Initial Ideals in the Exterior Algebra

Dominic Searles

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Department of Mathematics

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Abstract

In this thesis we investigate term orders, Gröbner bases, and initial ideals in the exterior algebra over a vector space of dimension n. We review properties of term orders and Gröbner bases, first in the familiar case of multivariate polynomial rings over algebraically closed fields, then in the exterior algebra. In the latter case, we investigate in particular computation of Gröbner bases and initial ideals with respect to noncoherent term orders.

Using properties of noncoherent term orders, we develop a construction method which allows us to find noncoherent initial ideals in the exterior algebra over a vector space of dimension $n \geq 6$, and we give some illustrative examples.

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Chapter 1

Introduction

This thesis concerns properties of term orders and Gröbner bases in the exterior algebra. We focus in particular on using Gröbner basis theory and properties of term orders to construct a *noncoherent initial ideal* in the exterior algebra over a vector space of dimension n.

For the purposes of this thesis, we will denote the polynomial ring in n variables over an algebraically closed field k by $S = k[x_1, \ldots, x_n] = k[\mathbf{x}]$.

Gröbner bases are important structures in commutative algebra and algebraic combinatorics. A Gröbner basis G is a specific type of generating subset of an ideal $I \subset R$, where R is typically the multivariate polynomial ring S, although the concept may be generalised to other structures such as the exterior algebra. Relative to some order $\prec$ taken on the monomials of S, a defining property of a

Gröbner basis G is that the ideal generated by the initial terms of the polynomials in I, called the *initial ideal of I with respect to* $\prec$, is itself generated by the initial terms of the polynomials in G. It is important to realise that if a different term order $\prec'$ is chosen on the monomials of S, Gröbner bases of I with respect to $\prec'$ may be very different from Gröbner bases of I with respect to the original term order $\prec$. Choice of term order is particularly important for computational considerations, as choosing a different term order on the monomials may radically affect both the time taken to compute a Gröbner basis, and the degrees and coefficients of the polynomials comprising the Gröbner basis.

Gröbner bases are a useful tool in a variety of situations. It is well known that they solve the ideal membership problem in S, that is, how to decide whether a specific polynomial is in an ideal $I \subseteq S$, given only a generating set for I. This property is used as a motivating example for the study of Gröbner bases in Chapter 2. Gröbner bases are also an effective tool for deciding whether two given ideals are equal. This is because even though Gröbner bases for an ideal with respect to a fixed term order are not in general unique, it can be shown that every ideal has a unique *reduced* Gröbner basis. Gröbner bases further give us a method for calculating the intersection of two ideals, and a method for solving systems of polynomial equations.

Gröbner bases may also be defined for ideals in the exterior algebra. However, the construction of Gröbner bases in the exterior algebra is more complicated than in S due the fact that the exterior algebra contains zero-divisors. The well-known Buchberger algorithm for computing Gröbner bases of ideals in a ring R implicitly assumes that if f and g are two polynomials in R, then the product of the initial monomials of f and g is the initial monomial of fg. While this is clearly true in S, it is not the case in rings with zero-divisors such as the exterior algebra.

There are also important differences between term orders on the monomials of $S = k[x_1, \ldots, x_n]$ and term orders on the monomials of the exterior algebra. The monomials of the exterior algebra on n variables are a proper finite subset of the monomials of $k[x_1, \ldots, x_n]$, namely those monomials which are square-free. This means that even though there are infinitely many term orders on $k[x_1, \ldots, x_n]$ for $n \geq 2$, there are only finitely many term orders on the exterior algebra. An interesting open problem, which we touch on in Chapter 3, is to provide bounds on the number of such term orders.

A natural assumption is that term orders on the monomials of exterior algebra may be defined by the restriction of term orders on the monomials of $k[x_1, \ldots, x_n]$ to the monomials of the exterior algebra. However this is false: when $n \geq 5$, there exist term orders on the monomials of the exterior algebra which cannot be extended to term orders on the monomials of S. Such orders are called *noncoherent*. Noncoherent term orders may be found in other structures, such as quotients of S by an Artinian monomial ideal.

Term orders on the monomials of the exterior algebra are in fact equivalent to *comparative probability orders*, which are important in decision theory. These orders have been studied in some depth, in particular Fishburn [8, 9] has devoted much recent effort to them. This relationship allows us to use concepts from comparative probability such as Fishburn's *cancellation conditions*, which enable us to decide whether a given term order is noncoherent, and give a set of binary comparisons from the order which implies the noncoherency of the order.

The existence of noncoherent term orders leads to an interesting phenomenon: noncoherent initial ideals. A noncoherent intial ideal is an initial ideal of an ideal I with respect to a noncoherent term order, which is not equal to any initial ideal of I with respect to any coherent term order. Maclagan [12] constructed a noncoherent initial ideal in a quotient of the polynomial ring S by a specific Artinian monomial ideal, and asked whether there exists a noncoherent initial ideal in the exterior algebra.

In Chapter 2 we compare how division works in univariate and multivariate polynomial rings. We then formally define Gröbner bases and present some of their elementary properties, demonstrating the well-known facts that Gröbner bases can be used to provide a simple proof of the famous Hilbert's Basis Theorem, and that Gröbner bases solve the ideal membership question in S. We introduce the famous Buchberger algorithm for computing Gröbner bases in S, and present a straightforward example of computing a Gröbner basis of an ideal in the two-variable polynomial ring $k[x, y]$.

In Chapter 3 we introduce the exterior algebra, and examine how term orders are defined in this context. We discuss the notion of coherency of term orders, give an example of a noncoherent term order, and define cancellation conditions for term orders. We reproduce [12] a method which may be used to compute Gröbner bases in the exterior algebra, and conclude with a simple example on three variables which illustrates the difference between computing Gröbner bases in S and in the exterior algebra.

In Chapter 4 we introduce the concept of a noncoherent initial ideal. We present a method which, given a noncoherent term order, enables us to construct an ideal in the exterior algebra on six or more variables which possesses a noncoherent initial ideal, thus answering Maclagan's question affirmatively. We then give two examples of ideals constructed via this process, one of which is homogeneous, while the other demonstrates the use of a *lexicographic extension* of a term order during the construction process.

In Chapter 5, we present some unsolved problems and comment on possibilities for future research.

Chapter 2

Gröbner Bases

2.1 Multivariate Division

Let k be an algebraically closed field, and let $S = k[x_1, \ldots, x_n] = k[\mathbf{x}]$ denote the polynomial ring in n variables over k. For simplicity, the monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ where $a_i \in \mathbb{N}$ for $1 \leq i \leq n$, will be denoted by $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} \in \mathbb{N}^n$.

When $n = 1$, S is equal to the familiar univariate polynomial ring $k[x]$. Any polynomial $f \in k[x]$ has the form $a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$, where $a_m \neq 0$, $a_i \in k$, and $deg(f) = m$, where $deg(f)$ denotes the degree of f. For any two polynomials $f, g \in k[x]$ with $deg(f) \geq deg(g)$, one can *divide* f by g to find $f = hg + r$ for unique $h, r \in k[x]$, where r is the remainder and $deg(r) < deg(g)$. If $r = 0$, we say *g divides f.*

The standard division algorithm in $k[x]$ works as follows: Let ax^s be the term of highest degree in g and bx^t be the term of highest degree in f, where $a, b \in k$ with $a, b \neq 0$. Let $f = hg + r$, where $h = 0$ and $r = f$. If $t < s$ we are done. Otherwise, let

$$r = r - \tfrac{b}{a}x^{t-s}g, \quad \text{and} \quad h = h + \tfrac{b}{a}x^{t-s}.$$

Repeat until $t < s$. Note that during each iteration, this process cancels the term of highest degree in r. When the process terminates, we have $f = hg + r$, where r is the remainder.

