
# Can We Win the Clone War?

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# Jedi Knights

Edith Elkind | Piotr Faliszewski

helped me to fight the clone wars.

# Voting as a group decision making tool

Collective decisions are often made by aggregating the preferences of individual agents by means of voting:

- each agent ranks the available alternatives, and
- a voting rule is used to select one or more winners.

# Decision making environment

The structure of the set of alternatives may be quite complex.

E.g., the set of possible plans of actions, may be huge, however some plans may be very similar to each other.

It may be reasonable to establish, first, which plans differ fundamentally, and which can be viewed as minor variations of each other.

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It may be reasonable to establish, first, which plans differ fundamentally, and which can be viewed as minor variations of each other.

A family on holiday, having to rent a car, may face the options:

- an automatic white car;
- an automatic blue car;
- an automatic red car;
- a manual blue car;
- a manual red car.

# Renting a car quandary

The preferences of husband and wife may be:

| husband | wife |
|---|---|
| manual blue | automatic white |
| manual red | automatic red |
| automatic white | automatic blue |
| automatic blue | manual red |
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⇓

| husband | wife |
|---|---|
| manual | automatic |
| automatic | manual |

The question to resolve is: manual or automatic?

# Clones in consumer surveys and recommenders systems

Suppose an electronics website runs a competition for the best digital camera by asking consumers to vote for their two favorite models from a given list.

Suppose preferences of consumers are:

$$
\begin{aligned}
60\% \quad &: \quad \textit{Sony} > \textit{Nikon} > \textit{Canon} \\
40\% \quad &: \quad \textit{Canon} > \textit{Nikon} > \textit{Sony}
\end{aligned}
$$

If every brand is represented by a single camera, then Nikon wins; if by two cameras, then Sony wins and Nikon will have no votes!



# Most voting rules are sensitive to cloning

In a Borda election with the set of candidates $C = \{a, b, c, d\}$, and there are four voters, whose preference orders are:

$$R_1 : a \succ c \succ b \succ d$$
$$R_2 : a \succ c \succ b \succ d$$
$$R_3 : a \succ c \succ b \succ d$$
$$R_4 : d \succ c \succ b \succ a$$

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$$
\begin{array}{ll}
R_1 : a \succ c \succ b \succ d & Sc_B(a) = 9 \\
R_2 : a \succ c \succ b \succ d & Sc_B(b) = 4 \\
R_3 : a \succ c \succ b \succ d & Sc_B(c) = 8 \\
R_4 : d \succ c \succ b \succ a & Sc_B(d) = 3
\end{array}
$$

The winner here is $a$ with 9 points.

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$$R_3 : a \succ c \succ b \succ d \qquad Sc_B(c) = 8$$
$$R_4 : d \succ c \succ b \succ a \qquad Sc_B(d) = 3$$

The winner here is $a$ with 9 points. However, replacing $b$ with three clones $b_1, b_2, b_3$ is a successful manipulation in favor of $c$:

$$R'_1 : a \succ c \succ b_{i_1} \succ b_{j_1} \succ b_{k_1} \succ d \qquad Sc_B(a) = 15$$
$$R'_2 : a \succ c \succ b_{i_2} \succ b_{j_2} \succ b_{k_2} \succ d \qquad Sc_B(b_{i_s}) = 4 - 12$$
$$R'_3 : a \succ c \succ b_{i_3} \succ b_{j_3} \succ b_{k_3} \succ d \qquad Sc_B(c) = 16$$
$$R'_4 : d \succ c \succ b_{i_4} \succ b_{j_4} \succ b_{k_4} \succ a \qquad Sc_B(d) = 5$$

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The Green party candidate was deliberately cloned.

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- what the real alternatives are, and,
- what the real issues are.

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However, revealing the structure of clones existing in the profile may help clarifying:

- what the real alternatives are, and,
- what the real issues are.

Our goal is to learn what shape clone structures may take.

