New Bounds for Simple Games

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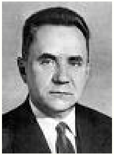
(joint work with Tatiana Gvozdeva)

Back in the USSR

Ustinov

Brezhnev



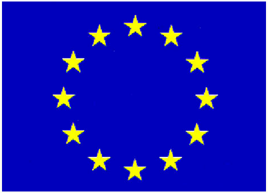
Kosygin

The three top state officials, the President, the Prime Minister, and the Minister of Defence, all had “nuclear suitcases”. Any two of them could authorise a launch of a nuclear warhead. No one could do it alone.

US Senate

United States Senate rules permit a senator, or a number of senators, to speak for as long as they wish and on any topic they choose, unless a supermajority of the Senate (60 Senators) brings debate to a close by invoking cloture.

The European Economic Community

In 1958, the Treaty of Rome established the following voting system. The voters were: France, Germany, Italy, Belgium, the Netherlands and Luxembourg.

- France, Germany and Italy got 4 votes each,
- Belgium and the Netherlands got two votes,
- Luxembourg was given one vote.

Passage requires at least 9 of the 17 possible votes.

UN Security Council

The 15 member UN Security Council consists of five permanent and 10 non-permanent countries. A passage requires:

- approval of at least nine countries,
- subject to a veto by any one of the permanent members.

Simple Games

A simple game is a mathematical object that is used to describe the distribution of power among coalitions of players.

They have also been studied in a variety of other mathematical contexts under various names, e.g.:

- boolean or switching functions,
- threshold functions,
- hypergraphs,
- coherent structures,
- Sperner systems,
- abstract simplicial complexes.

A number of results have been discovered several times.

Definition of a Simple Game

The set $P = \{1, 2, \ldots, n\}$ denotes the set of players.

Definition

A simple game is a pair $G = (P, W)$, where W is a subset of the power set 2^P, different from $\emptyset$, which satisfies the monotonicity condition:

if $X \in W$ and $X \subset Y \subseteq P$, then $Y \in W$.

Coalitions from W are called winning. We also denote

$$L = 2^P \setminus W$$

and call coalitions from L losing.

Significant Publications

- von Neumann, J., and O. Morgenstern (1944) *Theory of games and economic behavior*. Princeton University Press. Princeton. NJ
- Shapley, L.S (1962) *Simple games: an outline of the descriptive theory*. Behavioral Science 7: 59–66.
- Winder, R. *Threshold Logic*, Doctoral Thesis, Princeton University, Princeton, 1962.
- Muroga, S. *Threshold logic and Its Applications*. Wiley Interscience, New York, 1971.
- Taylor, A.D., and W.S. Zwicker (1999) *Simple games*. Princeton University Press. Princeton. NJ.

Weighted Majority Games

Definition

A simple game G is called a weighted majority game if there exists a weight function $w\colon P \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of all non-negative reals, and a real number q, called quota, such that

$$X \in W \iff \sum_{i \in X} w_i \geq q.$$

Such game is denoted

$$[q; w_1, \ldots, w_n].$$

Weights for Games in Examples

Nuclear suitcases game:

$$[2; 1, 1, 1].$$

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$$[9; 4, 4, 4, 2, 2, 1].$$

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European Community game:

$$[9; 4, 4, 4, 2, 2, 1].$$

UN Security Council game:

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

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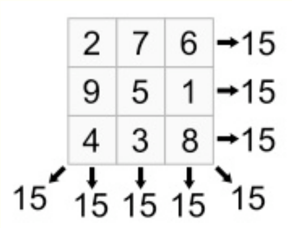
Does every simple game have weights?

Rigid Magic Squares

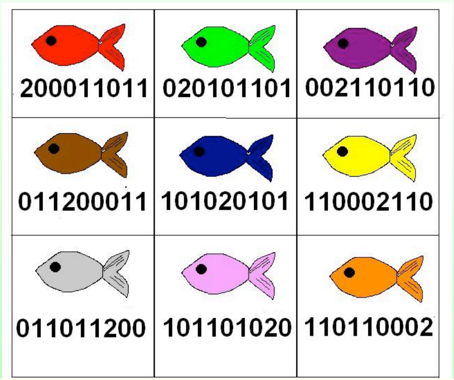
On the right you see a magic square. A rigid magic square will for some q have:

- The sum in every row and in every column is equal to q.
- No other subset of the numbers has the sum equal to q.