It is well known that for any ideal $I \subseteq k[x]$, if $I = \langle f_1, f_2, \ldots f_s \rangle$, then $I = \langle gcd(f_1, f_2, \ldots f_s) \rangle$. The *gcd* of any number of polynomials can be computed via the Euclidean algorithm, as $k[x]$ is a Euclidean domain, and thus to determine if a polynomial g is in the ideal generated by $f_1, f_2, \ldots f_s$, one need only compute the *gcd* of the generators and check whether this divides g, using the standard division algorithm.

For polynomials in one variable, the monomials are ordered by their degree, meaning the 'greatest' monomial in a polynomial is the one of greatest degree. However, in polynomials of more than one variable it is no longer obvious which monomial is 'greatest' and therefore it is unclear which monomial should be cancelled first when dividing. The notion of a *term order* on the monomials of $k[x_1, \ldots, x_n]$ is therefore necessary for division in a multivariate polynomial ring.

Definition 2.1. *A term order (or monomial order) is a total (complete, anti-symmetric, transitive) order on the monomials of S which satisfies the following two conditions:*

- $1 = \mathbf{x}^{\mathbf{0}} \prec \mathbf{x}^{\mathbf{u}}$ *for all* $\mathbf{u} \neq \mathbf{0}$.
- *If* $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ *then* $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \prec \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$ *for all* $\mathbf{c}$.

Equivalently, we can represent this order as an order on the exponent vectors of the monomials, satisfying the conditions:

- $\mathbf{0} \prec \mathbf{u}$ *for all* $\mathbf{u} \neq \mathbf{0}$.
- *If* $\mathbf{a} \prec \mathbf{b}$ *then* $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{c}$ *for all* $\mathbf{c}$.

One of the most commonly encountered term orders is the *lexicographic order*, with respect to a fixed order on the variables. Let us fix $x_n \prec x_{n-1} \prec \ldots \prec x_1$. In the lexicographic order, $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ if and only if the leftmost non-zero term in the vector $\mathbf{b} - \mathbf{a}$ is positive.

Example 2.2. *In $k[x, y, z]$ with the lexicographic order and $z \prec y \prec x$, we have*

$$y^{999}z^{10000} \prec y^{1000} \prec xy^3z^3 \prec x^2 \prec x^2y^2 \prec x^3$$

Given a term order $\prec$ on S, we may define the *leading term* and the *initial (or senior) monomial* of a polynomial f in S.

Definition 2.3. *The leading term of f, denoted $LT_\prec(f)$, is the greatest term of f with respect to $\prec$, while the initial monomial of f, denoted $in_\prec(f)$, is the monomial $\mathbf{x}^\mathbf{a}$ in $LT_\prec(f)$.*

Example 2.4. *Let $f = \frac{1}{2}x^3 + 4x^2y^2 - 3xy^3z^3$, where $f \in \mathbb{C}[x, y, z]$, and let $\prec$ be the lexicographic order with $z \prec y \prec x$. Then $LT_\prec(f) = \frac{1}{2}x^3$, and $in_\prec(f) = x^3$.*

The *multivariate division algorithm* approximates a division algorithm for multivariate rings. Note that polynomials in more than one variable do not form a Euclidean domain, thus it is impossible to construct a true division algorithm. For a given polynomial $g \in k[x_1, \ldots, x_n]$, a term order $\prec$, and an *ordered* set of polynomials $\{f_1, \ldots, f_m\}$, the multivariate division algorithm reduces g with respect to $\{f_1, \ldots, f_m\}$:

Set $a_1 = \ldots = a_m = 0$, $r = 0$, and $s = g$. Let

$$g = a_1 f_1 + \ldots + a_m f_m + r + s$$

The algorithm proceeds as follows: If $s = 0$ we are done. Otherwise, if $LT_\prec(s)$ is divisible by $LT_\prec(f_i)$ for some i, then take the smallest such i and let

$$s = s - \tfrac{LT_\prec(s)}{LT_\prec(f_i)} f_i, \quad \text{and} \quad a_i = a_i + \tfrac{LT_\prec(s)}{LT_\prec(f_i)}.$$

Repeat this process until $LT_\prec(s)$ is not divisible by $LT_\prec(f_i)$ for any i. If this situation occurs, then let

$$s = s - LT_{\prec}(s), \quad \text{and} \quad r = r + LT_{\prec}(s).$$

and continue the process from the beginning. Note that each iteration reduces the size of $LT_{\prec}(s)$. The process terminates when $s = 0$, and we have $g = \sum_{i=1}^{m} a_i f_i + r$, where r is the remainder.

A serious problem with multivariate division is that the remainder may be different if the order on $\{f_1, \ldots, f_m\}$ is changed. It is even possible for the remainder to be zero in some cases and non-zero in others, as the following example demonstrates.

Example 2.5. *Let $S = k[x, y]$, $g = x^2 - xy^2$, $f_1 = x - y^2, f_2 = x^2 - y$, and let $\prec$ be the lexicographic order with $y \prec x$. Note that $g = xf_1$, so $g \in \langle f_1, f_2 \rangle$. Then $g - xf_1 = 0$, so g reduces to zero immediately when divided by the ordered set $\{f_1, f_2\}$.*

However, if we divide by $\{f_2, f_1\}$ instead, we find $g - f_2 = -xy^2 + y$. This is no longer divisible by f_2, so we now divide by f_1, giving $(-xy^2 + y) - (-y^2)f_1 = -y^4 + y$, which is not divisible by either f_2 or f_1, and thus cannot be reduced further.

This raises questions about how to solve the ideal membership problem in S, i.e. how to decide whether a given polynomial is in an ideal I, given only an arbitrary generating set for I. However, as mentioned in the introduction, Gröbner bases resolve this issue. If $\{f_1, \ldots, f_m\}$ is a Gröbner basis for I, then remainders on division by $\{f_1, \ldots, f_m\}$ are unique regardless of the order in which the polynomials $f_1, \ldots, f_m$ are given.

2.2 Properties of Gröbner Bases

There is much literature available on Gröbner basis theory. A straightforward introduction to Gröbner bases is given in chapters 11 and 12 of *Lectures in Geometric Combinatorics* [16]. In this section we will give an overview of some of the key concepts.

Definition 2.6. *Let I be an ideal of S. I is a monomial ideal if there is a generating set of I which consists only of monomials.*

Definition 2.7. *Let I be an ideal of S. The initial ideal of I with respect to a term order $\prec$ is the monomial ideal*

$$in_{\prec}(I) = \langle in_{\prec}(f) : f \in I \rangle.$$

Definition 2.8. *A subset G of I is a Gröbner basis for I with respect to a term order $\prec$ if the ideal generated by the set $\{in_{\prec}(g) : g \in G\}$ of initial monomials of polynomials in G is equal to $in_{\prec}(I)$.*

Definition 2.9. *If $\{in_{\prec}(g) : g \in G\}$ is the unique minimal generating set of $in_{\prec}(I)$, then G is a minimal Gröbner basis of I with respect to $\prec$. A minimal Gröbner basis is reduced if the coefficient of the leading term of every $g \in G$ is*

1, and no non-initial term of any $g \in G$ is divisible by any element of $\{in_{\prec}(g) : g \in G\}$.

We now present some basic properties of Gröbner bases, showing firstly that Gröbner bases may be used to provide a simple proof of Hilbert's Basis Theorem, and concluding with a proof that Gröbner bases solve the ideal membership problem in S.

Lemma 2.10. Dickson's Lemma.