# Clone Sets and Clone Structures

Our set of alternatives is $[m] = \{1, 2, \ldots, m\}$.

## Definition (Tideman, 1987)

Let $\mathcal{R} = (\succ_1, \ldots, \succ_n)$ be a profile on $[m]$. We say that a non-empty subset $C \subseteq [m]$ is a clone set for $\mathcal{R}$ if every $a \in [m] \setminus C$ and every $c, c' \in C$

$$c \succ_i a \Longrightarrow c' \succ_i a \quad \text{and} \quad a \succ_i c \Longrightarrow a \succ_i c'$$

for every $i = 1, 2, \ldots, n$.

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## Definition

Let $\mathcal{R} = (\succ_1, \ldots, \succ_n)$ be as above and let $\mathcal{C}(\mathcal{R}) \subseteq 2^{[m]}$ be a family of all clone sets for $\mathcal{R}$. We call $\mathcal{C}(\mathcal{R})$ the clone structure on $[m]$ associated with $\mathcal{R}$.

# A Simple Example

**Example**

For $\mathcal{R} = (\succ_1, \succ_2)$ where

$$1 \succ_1 3 \succ_1 2$$
$$3 \succ_2 2 \succ_2 1$$

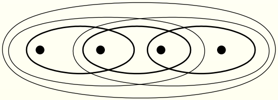
the clone structure will be

$$\mathcal{C}(\mathcal{R}) = \{\{1\},\ \{2\},\ \{3\},\ \{2,3\},\ \{1,2,3\}\}.$$

# An Important Example

## Example (String of sausages)

Let $\mathcal{R}$ consist of a single linear order $R_1 : 1 \succ 2 \succ \cdots \succ m$.
Then $\mathcal{C}(\mathcal{R}) = \{[i,j] \mid i \leq j\}$, where $[i,j] = \{i, i+1, \ldots, j\}$.



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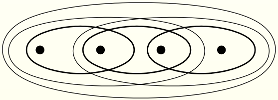


The only nontrivial automorphism is a reflection.

## Proposition

*Any clone structure over* $[m]$ *consists of at most* $\frac{m(m+1)}{2}$ *sets.*

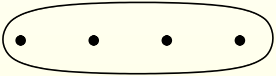
# Another Important Example

**Example (Fat sausage)**

Let $\mathcal{R}$ be a cyclic profile on $[m]$, i.e., $\mathcal{R} = (R_1, \ldots, R_m)$, and $i$-th order is given by

$$R_i : i \succ i+1 \succ \cdots \succ m \succ 1 \succ \cdots \succ i-1.$$

Then $\mathcal{C}(\mathcal{R}) = \{[m]\} \cup \{\{i\} \mid i \in [m]\}$.



The group of automorphisms is the whole $S_m$.

# Four Necessary Conditions

Given two sets $X, Y \subseteq [m]$ are said to intersect non-trivially and write $X \bowtie Y$ if $X \cap Y \neq \emptyset$, $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$.

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## Proposition

*Let $\mathcal{R}$ be a profile on $[m]$. Then $\mathcal{C}(\mathcal{R})$ satisfies:*

(1) $\{i\} \in \mathcal{C}(\mathcal{R})$ *for any* $i \in [m]$*;*

(2) $\emptyset \notin \mathcal{C}$ *and* $[m] \in \mathcal{C}(\mathcal{R})$*;*

(3) *if* $C_1$ *and* $C_2$ *are in* $\mathcal{C}(\mathcal{R})$ *and* $C_1 \cap C_2 \neq \emptyset$*, then* $C_1 \cap C_2$ *and* $C_1 \cup C_2$ *are also in* $\mathcal{C}(\mathcal{R})$*;*

(4) *if* $C_1$ *and* $C_2$ *are in* $\mathcal{C}(\mathcal{R})$ *and* $C_1 \bowtie C_2$*, then* $C_1 \setminus C_2$ *and* $C_2 \setminus C_1$ *are also in* $\mathcal{C}(\mathcal{R})$*.*

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These are not sufficient since $2^{[m]} \setminus \{\emptyset\}$ satisfies these and it has more than $\frac{m(m+1)}{2}$ sets.