Such number q will be called a threshold.

A Rigid Magic Square



The quota is

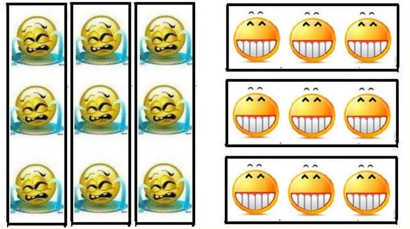
$$q = 222222222.$$

Gabelman's game Gab_n

Example

Let us take an $n \times n$ rigid magic square with threshold q and n^2 of players, one for each cell. We assign to a player the weight in his cell.

- Coalitions whose weight is $> q$ are winning.
- Coalitions whose weight is $< q$ are losing.
- Rows are winning.
- Columns are losing.



No system of weights can be found for this game.

Trading transform. Example

Definition

The sequence of coalitions

$$\mathcal{T} = (X_1, \dots, X_j; Y_1, \dots, Y_j)$$

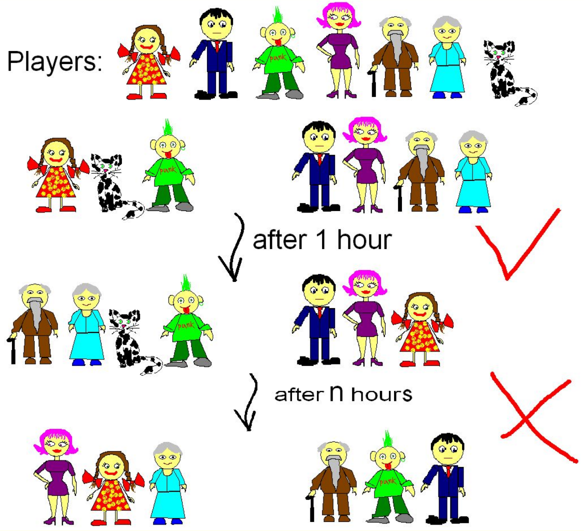
is called a trading transform if the coalitions $X_1, \dots, X_j$ can be converted into the coalitions $Y_1, \dots, Y_j$ by rearranging players.

In Gabelman's game Gab_3 with 9 players

$$\mathcal{T} = (Row_1, Row_2, Row_3; Col_1, Col_2, Col_3)$$

is a trading transform.

Yet another example of trading transform



A criterion of weightedness

Definition

A simple game G is k-trade robust if for all $j \leq k$ no trading transform

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

exists where $X_1, \ldots, X_j$ are winning and $Y_1, \ldots, Y_j$ are losing. It is said to be trade robust if it is k-trade robust for every k.

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Theorem (Taylor & Zwicker, 1992)

For a simple game G the following is equivalent:

1. *G is weighted.*
2. *G is trade robust.*
3. *G is 2^{2^n}-trade robust.*

Function f

Definition

Let G be a simple game and

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Definition

If G is weighted we set $f(G) = \infty$. Otherwise, $f(G)$ is the length of the shortest certificate of non-weightedness. For games with n players we define

$$f(n) = \max_{f(G) \neq \infty} f(G).$$

Bounds on function f

In terms of the function f the results known before us can be summarised as follows:

$$\lfloor \sqrt{n} \rfloor \le f(n) \le 2^{2^n}.$$

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Theorem (Gvozdeva-Slinko, 2009)

$$\left\lfloor \frac{n}{2} \right\rfloor \le f(n) \le 2^{\frac{1}{2}(n+1)\log_2 n}.$$

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$$\left\lfloor \frac{n}{2} \right\rfloor \leq f(n) \leq 2^{\frac{1}{2}(n+1)\log_2 n}.$$

Our proof for the lower bound uses results of Fishburn and Conder & Slinko on comparative probability orders.

The Idea of the Lower Bound

Consider weights $(w_1, w_2, w_3, w_4, w_5) = (1, 2, 5, 6, 10)$. Then:

Equality	Total weight
$13 \sim 4$	6
$14 \sim 23$	7
$25 \sim 134$	12
$34 \sim 15$	11

$\longrightarrow$

Equality	Total weight
$136 \sim 46$	6+106=112
$147 \sim 237$	7+105=112
$258 \sim 1348$	12+100=112
$349 \sim 159$	11+101=112

We add : $(w_6, w_7, w_8, w_9) = (106, 105, 100, 101)$ and define

- Coalitions whose weight is > 112 are winning.
- Coalitions whose weight is < 112 are losing.
- 46, 237, 1348, 159 are winning.
- 136, 147, 258, 349 are losing.