Let U be a subset of $\mathbb{N}^n$. Then there is a finite subset of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq U$ such that

$$U \subseteq (\mathbf{v}_1 + \mathbb{N}^n) \cup \ldots \cup (\mathbf{v}_r + \mathbb{N}^n).$$

Proof. We prove this statement by induction on n. The case $n = 1$ is trivial. Suppose the statement is true for all $m < n$. Let $\pi : \mathbb{N}^n \to \mathbb{N}^{n-1}$ be the projection

$$\pi(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n).$$

Then, by the induction hypothesis, there exist vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ such that

$$\pi(U) \subseteq (\pi(\mathbf{u}_1 + \mathbb{N}^{n-1}) \cup \ldots \cup (\pi(\mathbf{u}_r + \mathbb{N}^{n-1}).$$

Let N be the largest number appearing in the first coordinate of $u_1, \ldots, u_r$. Define

$$U_i = \{u \in U : \text{the first coordinate of } u \text{ is } i\},\ 0 \leq i \leq N$$

and

$$U_{\geq N} = \{u \in U : \text{the first coordinate of } u \text{ is at least } N\}.$$

Then $U = U_0 \cup \ldots \cup U_{N-1} \cup U_{\geq N}$. It is clear that

$$U_{\geq N} \subseteq (u_1 + \mathbb{N}^n) \cup \ldots \cup (u_r + \mathbb{N}^n).$$

The first coordinate is fixed for every U_i so we may apply the induction hypothesis to it. We can see the number of vectors we need is at most $r + N$. $\square$

Theorem 2.11. *Let $I \subseteq S$ be an ideal, and $\prec$ any term order on S. Then I has a finite Gröbner basis with respect to $\prec$.*

Proof. Let

$$U = \{\mathbf{v} \in \mathbb{N}^n : \mathbf{x}^{\mathbf{v}} = in_{\prec}(f) \text{ for some } f \in I\} \subseteq \mathbb{N}^n.$$

By Lemma 2.9, there are finitely many polynomials $f_1, f_2, \ldots, f_m$ such that

$$U \subseteq (\mathbf{v}_1 + \mathbb{N}^n) \cup \ldots \cup (\mathbf{v}_m + \mathbb{N}^n),$$

where $\mathbf{x}^{\mathbf{v}_i} = in_{\prec}(f_i)$ for $1 \leq i \leq m$. Suppose that $f \in I$, with $in_{\prec}(f) = a\mathbf{x}^{\mathbf{w}}$. Then $\mathbf{w} = \mathbf{v}_j + \mathbf{v}$ for some $\mathbf{v} \in \mathbb{N}^n$, and some j such that $1 \leq j \leq m$. This means that $\mathbf{x}^{\mathbf{w}} = \mathbf{x}^{\mathbf{v}_j}\mathbf{x}^{\mathbf{v}}$, and therefore $in_{\prec}(f_j)$ divides $in_{\prec}(f)$. But this implies $\{f_1, f_2, \ldots f_m\}$ is a Gröbner basis for I.

$\square$

Lemma 2.12. *The Gröbner basis G of an ideal I is a basis of I.*

Proof. We reproduce the proof from [16]. By definition, $\langle G\rangle \subseteq I$, so we need to show that if $f \in I$, then $f \in \langle G\rangle$. Suppose not. Then we can assume without loss of generality that f is monic, and that among all polynomials of I which are not in $\langle G\rangle$, f has the smallest initial monomial with respect to $\prec$. But $f \in I$ implies that $in_{\prec}(f) \in in_{\prec}(I)$, which implies there is some $g \in G$ with the property that $in_{\prec}(g)$ divides $in_{\prec}(f)$. Suppose $in_{\prec}(f) = \mathbf{x}^{\mathbf{m}} in_{\prec}(g)$. Then $h = f - \mathbf{x}^{\mathbf{m}} g$ is a polynomial in I with smaller initial monomial than f. Thus, by our assumption, $h \in \langle G\rangle$, which implies that $h + \mathbf{x}^{\mathbf{m}} g = f \in \langle G\rangle$, which is a contradiction. Thus $I \subseteq \langle G\rangle$. □

Theorem 2.13. Hilbert's Basis Theorem. *Every ideal I in S has a finite generating set.*

Proof. By Theorem 2.11, every ideal I in S has a finite Gröbner basis, and by Lemma 2.12, a Gröbner basis of I with respect to any term order is a basis of I. Thus I possesses a finite basis, that is, I has a finite generating set. □

The following two results are reproduced from [16].

Theorem 2.14. *If $\{f_1, \ldots, f_m\}$ is a Gröbner basis for I, then the multivariate division algorithm will return the same remainder regardless of the order chosen on $\{f_1, \ldots, f_m\}$.*

Proof. Suppose $\{f_1, \ldots, f_m\}$ is a Gröbner basis for I, and suppose that if we divide a polynomial $g \in S$ by $\{f_1, \ldots, f_m\}$ with the elements ordered in two different ways, then we obtain two remainders, $r_1, r_2 \in S$. Then $f = \sum a_i f_i + r_1 = \sum a_i^{'} f_i + r_2$, so $r_1 - r_2 = \sum a_i^{'} f_i - \sum a_i f_i \in I$, and no term of $r_1 - r_2$ is divisible by $in_{\prec}(f_i)$ for any f_i in the Gröbner basis. But this implies $r_1 - r_2 = 0$, as otherwise $0 \neq in_{\prec}(r_1 - r_2) \in in_{\prec}(I)$ and some $in_{\prec}(f_i)$ would divide it. □

Corollary 2.15. *Gröbner bases solve the ideal membership problem in S, that is, a polynomial $f \in S$ is in I if and only if its remainder after division by a Gröbner basis G of I is zero.*

Proof. Firstly, assume $f \in S$ has remainder zero after division by G. Then f can be written as a combination of elements of the Gröbner basis, all of which are in I, thus $f \in I$. Now, suppose $f \in I$. Then as G is a basis for I, f can be written as a combination of elements of the Gröbner basis. Subtracting this combination from f gives zero, implying zero is a remainder (and thus the unique remainder) on division by G. □

2.3 The Buchberger Algorithm

Given a term order $\prec$ and a generating set $\{f_1, \ldots, f_m\}$ for an ideal I, the standard way to compute a Gröbner basis is via the *Buchberger algorithm*, due to Bruno Buchberger [1]. The algorithm requires the definition of the S-polynomial of a pair of polynomials $f_i, f_j \in \{f_1, \ldots, f_m\}$, denoted S_{f_i,f_j}.

Definition 2.16. *Let m_{ij} be the least common multiple of the leading terms of f_i and f_j. Then*

$$S_{f_i,f_j} = \frac{m_{ij}}{LT_{\prec}(f_i)}(f_i) - \frac{m_{ij}}{LT_{\prec}(f_j)}(f_j).$$

The Buchberger algorithm uses the fact that a set of polynomials $G = \{g_1, \ldots, g_m\}$ is a Gröbner basis with respect to $\prec$ if and only if the remainder of every S-polynomial on division by G is zero. We omit the proof, which can be found in [3]. The algorithm works as follows:

Let $G = \{f_1, \ldots, f_m\}$, the given set of generators of I. Calculate the S-polynomials S_{f_i,f_j} for all possible pairs $f_i, f_j \in G$, then find the remainder of each S-polynomial on division by G. Whenever this results in a non-zero remainder r, add r to the set G, and calculate all new S-polynomials formed due to the addition of r. Continue this process until every S-polynomial reduces to zero. G will then be a Gröbner basis for I with respect to $\prec$.

