Simple Games beyond Weighted Majority Games

Arkadii Slinko

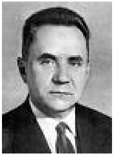
Department of Mathematics
The University of Auckland

(joint work with Tatiana Gvozdeva, Lane Hemaspaandra and Marston Conder)

Back in the USSR


Ustinov


Brezhnev


Kosygin

The three top state officials, the President, the Prime Minister, and the Minister of Defence, all had “nuclear suitcases”. Any two of them could authorise a launch of a nuclear warhead. No one could do it alone.

US Senate

United States Senate rules permit a senator, or a number of senators, to speak for as long as they wish and on any topic they choose, unless a supermajority of the Senate (60 Senators) brings debate to a close by invoking cloture.

UN Security Council

The 15 member UN Security Council consists of five permanent and 10 non-permanent countries. A passage requires:

- approval of at least nine countries,
- subject to a veto by any one of the permanent members.

Simple Games

The set $P = \{1, 2, \ldots, n\}$ denotes the set of players.

Definition

A simple game is a pair $G = (P, W)$, where W is a subset of the power set 2^P, different from $\emptyset$, which satisfies the monotonicity condition:

if $X \in W$ and $X \subset Y \subseteq P$, then $Y \in W$.

Coalitions from W are called winning. We also denote

$$L = 2^P \setminus W$$

and call coalitions from L losing.

Significant Publications

Simple games are far from being simple. They are discussed in:

- von Neumann, J., and O. Morgenstern (1944) *Theory of games and economic behavior*. Princeton University Press. Princeton. NJ
- Shapley, L.S (1962) *Simple games: an outline of the descriptive theory*. Behavioral Science 7: 59–66
- Taylor, A.D., and W.S. Zwicker (1999) *Simple games*. Princeton University Press. Princeton. NJ

Weighted Majority Games

Definition

A simple game G is called a weighted majority game if there exists a weight function $w\colon P \to \mathcal{R}^+$, where $\mathcal{R}^+$ is the set of all non-negative reals, and a real number q, called quota, such that

$$X \in W \iff \sum_{i \in X} w_i \geq q.$$

Such game is denoted

$$[q; w_1, \ldots, w_n].$$

Weights for Games in Examples

Nuclear suitcases game:

$$[2; 1, 1, 1].$$

American Senate game:

$$[60; 1, 1, 1, \ldots, 1].$$

UN Security Council game:

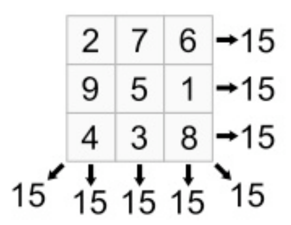
$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

Does every game have weights?

Rigid Magic Squares

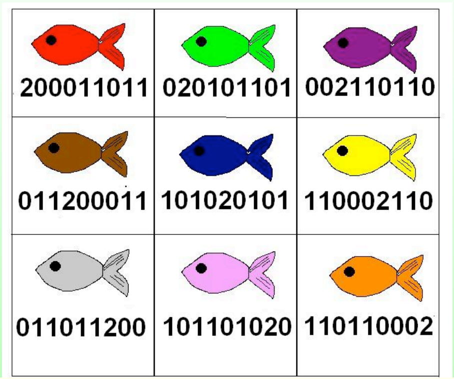
On the right you see a magic square. A rigid magic square will have:

- The sum in every row and in every column is equal to q.
- No other subset of the numbers has the sum equal to q.



Such number q will be called a threshold.

A Rigid Magic Square



The quota is

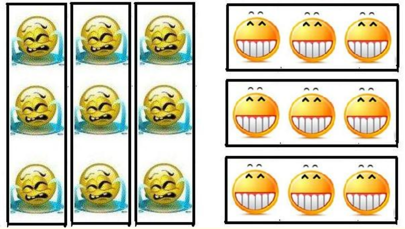
$$q = 222222222.$$

Gabelman's game Gab_n

Example

Let us take an $n \times n$ rigid magic square with threshold q and n^2 of players, one for each cell. We assign to a player the weight in his cell.

- Coalitions whose weight is $> q$ are winning.
- Coalitions whose weight is $< q$ are losing.
- Rows are winning.
- Columns are losing.



No system of weights can be found for this game.

