

# Mathematics of Proportional Representation

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## The article

This talk is based on the paper

*“Proportional Representation and Strategic Voters”*

written jointly with **Shaun White**.

This paper is to appear in

*“Journal of Theoretical Politics”*.

View this paper and my other research on my Social Sciences Research Network author page:

<http://ssrn.com/author=331772>

## Aims of this paper

- This paper was initially motivated by a desire to explain the behaviour of voters at the New Zealand general election held **September 17th, 2005**.
- Some NZ voters voted insincerely even though their doing so could have cost their most preferred party seats.
- We analyse the election and present two models that account for the behaviour observed in the election.
- We investigate opportunities for strategic voting under **proportional representation (PR)**, other than those that emerge due to rounding.

# Electoral system of New Zealand

- In 1993 New Zealand adopted **mixed member proportional system (MMP)** after many years of first-past-the-post (FPP).
- Every NZer has a **party vote** and an **electoral vote**. Party vote determines the proportionality of the House. Electoral vote determines the local representative.
- Parties that receive either
  - **5% of the party vote** or
  - **an electoral seat**are entitled to share 120 seats in the House proportionally to their party vote.
- The seats are allocated using the **Sainte-Lague formula**: firstly to electoral MPs, then parties fill their remaining quotas from their party lists.

# The 2005 NZ general election

- The election took place **September 17th, 2005**.
- The two opinion polls closest to the election gave the following results (party vote):

Poll	Date	Labour	Nats	NZ First	Greens
TVNZ	15/09	38%	41%	5.5%	5.1%
Herald	16/09	44.6%	37.4%	4.5%	4.6%

- The margin of error was  $\pm 3\%$ .
- The Green party were not expected to win an electoral seat, and NZ First were expected to win at most one.
- As it turned out neither the Green party nor NZ First won an electoral seat but both passed the threshold.

## The 2005 NZ general election

- Anecdotal evidence (reports to the authors) has suggested that some voters with preferences

Labour > Greens > ...

may have cast their vote for the Greens.

- Greens received 5.3% of the party vote while Labour got 41.1%.
- National received 39.1% and NZ First 5.7%.
- Hypothetically, about 0.4% of the electorate who intended to vote Labour voted strategically for the Greens.

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- National received 39.1% and NZ First 5.7%.
- Hypothetically, about 0.4% of the electorate who intended to vote Labour voted strategically for the Greens.
- Thus we assume that the Greens would receive 4.9% of the vote if everybody voted sincerely and Labour would receive 41.5%.

## The seats and the Shapley-Shubik power indices

Party	Hypothesised		Actual	
	Seats	SS	Seats	SS
Labour	54	0.414	50	0.324
National	50	0.214	48	0.262
NZ First	7	0.214	7	0.143
Green Party	0	0.0	6	0.110
Maori Party	4	0.081	4	0.076
United Future	3	0.048	3	0.043
ACT	2	0.014	2	0.029
Progressive	1	0.014	1	0.014

Labour and Greens could not form a government and, as a result, the **strange bedfellows phenomenon** had occurred with parties coloured purple forming the government.

## Seat and power maximising voters

- A **seat maximiser** derives utility from the allocation of seats. If there are  $m$  parties, she has a vector of utilities  $u_1, \dots, u_m$ , where  $u_i$  is the utility of a seat won by the  $i$ th party. Her total utility is

$$U(s_1, \dots, s_m) = u_1 s_1 + u_2 s_2 + \dots + u_m s_m,$$

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- A **power maximiser's** utility is derived from the power of parties measured by some power index. The total utility of the parliament will in this case be

$$U(s_1, \dots, s_m) = u_1 p_1 + u_2 p_2 + \dots + u_m p_m,$$

where  $p_i$  is the power of the  $i$ th party.