Example 2.17. *Let $f_1 = x - y^2$, $f_2 = x^2 - y$. We will use the Buchberger algorithm to compute the Gröbner basis of the ideal $I = \langle f_1, f_2 \rangle \subset k[x, y]$ with respect to the lexicographic order with $y \prec x$.*

Observe first that $LT_{\prec}(f_1) = x$ and $LT_{\prec}(f_2) = x^2$. There are only two polynomials generating I, so to begin with there is only one S-pair, S_{f_1,f_2} We calculate:

$S_{f_1,f_2} = xf_1 - 1f_2 = x^2 - xy^2 - (x^2 - y) = -xy^2 + y.$

We now reduce S_{f_1,f_2} *as much as possible with respect to* $\{f_1, f_2\}$*:*

$(-xy^2 + y) - (-y^2)f_1 = -y^4 + y.$

$f_3 = -y^4 + y$ *cannot be reduced any further, so it must be added to the set containing* f_1 *and* f_2*. The addition of* f_3 *means* S_{f_1,f_2} *will now reduce to zero, but we must check* S_{f_1,f_3} *and* S_{f_2,f_3}*. Observe that* $LT_{\prec}(f_3) = -y^4$*.*

$S_{f_1,f_3} = (-y^4)f_1 - xf_3 = -xy^4 + y^6 + xy^4 - xy = -xy + y^6.$

$(-xy + y^6) - (-y)f_1 = y^6 - y^3.$

$(y^6 - y^3) - (-y^2)f_3 = 0,$ *so* S_{f_1,f_3} *reduces to zero.*

$S_{f_2,f_3} = (-y^4)f_2 - x^2 f_3 = -x^2y^4 + y^5 + x^2y^4 - x^2y = -x^2y + y^5.$

$(-x^2y + y^5) - (-y)f_2 = y^5 - y^2.$

$(y^5 - y^2) - (-y)f_3 = 0,$ *so* S_{f_2,f_3} *reduces to zero.*

As all S*-polynomials reduce to zero with respect to the set of polynomials* $\{f_1, f_2, f_3\}$ $= \{x - y^2, x^2 - y, -y^4 + y\}$*, this set is a Gröbner basis for* I *with respect to* $\prec$*. Note that this Gröbner basis is not minimal, as* $in_{\prec}(f_1) = x$ *divides* $x^2 = in_{\prec}(f_2)$*. We may delete* f_2 *to get a minimal Gröbner basis* $\{x - y^2, -y^4 + y\}$*, and multiply* f_3 *by* -1 *to obtain the reduced Gröbner basis* $\{x - y^2, y^4 - y\}$*.*

Chapter 3

The Exterior Algebra

The exterior algebra, also called the Grassmann algebra, is important in various fields. It is used widely in both differential and algebraic geometry and in multilinear algebra, and it also plays a role in other areas such as representation theory.

Let k be an infinite field of characteristic $\neq 2$, and V a vector space of dimension n over k, with ordered basis $\{X_1, \ldots X_n\}$.

Definition 3.1. *The exterior algebra of V over k is the quotient of the free associative (non-commutative) polynomial ring $k\langle X_1, \ldots, X_n\rangle$ by the anticommutator ideal $I = \langle X_iX_j + X_jX_i : 1 \leq i \leq j \leq n\rangle$.*

The multiplication in the exterior algebra is often called the *wedge product* or *exterior product*, and is typically denoted $\wedge$. It is associative and bilinear, and has the following properties:

- $x \wedge x = 0$ for all $x \in V$.

This implies that

- $x \wedge y = -y \wedge x$ for all $x, y \in V$
- If σ is a permutation of $\{1, 2, \ldots, n\}$, then $x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} = sgn(\sigma) x_1 x_2 \ldots x_n$, where $sgn(\sigma)$ denotes the signature of the permutation σ
- $x_1 \wedge \ldots \wedge x_k = 0$ if $x_1, \ldots, x_k$ are linearly dependent in V.

For a more intuitive definition, the exterior algebra (on n variables) can be thought of as the algebra generated by the exterior product $\wedge$ (as defined above) on a vector space of dimension n over k.

For the sake of brevity, from this point we will omit the wedge product notation, and write xy for $x \wedge y$ in the exterior algebra.

The set of monomials M_{ext} of the exterior algebra on n variables is

$$M_{ext} = \bigcup_{k=0}^{n} \{x_{i_1} x_{i_2} \ldots x_{i_k} : 1 \leq i_1 < i_2 \ldots < i_k \leq n\}$$

where the $k = 0$ case is the set $\{1\}$ containing only the unity element.

We may observe that M_{ext} is exactly the set of square-free monomials of S, and there are $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ of them. This set forms a basis of the exterior algebra, so the exterior algebra (on n variables) has dimension 2^n.

3.1 Term Orders on the Exterior Algebra

Let $[n]$ denote the n-element set $\{1, 2, \ldots, n\}$. There is a natural bijection between the set of all subsets of $[n]$, denoted $2^{[n]}$, and the set M_{ext} of monomials of the exterior algebra on n variables. This bijection $\varphi : 2^{[n]} \to M_{ext}$ is given by

$$\varphi(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ \Pi_{i \in A} x_i & \text{if } A \neq \emptyset \end{cases}$$

for all $A \subseteq [n]$.

It is clear we may also effectively map an order on $2^{[n]}$ to an order on M_{ext} by defining for all $A, B \subseteq [n]$:

$$A \prec B \iff \varphi(A) \prec \varphi(B).$$

Where there is no ambiguity, we will identify these two orders.

Therefore, considering term orders on the monomials of the exterior algebra on n variables is equivalent to considering certain orders on $2^{[n]}$. Maclagan [11] calls these *Boolean term orders*.

Definition 3.2. *A Boolean term order is a total order on the subsets of* $[n]$ *such that:*

- $\emptyset \prec A$ *for all* $\emptyset \neq A \subseteq [n]$

- $A \prec B \iff A \cup C \prec B \cup C$, *for all* $A, B, C \subseteq [n]$ *with the condition that* $(A \cup B) \cap C = \emptyset$.

For brevity, instead of writing $\{a_{i_1}, a_{i_2}, \ldots, a_{i_j}\}$ for the subset $A \subseteq [n]$ we will write $a_{i_1} a_{i_2} \ldots a_{i_j}$.

Example 3.3. *The following order is a Boolean term order on the subsets of a three-element set:*

$$\emptyset \prec 1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 23 \prec 123$$

Applying φ *to this order gives us the following equivalent order on the monomials of the exterior algebra on 3 variables:*

$$1 \prec x_1 \prec x_2 \prec x_1x_2 \prec x_3 \prec x_1x_3 \prec x_2x_3 \prec x_1x_2x_3.$$

It is worth noting that Boolean term orders are equivalent to *comparative probability orders*. Comparative probability orders are used in mathematical economics and other disciplines to analyse preferences. Their study dates back to fundamental work done by Bruno de Finetti [5], and in the context of comparative probability the second condition in Definition 3.2 is called *de Finetti's axiom*. The comparison $A \cup C \prec B \cup C$ derived from a known comparison $A \prec B$ via de Finetti's axiom is sometimes referred to as a *de Finetti consequence* of the comparison $A \prec B$. We will make use of this idea in Chapter 4.

Definition 3.4. *A Boolean term order $\prec$ is coherent if there is a weight vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ such that*

$$A \prec B \iff \textstyle\sum_{i \in A} w_i < \sum_{j \in B} w_j$$

The equivalent concept in the context of comparative probability is that of an *(additively) representable* comparative probability order. In the context of Gröbner basis term orders on monomials of the exterior algebra on n variables, a term order is coherent if it can be extended to a term order on all the monomials of $k[x_1, \ldots, x_n]$.

Following Maclagan [11], we will henceforth assume the elements are ordered

$$1 \prec 2 \prec \ldots \prec n.$$

Note this is the opposite to the order we assumed on the variables of $k[x_1, \ldots, x_n]$ in Chapter 2, however in this context, this order is more convenient.

This assumption reduces the number of possible orders under consideration by a factor of $n!$. Following [4], we denote the set of Boolean term orders under this condition by $\mathcal{P}_n^*$, and the subset of coherent orders by $\mathcal{L}_n^*$.

We can observe that while there are infinitely many term orders on $k[x_1, \ldots, x_n]$ for $n \geq 2$, there are only finitely many Boolean term orders as the exterior algebra over n variables has only a finite number of monomials. For $n \leq 7$, the enumerated number of Boolean term orders is:

n	1	2	3	4	5	6	7
$\lvert\mathcal{P}_n^*\rvert$	1	1	2	14	546	169444	560043206
$\lvert\mathcal{L}_n^*\rvert$	1	1	2	14	516	124187	214580603

It is not known how many Boolean term orders there are in general. Some bounds, however, have been found on both the total number of term orders and the number of coherent term orders. See [7] for more details.