Trading transform

Definition

The sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

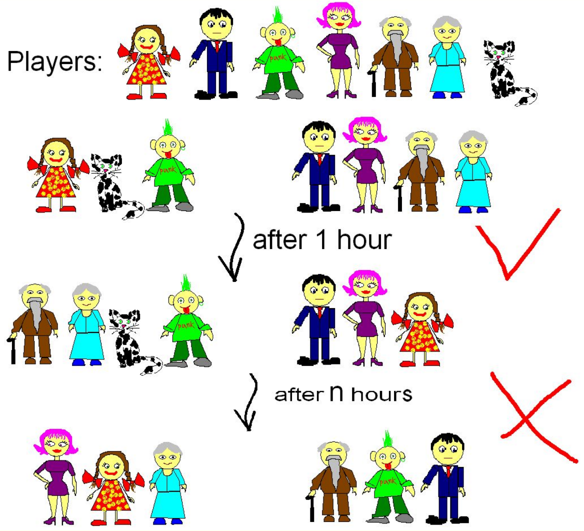
is called a trading transform if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players.

In Gabelman's game Gab_3 with 9 players

$$\mathcal{T} = (Row_1, Row_2, Row_3; Col_1, Col_2, Col_3)$$

is a trading transform.

Yet another trading transform



A criterion of weightedness

Definition

A simple game is k-trade robust if for all $j \leq k$ no trading transform

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

exists where $X_1, \ldots, X_j$ are winning and $Y_1, \ldots, Y_j$ are losing.

Theorem (Taylor & Zwicker, 1992)

For a simple game G the following is equivalent:

1. *G is weighted.*
2. *G is trade robust.*
3. *G is 2^{2^n}-trade robust.*

Function f

Definition

Let G be a simple game and

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

a trading transform where $X_1, \ldots, X_j$ are winning and $Y_1, \ldots, Y_j$ are losing (so G is not j-trade robust). Then we call $\mathcal{T}$ a certificate of non-weightedness.

Definition

If G is weighted we set $f(G) = \infty$. Otherwise $f(G)$ is the length of the shortest certificate of non-weightedness. For games with n players we define

$$f(n) = \max_{f(G) \neq \infty} f(G).$$

Bounds on function f

In terms of the function f the results known to date can be summarised as follows:

$$\lfloor \sqrt{n} \rfloor \leq f(n) \leq 2^{2^n}.$$

Theorem (Gvozdeva-Slinko, 2009)

$$\left\lfloor \frac{n}{2} \right\rfloor \leq f(n) \leq (n+1)2^{\frac{1}{2}n\log_2 n}.$$

The idea of the proof for the lower bound can be illustrated on the following example.

The Idea of the Lower Bound

Consider weights $(w_1, w_2, w_3, w_4, w_5) = (1, 2, 5, 6, 10)$. Then:

Equality	Total weight
$13 \sim 4$	6
$14 \sim 23$	7
$25 \sim 134$	12
$34 \sim 15$	11

$\longrightarrow$

Equality	Total weight
$136 \sim 46$	6+106=112
$147 \sim 237$	7+105=112
$258 \sim 1348$	12+100=112
$349 \sim 159$	11+101=112

We add : $(w_6, w_7, w_8, w_9) = (106, 105, 100, 101)$ and define

- Coalitions whose weight is > 112 are winning.
- Coalitions whose weight is < 112 are losing.
- 46, 237, 1348, 159 are winning.
- 136, 147, 258, 349 are losing.

This gives us $f(9) \geq 4$.

Rough weights

Definition

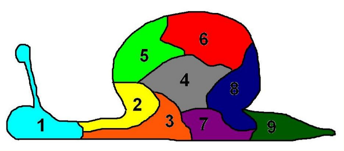
A simple game G is called roughly weighted if there exists a weight function $w\colon P \to \mathcal{R}^+$, not identically equal to zero, and a non-negative real number q, called quota, such that for $X \in 2^P$

$$\sum_{i \in X} w_i > q \Longrightarrow X \in W,$$
$$\sum_{i \in X} w_i < q \Longrightarrow X \in L.$$

We say $[q; w_1, \ldots, w_n]$ is a rough voting representation for G.

An Example of Roughly Weighted Majority Game

This Kingdom has 9 provinces. A passage requires approval of at least three provinces, not all of which are neighbours.

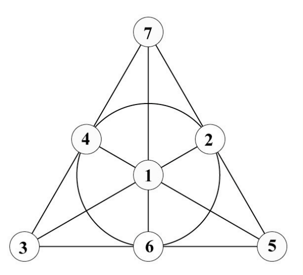


We assign weight 1 to every province. Then:

- Coalitions whose weight is > 3 are winning.
- Coalitions whose weight is < 3 are losing.

Gabelman's games are not weighted but they are roughly weighted. So are our examples. Does every simple game have rough weights?