## One More Necessary Condition

Given a profile $\mathcal{R}$ over $[m]$ and a set $X \in \mathcal{C}(\mathcal{R})$, we say that a set $Z \in \mathcal{C}(\mathcal{R})$ is a proper minimal clone superset of $X$ if $X \subsetneq Z$, and there is no set $Y \in \mathcal{C}(\mathcal{R})$ such that $X \subsetneq Y \subsetneq Z$.

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*For any profile $\mathcal{R}$ on $[m]$, each $X \in \mathcal{C}(\mathcal{R})$ has at most two proper minimal clone supersets in $\mathcal{C}(\mathcal{R})$.*

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## Proposition

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Note, however, that for $m = 3$ the set family $2^{[m]} \setminus \{\emptyset\}$ satisfies the conclusion of this Proposition and also (1)–(4). Yet, it is obviously not a clone structure: its cardinality is still too large:

$$2^3 - 1 > \frac{3 \cdot (3 + 1)}{2}.$$

# And One More

## Definition

A set family $\{A_0, \ldots, A_{k-1}\}$ is a bicycle chain if $k \geq 3$ and for all $i = 0, \ldots, k-1$ it holds that (1) $A_{i-1} \bowtie A_i$; (2) $A_{i-1} \cap A_i \cap A_{i+1} = \emptyset$; (3) $A_i \subseteq A_{i-1} \cup A_{i+1}$, where all indices are modulo $k$.

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## Proposition

*A clone structure over $[m]$ does not contain a bicycle chain.*

# Now we have enough axioms

**Theorem**

*A family $\mathcal{F}$ of subsets of $[m]$ is a clone structure if and only if it satisfies the following axioms:*

- A1. $\{i\} \in \mathcal{F}$ *for any* $i \in [m]$.
- A2. $\emptyset \notin \mathcal{F}$ *and* $[m] \in \mathcal{F}$.
- A3. *if* $C_1$ *and* $C_2$ *are in* $\mathcal{F}$ *and* $C_1 \cap C_2 \neq \emptyset$, *then* $C_1 \cup C_2$ *and* $C_1 \cap C_2$ *are also in* $\mathcal{F}$.
- A4. *if* $C_1$ *and* $C_2$ *are in* $\mathcal{F}$ *and* $C_1 \bowtie C_2$, *then* $C_1 \setminus C_2$ *and* $C_2 \setminus C_1$ *are also in* $\mathcal{F}$.
- A5. *Each* $C \in \mathcal{F}$ *has at most two proper minimal supersets in* $\mathcal{F}$.
- A6. $\mathcal{F}$ *does not contain a bicycle chain.*

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A6. $\mathcal{F}$ *does not contain a bicycle chain.*

In the proof the concept of a subfamily plays an essential role.

# The idea of a subfamily

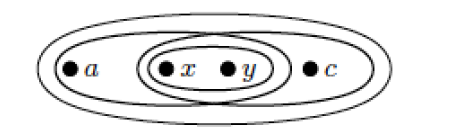
## Example

Consider set families

$$\mathcal{D} = \{\{x\}, \{y\}, \{x, y\}\},$$

$$\mathcal{C} = \{\{a\}, \{x\}, \{y\}, \{x, y\}, \{c\}, \{a, x, y\}, \{x, y, c\}, \{a, x, y, c\}\}.$$

Then $\mathcal{D}$ would be naturally called a subfamily of $\mathcal{C}$.




Subfamily is something that can be collapsed and replaced with one element. In our case $\mathcal{D}$ can be collapsed into $b$:

$$\mathcal{C}(\mathcal{D} \to b) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

# Subfamilies

## Definition

Let $\mathcal{F}$ be a family of subsets on a finite set $F$. A subset $\mathcal{E} \subseteq \mathcal{F}$ is a subfamily of $\mathcal{F}$ if there is a set $E \in \mathcal{F}$ such that

1. $\mathcal{E} = \{X \in \mathcal{F} \mid X \subseteq E\}$;
2. for any $Y \in \mathcal{F} \setminus \mathcal{E}$ we have either $E \subseteq Y$ or $Y \cap E = \emptyset$.