Definition 3.5. *The lexicographic extension of a Boolean term order* $\prec$ *on the subsets of* $[n]$ *is the order* $\prec'$ *on the subsets of* $[n+1]$ *defined by letting the first* 2^n *subsets of* $[n+1]$ *appearing in* $\prec'$ *be the subsets of* $[n]$*, ordered by* $\prec$*, and the remaining* 2^n *subsets of* $[n+1]$ *be ordered by the rule*

$$A \prec B \iff (A \setminus \{n+1\}) \prec (B \setminus \{n+1\}).$$

Note the central comparison of the lexicographic extension of $\prec$ *will always be* $12\ldots n \prec n+1$.

Example 3.6. *The lexicographic extension of the order*

$\prec = \emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 23 \prec 123$

is the order

$\prec' = \emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 23 \prec 123 \prec 4 \prec 14 \prec 24 \prec 34 \prec 124 \prec 134 \prec 234 \prec 1234.$

This method of creating an order on $2^{[n+1]}$ from an order on $2^{[n]}$ will be useful in Chapter 4.

3.2 Noncoherent Term Orders

While representation by a weight vector for a Boolean term order seems natural, when $n \geq 5$ we encounter term orders which cannot be represented by a weight vector. Such orders are called *noncoherent*.

The question of whether all Boolean term orders are coherent was first asked in terms of comparative probability. In 1951, de Finetti [6] raised the question as to whether all comparative probability orders are additively representable, and this was answered in the negative by Kraft, Pratt, and Seidenberg [10] in 1959.

Example 3.7. *A noncoherent Boolean term order for $n = 5$ is*

$\emptyset \prec 1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 4 \prec 14 \prec 23 \prec 123 \prec 24 \prec 124 \prec 5 \prec 34 \prec 15 \prec 25 \prec 134 \prec 234 \prec 125 \prec 1234 \prec 35 \prec 135 \prec 45 \prec 145 \prec 235 \prec 1235 \prec 245 \prec 1245 \prec 345 \prec 1345 \prec 2345 \prec 12345$

To see that this term order is not coherent, we observe that the order contains the following comparisons:

$13 \prec 4$

$14 \prec 23$

$34 \prec 15$

$25 \prec 134$

Note that the quantity on the left hand sides is equal to the quantity on the right hand sides. If we now take this order to be an order on the monomials of the exterior algebra, we have:

$x_1x_3 \prec x_4$

$x_1x_4 \prec x_2x_3$

$x_3x_4 \prec x_1x_5$

$x_2x_5 \prec x_1x_3x_4$

We now multiply all the left hand sides together and all the right hand sides together, which gives us $x_1^2x_2x_3^2x_4^2x_5 \prec x_1^2x_2x_3^2x_4^2x_5$*, a contradiction. Therefore this order cannot be extended to an order on all monomials in 5 variables, and thus it is noncoherent.*

The method of giving a certificate for the noncoherency of a Boolean term order has been given much attention by Fishburn [8, 9]. Let $(A_1, \ldots, A_M) =_0 (B_1, \ldots, B_M)$ mean that $A_j, B_j \in 2^{[n]}$ for all j and, for every $i \in \{1, \ldots, n\}$, $|\{j : i \in A_j\}| = |\{j : i \in B_j\}|$. Fishburn quotes the following axiom from Kraft, Pratt, and Seidenberg which, when added to the defining axioms of a Boolean term order, results in a set of axioms which are both necessary and sufficient for coherency:

For all $M \geq 2$ *and all* $A_j, B_j \in 2^{[n]}$*, if* $(A_1, \ldots, A_M) =_0 (B_1, \ldots, B_M)$ *and* $A_j \prec B_j$ *for all* $j < M$*, then it is not the case that* $A_M \prec B_M$*.*

Violation of this axiom for a certain M implies the failure of the M^{th} *cancellation condition*, denoted C_M, and means the order is noncoherent.

Example 3.8. *The four comparisons* $13 \prec 4$*,* $14 \prec 23$*,* $34 \prec 15$*,* $25 \prec 134$ *quoted earlier comprise a failure of* C_4*, implying that any order containing this set of comparisons is noncoherent.*

For a Boolean term order to be coherent, it is necessary and sufficient that all cancellation conditions C_M are satisfied. Thus to show an order is noncoherent, it is enough to present a failure of C_M for some M.

Fishburn demonstrated [8] that the cancellation conditions C_2 and C_3 are implied by de Finetti's axiom and properties of term orders, and thus cannot fail. Hence C_4 is the first non-trivial cancellation condition. The construction process in Chapter 4 will focus on failures of C_4.

3.3 Gröbner Basis Computation in the Exterior Algebra

To define and compute Gröbner bases in the exterior algebra, we use the following work of Maclagan [12].

Definition 3.9. *A monomial ideal $I \subseteq S = k[x_1, \ldots, x_n]$ is an Artinian monomial ideal if $x_1^{d_1}, \ldots, x_n^{d_n}$ are among its minimal generators (the unique smallest set of monomials that generates I) for some strictly positive $d_1, \ldots, d_n$.*

Let I be an Artinian monomial ideal in S. Then S/I is a finite-dimensional k-vector space with basis the set of images of monomials of S not in I, denoted M. These monomials are called *standard monomials.*

Definition 3.10. *A term order $\prec$ on M is a total order on M which satisfies:*

- $1 = \mathbf{x}^{\mathbf{0}} \prec \mathbf{x}^{\mathbf{u}}$ *for all* $\mathbf{x}^{\mathbf{u}} \neq 1$ *in* M.
- *If* $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ *then* $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \prec \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$ *whenever* $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}}$ *and* $\mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$ *are both in* M.

Note that a Boolean term order is a term order on S/I when I is generated by the squares of the variables.

Gröbner basis theory in S/I is similar to Gröbner basis theory on S, with some modifications. A set $G = \{g_1, \ldots, g_l\}$ is a Gröbner basis for an ideal $J \subseteq S/I$ with respect to a term order $\prec$ if $in_\prec(G) = \{in_\prec(g_1), \ldots, in_\prec(g_l)\}$ generates $in_\prec(J)$. We may now give the method for calculating a Gröbner basis in S/I. The definition of S-polynomials in S/I is analagous to the definition in the polynomial ring S.

Definition 3.11. *Let m_{ij} be the least common multiple of the leading terms of g_i and g_j. An S-polynomial, S_{g_i,g_j} is the polynomial $\frac{m_{ij}}{LT_\prec(g_i)}(g_i) - \frac{m_{ij}}{LT_\prec(g_j)}(g_j)$.*

When calculating a Gröbner basis in S/I, we require the additional concept of a T*-polynomial.*

Definition 3.12. *A T-polynomial, $T_{g_i,\mathbf{x}^\mathbf{a}}$, where $\mathbf{x}^\mathbf{a} in_\prec(g_i) \in I$, is the polynomial $\mathbf{x}^\mathbf{a} g_i$.*

Theorem 3.13. *A set of polynomials $G = \{g_1, \ldots, g_l\} \subseteq S/I$ is a Gröbner basis for the ideal $\langle G \rangle$ they generate if and only if all S-polynomials and T-polynomials reduce to zero with respect to G.*

The proof of this theorem may be found in [12].

This defines Gröbner basis computation for quotients of S by Artinian monomial ideals, which are commutative, contain only finitely many monomials, and contain zero-divisors. While the exterior algebra is not commutative, it is anti-commutative, i.e. commutative modulo a negative sign. Changes in sign do not alter the working of the division algorithm, and therefore the same theorem holds for the exterior algebra.

The necessity of the T-polynomials is due to the fact that in an algebra with zero-divisors, it is not always the case that if a polynomial reduces to zero with respect to a generating set of polynomials, then all monomial multiples of that polynomial reduce to zero as well.