This gives us $f(9) \geq 4$. Gabelman's example gives $f(9) \geq 3$.

The Ideal of a Game

Let $T = \{-1, 0, 1\}$ and $T^n = T \times T \times \ldots \times T$ (n times).

Definition

Let $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$, where the only nonzero element 1 is in the ith position. Then a subset $I \subseteq T^n$ will be called an ideal in T^n if for any $i = 1, 2, \ldots, n$

$$(\mathbf{v} \in I \text{ and } \mathbf{v} + \mathbf{e}_i \in T^n) \Longrightarrow \mathbf{v} + \mathbf{e}_i \in I.$$

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Any game $G = (P, W)$ is associated with an ideal. For any pair (X, Y), where $X \in W$ and $Y \in L$, we define

$$\mathbf{v}_{X,Y} = \chi(X) - \chi(Y) \in T^n,$$

where $\chi(X)$ and $\chi(Y)$ are the characteristic vectors of X and Y, respectively. The set of all such vectors we will denote $I(G)$ and call the ideal of G.

The Idea of the Upper Bound

Proposition

Let G be a game for which all coalitions $X_1, \ldots, X_j$ *are winning and all coalitions* $Y_1, \ldots, Y_j$ *are losing. Then the sequence*

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

is a certificate of non-weightedness iff

$$\mathbf{v}_{X_1,Y_1} + \ldots + \mathbf{v}_{X_j,Y_j} = \mathbf{0}.$$

This reduces the problem to Linear Algebra. Further details are technical.

Rough weights

Definition

A simple game G is called roughly weighted if there exists a weight function $w\colon P \to \mathbb{R}^+$, not identically equal to zero, and a positive real number q, called quota, such that for $X \in 2^P$

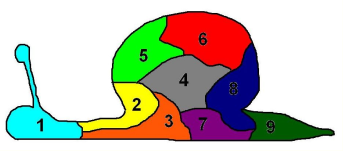
$$\sum_{i \in X} w_i > q \Longrightarrow X \in W,$$

$$\sum_{i \in X} w_i < q \Longrightarrow X \in L.$$

We say $[q; w_1, \ldots, w_n]$ is a rough voting representation for G.

An Example of Roughly Weighted Majority Game

This Kingdom has 9 provinces. A passage requires approval of at least three provinces, not all of which are neighbours.



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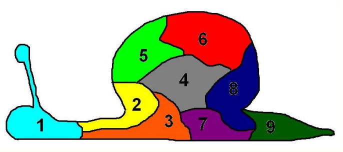

We assign weight 1 to every province. Then:

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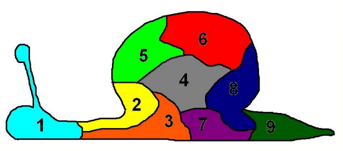

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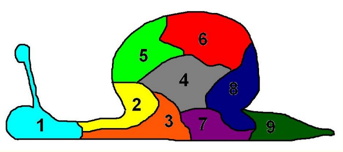

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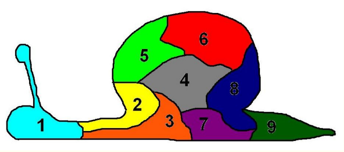

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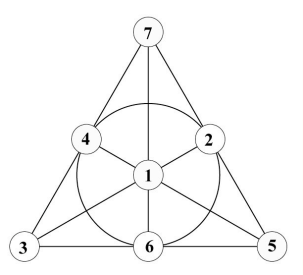
Gabelman's games are not weighted but they are roughly weighted. So are our examples. Does every simple game have rough weights?

The Fano plane game



We take $P = \{1, 2, \ldots, 7\}$ and the lines $X_1, \ldots, X_7$ as minimal winning coalitions:

$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}, \{2, 4, 6\}.$

Then the sequence

$$\mathcal{T} = (X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset)$$

is a certificate of non-weightedness of G. But it actually shows more: the absence of rough weights.

