Q^y = \{x \in X : (x, y) \in Q\}.$$

If f is a function defined on R to $\overline{\mathbb{R}}$, and $x \in X$, then the **x-section of f** if the function f_x defined on Y by

$$f_x(y) = f(x, y), \quad y \in Y.$$

Similarly, if $y \in Y$, then the **y-section of f** is the function f^y defined on X by

$$f^y(x) = f(x, y), \quad x \in X.$$

Lemma 10.2. If $Q \subset \mathscr{S} \times \mathscr{T}$, then

1. $Q_x \in \mathscr{T}, Q^y \in \mathscr{S}$,
2. $f : X \times Y \to \overline{\mathbb{R}}$ is $\mathscr{S} \times \mathscr{T}$-measurable, then
   - $f(\cdot, y) : X \to \overline{\mathbb{R}}$ is $\mathscr{S}$-measurable,
   - $f(x, \cdot) : Y \to \overline{\mathbb{R}}$ is $\mathscr{T}$-measurable.

Definition. Let $\mathscr{M}$ be a collection of subsets of a set Z, $\mathscr{M}$ is a **monotone class** if

- $\{E_n\}$ is expanding in $\mathscr{M}$ implies that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{M}$;
- $\{E_n\}$ is contracting in $\mathscr{M}$ implies that $\bigcap_{n=1}^{\infty} E_n \in \mathscr{M}$.

Example 10.1. The collection of all intervals in $\mathbb{R}$ is a monotone class, but not a σ-algebra .

Theorem 10.2. TONELLI'S THEOREM
Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ be σ-finite measure spaces with product measure $(X \times Y, \mathscr{S} \times \mathscr{T}, \pi)$. If $f : X \times Y \to \overline{\mathbb{R}}, f \geq 0$, f is $\mathscr{S} \times \mathscr{T}$-measurable, then

1. the function $x \mapsto \int_Y f(x, y)\,\mathrm{d}\nu(y)$ is $\mathscr{S}$-measurable, and
$$\int_X \left(\int_Y f(x, y)\,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_{X \times Y} f\,\mathrm{d}\pi(x, y),$$
2. the function $y \mapsto \int_X f(x, y)\,\mathrm{d}\mu(x)$ is $\mathscr{T}$-measurable, and
$$\int_Y \left(\int_X f(x, y)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y) = \int_{X \times Y} f\,\mathrm{d}\pi(x, y).$$

That is
$$\begin{aligned} \int_X \left(\int_Y f(x, y)\,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) &= \int_{X \times Y} f\,\mathrm{d}\pi(x, y) \\ &= \int_Y \left(\int_X f(x, y)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y). \end{aligned}$$

Theorem 10.3. MONOTONE CLASS THEOREM
If $\mathscr{A}$ is an algebra of sets, then the σ-algebra $\mathscr{S}$ generated by $\mathscr{A}$ coincides with the monotone class $\mathscr{M}$ generated by $\mathscr{A}$.

Lemma 10.3. Tounelli for $f = \chi_Q$
If $f = \chi_Q, Q \in \mathscr{S} \times \mathscr{T}$, then Tonelli holds for f.

Remark 10.3. If (f_n) is an increasing sequence of non-negative, measurable functions for which 1 and 2 of Tonelli holds, then 1 and 2 of Tonelli holds for $f = \lim f_n$.

Claim. Let $\mathscr{M} := \{Q \subset X \times Y :$ Tonelli holds for $\chi_Q\}$, note $\mathscr{A}_0 \subset \mathscr{M}$. Then $\mathscr{M}$ is a monotone class.

Remark 10.4. Recall each f non-negative, measurable function can be approximated by an increasing sequence of non-negative, simple, measurable functions. By linearity, Tonelli holds for these approximations, hence Tonelli holds for f.
In the general case, $X = \bigcup X_i, Y = \bigcup Y_i$, where X_i, Y_i are expanding sets of finite measures, then $f|_{X_i \times Y_i}$ satisfies Tonelli, hence so does the limit $f = f|_{X \times Y}$.

Theorem 10.4. FUBINI'S THEOREM
Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ be σ-finite measures with product measure $(X \times Y, \mathscr{S} \times \mathscr{T}, \pi)$ and $f : X \times Y \to \overline{\mathbb{R}}$ be π-integrable, i.e. $\int f^+ \,\mathrm{d}\pi, \int f^- \,\mathrm{d}\pi < \infty$ implies $\int |f| \,\mathrm{d}\pi < \infty$.

1. There is a set Y_0 with $\nu(Y_0) = 0$, such that $f(\cdot, y)$ is μ-integrable over $X, \forall y \in Y \backslash Y_0$.
   Further, if $f(\cdot, y) := 0$ on Y_0 (or some other choice)
   $$y \mapsto \int_X f(x, y) \,\mathrm{d}\mu(x)$$
   is ν-integrable over Y, and
   $$\int_Y \left(\int_X f(x, y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y) = \int_{X \times Y} f \,\mathrm{d}\pi(x, y)$$

2. If either of the iterated integrals
   $$\int_Y \left(\int_X |f| \,\mathrm{d}\mu \right) \mathrm{d}\nu, \quad \int_X \left(\int_Y |f| \,\mathrm{d}\nu \right) \mathrm{d}\mu$$
   is finite, then
   $$\int_Y \int_X f \,\mathrm{d}\mu \,\mathrm{d}\nu = \int_X \int_Y f \,\mathrm{d}\nu \,\mathrm{d}\mu.$$

Example 10.2. It is not always possible to change the order of integration
Let
$$f(x, y) := \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = f(x, y).$$

Example 10.3. σ-finite is necessary:
Let μ be Lebesgue measure on $[0, 1]$, ν be counting measure on $[0, 1]$, let

$$f(x, y) := \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

ν is not σ-finite and the iterated integrals are not the same.