The set $E$ is called the support of $\mathcal{E}$. $\mathcal{E}$ is called a proper subfamily of $\mathcal{F}$ if $E$ is a proper subset of $F$.

# Composition of Families

Composition of families does exactly the opposite to collapsing.
E.g., we can undo collapsing $\mathcal{D}$ in the previous example.

## Example

Consider set families $\mathcal{D} = \{\{x\}, \{y\}, \{x, y\}\}$ and
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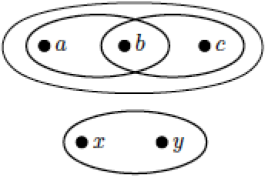
Consider set families $\mathcal{D} = \{\{x\}, \{y\}, \{x, y\}\}$ and $\mathcal{C} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$

Then we can replace $\{b\}$ with $\mathcal{D}$ and in other sets $b$ with $x, y$. This results in a family of subsets $\mathcal{C}(b \to \mathcal{D})$ given by

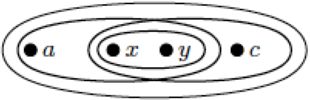
$$\{\{a\}, \{x\}, \{y\}, \{x, y\}, \{c\}, \{a, x, y\}, \{x, y, c\}, \{a, x, y, c\}\}.$$

# Composition of structures. Example

On the previous slide two families were composed over *b*:




(a) Before embedding.




(b) After embedding.

# Towards the Main Theorem

A crucial step towards the main theorem:

**Proposition**

*Let $\mathcal{E}$ and $\mathcal{F}$ be families of subsets on disjoint sets $E$ and $F$, respectively, that are clone structures. Then for any $e \in E$ the composition $\mathcal{E}(e \to \mathcal{F})$ is also a clone structure.*

We illustrate the proof on the following slide.

# Idea of the proof

Consider two clone structures

$$\mathcal{C} = \{\{1\},\ \{2\},\ \{3\},\ \{2,3\},\ \{1,2,3\}\}, \quad \mathcal{D} = \{\{4\},\{5\},\{4,5\}\}$$

and show that $\mathcal{C}(3 \to \mathcal{D})$ is also a clone structure.

We know $\mathcal{C} = \mathcal{C}(\mathcal{R})$, where $\mathcal{R} = (R_1, R_2)$ with

$$\begin{aligned} R_1 &: 1 \succ 3 \succ 2 \\ R_2 &: 3 \succ 2 \succ 1 \end{aligned}$$

We obtain $\mathcal{C}(3 \to \mathcal{D})$ by cloning 3:

$$\begin{aligned} R_1' &: 1 \succ 4 \succ 5 \succ 2 \\ R_2' &: 5 \succ 4 \succ 2 \succ 1 \end{aligned}$$

Indeed we obtain the clone structure

$$\mathcal{C}(\mathcal{R}') = \{\{1\},\ \{2\},\ \{4\},\ \{5\},\{4,5\},\ \{2,4,5\},\ \{1,2,4,5\}\}.$$

which is exactly $\mathcal{C}(3 \to \mathcal{D})$.

# A Crucial Step Towards the Main Theorem

## Definition

A family of subsets is indecomposable if it does not have proper subfamilies.

The crucial steps towards the main theorem is the following proposition:

## Proposition

*Any indecomposable family of subsets satisfying A1–A6 is either a string of sausages or a fat sausage and hence is a clone structure.*

# Proof of the Main Theorem

Suppose family $\mathcal{F}$ satisfies A1–A6. If it is indecomposable, then $\mathcal{F}$ is a clone structure.