The Fano plane game



We take $P = \{1, 2, \ldots, 7\}$ and the lines $X_1, \ldots, X_7$ as minimal winning coalitions:

$$\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,5,7\},$$
$$\{3,4,7\}, \{3,5,6\}, \{2,4,6\}.$$

Then the sequence

$$\mathcal{T} = (X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset)$$

is a certificate of non-weightedness of G. But it shows more: the absence of rough weights.

A criterion of rough weightedness

Theorem (Gvozdeva-Slinko, 2009)

A game G is roughly weighted if for no j there exists a certificate of non-weightedness of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset). \qquad (\star)$$

Certificates of the form $(\star)$ we call potent.

This theorem leads to the introduction of another function.

Definition

Let the number of players be n. If G is roughly weighted, then $g(G) = \infty$. Else, let $g(G)$ be the length of the shortest potent certificate of non-weightedness of the form, and

$$g(n) = \max_{g(G) \neq \infty} g(G).$$

Bounds for g

In particular, $g(Fano) = 8$, while $f(Fano) = 2$. In particular,

$$g(7) \geq 8.$$

Theorem (Gvozdeva-Slinko, 2009)

For $n \geq 5$

$$2n + 3 \leq g(n) < (n + 1)2^{\frac{1}{2} n \log_2 n}.$$

More Definitions

Definition

A simple game G is called proper if

$$X \in W \Longrightarrow X^c \in L,$$

strong if

$$X \in L \Longrightarrow X^c \in W,$$

and a constant sum game if G is both proper and strong.

- Nuclear sutcases game and EEC: constant sum games
- American Senat and UN Security Council: proper but not strong.
- Gamelman's game: strong but not proper.

Weightedness of Small Games

Theorem (Shapley, 1962)

The following games are weighted:

- *every game with 3 or less players,*
- *every strong or proper game with 4 or less players,*
- *every constant sum game with 5 or less players.*

Rough Weightedness of Small Games

Theorem (Gvozdeva-Slinko, 2009)

The following games are roughly weighted:

- *every game with 4 or less players,*
- *every strong or proper game with 5 or less players,*
- *every constant sum game with 6 or less players.*

The $\mathcal{A}$-Hierarchy

Definition

Let q be a rational number. A game G belongs to the class $\mathcal{A}_q$ of $\mathcal{A}$-hierarchy if G possesses a potent certificate of nonweightedness

$$(X_1, \ldots, X_m; Y_1, \ldots, Y_m),$$

with ℓ grand coalitions among $X_1, \ldots, X_m$ and ℓ empty coalitions among $Y_1, \ldots, Y_m$, such that $q = \ell/m$. If α is irrational, we set $\mathcal{A}_\alpha = \bigcap_{q<\alpha} \mathcal{A}_q$.

The larger the parameter α the more power is given to some relatively small coalitions.

The $\mathcal{A}$-Hierarchy

Theorem

If $0 < \alpha < \beta < \frac{1}{2}$, then $\mathcal{A}_\alpha \supsetneq \mathcal{A}_\beta$. Every simple non-weighted game belongs to one of the classes $\mathcal{A}_\alpha$ for $\alpha \in [0, \frac{1}{2})$.

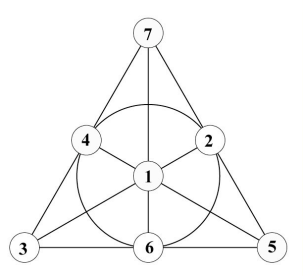
Definition

A game G is critical for $\mathcal{A}_\alpha$ if it belongs to $\mathcal{A}_\alpha$ but does not belong to any $\mathcal{A}_\beta$ for $\beta > \alpha$.

Example

$\mathcal{A}_0$ is comprised of non-weighted roughly weighted games. Fano is critical for $\mathcal{A}_{1/8}$.

Two thresholds for the Fano plane game



We take $P = \{1, 2, \dots, 7\}$ and the lines $X_1, \dots, X_7$ as minimal winning coalitions:

$\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,5,7\},$
$\{3,4,7\}, \{3,5,6\}, \{2,4,6\}.$

Then we can assign weight one to every player and tell, that

- Coalitions whose weight is > 4 are winning.
- Coalitions whose weight is < 3 are losing.
- Coalition whose weight is 3 is winning if it is a line.
- Coalition whose weight is 4 is winning if it contains a line.