Example 3.14. *Let $f = x_1x_2 + x_3$, let $\prec$ be a term order such that $x_3 \prec x_1x_2$, and let*

$$K = \langle x_1x_2 + x_3 \rangle$$

be the principal ideal generated by f in the exterior algebra on three variables.

As K is principal, there are no S-polynomials to consider. But $\{x_1x_2 + x_3\}$ is not a Gröbner basis for K, as $in_{\prec}(x_1x_2+x_3) = x_1x_2$ does not generate the initial ideal $in_{\prec}(K)$. This is because the T-polynomials

$$T_{f,x_1} = x_1(x_1x_2 + x_3) = x_1x_3 \quad \textit{and} \quad T_{f,x_2} = x_2(x_1x_2 + x_3) = x_2x_3$$

are also monomials in K, and thus appear in $in_{\prec}(K)$, but are not generated by x_1x_2.

Therefore, taking T-polynomials into account, we find a Gröbner basis for K is

$$\{x_1x_2 + x_3,\ x_1x_3,\ x_2x_3\}$$

and the initial ideal $in_{\prec}(K)$ is

$$\langle x_1x_2,\ x_1x_3,\ x_2x_3\rangle.$$

Chapter 4

Noncoherent Initial Ideals in the Exterior Algebra

The previous sections give us sufficient background to work with term orders, Gröbner bases, and initial ideals in the exterior algebra.

Definition 4.1. *A noncoherent initial ideal of an ideal I is an initial ideal of I with respect to a noncoherent term order, which is not equal to the initial ideal of I with respect to any coherent term order.*

Maclagan [12] demonstrated the existence of a noncoherent initial ideal of an ideal J in a quotient of $k[x_1, x_2, x_3, x_4]$ by a specific Artinian monomial ideal I, and asked whether a noncoherent initial ideal exists in the exterior algebra. We will give a construction theorem that produces noncoherent intial ideals in the exterior algebra on n variables where $n \geq 6$, and demonstrate some simple examples.

4.1 A Construction Theorem

Let $n \geq 5$, and $A_1, \ldots, A_4, B_1, \ldots, B_4$ be subsets of $[n]$ such that in some Boolean term order $\prec$, we have $(A_1, \ldots, A_4) =_0 (B_1, \ldots, B_4)$ and $A_1 \prec B_1, A_2 \prec B_2, A_3 \prec B_3, A_4 \prec B_4$. Then $\prec$ is noncoherent, with $\{A_i \prec B_i : 1 \leq i \leq 4\}$ comprising a failure of the cancellation condition C_4.

We now recall the definition of the bijection $\varphi : 2^{[n]} \rightarrow M_{ext}$ given in Chapter 3:

$$\varphi(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ \Pi_{i \in A} x_i & \text{if } A \neq \emptyset \end{cases}$$

for all $A \subseteq [n]$.

As before, we will identify the Boolean term order $\prec$ on subsets of $[n]$ with the equivalent term order on monomials of the exterior algebra on n variables.

Theorem 4.2. *Suppose the eight subsets $\{A_i : 1 \leq i \leq 4,\ B_i : 1 \leq i \leq 4\}$ which form the four comparisons comprising a failure of C_4 have the additional property that none of them is contained in (or equal to) any of the other seven. Then if we take the ideal*

$$I = \langle\ \varphi(B_1) - \varphi(A_1),\ \varphi(B_2) - \varphi(A_2),\ \varphi(B_3) - \varphi(A_3),\ \varphi(B_4) - \varphi(A_4)\ \rangle$$

and the term order $\prec$, the initial ideal $in_{\prec}(I)$ is a noncoherent initial ideal in the exterior algebra on n variables.

Proof. We first note that only $\prec$, or another noncoherent term order, chooses (respectively) $\varphi(B_1),\ \varphi(B_2),\ \varphi(B_3),\ \varphi(B_4)$ as the initial monomials of the four generators of I. If $\prec_c$ is any coherent term order, then it must be the case that for some i, $B_i \prec_c A_i$, and thus $\varphi(A_i) \in in_{\prec_c}(I)$. It is therefore sufficient to show that $in_{\prec}(I)$ does not contain any of $\varphi(A_1),\ \varphi(A_2),\ \varphi(A_3),\ \varphi(A_4)$.

We therefore need to establish that generating sets of $in_{\prec}(I)$ do not contain divisors of any $\varphi(A_i)$, $1 \leq i \leq 4$. Note that a Gröbner basis of I with respect to $\prec$ will comprise the four generators of I plus certain T-polynomials and the remainders of S-polynomials on division by the generators of I, so we can establish this by examining the properties of the S-polynomials and T-polynomials created during the calculation of a Gröbner basis of I.

Let r be the remainder on reducing an S-polynomial with respect to the generators of I. In general, r may not be zero. But any monomial part of a nonzero r (and thus $in_{\prec}(r)$) will be a multiple of a monomial part of one of the generators of I - that is, it will be of the form $\mathbf{x}^{\mathbf{a}} \cdot \varphi(A_i)$ or $\mathbf{x}^{\mathbf{a}} \cdot \varphi(B_i)$ for some $\varphi(A_i)$ or $\varphi(B_i)$, where $1 \leq i \leq 4$ and $1 \neq \mathbf{x}^{\mathbf{a}}$ is some monomial in the exterior algebra on n variables. None of the $\varphi(A_i)$ or $\varphi(B_i)$ divides any of the other $\varphi(A_i)$ or $\varphi(B_i)$, so $in_{\prec}(r)$ will also not divide any of these - in particular, it will not divide any of the $\varphi(A_i)$. Therefore, the non-initial monomials of the four generators (the

$\varphi(A_i))$ cannot be generated by the initial monomial of the remainder created by reducing an S-polynomial.

Any nonzero T-polynomial of a generator has the form $\mathbf{x^a} \cdot \varphi(A_i)$, where $1 \neq \mathbf{x^a}$ is some monomial in the exterior algebra on n variables. As the generators of I each consist of exactly two monomial parts, each of these T-polynomials is itself a monomial, and thus is equal to its initial monomial. All we need to do then is observe that as $\varphi(A_i)$ does not divide $\varphi(A_j)$ for $1 \leq i, j \leq 4$, $i \neq j$, then $\mathbf{x^a} \cdot \varphi(A_i)$ does not divide $\varphi(A_j)$ for any $1 \leq i, j \leq 4$. The other case to consider is T-polynomials of a remainder r of an S-polynomial. This has a maximum of two monomial parts, and and any one of these is a multiple of a monomial part of one of the generators, so any nonzero T-polynomial of r will likewise be a multiple of a monomial part of one of the generators, and so will not divide any $\varphi(A_i)$. Therefore, no T-polynomial can generate any of the $\varphi(A_i)$.

By definition, the initial ideal $in_{\prec}(I)$ is generated by the set of initial monomials of polynomials in a Gröbner basis. A Gröbner basis will consist of the four generators of I, various T-polynomials, and remainders of S-polynomials that did not reduce to zero during the calculation, so $in_{\prec}(I)$ is generated by the initial monomials of these polynomials. We have shown that none of the initial monomials of T-polynomials or of remainders of S-polynomials generates any of the $\varphi(A_i)$. Neither does any of $\varphi(B_1), \ldots, \varphi(B_4)$, the initial monomials of the generators with respect to $\prec$, as we have assumed that none of the eight subsets

$\{\varphi(A_i), \varphi(B_i) : 1 \leq i \leq 4\}$ divides any of the other seven. Therefore, no divisor of any of the $\varphi(A_i)$ is in $in_{\prec}(I)$, so none of the $\varphi(A_i)$ is in $in_{\prec}(I)$, but $in_{\prec_c}(I)$ must contain at least one of the $\varphi(A_i)$ for any coherent term order $\prec_c$.