## Seat and power maximising voters

- A seat maximising voter, who prefers Labour to the Greens, would prefer the actual outcome to the hypothesised one if

$$u(\text{Greens}) > 0.667 \cdot u(\text{Labour}).$$

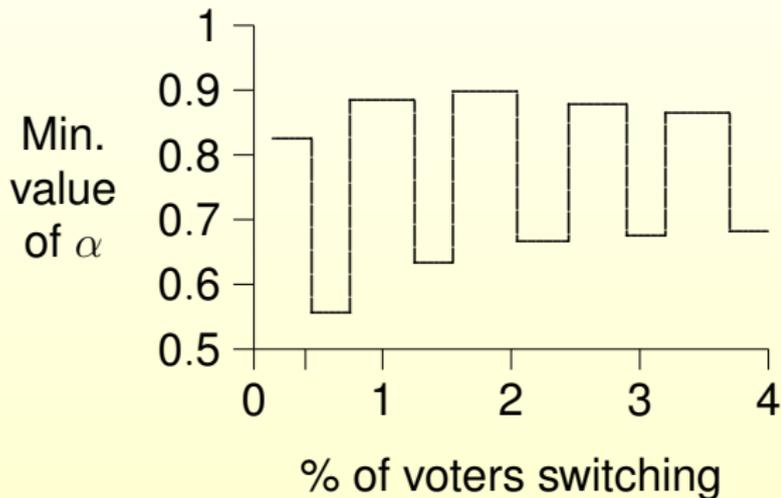
- A power maximising voter, who prefers Labour to the Greens, would prefer the actual outcome to the hypothesised one if

$$u(\text{Greens}) > 0.826 \cdot u(\text{Labour}).$$

## Sainte-Lague formula in action

Let  $\alpha = u(\text{Greens})/u(\text{Labour})$  for a power maximising voter.

The following graph shows for which values of  $\alpha$  voting for the Greens become profitable for a Labour voter:



The graph for seat maximising voters is similar.

## A little history

The subject of manipulability of proportional representation (PR) beyond rounding, is a contentious issue in political science literature.

- **Duverger (1954)** dismissed the possibility of strategic voting in PR.
- He was criticised for that by **Leys (1959)** and **Sartori (1968)** who believed that “**the wasted vote logic** must be applicable to certain kinds of PR systems.”
- **Bowler and Lanoue (1992)** considered that “under proportional representation ... voting sincerely is a dominant strategy”
- They were criticised for that by **Cox (1997)**, who believed that the manipulability of PR is implied by the GS theorem.

# Main assumptions

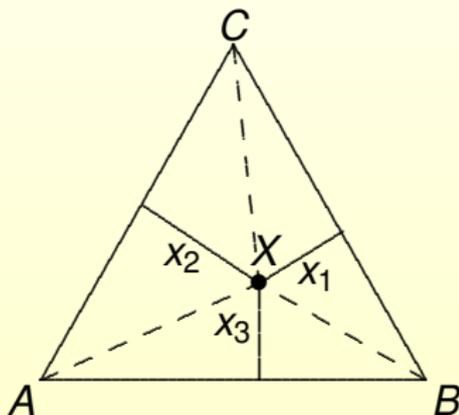
- The world with the rounding is far too complicated. Voters are unlikely to make these calculations.
- From now on we ignore rounding and **allow fractional seats**.
- To understand incentives for strategic voting we restrict ourselves with just **three** parties which will be  $A, B, C$ .

## Geometric representation of the results

The results may be represented by a point  $\mathbf{x}$  of the simplex  $S^{m-1}$ :

$$X = (x_1, \dots, x_m), \quad x_1 + \dots + x_m = 1,$$

where the **homogeneous barycentric coordinate**  $x_i$  is the fraction of the vote the  $i^{\text{th}}$  party has won. In case of three parties:

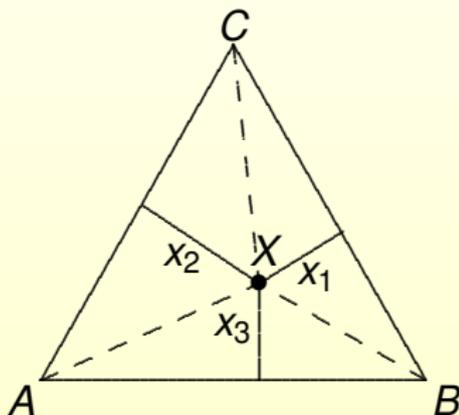


## Geometric representation of parliaments

A parliament is also represented by a point  $\mathbf{x}$  of the simplex  $S^{m-1}$ :

$$X = (x_1, \dots, x_m), \quad x_1 + \dots + x_m = 1,$$

where the **homogeneous barycentric coordinate**  $x_i$  is the fraction of the seats the  $i^{\text{th}}$  party has won.