A criterion of rough weightedness

Theorem (Gvozdeva-Slinko, 2009)

A game G is roughly weighted if for no j there exists a certificate of non-weightedness of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset). \qquad (\star)$$

We will call such a certificate potent.

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We will call such a certificate potent.

In the ideal, $(\star)$ is equivalent to

$$\sum_{i=1}^{j} \mathbf{v}_{X_i, Y_i} + \mathbf{v}_{P,\emptyset} = \mathbf{0}.$$

where $\mathbf{v}_{P,\emptyset} = (1, 1, \ldots, 1)$.

Function g

This theorem leads to the introduction of another function g.

Definition

Let the number of players be n. If G is roughly weighted, then $g(G) = \infty$. Else, let $g(G)$ be the length of the shortest potent certificate of non-weightedness and define

$$g(n) = \max_{g(G) \neq \infty} g(G).$$

$g(Fano) = 8$

We saw that $g(Fano) \leq 8$. However, it cannot be smaller than 8. Suppose

$$\sum_{i=1}^{j} \mathbf{v}_{X_i,Y_i} + (1, 1, \ldots, 1) = \mathbf{0}.$$

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In particular,

$$g(7) \geq 8.$$

Bounds for g

Theorem (Gvozdeva-Slinko, 2009)

For $n \geq 5$

$$2n + 3 \leq g(n) < 2^{\frac{1}{2}(n+1)\log_2 n}.$$

The lower bound is proved by constructing a series of examples. The upper bound is the same as for function f.

The lower bound for $g(n)$

Let us define a game $G_{n,2} = ([n], W)$ for which the following holds:

- $\{1, 2\} \in W$ and $\{3, 4, 5\} \in W$,
- If $|S| > 3$, then $S \in W$.

Note that all losing coalitions have cardinality of at most 3.

The trading transform

$$\mathcal{T} = \{\{1,2\}^n, \{3,4,5\}^{n+2}, P; \{2,3,5\}^3, \{2,3,4\}^3, \underbrace{\{2,3,6\}, \ldots, \{2,3,n\}}_{n-5}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}^{n-1}, \emptyset\}$$

is a potent certificate of non-weightedness of length $2n + 3$.

More Definitions

Definition

A simple game G is called proper if

$$X \in W \Longrightarrow X^c \in L,$$

strong if

$$X \in L \Longrightarrow X^c \in W,$$

and a constant sum game if G is both proper and strong.

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- Nuclear sutcases game and EEC: constant sum games
- American Senat and UN Security Council: proper but not strong.
- Gamelman's game: strong but not proper.

Cyclic games

Definition

A game with n players is cyclic if the characteristic vectors of minimal winning coalitions consist of a vector $\mathbf{w} \in \mathbb{Z}_2^n$ and all its cyclic permutations. We will denote it $C(\mathbf{w})$.

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Theorem

Let the Hamming weight of $\mathbf{w} \in \mathbb{Z}_2^n$ is smaller than $n/2$. Then, if $C(\mathbf{w})$ is proper, it is not roughly weighted.

Proof: The sequence

$$\mathcal{T} = (X_1, \ldots, X_n, \underbrace{P, \ldots, P}_{n-2k}; X_1^c, \ldots, X_n^c, \underbrace{\emptyset, \ldots, \emptyset}_{n-2k})$$

is a potent certificate of non-weightedness.

Projective Games

Let $PG(n, q)$, where $q = p^r$, be the projective n-dimensional space for a prime p. After Richardson (1956) we define projective game

$$Pr_{n,q} = (PG(n, q), W),$$

where W is defined by the set of minimal winning coalitions:

$$W^m = \{\text{all } (n-1)\text{-dimensional subspaces of } PG(n, q)\}.$$

Theorem

Any projective game is not roughly weighted.

Proof: By Singer's theorem $Pr_{n,q}$ is cyclic.

Weightedness of Small Games

Theorem (Shapley, 1962)

The following games are weighted:

- *every game with 3 or less players,*
- *every strong or proper game with 4 or less players,*
- *every constant sum game with 5 or less players.*

Rough Weightedness of Small Games

Theorem (Gvozdeva-Slinko, 2009)

The following games are roughly weighted:

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- Applications to threshold abstract simplicial complexes;
- Applications to effectivity functions.

Thank you for your attention!