Therefore, $in_{\prec}(I)$ is a noncoherent initial ideal. $\square$

4.2 Examples

The proof in the previous section does not actually demonstrate the existence of an order $\prec$ containing four comparisons $A_i \prec B_i$, with $(A_1, \ldots, A_4) =_0 (B_1, \ldots, B_4)$ and satisfying the property that none of these eight subsets is contained in any of the other seven.

In this section we will see that if we take a noncoherent order that violates the cancellation condition C_4, we can use the subsets of the C_4 violation to generate another set of comparisons which still violates C_4 and has the desired pairwise noncontainment property. This is achieved by taking de Finetti consequences of some of the comparisons. The new set of comparisons will either be in the original order on $2^{[n]}$ or in another order on $2^{[n+k]}, k \leq 4$ which can be obtained from the original order by use of lexicographic extensions.

We will provide some examples of such orders and the noncoherent initial ideals that may be generated from them.

Example 4.3. *We take the following Boolean term order on the subsets of a* *six-element set:*

$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 4 \prec 5 \prec 14 \prec 6 \prec 23 \prec_* 15 \prec 16 \prec 123 \prec 24 \prec 25 \prec_* 34 \prec 124 \prec 35 \prec_* 26 \prec \mathbf{125} \prec \mathbf{134} \prec 36 \prec \mathbf{135} \prec \mathbf{126} \prec 45 \prec 136 \prec 46 \prec \mathbf{234} \prec \mathbf{145} \prec 56 \prec \mathbf{146} \prec_* \mathbf{235} \prec 1234 \prec 236 \prec 156 \prec 1235 \prec 245 \prec 1236 \prec 345 \prec 246 \prec 1245 \prec 256 \prec 346 \prec 1345 \prec 1246 \prec 356 \prec 1256 \prec 1346 \prec 1356 \prec 456 \prec 2345 \prec 2346 \prec 1456 \prec 12345 \prec 2356 \prec 12346 \prec 12356 \prec 2456 \prec 3456 \prec 12456 \prec 13456 \prec 23456 \prec 123456$

Let us label the comparisons marked with an asterisk as follows:

$A_1' = 25 \prec 34 = B_1'$

$A_2' = 35 \prec 26 = B_2'$

$A_3' = 23 \prec 15 = B_3'$

$A_4' = 146 \prec 235 = B_4'$

We may observe $(A_1', \ldots, A_4') =_0 (B_1', \ldots, B_4')$, *as the sum on the left hand side of the four comparisons equals the sum on the right hand side, so these comparisons comprise a failure of the cancellation condition* C_4, *implying this order is noncoherent.*

However, some of the eight subsets in these comparisons are contained within others. To get around this problem, we take de Finetti consequences of the first three comparisons listed above to form the set of comparisons marked in bold:

$A_1 = 125 \prec 134 = B_1$

$A_2 = 135 \prec 126 = B_2$

$A_3 = 234 \prec 145 = B_3$

$A_4 = 146 \prec 235 = B_4$

These clearly satisfy the pairwise noncontainment property we desire, as all eight of these sets have equal cardinality and are not equal to any of the other seven. We may observe they still comprise a C_4 failure because the same quantity has been added to both the left and right hand sides, ensuring $(A_1, \dots, A_4) =_0 (B_1, \dots, B_4)$.

We now apply φ to the above order to obtain the following comparisons in a term order $\prec$ on the monomials of the exterior algebra on six variables:

$\varphi(A_1) = x_1x_2x_5 \prec x_1x_3x_4 = \varphi(B_1)$

$\varphi(A_2) = x_1x_3x_5 \prec x_1x_2x_6 = \varphi(B_2)$

$\varphi(A_3) = x_2x_3x_4 \prec x_1x_4x_5 = \varphi(B_3)$

$\varphi(A_4) = x_1x_4x_6 \prec x_2x_3x_5 = \varphi(B_4)$

Applying φ preserves noncoherency, so this set of comparisons comprises a failure of C_4 for $\prec$. It is possible this set of comparisons may also appear in some noncoherent order which is not equal to $\prec$, but never in any coherent order.

Following the process outlined in the previous section, let

$$g_i = \varphi(B_i) - \varphi(A_i),\ 1 \leq i \leq 4$$

and let

$$I = \langle g_1, g_2, g_3, g_4 \rangle =$$
$$\langle x_1x_3x_4 - x_1x_2x_5,\ x_1x_2x_6 - x_1x_3x_5,\ x_1x_4x_5 - x_2x_3x_4,\ x_2x_3x_5 - x_1x_4x_6 \rangle$$

We claim that $in_{\prec}(I)$ *is a noncoherent initial ideal. As at least one of the monomials* $x_1x_2x_5$, $x_1x_3x_5$, $x_2x_3x_4$, $x_1x_4x_6$ *must appear in* $in_{\prec_c}(I)$ *where* $\prec_c$ *is any coherent order, it will suffice to show that none of these monomials is in* $in_{\prec}(I)$*.*

To find $in_{\prec}(I)$*, we first calculate a Gröbner basis* G *of* I *with respect to* $\prec$*, using the process given in Chapter 3. The calculation is somewhat involved and tedious, so we omit it here. We find*

$$G = \{x_1x_3x_4 - x_1x_2x_5,\ x_1x_2x_6 - x_1x_3x_5,\ x_1x_4x_5 - x_2x_3x_4,$$
$$x_2x_3x_5 - x_1x_4x_6,\ x_1x_3x_5x_6,\ x_2x_3x_4x_6\}$$

and thus

$$in_{\prec}(I) = \langle x_1x_3x_4,\ x_1x_2x_6,\ x_1x_4x_5,\ x_2x_3x_5,\ x_1x_3x_5x_6,\ x_2x_3x_4x_6 \rangle$$

It is easy to observe that none of the four monomials $x_1x_2x_5$, $x_1x_3x_5$, $x_2x_3x_4$, $x_1x_4x_6$ *are in* $in_{\prec}(I)$*. Therefore,* $in_{\prec}(I) \neq in_{\prec_c}(I)$ *for any coherent term order* $\prec_c$*, and so* $in_{\prec}(I)$ *is a noncoherent initial ideal.*

Given a noncoherent Boolean term order on $2^{[n]}$, we may construct a noncoherent intial ideal on the exterior algebra on m variables, where $m > n$, by use of lexicographic extensions. This technique can guarantee construction of a noncoherent initial ideal on $m = n + 4$ variables, and may allow for construction of a noncoherent initial ideal on $n + 1 \leq m \leq n + 3$ variables. The following example uses a noncoherent order on 5 variables to construct a noncoherent initial ideal in the exterior algebra on 6 variables.

Example 4.4. *We take the following Boolean term order $\prec_5$ on 5 elements:*

$\emptyset \prec 1 \prec 2 \prec 12 \prec 3 \prec 13 \prec_* 4 \prec 14 \prec_* 23 \prec 123 \prec 24 \prec 124 \prec 5 \prec 34 \prec_* 15 \prec 25 \prec_* 134 \prec 234 \prec 125 \prec 1234 \prec 35 \prec 135 \prec 45 \prec 145 \prec 235 \prec 1235 \prec 245 \prec 1245 \prec 345 \prec 1345 \prec 2345 \prec 12345$

This order contains the comparisons

$A_1' = 13 \prec 4 = B_1'$

$A_2' = 14 \prec 23 = B_2'$

$A_3' = 34 \prec 15 = B_3'$

$A_4' = 25 \prec 134 = B_4'$

The sum on the left hand side of these four comparisons is equal to the sum on the right hand side, so $(A_1', \ldots, A_4') =_0 (B_1', \ldots, B_4')$, and thus they comprise a failure of the cancellation condition C_4, which implies this order is noncoherent.