The parliament is a **mixture** of parties!

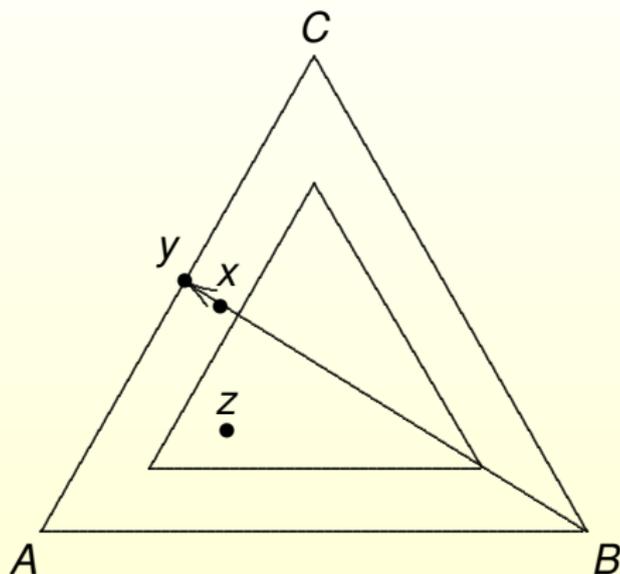
# Parliament seat allocation rules

- A *seat allocation rule* is any mapping

$$F_a: \mathcal{S}^{m-1} \rightarrow \mathcal{S}^{m-1}.$$

- Given a vector of scores  $sc \in \mathcal{S}^{m-1}$ , a seat allocation rule determines the distribution of seats in parliament  $\mathbf{x} = (x_1, \dots, x_m)$  by  $\mathbf{x} = F_a(sc)$ .
- Main examples of  $F_a$ :
  - Identity seat allocation rule (no threshold),
  - Threshold seat allocation rule.

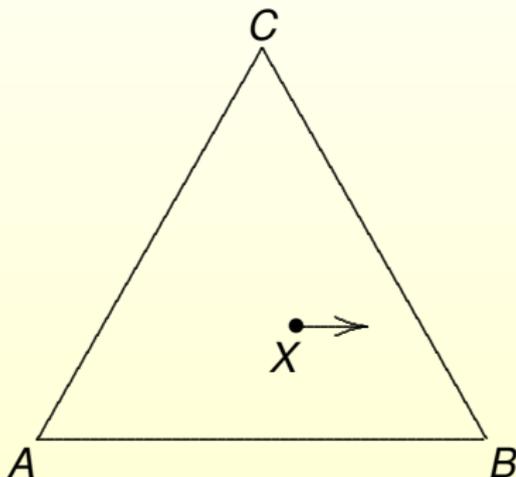
## Action of the threshold seat allocation rule



For the vector of normalised scores  $\mathbf{x}$  the second party is below the threshold, so:  $F_a(\mathbf{z}) = \mathbf{z}$ ,  $F_a(\mathbf{x}) = \mathbf{y}$ .

## How can the score be influenced?

If a group of like-minded voters with a sincere preference  $L : A > B > C$  change it to  $L' : B > A > C$ , this causes a **horizontal shift**:



The score of  $C$  remains the same, the score of  $B$  increases and the score of  $A$  decreases.

## Types of Manipulability

We have ordinal preferences, say  $A > B > C$ , and also utilities  $u_1 > u_2 > u_3$ . Utilities can vary but as long as they are ordered as shown, voters will be in the same ordinal type.

Suppose voters of this type vote strategically submitting their preferences as  $B > A > C$  (that is vote for  $B$  instead of  $A$ ) and they manage to change the result of the election.