$\mathcal{B}$-Hierarchy

Definition

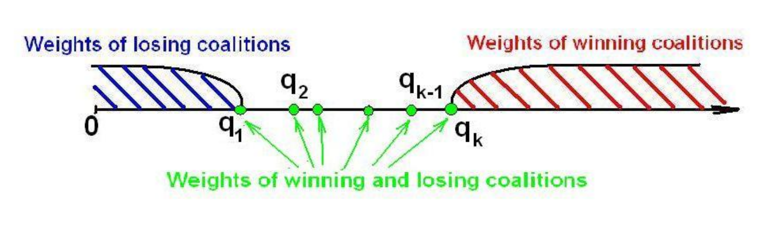
A simple game $G = (P, W)$ belongs to $\mathcal{B}_k$ if there exist real numbers $0 < q_1 \leq q_2 \leq \ldots \leq q_k$, called thresholds, and a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that

(a) if $\sum_{i \in X} w(i) > q_k$, then X is winning,

(b) if $\sum_{i \in X} w(i) < q_1$, then X is losing,

(c) if $q_1 \leq \sum_{i \in X} w(i) \leq q_k$, then

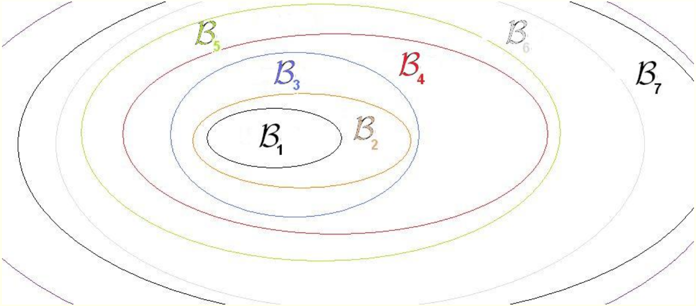
$$w(X) = \sum_{i \in X} w(i) \in \{q_1, \ldots, q_k\}.$$



$\mathcal{B}$-Hierarchy

Theorem

For every natural number k $\in \mathbb{N}^+$, there exists a game in $\mathcal{B}_{k+1} \setminus \mathcal{B}_k$. Every simple game belongs to one of the classes of this hierarchy.



Example

Weighted and roughly weighted games form the class $\mathcal{B}_1$. Fano is critical for $\mathcal{B}_2$.

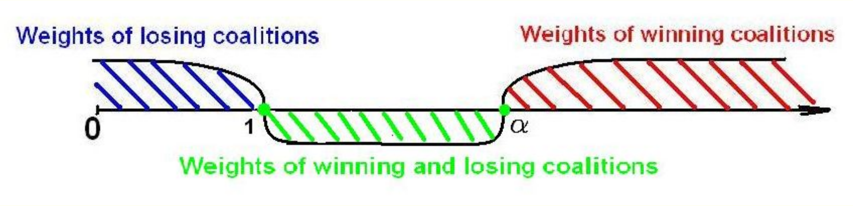
A faculty vote

There are 99 academics in the Science Faculty. If neither side controls a $2/3$ majority, then the Dean would decide the outcome as he wished. Here we need 33 thresholds so the game is in $\mathcal{B}_{33}$.

$\mathcal{C}$-Hierarchy

Definition

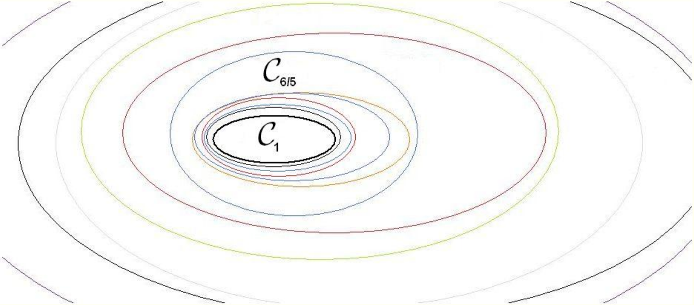
We say that a simple game $G = (P, W)$ is in the class $\mathcal{C}_\alpha$, $\alpha \in \mathbb{R}^{\geq 1}$, if there exists a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that for $X \in 2^P$ the condition $w(X) > \alpha$ implies that X is winning, and $w(X) < 1$ implies X is losing.



$\mathcal{C}$-Hierarchy

Theorem

For each $1 \leq \alpha < \beta$, it holds that $\mathcal{C}_\alpha \subsetneq \mathcal{C}_\beta$. Every simple game belongs to one of the classes $\mathcal{C}_\alpha$.



Example

Fano game is critical for $\mathcal{C}_{4/3}$ and the faculty vote game is critical for $\mathcal{C}_2$.