We now take the lexicographic extension of this Boolean term order, to form the following Boolean term order $\prec_6$:

$\emptyset \prec 1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 4 \prec 14 \prec 23 \prec \mathbf{123} \prec \mathbf{24} \prec 124 \prec 5 \prec 34 \prec$

15$\prec$ **25** $\prec$ **134** $\prec$ 234 $\prec$ 125 $\prec$ 1234 $\prec$ 35 $\prec$ 135 $\prec$ 45 $\prec$ 145 $\prec$ 235 $\prec$ 1235 $\prec$ 245 $\prec$ 1245 $\prec$ 345 $\prec$ 1345 $\prec$ 2345 $\prec$ 12345 $\prec$ 6 $\prec$ 16 $\prec$ 26 $\prec$ 126 $\prec$ 36 $\prec$ 136 $\prec$ 46$\prec$ **146** $\prec$ **236** $\prec$ 1236 $\prec$ 246 $\prec$ 1246 $\prec$ 56$\prec$ **346** $\prec$ **156** $\prec$ 256 $\prec$ 1346 $\prec$ 2346 $\prec$ 1256 $\prec$ 12346 $\prec$ 356 $\prec$ 1356 $\prec$ 456 $\prec$ 1456 $\prec$ 2356 $\prec$ 12356 $\prec$ 2456 $\prec$ 12456 $\prec$ 3456 $\prec$ 13456 $\prec$ 23456 $\prec$ 123456

The four bolded comparisons

$A_1 = 123 \prec 24 = B_1$

$A_2 = 25 \prec 134 = B_2$

$A_3 = 146 \prec 236 = B_3$

$A_4 = 346 \prec 156 = B_4$

are carefully chosen de Finetti consequences of the failure of C_4 for $\prec_5$. Therefore, any term order in which these above four comparisons appear (in particular $\prec_6$) must be noncoherent. Observe that we now have the property we desire, that is, none of the above eight subsets are contained in any of the other seven.

As before, we apply φ to the above Boolean term order $\prec_6$. This creates a term order $\prec$ on the monomials of the exterior algebra on six variables, containing the comparisons

$\varphi(A_1) = x_1x_2x_3 \prec x_2x_4 = \varphi(B_1)$

$\varphi(A_2) = x_2x_5 \prec x_1x_3x_4 = \varphi(B_2)$

$\varphi(A_3) = x_1x_4x_6 \prec x_2x_3x_6 = \varphi(B_3)$

$\varphi(A_4) = x_3x_4x_6 \prec x_1x_5x_6 = \varphi(B_4)$

We now take the ideal

$$I = \langle x_2x_4 - x_1x_2x_3,\ x_1x_3x_4 - x_2x_5,\ x_2x_3x_6 - x_1x_4x_6,\ x_1x_5x_6 - x_3x_4x_6\rangle$$

in the exterior algebra on six variables. As before, to show $in_{\prec}(I)$ *is a noncoherent initial ideal, it suffices to show that* $in_{\prec}(I)$ *does not contain any of the monomials* $x_1x_2x_3$, x_2x_5, $x_1x_4x_6$, $x_3x_4x_6$, *at least one of which must be in* $in_{\prec_c}(I)$ *for any coherent term order* $\prec_c$.

We now compute a Gröbner basis G *of* I *with respect to* $\prec$:

$$G = \{x_2x_4 - x_1x_2x_3,\ x_1x_3x_4 - x_2x_5,\ x_2x_3x_6 - x_1x_4x_6,$$
$$x_1x_5x_6 - x_3x_4x_6,\ x_1x_2x_5,\ x_2x_3x_5,\ x_2x_5x_6,\ x_3x_4x_5x_6\}$$

and thus

$$in_{\prec}(I) = \langle x_2x_4,\ x_1x_3x_4,\ x_2x_3x_6,\ x_1x_5x_6,\ x_1x_2x_5,\ x_2x_3x_5,\ x_2x_5x_6,\ x_3x_4x_5x_6\rangle.$$

It is easy to observe that none of the four monomials $x_1x_2x_3$, x_2x_5, $x_1x_4x_6$, $x_3x_4x_6$ *are in* $in_{\prec}(I)$. *Therefore* $in_{\prec}(I) \neq in_{\prec_c}(I)$ *for any coherent term order* $\prec_c$, *and so* $in_{\prec}(I)$ *is a noncoherent initial ideal.*

Chapter 5

Conclusions and Remaining Questions

The previous section demonstrates examples of noncoherent initial ideals in the exterior algebra on six variables, and methods to extend the construction to an arbitrary number of variables greater than six. It is known [8] that there are no noncoherent term orders on subsets of n-element sets where $n \leq 4$. This raises a question with regards to minimality of noncoherent initial ideals, specifically whether there exists a noncoherent initial ideal in the exterior algebra on 5 variables.

With the help of Fishburn's classification of all possible C_4 failures for comparative probability orderings on the subsets of a five-element set, we may demonstrate that our construction theorem is unable to produce an ideal in the exterior algebra on 5 variables which possesses a noncoherent intial ideal, as follows:

Theorem 5.1. **(Fishburn)** *Let the dual of a comparison $A \succ B$ be $B \succ A$ and the dual of a set of comparisons be the set of their duals. Let $n = 5$, with the elements ordered $1 \succ 2 \succ 3 \succ 4 \succ 5$. Then an order is noncoherent if and only if it satisfies one of I through V or the dual of one of I through V:*

I	II	III	IV	V
$245 \succ 13$	$235 \succ 14$	$234 \succ 15$	$235 \succ 14$	$234 \succ 15$
$15 \succ 24$	$15 \succ 23$	$1 \succ 235$	$1 \succ 345$	$1 \succ 245$
$34 \succ 25$	$34 \succ 35$	$25 \succ 34$	$34 \succ 25$	$25 \succ 34$
$2 \succ 45$	$2 \succ 35$	$35 \succ 2$	$45 \succ 3$	$45 \succ 2$

Theorem 5.2. *The construction theorem from Chapter 4 cannot produce an ideal in the exterior algebra on 5 variables which possesses a noncoherent initial ideal.*

Proof. First note that for each of the five C_4 failures in the table, none of them satisfies the pairwise noncontainment property we desire. Therefore, for each C_4 failure, we must take de Finetti consequences of at least one of the comparisons.

Observe all of the C_4 failures contain the central comparison of the order. The union of the two subsets comprising the central comparison is always the whole set $[n]$. This means that in the case where $n = 5$, any one-element subset must be contained in one of these two central subsets, and any four-element subset must contain one of these two central subsets. Therefore, to satisfy the pairwise non-containment property, all of the eight subsets comprising the failure of C_4 must contain either two or three elements.

The C_4 failures (III), (IV), and (V) all contain a comparison between a one-element subset and a three-element subset. As we cannot have a one-element subset, we must take a de Finetti consequence of this comparison, but this will turn the three-element subset into a four-element subset, which is also not allowed. Hence none of these C_4 failures can be used in the construction.

For (I), we can observe that the subset 24 in the second comparison is contained in the subset 245 in the first (central) comparison. Thus we must take a de Finetti consequence of the second comparison - our only option is to form $135 \prec 234$. But then 135 contains the subset 13 from the first comparison. This comparison already contains all five elements, so we cannot take a de Finetti consequence of it without introducing a sixth element. Thus (I) cannot be used. The case for (II) is similar.

□

Therefore, if a noncoherent initial ideal exists in the exterior algebra on five variables, a different method must be used to construct it.

Another question raised by the existence of noncoherent initial ideals is how a *Gröbner fan* may be defined for an ideal I in the exterior algebra which possesses a noncoherent initial ideal $in_{\prec}(I)$, as $in_{\prec}(I) \neq in_w(I)$ for any weight vector w. For more information on Gröbner fans, see Chapter 14 of [16] or Chapter 2 of [15].

There also remain some interesting questions relating specifically to Boolean term orders. We reproduce two of these here.

- Can we find improved bounds on both the number of coherent orders and the total number of orders?
- Does the ratio of the number of coherent term orders to the total number of term orders tend to zero as n increases?

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