We say that the system provides voters with **strong incentive** to manipulate if **all** voters of this type benefit from the misrepresentation and **weak incentive** if **some** voters of this type will benefit and some will not.

The voting rule is **micro-manipulable** if an arbitrary small percentage of the population can manipulate.

# Seat maximisers under PR without a threshold

## Theorem

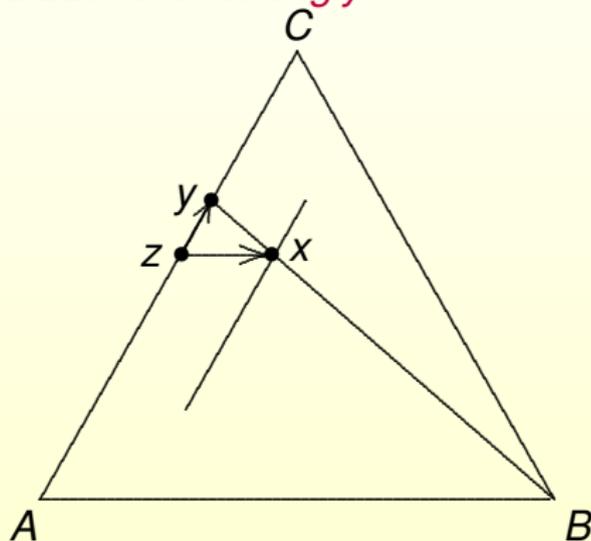
*For PR without threshold a seat maximiser has **no incentives to manipulate**.*

This is probably what [Bowler and Lanoue \(1992\)](#) had in mind.

# Seat maximisers under PR with a threshold

## Theorem

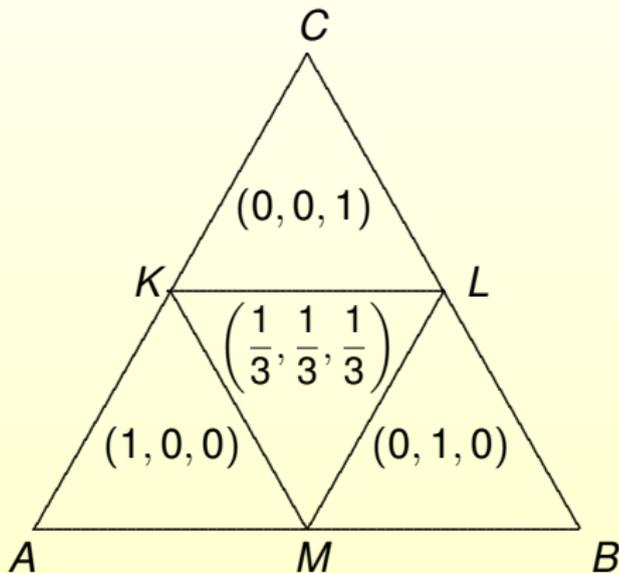
PR with a threshold is always micro-manipulable by seat maximising voters but *never strongly*.



When scores are approaching **x** on the threshold, parliaments tend to **y**.

## Indices of voting power for $m = 3$ parties

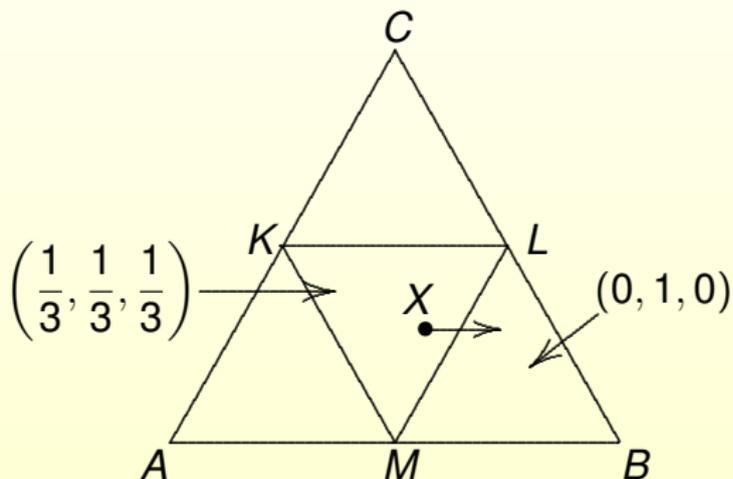
Let the parties be  $A$ ,  $B$ , and  $C$ . Regardless of the voting power index used, *four regions emerge*, where the vectors of indices of power for parliaments will be constant:



# Power maximisers and pure PR

## Theorem

Let the parliament choosing rule  $F$  be pure PR. Then  $F$  is *manipulable by power maximisers but never strongly*.