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Suppose family $\mathcal{F}$ satisfies A1–A6. If it is indecomposable, then $\mathcal{F}$ is a clone structure.

If not, it contains a proper subfamily $\mathcal{E}$. Let $z$ be an element which is not in the base set of $\mathcal{F}$. Then $\mathcal{F}' = \mathcal{F}(\mathcal{E} \to z)$ satisfies A1–A6, hence a clone structure by the induction hypothesis.

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So is $\mathcal{E}$.

However, then by the previous proposition,

$$\mathcal{F} = \mathcal{F}'(z \to \mathcal{E})$$

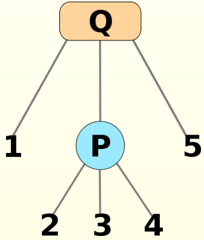
also a clone structure.

# *PQ*-trees

PQ-trees were introduced by Booth and Lueker (1976).

A PQ-tree $T$ over a set $A = \{a_1, \ldots, a_n\}$ is a class of equivalent ordered trees over $A$ such that:

- The leaves of the tree correspond to the elements of $A$.
- Each internal node is either of type P or of type Q.
- If a node is of type P, then its children can be permuted arbitrarily.
- If a node is of type Q, then the order of its children can only be reversed.

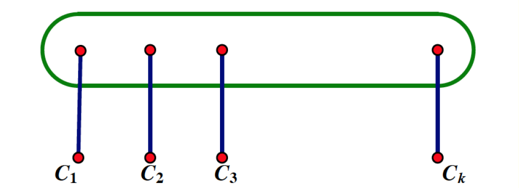


# Clone Structures and *PQ*-trees

## Theorem

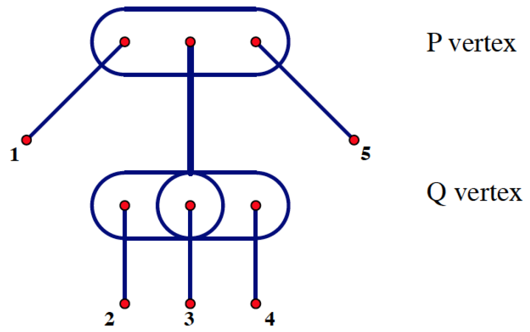
*Clone structures are in 1-1 corresponcence with PQ-trees.*

## Proof:




Collapsing all maximal subfamilies $C_1, \ldots, C_k$ we get either a fat sausage or a string of sausages which is either a P-vertex or a Q-vertex. This will be the root of the tree.

# Clone Structure as a PQ-tree. Example




Any fat sausage gives us a P-vertex and string of sausages gives us a Q-vertex.

# Implementability of Clone Structures 1

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## Proposition

*For $m > 3$ a fat sausage is 2-implementable.*

Proof: Suppose e.g., that $m = 2k$. For convenience, set $x_i = i$, $y_i = k + i$ for $i = 1, \ldots, k$. We define $\mathcal{R} = (R_1, R_2)$ as follows.

$$R_1 : x_1 \succ \ldots \succ x_k \succ y_1 \succ \ldots \succ y_k,$$
$$R_2 : y_1 \succ x_1 \succ y_2 \succ x_2 \succ \ldots \succ y_k \succ x_k.$$

Then no proper clone set $D$ exists. Looking at $R_1$: three cases to consider:

- $D \subseteq \{x_1, \ldots, x_k\}$;
- $D \subseteq \{y_1, \ldots, y_k\}$;
- $D \supseteq \{x_k, y_1\}$.

All cases are impossible if we take $R_2$ into account.

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## Proposition

*Any indecomposable clone structure over* $[m]$ *is* 3*-implementable.*

## Proof

- A string of sausages is 1-implementable;
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*Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be two 3-implementable clone structures on disjoint sets* $E$ *and* $F$*, respectively. Then for any* $e \in E$ *the composition* $\mathcal{E}(e \to \mathcal{F})$ *is also 3-implementable.*

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## Theorem

*Any clone structure is* 3*-implementable.*

# Single-Peaked Preferences

One of the assumptions often made in Political Science is that the ideological spectrum is one-dimensional.


This means that there exist a linear order $>$ over $C$ (the societal axis). We say that an order $\succ$ over $C$ is compatible with $>$ if for all $c, d, e \in C$ such that either $c > d > e$ or $e > d > c$, it holds that $c \succ d \implies d \succ e$.

# Two definitions

**Definition**
Let $\mathcal{R} = (\succ_1, \ldots, \succ_n)$ be a profile over $C$. It is called single-peaked if there is a societal axis such that $\succ_i$ is compatible with it for every $i = 1, \ldots, n$.

**Definition**
A clone structure $\mathcal{C}$ is single-peaked if it can be rimplemented by a single-peaked profile $\mathcal{R}$, that is $\mathcal{C} = \mathcal{C}(\mathcal{R})$.

# Some observations and open question

## Proposition

*Any fat sausage and string of sausages are single-peaked.*

# Some observations and open question

## Proposition

*Any fat sausage and string of sausages are single-peaked.*

For example, this profile, called a slide, is single-peaked and implements the fat sausage:

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $b$ | $b$ |
| $b$ | $a$ | $c$ | $c$ | $c$ |
| $c$ | $c$ | $a$ | $d$ | $d$ |
| $d$ | $d$ | $d$ | $a$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $a$ |

# Some observations and open question

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| $d$ | $d$ | $d$ | $a$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $a$ |

However, if $\mathcal{C}$ is a string of sausages over $\{a, b, c\}$ and $\mathcal{D}$ is a string of sausages over $\{u, v, w\}$. Then $\mathcal{C}(b \to \mathcal{D})$ is not single-peaked.

# Open Question

Open Question: Characterise single-peaked clone structures in terms of their *PQ*-trees.

# Single-crossing profiles

Let $\mathcal{R} = (\succ_1, \dots, \succ_n)$ be a profile over $C$ and let $\{a, b\} \subseteq C$. We say that $\mathcal{R}$ is single-crossing with respect to $\{a, b\}$ if there exist $1 \leq k \leq n$ such that $a \succ_i b$ for $i \in [1, \dots, k]$ and $b \succ_i a$ for $i \in [k+1, \dots, n]$ or $b \succ_i a$ for $i \in [1, \dots, k]$ and $a \succ_i b$ for $i \in [k+1, \dots, n]$.

A profile is single-crossing if it is single-crossing with respect to any pair. For example, any slide is single-crossing:

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $b$ | $b$ |
| $b$ | $a$ | $c$ | $c$ | $c$ |
| $c$ | $c$ | $a$ | $d$ | $d$ |
| $d$ | $d$ | $d$ | $a$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $a$ |

# Indecomposable structures are single-crossing

Any 1-implementable or 2-implementable clone structures are trivially single-crossing. The only indecomposable clone structure left is a fat sausage for $m = 3$.

We can use a slide to implement it:

$$\begin{matrix} a & b & b \\ b & a & c \\ c & c & a \end{matrix}$$

### Proposition

*Let $\mathcal{E}$ and $\mathcal{F}$ be two single-crossing clone structures on disjoint sets $E$ and $F$, respectively. Then for any $e \in E$ the composition $\mathcal{E}(e \to \mathcal{F})$ is also single-crossing.*

### Theorem

*Every clone structure is single crossing.*

# Shall we always declone?

Once you start decloning it is not clear where to stop.

However there is an exception: if you expected that the election will be single-peaked but it was not. You suspect that it was due to cloning.

We found a polynomial time algorithm that takes an election and outputs a single-peaked election collapsing as few clones as possible.

However decloning to a single-crossing election is NP-complete.

# Where published?

P. Faliszewski, E. Elkind and A. Slinko. Clone structures in voters' preferences. EC '12: Proceedings of the 13th ACM Conference on Electronic Commerce 2012: 496-513, ACM, New York, NY.

# And may the force be with you all!