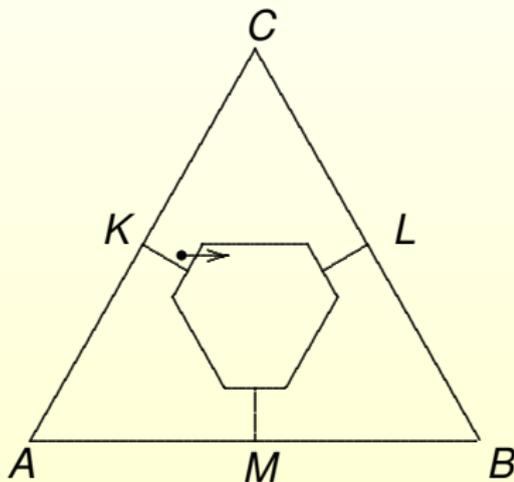


Only uncertainty averse voters will be better off.

# Strong manipulability of PR with threshold

## Theorem

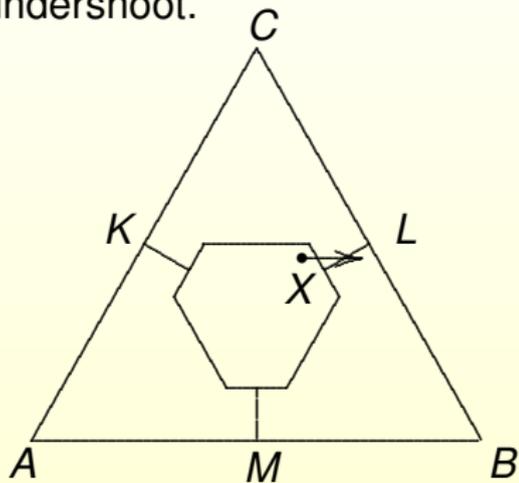
Let the parliament choosing rule be PR with a threshold. Then the rule is **strongly manipulable** by power maximising voters.



The horizontal arrow takes us from  $(0, 0, 1)$  to  $(1/3, 1/3, 1/3)$  which is profitable for all voters of  $A > B > C$  type.

## Undershooting phenomenon

Under PR with a threshold, manipulating uncertainty averse voters may undershoot.



From  $X$  uncertainty averse voters would manipulate to get from  $(1/3, 1/3, 1/3)$  to the region  $(0, 1, 0)$  but they may undershoot and end up in  $(0, 0, 1)$ .

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- We show that the incentives to manipulate may depend on the voter's attitude towards post-election uncertainty. An important distinction between weak and strong incentives to manipulate is made.
- An important distinction is made between seat maximising voters and power maximising voters. Their opportunities to manipulate are described for PR with and without threshold.

# Reflections on MMP. The threshold

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- By adopting (instead of designing) uncritically a voting system from Germany, New Zealand has got an electoral system which significantly distorts the society’s support for minor political parties.
- Do we really need a threshold? At least a threshold of 3% would be much more appropriate.

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- At the election 28.71% of voters gave their electorate vote and their party vote to different parties. This figure is high enough to suggest that a reasonable amount of insincere voting went on (e.g. Epsom).
- Voters in some constituencies (Epsom) get disproportionately large voting power.

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- Party boffins, not voters, determine who gets on the party list. List MPs are not accountable to the voters.
- Parliament gets a bunch of MPs who would be otherwise unelectable (e.g. porn stars in Italy).
- Urgent amendments to the current MMP are in order.

Thank you for your  
attention!